Quantization of Poisson Hopf algebras and moduli of flat connections

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Based on joint works with Ján Pulmann
and with David Li-Bland

Quantization of Poisson Hopf algebras

joint work with Ján Pulmann

Deformation quantization problem for Hopf algebras

Ingredients: a Poisson Hopf algebra

- a *commutative* Hopf algebra $(\mathcal{H}, m_0, \Delta_0, S_0, 1, \epsilon)$
- a compatible Poisson bracket $\{,\} : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$
  ($\Delta_0 : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ is a Poisson algebra morphism)

Typically $\mathcal{H} = C^\infty(G)$, in general $\mathcal{H}$ in a $\mathbb{Q}$-linear SMC

The problem

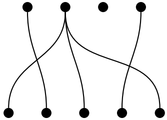
Find a universal deformation

$$m = \sum_{n=0}^{\infty} \hbar^n m_n \qquad \Delta = \sum_{n=0}^{\infty} \hbar^n \Delta_n \qquad S = \sum_{n=0}^{\infty} \hbar^n S_n$$

s.t. $(\mathcal{H}, m, \Delta, S, 1, \epsilon)$ is a Hopf algebra and $m_1 - m_1^{op} = \{,\}$
[For $\mathcal{H} = (U\mathfrak{g})^*$: Etingof-Kazhdan 1995]

The (symmetric) nerve of a group G

... is the functor (symmetric simplicial set)



$$N\colon \mathrm{FinSet}^{op} \to \mathrm{Set}$$

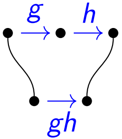
$$N(X) = \{g\colon X \times X \to G \mid g_{ij} g_{jk} = g_{ik} \ \&\ g_{ii} = 1 \, (\forall i, j, k \in X)\}$$

$$N(\bullet^n) \cong G^{n-1} \qquad \text{e.g.} \quad \bullet \xrightarrow{g_{12}} \bullet \xrightarrow{g_{23}} \bullet \xrightarrow{g_{34}} \bullet \quad (n = 4)$$

Segal condition

$N\colon \mathrm{FinSet}^{op} \to \mathrm{Set}$ is a nerve of a group iff $N(\bullet^n) \to N(\bullet\bullet)^{n-1}$ is a bijection and $|N(\emptyset)| = |N(\bullet)| = 1$

The product:
$N(\bullet\bullet) \times N(\bullet\bullet) \cong N(\bullet\bullet\bullet) \to N(\bullet\bullet)$



The (symmetric) nerve of a commutative Hopf algebra

... is a functor $N\colon \mathsf{FinSet} \to \mathsf{CommAlg}(\mathcal{C})$

($\mathcal{C}$ is a SMC, the Hopf algebra is $\mathcal{H} = N(\bullet\bullet) \in \mathcal{C}$)

= a *symmetric lax monoidal* functor $N\colon \mathsf{FinSet} \to \mathcal{C}$

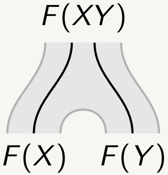
(the monoidal product in FinSet = disjoint union: $\bullet \otimes \bullet\bullet = \bullet\bullet\bullet$)

lax monoidal?

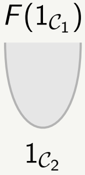
$F : \mathcal{C}_1 \to \mathcal{C}_2$ (monoidal categories)

$c_{X,Y}\colon F(X)F(Y) \to F(XY)$, $c_1\colon 1_{\mathcal{C}_2} \to F(1_{\mathcal{C}_1})$, *not invertible*

$c_{X,Y}$ =

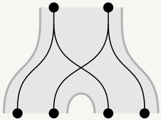


c_1 =



F sends algebras to algebras

e.g. $N(\bullet^n)$ is a (commutative) algebra:



The (symmetric) nerve of a commutative Hopf algebra

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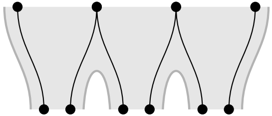
Segal condition

$N(\bullet\bullet)^{n-1} \to N(\bullet^n)$ is an iso
and $1_{\mathcal{C}} \to N() \to N(\bullet)$ are isos



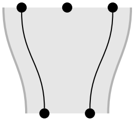
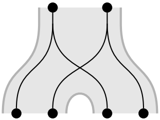
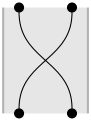
$N(\bullet^4)$

$\uparrow$

$N(\bullet\bullet)^3$

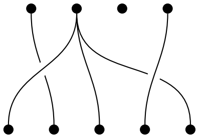
Theorem (The nerve of a commutative Hopf algebra)

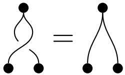
Commutative Hopf algebras in a SMC $\mathcal{C}$ are equivalent to symmetric lax-monoidal functors $N\colon \mathsf{FinSet} \to \mathcal{C}$ satisfying the Segal condition

$\mathcal{H} = N(\bullet\bullet)$, $\Delta =$ , $m =$ , $S =$ 

The (braided) nerve of a non-commutative Hopf algebra

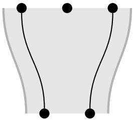
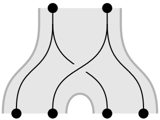
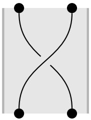
BrSet - “braided sets”:
(The BMC generated by a commutative algebra)





Theorem (The nerve of a Hopf algebra)

Hopf algebras (with invertible S) in a BMC $\mathcal{C}$ are equivalent to braided lax-monoidal functors N: BrSet $\to \mathcal{C}$ *satisfying the Segal condition*

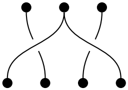
$\mathcal{H} = N(\bullet\bullet)$, $\Delta =$  , $m =$  , $S =$ 

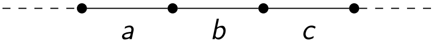
Constructing the nerve of a Hopf algebra

or Hopf holonomies in the plane

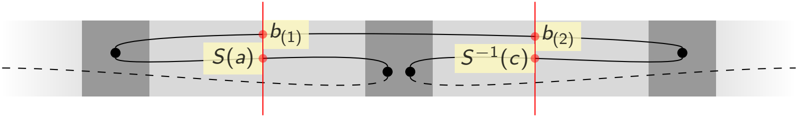
a Hopf algebra $\mathcal{H} \in \mathcal{C}$ $\rightsquigarrow$ a functor $N\colon \mathrm{BrSet} \to \mathcal{C}$

$N(\bullet^n) = \mathcal{H}^{n-1}$

$N($  $)\colon \mathcal{H}^3 \to \mathcal{H}^2$ is



$\longrightarrow$



$S(a)\, b_{(1)}$ $S^{-1}(c)\, b_{(2)}$

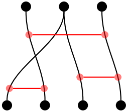
N is braided lax monoidal:

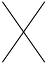
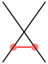
$N(\bullet^m)N(\bullet^n) = \mathcal{H}^{m-1}\mathcal{H}^{n-1} \to N(\bullet^{m+n-1}) = \mathcal{H}^{m+n-1}$: inserting 1

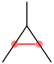
The semiclassical picture: FinSet + chord diagrams

Poisson Hopf algebras in terms of infinitesimal braids

ChordSet, the infinitesimally braided version of FinSet/BrSet:

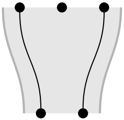
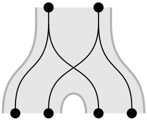
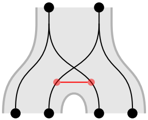


 $=$  $+ \frac{\epsilon}{2}$  $(\epsilon^2 = 0)$

 $= 0$

Theorem (The nerve of a Poisson Hopf algebra)

Poisson Hopf algebras in a linear SMC (or iBMC) $\mathcal{C}$ are equivalent to infinitesimally braided lax-monoidal functors $N\colon \mathrm{ChordSet} \to \mathcal{C}$ satisfying the Segal condition

$\mathcal{H} = N(\bullet\bullet)$, $\Delta =$ , $m =$ , $\{,\} =$ 

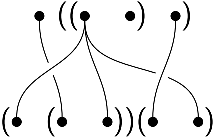
Quantization

Quantization of Poisson Hopf algebras

$$(\mathrm{Pa})\mathrm{BrSet} \xrightarrow{Q} \mathrm{ChordSet} \xrightarrow{\text{Poisson Hopf}} \mathcal{C}$$

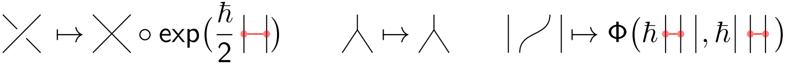
$$(\mathrm{Pa})\mathrm{BrSet} \xrightarrow{\text{Hopf}} \mathcal{C}$$

- (Pa)BrSet: parenthesized BrSet



- Quantization functor

$$Q\colon (\mathrm{Pa})\mathrm{BrSet} \to \mathrm{ChordSet}$$



$\Phi \in \mathbb{Q}\langle\langle X, Y \rangle\rangle$ a Drinfeld associator

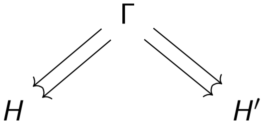
. . . and moduli of flat connections

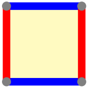
joint work with David Li-Bland

Lu-Weinstein double symplectic groupoid

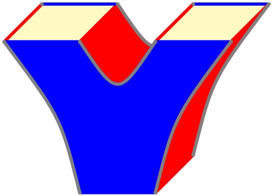
as a moduli space of flat connections on a square

Manin triple $H, H' \subset G \ \rightsquigarrow$ LW double symplectic groupoid



$\Gamma =$  (moduli space of flat G-connections)

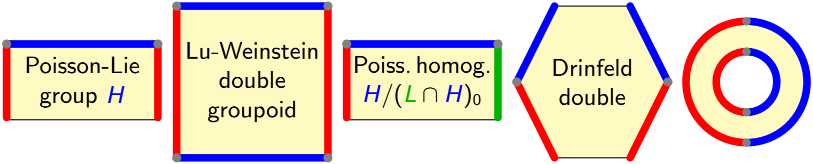
Products is Γ: horizontal and vertical gluings of squares



Poisson-Lie groups and their friends as moduli spaces

on surfaces with decorated boundaries

Flat G-bundles, with (flat) reductions of the structure group over the colored segments (to Lagrangian / coiso subgroups of G)



($H, H' \subset G$ a Manin triple, $L \subset G$ Lagrangian)

data at a vertex 

:

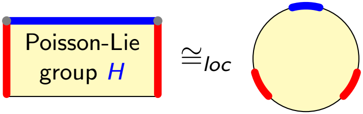
a common reduction to $(C \cap C')_0$ (used above)

or an element of $C \backslash G / C'$ (mutual position of reductions)

"Principle": every relevant Poisson manifold is a moduli space, every relevant coisotropic submanifold (or a Poisson map) comes from a cobordism (or from a surface embedding).

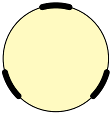
Deformation quantization of decorated moduli spaces

with isolated colored arcs

Not all possible decorations, but: 

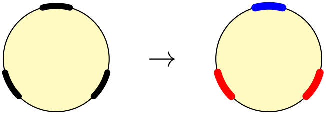
1. Quantize quasi-Poisson moduli spaces*:
   flat principal bundles trivialized at the black arcs

   (non-trivial associativity)



2. Take invariants w.r.t. the color coisotropic subgroups

   (ordinary associativity)



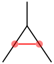
*[Alekseev, Kosmann-Schwarzbach, Meinrenken]

Algebras in $U\mathfrak{g}\text{-mod}_\hbar^\Phi$

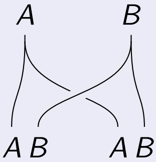
Deforming $U\mathfrak{g}$-mod - as a braided monoidal category

$\mathfrak{g}$ a Lie algebra, $t \in (S^2\mathfrak{g})^{\mathfrak{g}}$,  $= t$

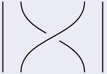
$\mathfrak{g} \curvearrowright M$ with coisotropic stabilizers $\Rightarrow$
$C^\infty(M) \in U\mathfrak{g}\text{-mod}_\hbar^\Phi$ is commutative & associative

 $= 0$

Tensor products of algebras in $U\mathfrak{g}\text{-mod}_\hbar^\Phi$



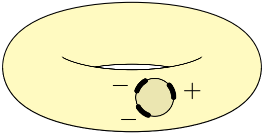
$$U(\mathfrak{g} \oplus \mathfrak{g})\text{-mod}_\hbar^\Phi \xrightarrow{\mathrm{res}_{\mathfrak{g}_{\mathrm{diag}}}} U\mathfrak{g}\text{-mod}_\hbar^\Phi \text{ is monoidal}$$

$m_{new} = m_{old} \circ J$, $\quad J =$  $\in (U\mathfrak{g})^{\otimes 4}$

$\mathfrak{c} \subset \mathfrak{g}$ coiso, $A \in U\mathfrak{g}\text{-mod}_\hbar^\Phi$ an algebra $\Rightarrow$ $A^{\mathfrak{c}} \in \mathrm{Vect}$ an algebra

Quantization of quasi-Poisson moduli spaces

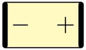
The problem (a 2d TQFT with a boundary):



$\rightsquigarrow$ algebra in $U(\mathfrak{g}^{n_+} \oplus \bar{\mathfrak{g}}^{n_-})\text{-mod}_\hbar^\Phi$

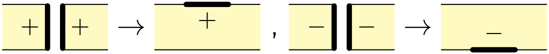
$(n_+ = 1, n_- = 2)$

Start with a commutative algebra:

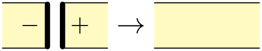
$k \times$ 

$\rightsquigarrow$ $C^\infty(G^k)$ with the original product

Apply J's, or take invariants:



: $m_{new} = m_{old} \circ J$

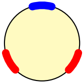


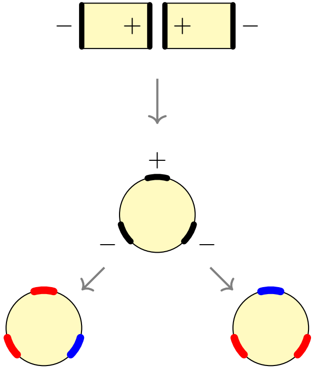
, or remove a thick arc : take G-invariants

J. Pulmann: independent of the cutting, up to coherent isos
+ functoriality under embeddings of surfaces

ideology: Chern-Simons with doubled surfaces

Quantization of Poisson Lie groups, again

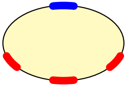
How to quantize the Poisson manifold  $\cong_{loc} H$:



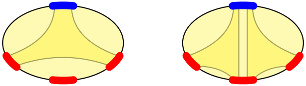
Etingof-Kazhdan   our method

(gives the product on $C^\infty(H)_\hbar$)

The nerve of $C^\infty(H)_\hbar$:

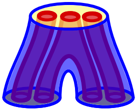
$N(\bullet\bullet\bullet) = C^\infty($  $)_\hbar$

The coproduct on $N(\bullet\bullet) = C^\infty(H)_\hbar$
from the functoriality w.r.t. embeddings:



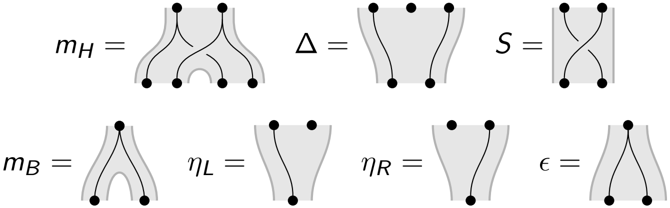
$$N(\bullet\bullet) \to N(\bullet\bullet\bullet) \xleftarrow{\sim} N(\bullet\bullet) \otimes N(\bullet\bullet)$$

Chern-Simons
with doubled surfaces



Some open problems

- Nerves of Hopf algebroids: $H = N(\bullet\bullet)$, $B = N(\bullet)$



  B is commutative, $\eta_{L,R}$ are (left/right) central -
  is there a picture with non-commutative B's?

- Nerves of higher groupoids: what kind of Poisson (or non-commutative) structures do we get from general $N\colon \mathrm{ChordSet} \to \mathcal{C}$ or $N\colon \mathrm{BrSet} \to \mathcal{C}$?

- What else can be extracted from moduli spaces and Chern-Simons theory?

THANKS!

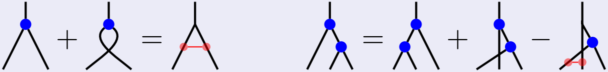
Appendix: Poisson and Poisson-Hopf algebras

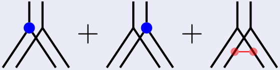
in infinitesimally braided categories (iBMCs)

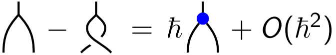
Poisson algebras

∼ [Alekseev Kosmann-Schwarzbach Meinrenken]

Commutative algebras with a biderivation  such that



Tensor (=fusion) product: 

Quantization in $\mathcal{C}_\hbar^\Phi$: 

Our method quantizes Poisson Hopf algebras in iBMCs

Example: Manin quadruples

[PŠ-Valach]

$$\mathfrak{d} = \mathfrak{h} \oplus \mathfrak{h}^* \oplus \mathfrak{g}, \quad \mathfrak{h}^\perp = \mathfrak{h} \oplus \mathfrak{g}, \quad \mathfrak{h}^{*\perp} = \mathfrak{h}^* \oplus \mathfrak{g}$$

$C^\infty(H)$ is a Poisson Hopf algebra in the iBMC $U\mathfrak{g}$-mod.

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