General covariance in field theory and the Batalin–Vilkovisky formalism

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Noether's approach to classical field theory is formulated in the context of the variational calculus. (This was the topic of last week's seminar, by Luca Vitalgiano.)

A defect of Noether's approach to field theory is that conservation laws only hold along the locus of solutions of the Euler-Lagrange equations ("on-shell"). This makes it difficult to formulate quantum field theory using her formalism.

In the 1980s, Batalin and Vilkovisky found a way to extend the conservation laws of Noether off-shell. They adjoin additional fields to the theory, which are auxilliary in character. Their approach may be summarized in a single Maurer–Cartan equation, the Batalin–Vilkovisky classical master equation

Noether always works with functionals, which are obtained by working modulo total derivatives. This makes it difficult to study boundary conditions in field theory. In the 1990s, in the context of general relativity, Soloviev found an approach to field theory which allows us to keep track of these total derivatives.

He refined the Poisson brackets introduced by Gelfand and Dikii to act on local expressions, in such a way that the Jacobi relation holds.

A final defect of Noether's approach is that it does not make manifest the underlying general covariance of the theory. That is, it is unclear how to handle gauge invariance with respect to diffeomorphisms of the world-sheet. This is important not just in the context of coupling to gravity, but also in the study of topological field theory.

In this talk, I will present an approach to general covariance in the context of Batalin–Vilkovisky formalism, in which the Maurer–Cartan equation of Batalin and Vilkovisky is replaced by a curved Maurer–Cartan equation.

Most of the results of this talk are taken from the article

Covariance in the Batalin–Vilkovisky formalism and the Maurer–Cartan equation for curved Lie algebras
Letters in Mathematical Physics, **109** (2019), 187–224

Differential graded symplectic supermanifolds

Let $\mathcal{M}$ be a graded supermanifold, with coordinates ξ^α. The ghost number of ξ^α is $\mathsf{gh}(\xi^\alpha) \in \mathbb{Z}$, and its parity is $\mathsf{p}(\xi^\alpha) \in \mathbb{Z}/2$.

The shifted cotangent bundle $T^*[-1]\mathcal{M}$ of $\mathcal{M}$ has coordinates ξ^α and ξ^+_α, where $\mathsf{gh}(\xi^+_\alpha) = -1 - \mathsf{gh}(\xi^\alpha)$, and $\mathsf{p}(\xi^+_\alpha) = 1 - \mathsf{p}(\xi^\alpha)$.

In today's talk, we simplify our formulas by considering fields in a single independent variable t, $\xi^\alpha(t)$ and $\xi^+_\alpha(t)$.

Let $\mathcal{O}_\infty(T^*[-1]\mathcal{M})$ be the space of functions on the jet-space of $T^*[-1]\mathcal{M}$. The coordinates on the jet-space are the coefficients of the Taylor series expansions

$$\xi^\alpha(t) = \sum_{k=0}^\infty \frac{t^k}{k!}\xi^\alpha_k$$

$$\xi^+_\alpha(t) = \sum_{k=0}^\infty \frac{t^k}{k!}\xi^+_{\alpha,k}.$$

Introduce the vector fields on the jet-space

$$\partial_{\alpha,k} = \frac{\partial}{\partial\xi^\alpha_k} \qquad \partial^\alpha_k = \frac{\partial}{\partial\xi^+_{\alpha,k}}.$$

The generator of infinitesimal time translation on the jet space is the vector field

$$\partial = \sum_{k=0}^{\infty} \left(\xi^{\alpha}_{k+1} \partial_{\alpha,k} + \xi^{+}_{k+1} \partial^{\alpha}_{k} \right) .$$

Vector fields that commute with ∂ are called evolutionary.

The space of functionals is the graded superspace

$$\mathcal{F} = \mathcal{O}_\infty / \partial \left[\mathcal{O}_\infty \right] .$$

Denote the image of a density $f \in \mathcal{O}_\infty$ in $\mathcal{F}$ by $\int f$.

The variational derivatives are infinite-order differential operators

$$\frac{\delta}{\delta \xi^{\alpha}} = \sum_{k=0}^{\infty} (-\partial)^k \partial^{\alpha}_{k} : \mathcal{F} \to \mathcal{O}_\infty, \qquad \frac{\delta}{\delta \xi^{+}_{\alpha}} = \sum_{k=0}^{\infty} (-\partial)^k \partial_{\alpha,k} : \mathcal{F} \to \mathcal{O}_\infty .$$

$\mathcal{F}$ is a graded Lie algebra with respect to the Batalin-Vilkovisky antibracket, of ghost number 1,

$$(\textstyle\int f, \int g) = \displaystyle\sum_\alpha (-1)^{(\mathsf{p}(f)+1)\,\mathsf{p}(\xi^\alpha)} \int \left(\frac{\delta f}{\delta \xi^\alpha} \frac{\delta g}{\delta \xi^+_\alpha} + (-1)^{\mathsf{p}(f)} \frac{\delta f}{\delta \xi^+_\alpha} \frac{\delta g}{\delta \xi^\alpha} \right)$$

The functional $\int \mathsf{D}$, where

$$\mathsf{D} = \xi^+_\alpha \partial \xi^\alpha,$$

lies in the centre of this graded Lie algebra. Consider the curved graded Lie algebra of power series in a formal variable u of ghost number 2, with curvature $\int u\mathsf{D}$.

The Hamiltonian vector field H_f (more properly $H_{\int f}$) associated to a functional $\int f \in \mathcal{F}$ is the vector field on the jet-space, or (graded) derivation of $\mathcal{O}_\infty$,

$$H_f = \sum_\alpha \sum_{k=0}^\infty (-1)^{(\mathsf{p}(f)+1)\,\mathsf{p}(\xi^\alpha)} \left(\partial^k \left(\frac{\delta f}{\delta \xi^\alpha} \right) \partial_k^\alpha + (-1)^{\mathsf{p}(f)} \partial^k \left(\frac{\delta f}{\delta \xi_\alpha^+} \right) \partial_{\alpha,k} \right)$$

This gives a morphism of graded Lie algebras from $\mathcal{F}$ to the graded Lie algebra of vector fields on the jet-space.

☡ This result is much harder to prove than in Poisson geometry, because the Hamiltonian vector field H_f does not equal the antibracket $\mathsf{ad}(f) = (\int f, -)$. In fact, $\mathsf{ad}(f)$ is a differential operator on $\mathcal{O}_\infty$ of infinite order, which differs from H_f by a total derivative.

Lemma

The Hamiltonian vector field associated to the functional $\int \mathsf{D}$ *is* $-\partial$.
More generally, the Hamiltonian vector field associated to the functional $\int \phi(t)\,\mathsf{D}$ *is* $-\phi(t)\partial$.

A one-form of ghost number 0 and even parity on $\mathcal{M}$

$$\nu = \nu_\alpha \, d\xi^\alpha$$

is a Liouville form if the two-form $d\nu$ is symplectic. (We may think of $-i\nu$ as a connection one-form on the prequantization line bundle L of $\mathcal{M}$.)

Denote the Poisson tensor associated to $d\nu$ by

$$\pi = (-1)^{|\xi^\alpha|(|\xi^\beta|+1)} \pi^{\alpha\beta} \partial_\alpha \wedge \partial_\beta .$$

Associate to the data of a graded symplectic supermanifold the functionals

$$S_0 = \sum_\alpha (-1)^{\mathsf{p}(\xi^\alpha)} \nu_\alpha \partial \xi^\alpha ,$$

where ∂ is differentiation with respect to t, and

$$S_1 = \frac{1}{2} \sum_{\alpha,\beta} \xi^+_\alpha \pi^{\alpha\beta} \xi^+_\beta .$$

Theorem (G, 2019)

The functional $\int S_u = \int (S_0 + uS_1)$ *satisfies the curved Maurer-Cartan equation*

$$\tfrac{1}{2}(\int S_u, \int S_u) = -\int u\mathsf{D}.$$

In other words,

$$(\int S_0, \int S_0) = 0, \qquad (\int S_0, \int S_1) = -\int \mathsf{D}, \qquad (\int S_1, \int S_1) = 0.$$

The Hamiltonian vector field associated to the functional $\int S_0$ makes $T^*[-1]\mathcal{M}$ into a differential graded supermanifold. The functional $\int S_1$ then expresses the general covariance of the theory.

Twisting

A differential on $\mathcal{M}$ is a Hamiltonian vector field with Hamiltonian W, of ghost number 1 and odd parity, satisfying

$$\{W, W\} = 0.$$

The AKSZ model is obtained by gauging the Maurer-Cartan element $\int S_u$ by the functional $z \int u^{-1} W$. Although the generator of this gauge transformation is singular, the resulting family of Maurer-Cartan elements is a well-defined pencil.

This twist gives a solution

$$S^W = S_0 + z(\textstyle\int S_1, \int W)$$

of the classical master equation

$$(\textstyle\int S^W, \int S^W) = 0,$$

together with an equation expressing covariance under reparametrization of the world-line

$$(\textstyle\int S^W, \int S_1) = -\int \mathsf{D}.$$

Quantum field theory in the Batalin–Vilkovisky framework

I do not have time to discuss the recent work of F. Bonechi, A. S. Cattaneo, J. Qiu and M. Zabzine. They show that to quantize solutions to the curved deformation of the classical Batalin–Vilkovisky master equation that we have discussed here, one should restrict to gauge conditions represented by Lagrangian submanifolds of the jet space of $T^*[-1]\mathcal{M}$ such that the restriction of the functional S_u to this Lagrangian submanifold is independent of u.

See the article

Equivariant Batalin-Vilkovisky formalism
J. Geom. Physics, **154** (2020) 103720

for further details.

The Soloviev bracket

In the remainder of the talk, I explain how to modify the curved Maurer–Cartan equation $(S_u, S_u) = -\int u\mathsf{D}$ to keep tally of integrations by parts.

Our main tool is the same horizontal de Rham complex that featured in last week's talk. First of all, we promote the space $\mathcal{F}$ of functionals to a sheaf over the world-line $\mathbb{R}$. By the Poincaré lemma, this sheaf has a resolution by the de Rham complex:

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{O}_\infty \xrightarrow{d} \mathcal{O}_\infty\, dt \longrightarrow \mathcal{F} \longrightarrow 0$$

By Soloviev, the horizontal de Rham complex is a differential graded Lie algebra, with (anti)bracket

$$\begin{aligned}(\!(f\,dt+\tilde{f},g\,dt+\tilde{g})\!) = \sum_{\alpha}(-1)^{(\mathsf{p}(f)+1)\,\mathsf{p}(\xi^\alpha)} \sum_{k,\ell=0}^{\infty} & \Big\{\Big(\big(\partial^\ell\partial_{\alpha,k}f\big)\big(\partial^k\partial^\alpha_\ell g\big) + (-1)^{\mathsf{p}(f)}\big(\partial^\ell\partial^\alpha_k f\big)\big(\partial^k\partial_{\alpha,\ell}g\big)\Big)\,dt \\ & - (-1)^{\mathsf{p}(g)}\Big(\big(\partial^\ell\partial_{\alpha,k}\tilde{f}\big)\big(\partial^k\partial^\alpha_\ell g\big) + (-1)^{\mathsf{p}(f)}\big(\partial^\ell\partial^\alpha_k\tilde{f}\big)\big(\partial^k\partial_{\alpha,\ell}g\big)\Big) \\ & + \Big(\big(\partial^\ell\partial_{\alpha,k}f\big)\big(\partial^k\partial^\alpha_\ell\tilde{g}\big) + (-1)^{\mathsf{p}(f)}\big(\partial^\ell\partial^\alpha_k f\big)\big(\partial^k\partial_{\alpha,\ell}\tilde{g}\big)\Big)\Big\}\end{aligned}$$

Assign dimension d_α to the field ξ^α, and dimension $1 - d_\alpha$ to the antifield ξ^+_α. World-line functions have dimension 0 and world-line 1-forms have dimension 1.

Lift the curvature $\int u\mathsf{D}$ to the horizontal de Rham complex:

$$u\mathsf{D} = u \sum_\alpha \bigl(\partial\xi^\alpha \xi^+_\alpha - d_\alpha \partial(\xi^\alpha \xi^+_\alpha) \bigr)$$

The differential

$$d_u(f\,dt + \tilde{f}) = (-1)^{\mathsf{p}(\tilde{f})} \partial\tilde{f}\,dt + u(-1)^{\mathsf{p}(f)} \sum_{\alpha,k} \bigl((1 - d_\alpha) \xi^+_{\alpha,k} \partial^\alpha_k f + d_\alpha \xi^\alpha_k \partial_{\alpha,k} f - f \bigr)$$

satisfies $d_u^2 = u(\!(\mathsf{D}, -)\!)$, and makes the horizontal de Rham complex, extended by the variable u of ghost number 2, into a curved differential graded Lie algebra.

The curved Maurer–Cartan equation in this curved differential graded Lie algebra takes the form

$$d_u S_u + \frac{1}{2}(\!(S_u, S_u)\!) = -u\mathsf{D}.$$

In the case of a symplectic graded supermanifold $\mathcal{M}$, this equation has the solution

$$S_0 = \sum_\alpha (-1)^{\mathsf{p}(\xi^\alpha)} \nu_\alpha \partial \xi^\alpha \, dt,$$

$$S_1 = \frac{1}{2} \sum_{\alpha,\beta} \big(\xi^+_\alpha \pi^{\alpha\beta} \xi^+_\beta \, dt + (-1)^{\mathsf{p}(\xi^\alpha)} \nu_\alpha \pi^{\alpha\beta} \xi^+_\beta \big).$$

Chern–Simons theory

The curved Maurer–Cartan equation can also be used to study higher dimensional models, as long as the fields transform as tensors, and not spinors. The problem of incorporating spinors into this approach remains a problem for future research; this may only be possible if the diffeomorphism group of the world-sheet is replaced by its isometry group, and general vector fields by Killing vector fields.

In Chern–Simons, there are three independent variables $\{t_1, t_2, t_3\}$, and three formal parameters $\{u_1, u_2, u_3\}$. The fields of the theory are a matrix of one-forms $A_1\,dt_1 + A_2\,dt_2 + A_3\,dt_3$, which has ghost number 0 and even parity, and a ghost c, which has ghost number 1 and odd parity.

We have

$$S_0 = \tfrac{1}{2}\Big(\sum_{i,j,k} \varepsilon^{ijk}\,\mathrm{Tr}\big(A_i\partial_j A_k\big) + \sum_i \mathrm{Tr}\big(c\partial_i A_i^+\big) + \sum_i \mathrm{Tr}\big(A_i^+\partial_i c\big)\Big)\,dt^1dt^2dt^3,$$

$$S_1 = \tfrac{1}{2}\Big(\sum_{i,j,k} \varepsilon^{ijk}u_i\,\mathrm{Tr}\big(A_j^+A_k^+\big) - \sum_i u_i\,\mathrm{Tr}\big(A_ic^+\big)\Big)\,dt^1dt^2dt^3$$

$$+ \tfrac{1}{2}\Big(\sum_j \mathrm{Tr}\big(A_jA_j^+\big) - \mathrm{Tr}\big(cc^+\big)\Big)\big(u_1dt^2dt^3 - u_2dt^1dt^3 + u_3dt^1dt^2\big).$$

This is the abelian Chern–Simons theory. It deforms to the non-abelian Chern–Simons theory by a twisting procedure similar to the one applied in the one-dimensional toy model, with

$$W = \tfrac{1}{6}\sum_{i,j,k} \varepsilon^{ijk}u_i^{-1}\,\mathrm{Tr}\big(c\,[A_j,A_k]\big)\,dt^1dt^2dt^3.$$

The resulting curved Maurer–Cartan element of the horizontal de Rham complex has the wrong value for the curvature D_i, and assigns the wrong world-sheet dimensions to the fields of the theory:

$$\mathsf{D}_i = \sum_j \operatorname{Tr}\big(\partial_i A_j A_j^+\big) + \tfrac{1}{2} \operatorname{Tr}\big(\partial_i c c^+ - c \partial_i c^+\big).$$

(The field A_i is a scalar field, while c is a half-density.)

This may be corrected for by twisting once more, by the element

$$V = \sum_i u_i \Big(\sum_j \iota(\partial_j) \operatorname{Tr}\big(A_i A_j^+\big) - \tfrac{1}{2} \iota(\partial_i) \operatorname{Tr}\big(cc^+\big) \Big) dt^1 dt^2 dt^3 .$$

Since $(\!(V, W)\!) = 0$, the composition of these two twists has the same effect as twisting by $V + W$.

Unlike the twist by W, the twist by V leaves no effect on descent to the curved differential graded Lie algebra of functionals. Another difference, of course, is that it is regular as a function of the equivariant parameters u_i, while W is singular.