Some examples of algebraic Poisson and symplectic groupoids

Jiang-Hua Lu

The University of Hong Kong

Global Poisson, September 24, 2020

(based on joint work with Victor Mouquin and Shizhuo Yu)

A class of polynomial Poisson algebras

Warm-up: systematical examples of polynomial Poisson brackets:

- Constant Poisson brackets on $\mathbb{C}[x_1, x_2, \ldots, x_l]$:

$$\{x_i, x_j\} = c_{ij}, \qquad c_{ij} \in \mathbb{C};$$

- Linear Poisson brackets:

$$\{x_i, x_j\} = \sum_k c_{ij}^{(k)} x_k, \qquad c_{ij}^{(k)} \in \mathbb{C};$$

- (Quadratic) log-canonical Poisson brackets:

$$\{x_i, x_j\} = c_{ij} x_i x_j, \qquad c_{ij} \in \mathbb{C};$$

- Higher order? Hard to make up in general.

A class of Poisson polynomial algebras

Theorem (Elek-L. 2012, 2019)

Every sequence of simple reflections

$$\mathbf{u} = (s_1, s_2, \ldots, s_l)$$

in a Weyl group gives rise to a polynomial Poisson structure on $\mathbb{C}^l$.

1. Each $\mathbb{C}[x_i, \ldots, x_j]$ for $1 \le i \le j \le l$ is a Poisson sub-algebra:
$$\{x_i, x_j\} = \underbrace{c_{ij} x_i x_j}_{\text{log-canonical terms}} + f_{ij}(x_{i+1}, \ldots, x_{j-1}).$$
2. Has a natural torus T-action by Poisson automorphisms.
3. Has finitely many T-leaves and thus also an open T-leaf.
4. Can be computed by a computer program (by Balazs Elek).

Example 1: Type G_2 with simple generators s_1, s_2: the sequence $\mathbf{u} = (s_1, s_2, s_1, s_2, s_1, s_2)$ gives a Poisson bracket on $\mathbb{C}[x_1, \ldots, x_6]$:

$$
\begin{aligned}
&\{x_1, x_2\} = -3x_1x_2, \quad \{x_1, x_3\} = -x_1x_3 - 2x_2,\\
&\{x_1, x_4\} = 0 - 6x_3^2, \quad \{x_1, x_5\} = x_1x_5 - 4x_3,\\
&\{x_1, x_6\} = 3x_1x_6 - 6x_5, \quad \{x_2, x_3\} = -3x_2x_3,\\
&\{x_2, x_4\} = -3x_2x_4 - 6x_3^3, \quad \{x_2, x_5\} = 0 - 6x_3^2,\\
&\{x_2, x_6\} = 3x_2x_6 - 18x_3x_5 + 6x_4,\\
&\{x_3, x_4\} = -3x_3x_4, \quad \{x_3, x_5\} = -x_3x_5 - 2x_4,\\
&\{x_3, x_6\} = 0 - 6x_5^2, \quad \{x_4, x_5\} = -3x_4x_5,\\
&\{x_4, x_6\} = -3x_4x_6 - 6x_5^3, \quad \{x_5, x_6\} = -3x_5x_6.
\end{aligned}
$$

Terms in red are the log-canonical terms.

Example 2: G_2 again, with $\mathbf{u} = (s_1, s_2, s_1, s_2, s_1, s_2, s_2, s_1, s_2, s_1, s_2, s_1)$, give a Poisson bracket on $\mathbb{C}[x_1, \ldots, x_{12}]$. Some examples:

$$\{x_1, x_8\} = -3x_1x_8 - 6x_7^3, \quad \{x_1, x_9\} = 0 - 6x_7^2,$$
$$\{x_1, x_{10}\} = 3x_1x_{10} - 18x_7x_9 + 6x_8, \quad \{x_1, x_{11}\} = 3x_1x_{11} - 6x_7,$$
$$\{x_1, x_{12}\} = 6x_1x_{12} - 6, \quad \{x_2, x_8\} = 0 - 6x_6x_7^3 + 6x_7^2,$$
$$\{x_2, x_9\} = x_2x_9 - 6x_6x_7^2 + 4x_7,$$
$$\{x_2, x_{10}\} = 3x_2x_{10} - 18x_6x_7x_9 + 6x_6x_8 + 6x_9,$$
$$\{x_2, x_{11}\} = 2x_2x_{11} - 6x_6x_7 + 2, \quad \{x_2, x_{12}\} = 3x_2x_{12} - 6x_6,$$
$$\{x_3, x_8\} = 3x_3x_8 - 18x_4x_6x_7^3 + 6x_5x_7^3 + 18x_4x_7^2,$$
$$\{x_3, x_9\} = 3x_3x_9 - 18x_4x_6x_7^2 + 6x_5x_7^2 + 12x_4x_7,$$
$$\{x_3, x_{10}\} = 6x_3x_{10} - 54x_4x_6x_7x_9 + 18x_4x_6x_8 + 18x_5x_7x_9 + 18x_4x_9 - 6x_5x_8 - 6,$$

$\exists$ open symplectic leaf given by $\phi_1(x_1, \ldots, x_{12}) \neq 0$, $\phi_2(x_1, \ldots, x_{12}) \neq 0$ for some $\phi_1, \phi_2 \in \mathbb{C}[x_1, \ldots, x_{12}]$ (with Lie-theoretical meaning).

Today's Talk:

- Some background on these polynomial Poisson structures;
- Construct algebraic symplectic groupoids for them.

Part I: Some basic concepts;

Part II: A general construction;

Part III: Specializing to one example;

Part IV: In the (grand) scheme of things ......

1. T-Leaves.

- If T is a torus, a T-Poisson manifold is a Poisson manifold (X, π_X) with a T-action preserving π_X.

- A T-orbit of symplectic leaves, or a T-leaf for short, of a T-Poisson manifold (X, π_X) is a sub-manifold L of X of the form

$$L = \bigcup_{t \in T} t\Sigma,$$

  where Σ is a symplectic leaf of (X, π_X), and the map

$$T \times \Sigma \longrightarrow L, \ \ (t, x) \longmapsto tx$$

  is a submersion.

2. Mixed Product Poisson Structures.

Let $Y = Y_1 \times \cdots \times Y_n$ with projections

$$p_i: \quad Y \longrightarrow Y_i, \qquad i \in [1, n].$$

A Poisson structure π on Y is called a mixed product if

- $p_i(\pi)$ is a well-defined Poisson structure on Y_i for each i;
- Equivalently, if each $p_i^* C^\infty(Y_i) \subset C^\infty(Y)$ is a Poisson sub-algebra.

In coordinates for $X \times Y$,

$$\pi = \sum_{i<j} \alpha_{ij}(x) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} + \sum_{k<l} \beta_{kl}(y) \frac{\partial}{\partial y_k} \wedge \frac{\partial}{\partial y_l} + \sum_{i,k} \gamma_{ik}(x, y) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_k}.$$

3. Poisson groupoids and symplectic groupoids.

- A Poisson groupoid is a Lie groupoid $\theta_\pm : \Gamma \rightrightarrows M$ with a Poisson structure π such that the graph of the groupoid multiplication

$$\{(\gamma, \gamma', \gamma\gamma') : \theta_-(\gamma) = \theta_+(\gamma')\} \subset \Gamma \times \Gamma \times \Gamma$$

  is a coisotropic sub-manifold of $(\Gamma \times \Gamma \times \Gamma, \pi \times \pi \times (-\pi))$. In such a case, $\pi_M := \theta_+(\pi) = -\theta_-(\pi)$ is a well-defined Poisson structure on M. Denote the Poisson groupoid as $(\Gamma, \pi) \rightrightarrows (M, \pi_M)$.

- A symplectic groupoid is a Poisson groupoid $(\Gamma, \pi) \rightrightarrows (M, \pi_M)$, where π is non-degenerate and $\dim \Gamma = 2 \dim M$;

- Basic problem in Poisson geometry: To construct interesting Poisson groupoids and symplectic groupoids.

Examples of Poisson groupoids:

1. Pair Poisson groupoid $(X \times X, \pi_X \times (-\pi_X)) \rightrightarrows (X, \pi_X)$;
2. A Poisson Lie group is a Poisson groupoid over a one point space;

Can combine these two examples to construct gauge Poisson groupoids:

- A Poisson action of a Poisson Lie group (G, π_G) is a Poisson map $(X \times G,\ \pi_X \times \pi_G) \to (X,\ \pi_X)$ that is also a right Lie group action. In such a case, have quotient Poisson manifolds

$$(X/Q,\ \pi_{X/Q}) \qquad \text{(assuming that } X/Q \text{ is smooth)},$$

where Q is a coisotropic subgroup of (G, π_G), and $\pi_{X/Q}$ is the projection of π_X to X/Q.

A General Construction, I: Gauge Poisson Groupoids

Given free proper Poisson Lie group action of (G,π_G) on (X,π_X), have

- $(X/G, \pi_{X/G})$, quotient of (X, π_X) by G;
- $((X \times X)/G, \pi)$, quotient of $\pi_X \times (-\pi_X)$ by diagonal G-action;
- $(X \times X)/G \rightrightarrows X/G$, gauge groupoid with

$$\begin{aligned}
&\text{source}: \quad (X \times X)/G \to X/G: \quad [x_1, x_2] \mapsto [x_1],\\
&\text{target}: \quad (X \times X)/G \to X/G: \quad [x_1, x_2] \mapsto [x_2],\\
&\text{unit}: \quad X/G \to (X \times X)/G, \quad [x] \mapsto [x, x],\\
&\text{inverse}: \quad (X \times X)/G \to (X \times X)/G: \quad [x_1, x_2] \mapsto [x_2, x_1],\\
&\text{multiplication}: \ [x_1, x_2] \cdot [x_3, x_4] = [x_1 g, x_4], \ \text{where } x_2 g = x_3.
\end{aligned}$$

Lemma (Reference?)

As above, $((X \times X)/G, \pi) \rightrightarrows (X/G, \pi_{X/G})$ is a Poisson groupoid.

A General Construction, II: Series of Quotient Poisson Manifolds:

- Given closed Poisson Lie sub-group $Q \subset (G, \pi_G)$ and Poisson action

$$(Q, \pi_G) \times (Y, \pi_Y) \longrightarrow (Y, \pi_Y), \ (q, y) \mapsto qy,$$

projection of $\pi_G \times \pi_Y$ is a well-defined Poisson structure on

$$G \times_Q Y \stackrel{\text{def}}{=} (G \times Y)/Q, \ \ (g, y)q := (xq, \ q^{-1}y).$$

- Given closed Poisson Lie sub-groups $Q_j \subset (G, \pi_G)$, $j \in [1, n]$, have successive quotients G/Q_n, $G \times_{Q_{n-1}} (G/Q_n)$, $\cdots$, $\cdots$, $\cdots$,

$$Z_n = G \times_{Q_1} \times \cdots \times G \times_{Q_{n-1}} G/Q_n,$$

the quotient of G^n by $Q_1 \times \cdots \times Q_{n-1} \times Q_n$ by action

$$(g_1, g_2, \ldots, g_n) \cdot (q_1, q_2, \ldots, q_n) = (g_1 q_1, q_1^{-1} g_2 q_2, \ldots, q_{n-1}^{-1} g_n q_n),$$

with quotient Poisson structure π_{Z_n}, projection of $(\pi_G)^n$ on G^n.

Combining the two constructions: (Summary of Part II)

A closed Poisson Lie sub-group $Q \subset (G, \pi_G)$ gives a series: for $n \geq 1$,

$$X_n = \overbrace{G \times_Q \cdots \times_Q G}^{n}, \qquad Y_n = \overbrace{G \times_Q \cdots \times_Q G}^{n}/Q = X_n/Q, \qquad \text{and}$$

$$X_{2n} = \overbrace{G \times_Q \cdots \times_Q G}^{2n} = \overbrace{G \times_Q \cdots \times_Q G}^{n} \times_Q \overbrace{G \times_Q \cdots \times_Q G}^{n} = X_n \times_Q X_n.$$

Theorem (L.-Mouquin-Yu 2020)

For any $n \geq 1$, $(X_{2n}, \pi_{X_{2n}}) \rightrightarrows (Y_n, \pi_{Y_n})$ is a Poisson groupoid with

- *source* $\theta_+ : X_{2n} \to Y_n$, $[g_1, \ldots, g_{2n}]_{X_{2n}} \mapsto [g_1, \ldots, g_n]_{Y_n}$;
- *target* $\theta_- : X_{2n} \to Y_n$, $[g_1, \ldots, g_{2n}]_{X_{2n}} \mapsto [g_{2n}^{-1}, \ldots, g_{n+1}^{-1}]_{Y_n}$;
- *unit* : $Y_n \to X_{2n}$, $[g_1, \ldots, g_n]_{Y_n} \mapsto [g_1, \ldots, g_n, g_n^{-1}, \ldots, g_1^{-1}]_{X_{2n}}$;
- *inverse* : $X_{2n} \to X_{2n}$, $[g_1, \ldots, g_{2n}]_{X_{2n}} \mapsto [g_{2n}^{-1}, \ldots, g_1^{-1}]_{X_{2n}}$;
- *product: for* $\gamma = [g_1, \ldots, g_{2n}]_{X_{2n}}$ *and* $\gamma' = [g_1', \ldots, g_{2n}']_{X_{2n}}$ *composable,*

$$\gamma\gamma' = [g_1, \ \ldots, \ g_n, \ g_{n+1} \cdots g_{2n} g_1' \cdots g_n' g_{n+1}', \ g_{n+2}', \ldots, g_{2n}']_{X_{2n}}.$$

Fix

- G: connected complex semi-simple Lie group with $\mathfrak{g} = \mathrm{Lie}(G)$;
- (B, B_-): pair of opposition Borel sub-groups of G;
- $T = B \cap B_-$: maximal torus of G;
- $W = N_G(T)$: Weyl group of (G, T);
- $\langle\, ,\, \rangle_{\mathfrak{g}}$: invariant symmetric non-degenerate blinear form on $\mathfrak{g}$.

Well-known fact: Above data give rise to standard classical r-matrix $r_{\mathrm{st}} \in \mathfrak{g} \otimes \mathfrak{g}$ and Poisson Lie group (G, π_{st}), where

$$\pi_{\mathrm{st}} = r_{\mathrm{st}}^L - r_{\mathrm{st}}^R.$$

Both B and B_- are Poisson Lie sub-groups of (G, π_{st}).

Now apply the general construction in Part II to the pair $B \subset (G, \pi_{\mathrm{st}})$.

Part III: Specializing to an example

Apply Part II to $B \subset (G, \pi_{\mathrm{st}})$: For any integer $n \geq 1$, have

$$F_n = \overbrace{G \times_B \cdots \times_B G}^{n} / B \quad \text{and} \quad \widetilde{F}_{2n} = \overbrace{G \times_B \cdots \times_B G}^{2n}$$

with respective quotient Poisson structures π_n and $\widetilde{\pi}_{2n}$. Introduce

$$\Gamma_{2n} = \{[g_1, g_2, \ldots, g_{2n}]_{\widetilde{F}_{2n}} : g_1 g_2 \cdots g_{2n} \in B_-\} \subset \widetilde{F}_{2n}.$$

Theorem (L. -Mouquin-Yu, 2020)

For each $n \geq 1$, $(\widetilde{F}_{2n}, \widetilde{\pi}_{2n})$ is a Poisson groupoid over (F_n, π_n) with

$$\begin{aligned}
&\textit{source}\ \theta_+ : \widetilde{F}_{2n} \to F_n,\ [g_1, \ldots, g_{2n}]_{\widetilde{F}_{2n}} \mapsto [g_1, \ldots, g_n]_{F_n};\\
&\textit{target}\ \theta_- : \widetilde{F}_{2n} \to F_n,\ [g_1, \ldots, g_{2n}]_{\widetilde{F}_{2n}} \mapsto [g_{2n}^{-1}, \ldots, g_{n+1}^{-1}]_{F_n};\\
&\textit{unit}, \quad \textit{inverse}, \quad \textit{multiplication}, \cdots, \cdots, \cdots.
\end{aligned}$$

Furthermore, Γ_{2n} is a wide Poisson sub-groupoid of $(\widetilde{F}_{2n}, \widetilde{\pi}_{2n}) \rightrightarrows (F_n, \pi_n)$.

Today's main objects of interests: $(\Gamma_{2n}, \widetilde{\pi}_{2n}) \rightrightarrows (F_n, \pi_n)$.

A name for $(\Gamma_{2n}, \widetilde{\pi}_{2n}) \rightrightarrows (F_n, \pi_n)$: Let $N \subset B$ be the unipotent radical.

- $\mathcal{B} = F_1 = G/B$: flag variety of G, with Poisson structure π_1;
- $\mathcal{A} := G/N$: decorated flag variety of G, with Poisson structure $\pi_1' := \pi_{G/N}$. Let $\mathcal{A}^o = B_- N/N \subset \mathcal{A}$ (open).
- As varieties, have isomorphism $\widehat{\Theta}_{2n} : \Gamma_{2n} \cong \mathcal{C}_{2n} := \mathcal{B}^{2n-1} \times \mathcal{A}^o$:

  $$\widehat{\Theta}_{2n}([g_1, g_2, \ldots, g_{2n}]_{\widetilde{F}_{2n}}) = (g_1.B, \ \ldots, g_1 \cdots g_{2n-1}.B, \ g_1 g_2 \cdots g_{2n}.N).$$

  Similarly, have isomorphism $\Theta_n : F_n \cong \mathcal{B}^n$.
- Define $\overline{\pi}_n = \Theta_n(\pi_n) \in \mathfrak{X}^2(\mathcal{B}^n)$ and $\widehat{\pi}_{2n} = \widehat{\Theta}_{2n}(\widetilde{\pi}_{2n}) \in \mathfrak{X}^2(\mathcal{C}_{2n})$.
- Call $(\Gamma_{2n}, \widetilde{\pi}_{2n}) \rightrightarrows (F_n, \pi_n)$ or $(\mathcal{C}_{2n}, \widehat{\pi}_{2n}) \rightrightarrows (\mathcal{B}^n, \overline{\pi}_n)$ the n'th configuration Poisson groupoid of flags of G.

Theorem (L.-Mouquin, 2016)

1. $\overline{\pi}_n$ *is a mixed product of n-copies of* π_1.
2. $\widehat{\pi}_{2n}$ *is a mixed product of* $(2n-1)$*-copies of* π_1 *and one* π_1'.

Part III: Specializing to an example

A decomposition of (F_n, π_n): For $\mathbf{u} = (u_1, \ldots, u_n) \in W^n$, let

$$\mathcal{O}^{\mathbf{u}} = (Bu_1B) \times_B \cdots \times_B (Bu_nB)/B \subset F_n.$$

Bruhat decomposition $G = \bigsqcup_{u \in W} BuB$ gives the disjoint union

$$F_n = \bigsqcup_{\mathbf{u} \in W^n} \mathcal{O}^{\mathbf{u}} \qquad \text{(of T-Poisson sub-manifolds).}$$

Lemma

Using root sub-groups and reduced words of each u_i, one has

$$\mathcal{O}^{\mathbf{u}} \; \cong \; \mathbb{C}^{l(\mathbf{u})}, \qquad l(\mathbf{u}) = l(u_1) + \cdots + l(u_n),$$

resulting in a polynomial Poisson structure $\pi_{\mathbf{u}}$ on $\mathbb{C}^{l(\mathbf{u})}$.

- $\mathcal{O}^{\mathbf{u}} \subset F_n$ is called a generalized Schubert (or Bruhat) cell;
- $\pi_n := \pi_n|_{\mathcal{O}^{\mathbf{u}}}$ is called the standard Poisson structure on $\mathcal{O}^{\mathbf{u}}$;
- $\mathcal{O}^{\mathbf{u}} \cong \mathbb{C}^{l(\mathbf{u})}$ gives Bott-Samelson coordinates on $\mathcal{O}^{\mathbf{u}}$;
- Earlier G_2 examples: $\mathbf{u} = (s_1, s_2, s_1, s_2, s_1, s_2)$ or $(\mathbf{u}, \mathbf{u}^{-1})$.

Part III: Specializing to an example

Goal of talk: Construct algebraic symplectic groupoids of $(\mathcal{O}^{\mathbf{u}}, \pi_n)$.

Some history:

- For $n = 1$, $\mathcal{O}^u = BuB/B \subset G/B$. L.-Mouquin (2018) proved that

  $$G^{u,u} = BuB \cap B_- uB_- \subset (G, \pi_{\mathrm{st}}) \quad \text{(double Bruhat cell)}$$

  is a Poisson groupoid over $(\mathcal{O}^u, \pi_1)$ and symplectic leaves of $G^{u,u}$ are symplectic groupoids over $(\mathcal{O}^u, \pi_1)$ (using Kogan-Zelevinsky);

- For arbitrary $\mathbf{u} \in W^n$, Mouquin (2019) showed that the generalized double Bruhat cell $G^{\mathbf{u},\mathbf{u}}$ is a Poisson groupoid (in many ways) over $(\mathcal{O}^{\mathbf{u}}, \pi_n)$. No statement on symplectic leaves.

- General phenomenon: For Poisson structures related to (G, π_{st}), T-leaves are easier to determine (and often finitely many), but symplectic leaves are trickier.

Main result for today's talk (Summary of Part III):

Recall the configuration Poisson groupoid $\theta_\pm : (\Gamma_{2n}, \widetilde{\pi}_{2n}) \rightrightarrows (F_n, \pi_n)$.
Recall $F_n = \bigsqcup_{\mathbf{u} \in W^n} \mathcal{O}^{\mathbf{u}}$. For $\mathbf{u} \in W^n$, have full sub-groupoid over $\mathcal{O}^{\mathbf{u}}$

$$\Gamma^{\mathbf{u}} = \theta_+^{-1}(\mathcal{O}^{\mathbf{u}}) \cap \theta_-^{-1}(\mathcal{O}^{\mathbf{u}}) \rightrightarrows \mathcal{O}^{\mathbf{u}}.$$

Fact: $\Gamma^{\mathbf{u}}$ is a single T-leaf of $(\Gamma_{2n}, \widetilde{\pi}_{2n})$.

Theorem (L.-Mouquin-Yu, 2020)

- *For each $\mathbf{u} \in W^n$, $(\Gamma^{\mathbf{u}}, \widetilde{\pi}_{2n}) \rightrightarrows (\mathcal{O}^{\mathbf{u}}, \pi_n)$ is a Poisson groupoid;*
- *the symplectic leaf $\Sigma^{\mathbf{u}}$ of $\widetilde{\pi}_{2n}$ through the identity section $\mathcal{O}^{\mathbf{u}} \hookrightarrow \Gamma^{\mathbf{u}}$ is a symplectic groupoid $(\Sigma^{\mathbf{u}}, \widetilde{\pi}_{2n}) \rightrightarrows (\mathcal{O}^{\mathbf{u}}, \pi_n)$;*
- *isomorphisms $G^{\mathbf{u},\mathbf{u}} \cong \Gamma^{\mathbf{u}}$ give Mouquin's Poisson groupoids $G^{\mathbf{u},\mathbf{u}} \rightrightarrows \mathcal{O}^{\mathbf{u}}$.*

Part IV: In the (grand) scheme of things

Why are the $(\mathcal{O}^{\mathbf{u}}, \pi_n)$'s interesting? Why their symplectic groupoids?

1. Generalized Schubert cells as basic building blocks

- The decomposition

$$(F_n, \pi_n) = \bigsqcup_{\mathbf{u} \in W^n} (\mathcal{O}^{\mathbf{u}}, \pi_n)$$

gives a paving of (F_n, π_n) by generalized Schubert cells.

Theorem (L.-Yu, 2019)

Many Poisson homogeneous spaces of $(G, \pi_{\rm st})$, including

$$(G, \pi_{\rm st}), \quad (G/B, \pi_{G/B}), \quad (G/T, \pi_{G/T}),$$

have open covers by (T-mixed products of) generalized Schubert cells.

2. A prominent feature of $(\mathcal{O}^{\mathbf{u}}, \pi_n), (\Sigma^{\mathbf{u}}, \widetilde{\pi}_{2n}) \subset (\Gamma^{\mathbf{u}}, \widetilde{\pi}_{2n})$:

- A symmetric T-Poisson CGL, defined by Goodearl-Yakimov, is a polynomial Poisson algebra $(A = \mathbb{C}[x_1, x_2, \ldots, x_l], \{\,,\,\})$ with a T-action by Poisson algebra automorphisms, such that

  1. each x_i is a weight vector with weight $\lambda_i \in \mathrm{Hom}_{\mathbb{C}}(T, \mathbb{C}^\times)$;
  2. $\exists\, h_1, \ldots, h_l \in \mathrm{Lie}(T)$ s.t. $\lambda_i(h_i) \neq 0$ for all $i \in [1, l]$, and

$$\{x_i, x_j\} \in \lambda_i(h_j) x_i x_j + \mathbb{C}[x_{i+1}, \ldots, x_{j-1}], \quad 1 \leq i < j \leq l.$$

- Can extend the Poisson structure π from $\mathbb{C}^l$ to $\pi \bowtie 0$ on $\mathbb{C}^l \times T$ via

$$\{x_i, \xi_j\} = c_{ij} x_i \xi_j, \quad \{\xi_j, \xi_k\} = 0, \qquad i \in [1, l],\ j, k \in [1, \dim T].$$

Theorem (Elek-L. 2012, 2019)

$(\mathcal{O}^{\mathbf{u}}, \pi_n)$ is a symmetric T-Poisson CGL in Bott-Samelson coordinates.

- For $\mathbf{u} = (u_1, \ldots, u_n) \in W^n$, define $\mathbf{u}^{-1} = (u_n^{-1}, \ldots, u_1^{-1})$, and

$$\mathcal{O}_e^{(\mathbf{u},\mathbf{u}^{-1})} = \{[g_1, g_2, \ldots, g_{2n}]_{F_{2n}} \in \mathcal{O}^{(\mathbf{u},\mathbf{u}^{-1})} : g_1 g_2 \cdots g_{2n} \in B_- B\}.$$

Lemma

For any $\mathbf{u} \in W^n$, $\mathcal{O}_e^{(\mathbf{u},\mathbf{u}^{-1})}$ is an open symplectic leaf of $(\mathcal{O}^{(\mathbf{u},\mathbf{u}^{-1})}, \pi_{2n})$.

Theorem (L.-Mouquin-Yu, 2020)

For any $\mathbf{u} = (u_1, \ldots, u_n) \in W^n$, have Poisson isomorphism

$$I: \ (\Gamma^{\mathbf{u}}, \widetilde{\pi}_{2n}) \cong (\mathcal{O}_e^{(\mathbf{u},\mathbf{u}^{-1})} \times T, \pi_{2n} \bowtie 0),$$

and $I(\Sigma^{\mathbf{u}}) \subset \mathcal{O}_e^{(\mathbf{u},\mathbf{u}^{-1})} \times T$ is a finite cover of $\mathcal{O}_e^{(\mathbf{u},\mathbf{u}^{-1})}$ by projection.

Conclusion:

- For each $\mathbf{u} \in W^n$, $(\Gamma^{\mathbf{u}}, \widetilde{\pi}_{2n})$ and $(\Sigma^{\mathbf{u}}, \widetilde{\pi}_{2n})$ are respectively localizations and specializations of symmetric Poisson CGLs.

Example: For G_2 with $(\mathbf{u}, \mathbf{u}^{-1}) = (s_1, s_2, s_1, s_2, s_1, s_2, s_2, s_1, s_2, s_1, s_2, s_1)$:

1. recall polynomial Poisson structure π on $\mathcal{O}^{(\mathbf{u},\mathbf{u}^{-1})} \cong \mathbb{C}^{12}$;
2. open symplectic leaf $\mathcal{O}_e^{(\mathbf{u},\mathbf{u}^{-1})} \subset \mathbb{C}^{12}$ defined by
$$\phi_1(x_1, \ldots, x_{12}) \neq 0 \quad \text{and} \quad \phi_1(x_1, \ldots, x_{12}) \neq 0;$$
3. extend π to $\pi \bowtie 0$ on $\mathbb{C}^{12} \times \mathbb{C}^2$ by
$$\{\xi_1, \xi_2\} = 0, \quad \{x_i, \xi_j\} = \lambda_{ij} x_i \xi_j, \quad i \in [1, 12], \; j = 1, 2.$$
4. $\Gamma^{\mathbf{u}} \cong \mathcal{O}_e^{(\mathbf{u},\mathbf{u}^{-1})} \times T \subset \mathbb{C}^{14}$ is given by
$$\phi_1(x_1, \ldots, x_{12}) \neq 0, \;\; \phi_1(x_1, \ldots, x_{12}) \neq 0, \;\; \xi_1 \neq 0, \;\; \xi_2 \neq 0;$$
5. symplectic leaf $\Sigma^{\mathbf{u}} \subset \Gamma^{\mathbf{u}} \subset \mathbb{C}^{12} \times (\mathbb{C}^\times)^2$ is given by
$$\xi_1^2 = \phi_1(x_1, \ldots, x_{12}), \qquad \xi_2^2 = \phi_2(x_1, \ldots, x_{12}),$$
a 4-to-1 cover of the symplectic leaf $\mathcal{O}_e^{(\mathbf{u},\mathbf{u}^{-1})} \cong \mathbb{C}^{12}_{\phi_1\phi_2 \neq 0}$.

3. In connection with cluster algebras

- Goodearl-Yakimov theory (2018): Every symmetric T-Poisson CGL extension is naturally a cluster algebra.

- L. Shen and D. Weng defined certain configuration spaces of flags $\mathrm{Conf}_e^{\mathbf{v}}(\mathcal{A})$ for any $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in W^n$ and showed that they are cluster varieties.

- Fact. For each $\mathbf{u} \in W^n$, $\Gamma^{\mathbf{u}} \cong \mathrm{Conf}_e^{(\mathbf{u},\mathbf{u}^{-1})}(\mathcal{A})$.

4. In connection with integrable systems

Theorem (L.-Mi, 2018)

There is a completely integrable system on $(\mathcal{O}_e^{(\mathbf{u},\mathbf{u}^{-1})}, \pi_{2n})$ all of whose Hamiltonian flows are defined on $\mathbb{C}$.

Summary

Summary of today's talk

1. Poisson Lie sub-group $Q \subset (G, \pi_G)$ gives series of Poisson gorupoids;
2. Applying to $B \subset (G, \pi_{\rm st})$, get the configuration Poisson groupoids
$$\theta_\pm : \ (\Gamma_{2n}, \widetilde{\pi}_{2n}) \rightrightarrows (F_n, \pi_n), \qquad n \geq 1;$$
3. $F_n = \bigsqcup_{\mathbf{u} \in W^n} \mathcal{O}^{\mathbf{u}}$, decomposition into generalized Schubert cells
4. Each $\mathcal{O}^{\mathbf{u}} \cong \mathbb{C}^{l(\mathbf{u})}$ gives rise to a polynomial Poisson algebra;
5. Each $\Gamma^{\mathbf{u}} := \theta_+^{-1}(\mathcal{O}^{\mathbf{u}}) \cap \theta_-^{-1}(\mathcal{O}^{\mathbf{u}})$ is a single T-leaf of $(\Gamma_{2n}, \widetilde{\pi}_{2n})$.

Main results: For each $\mathbf{u} \in W^n$,

- $(\Gamma^{\mathbf{u}}, \widetilde{\pi}_{2n}) \rightrightarrows (\mathcal{O}^{\mathbf{u}}, \pi_n)$ is a Poisson groupoid;
- the symplectic leaf $\Sigma^{\mathbf{u}}$ through the identity section $\mathcal{O}^{\mathbf{u}} \hookrightarrow \Gamma^{\mathbf{u}}$ is a symplectic groupoid $(\Sigma^{\mathbf{u}}, \widetilde{\pi}_{2n}) \rightrightarrows (\mathcal{O}^{\mathbf{u}}, \pi_n)$.

Part IV: In the (grand) scheme of things

Summary cont'd: $(\Sigma^{\mathbf{u}} \subset \Gamma^{\mathbf{u}}, \widetilde{\pi}_{2n}) \rightrightarrows (\mathcal{O}^{\mathbf{u}}, \pi_n)$ as the meeting ground

1. (This talk) Algebraic Poisson/symplectic groupoids;
2. (This talk) Localizations/specializations of symmetric Poisson CGLs
3. (Goodearl-Yakimov theory, Shen-Weng) Poisson cluster varieties;
4. (L.-Mi) Integrable systems with complete Hamiltonian flows;
5. (Mi, Mouquin, work in progress) Systematic quantization;
6. (Remark:) $\Gamma^{\mathbf{u}} \rightrightarrows \mathcal{O}^{\mathbf{u}}$ is isomorphic to a sub-groupoid of product of

$$\mathcal{O}^{\mathbf{u}} \times \mathcal{O}^{\mathbf{u}} \rightrightarrows \mathcal{O}^{\mathbf{u}} \ \text{(pair groupoid)} \quad \text{and} \quad B \times B_- \rightrightarrows \{e\} \ \text{(group)}.$$

Question: How do these structures interact?

References

1. B. Elek and J.-H. Lu, Bott-Samelson varieties and Poisson-Ore extensions, *IMRN*, 2019.
2. K. Goodearl and M. Yakimov, Cluster algebras on nilpotent Poisson algebras, arXiv:1801.01963v2.
3. J.-H. Lu and Y. Mi, *Generalized Bruhat cells and completeness of Hamiltonian flows of Kogan-Zelevinsky integrable systems,* Kostant memorial volume, Progress in Mathematics, **326** (2018).
4. J.-H. Lu and V. Mouquin, Mixed product Poisson structures associated to Poisson Lie groups and Lie bialgebras, *IMRN*, 2016.
5. J.-H. Lu and V. Mouquin, On the T-leaves of some Poisson structures related to products of flag varieties, *Adv. Math.*, 2017).
6. J.-H. Lu and V. Mouquin, Double Bruhat cells and symplectic groupoids, *Trans. Groups*, 2018.
7. J.-H. Lu, V. Mouquin, and S. Yu, Configuration spaces, generalized double Bruhat cells, and symplectic groupoids, to appear in archive.
8. J.-H. Lu and S. Yu, Bott-Samelson atlases, total positivity, and Poisson structures on some homogeneous spaces, to appear in *Selecta.*
9. V. Mouquin, Local Poisson groupoids over mixed product Poisson structures and generalized double Bruhat cells, arXiv:1908.04044.
10. L. Shen, D. Weng, Cluster structures on double Bott-Samelson cells, arXiv: arXiv:1904.07992.