Some topological properties of monotone complexity one spaces

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Based on:

“*On topological properties of positive complexity one spaces*"
(with D. Sepe), Transformation Groups

and

“*Tall and monotone complexity one spaces of dimension six*"
(with I. Charton and D. Sepe), in preparation.

Positive monotone vs. Fano

(M, ω): compact symplectic manifold of dimension $2n$

J: almost complex structure compatible with ω
($\omega(\cdot, J\cdot)$ is a Riemannian metric)

c_1: first Chern class of (TM, J) $\rightsquigarrow$ (TM, ω)

Definition

A symplectic manifold (M, ω) is called **(positive) monotone** if

$$c_1 = \lambda[\omega] \quad (\text{with } \lambda > 0)$$

Henceforth consider *positive monotone symplectic manifolds*

Positive monotone vs. Fano

Positive monotone symplectic manifolds $\leftsquigarrow$ Fano varieties:

Fano variety: smooth complex variety Y s.t.
the anticanonical line bundle $\mathcal{L} = -K_Y$ (where $K_Y = \det(T^*M)$) is *ample*:

$$\exists \quad j\colon Y \to \mathbb{C}P^N \quad \text{and} \quad k >> 0 \text{ s.t. } \mathcal{L}^k = j^*\mathcal{O}(1)$$

Endow Y with $j^*(\omega_{FS}) \rightsquigarrow Y$ is *positive monotone*

Facts:

Fano varieties are simply connected and their Todd genus Td is 1.

(Example: $\dim_{\mathbb{C}}(Y) = 1 \implies Td(Y) = \frac{c_1}{2}[Y]$,
$\dim_{\mathbb{C}}(Y) = 2 \implies Td(Y) = \frac{c_1^2+c_2}{12}[Y]$,
$\dim_{\mathbb{C}}(Y) = 3 \implies Td(Y) = \frac{c_1c_2}{24}[Y]$)

Positive monotone vs. Fano

When is a positive monotone symplectic manifold (M, ω) diffeomorphic to a Fano variety?

- $\dim(M) = 2, 4$: always (McDuff, Gromov, Taubes)
- $\dim(M) \geq 12$: not always (Fine–Panov, Reznikov)

What if one assumes that (M, ω) has symmetries?

Symmetries

(M, ω): compact symplectic manifold of dimension $2n$
T: compact torus of dimension d

Assume $T \circlearrowright M$ is *Hamiltonian*:

$\exists\ \psi\colon M \to Lie(T)^*$ (*moment map*) s.t.

- ψ is T-invariant
- $\forall \xi \in Lie(T)$

$$d\langle \psi, \xi \rangle = -\iota_{X_\xi}\omega$$

Definition:

- *Hamiltonian T-space*: (M, ω, T, ψ), where the action is effective
- *complexity of* (M, ω, T, ψ): $\dim(M)/2 - \dim(T)$

Note: complexity is ≥ 0

Conjecture (Fine, Panov 2010)

Every positive monotone Hamiltonian S^1-space of dimension 6 is diffeomorphic to a Fano threefold

Theorem (Lindsay, Panov 2019)

Every positive monotone Hamiltonian S^1-space of dimension 6 is *simply connected* and has *Todd genus 1*

Results

Theorem (S., Sepe 2020)

If (M, ω, T, ψ) is a positive monotone complexity one space then M is simply connected, its Todd genus is 1 and its odd Betti numbers vanish.

Specialization to low dimensions (I. Charton, D. Sepe):

- $\dim(\mathbf{M}) = \mathbf{4}$, $\dim(\mathbf{T}) = \mathbf{1}$:
  the circle action extends to a T^2 action and (M, ω, T, ψ) is S^1-equivariantly symplectomorphic to a Fano two-fold with holomorphic $\mathbb{C}^*$-action
- $\dim(\mathbf{M}) = \mathbf{6}$, $\dim(\mathbf{T}) = \mathbf{2}$: if (M, ω, T, ψ) is *tall*
  the T^2 action extends to a T^3 action and (M, ω, T, ψ) is T^2-equivariantly symplectomorphic to a Fano three-fold with holomorphic $(\mathbb{C}^*)^2$-action. Moreover there are 20 such examples.

Proof of:

Theorem (S., Sepe 2020)

If (M, ω, T, ψ) is a positive monotone complexity one space then M is simply connected, its Todd genus is 1 and its odd Betti numbers vanish.

Proof of simple connectedness

Consequence of

(a) Theorem (Li)

Let (M, ω, T, ψ) be a compact Hamiltonian T-space. For any $\alpha \in \psi(M)$, $\pi_1(M) \simeq \pi_1(M_\alpha)$, where $M_\alpha = \psi^{-1}(\alpha)/T$ is the reduced space at α.

and

(b) Theorem (S., Sepe)

Let (M, ω, T, ψ) be a positive monotone complexity one space. Then the connected components of the fixed point set M^T are either points or spheres.

Proof of simple connectedness

How do (a) and (b) imply that $\pi_1(M)$ is trivial?

v is a vertex of $\psi(M) \implies \psi^{-1}(v)$ connected component of M^T.

Consider $M_v = \psi^{-1}(v)/T = \psi^{-1}(v)$.

$$\pi_1(M) \underset{(a)}{=} \pi_1(M_v) = \pi_1(\psi^{-1}(v)) \underset{(b)}{=} \begin{cases} \pi_1(pt) \\ \pi_1(S^2) \end{cases}$$

Proof of (b)

Observations:

- **(Local normal form – weights of the T action)**
  Around $p \in M^T$ there exist complex coordinates $z_1, \ldots, z_n$ on M and $\alpha_1, \ldots, \alpha_n \in \ell^* \subset Lie(T)^*$ s.t.

  $$T \ni \exp(\xi) * (z_1, \ldots, z_n) = (e^{2\pi i \alpha_1(\xi)} z_1, \ldots, e^{2\pi i \alpha_n(\xi)} z_n)$$

  and

  $$\psi_{lin}(z_1, \ldots, z_n) = \frac{1}{2} \sum_{j=1}^{n} \alpha_j |z_j|^2 + \psi(p)$$

- $C :=$ connected component of M^T

  $$\dim(C) \leq 2 * \text{complexity}$$

  (comes from $\mathrm{rank}_{\mathbb{C}}(N_C) \geq \dim(T)$ and effectiveness of the action). If complexity is 1, C is a point or a surface.

- If $\dim(C) = 2 * \text{complexity} \implies \psi(C)$ is a vertex of $\psi(M)$ (moment map is open onto its image)

Proof of (b)

Observations:

- If $\exists v$ vertex of $\psi(M)$ s.t. $\psi^{-1}(v)$ is a point $\overset{(a)}{\Longrightarrow}$ simple connectedness.
  Assume $\psi^{-1}(v)$ is a surface, for all vertices v of $\psi(M)$.

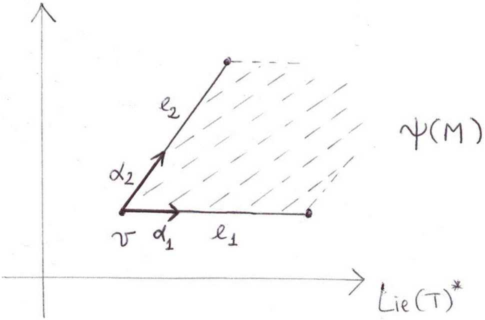
Duistermaat-Heckman density function $DH\colon \psi(M) \to \mathbb{R}$:

$$DH(\alpha) := \text{symplectic volume of } M_\alpha$$

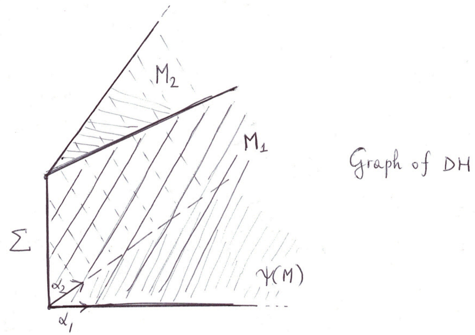
- DH attains its minimum *min* at a vertex v of $\psi(M)$
  (Cho, Kim $\Longrightarrow$ $\log(DH)$ is concave, $\psi(M)$ convex)

Proof of (b)

$\Sigma := \psi^{-1}(\min)$,
$\alpha_1, \ldots, \alpha_{n-1}$: weights of the T action on the normal bundle N_Σ
$e_1, \ldots, e_{n-1}$: corresponding edges in $\psi(M)$



- N_Σ splits as direct sum of line bundles $N_1 \oplus \cdots \oplus N_{n-1}$, T acts on N_i with weight α_i.
- $M_i := \psi^{-1}(e_i)$: compact symplectic 4-dimensional submanifold with a Hamiltonian S^1 action, $\Sigma \subset M_i$, for all $i = 1, \ldots, n-1$
- Normal bundle to Σ in M_i is N_i



Proof of (b)

- $DH\colon \psi(M_i) = [v, v'] \to \mathbb{R}$ restricted to $[v, v+\epsilon)$ is:

$$DH(x) = \int_\Sigma \omega - c_1(N_i)[\Sigma](x - v)$$

- DH attains its minimum at $v \implies$

$$c_1(N_i)[\Sigma] \leq 0 \quad \forall i = 1, \dots, n$$

Proof of (b)

- $c_1 = [\omega] \implies c_1[\Sigma] > 0$

$$\underbrace{c_1[\Sigma]}_{>0} = \underbrace{\sum_{i=1}^{n-1} c_1(N_i)[\Sigma]}_{\leq 0} + c_1(\Sigma)[\Sigma]$$

$\implies c_1(\Sigma)[\Sigma] > 0$, namely $\Sigma = S^2$.

Todd genus = 1

Hirzebruch genus: genus χ_y associated to the generating function

$$\frac{x(1+ye^{-x})}{1-e^{-x}}$$

Todd genus: Evaluation of χ_y at $y=0$.

- If S^1 acts on $M \implies$ "Localization of the Hirzebruch genus":

$$\chi_y(M) = \sum_{j=1}^{N} (-y)^{d_j} \chi_y(F_j)$$

  where: $F_1, \ldots, F_N$ connected components of M^{S^1}
  d_j number of negative weights in the normal bundle to F_j
- If action is Hamiltonian $\implies \exists j : F_j$ is minimum of the moment map and $d_j = 0$

Todd genus = 1

- Consider $S^1 \subset T$ s.t. $M^{S^1} = M^T$;
  Theorem (b) $\implies$ minimum F of the S^1 moment map is either a point or a sphere hence

$$Td(M) = \chi_0(M) = \chi_0(F) = 1.$$

Odd Betti numbers vanish

It follows from

$$H^*(M;\mathbb{R}) = \bigoplus_{j=1}^{N} H^{*-2d_j}(F_j;\mathbb{R})$$

□

Thank you!