Deformations, cohomologies and homotopy of relative Rota-Baxter Lie algebras

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Rota-Baxter operators on Lie algebras

The notion of Rota-Baxter operators on Lie algebras was introduced as the operator form of the classical Yang-Baxter equation.

Definition

A linear operator $T:\mathfrak{g}\longrightarrow\mathfrak{g}$ on a Lie algebra $\mathfrak{g}$ is called a Rota-Baxter operator (of weight λ) if the following condition is satisfied:

$$[T(x),T(y)]_{\mathfrak{g}}=T([T(x),y]_{\mathfrak{g}}+[x,T(y)]_{\mathfrak{g}}+\lambda[x,y]_{\mathfrak{g}}).$$

Semonov-Tian-Shansky proved that if there is an ad-invariant, non-degenerate, symmetric bilinear form on $\mathfrak{g}$, then a Rota-Baxter operator and a triangular r-matrix ($[r,r]=0$) are equivalent.

Example

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Let the Lie algebra $\mathfrak{g}$ *be the direct sum of two subalgebras* $\mathfrak{g}_1$ *and* $\mathfrak{g}_2$. *Then*

$$T(x_1, x_2) = -x_1, \quad \forall x_1 \in \mathfrak{g}_1, x_2 \in \mathfrak{g}_2$$

is a Rota-Baxter operator of weight 1.

Modified Yang-Baxter equations

Let $R \in \mathrm{End}\,(\mathfrak{g})$ be a solution of the **modified Yang-Baxter equation**:

$$[R(u), R(v)]_{\mathfrak{g}} = R([R(u), v]_{\mathfrak{g}}) + R([u, R(v)]_{\mathfrak{g}}) - [u, v]_{\mathfrak{g}},$$

from which Semenov-Tian-Shansky obtained an Infinitesimal Factorization Theorem for the Lie algebra $\mathfrak{g}$ with important applications to integrable systems. Under the transformation

$$R = \mathrm{Id} + 2T,$$

the operator R satisfies the modified Yang-Baxter equation if and only if the operator T is a Rota-Baxter operator of weight 1.

Relative Rota-Baxter operators on Lie algebras

Kupershmidt introduced the notion of a relative Rota-Baxter operator (also called an $\mathcal{O}$-operator) on a Lie algebra $\mathfrak{g}$ with respect to arbitrary representation.

Definition

A relative Rota-Baxter operator on a Lie algebra $(\mathfrak{g}, [-,-]_\mathfrak{g})$ with respect to a representation $(V;\rho)$ is a linear map $T: V \longrightarrow \mathfrak{g}$ satisfying the following quadratic constraint:

$$[Tu, Tv]_\mathfrak{g} = T\big(\rho(Tu)(v) - \rho(Tv)(u)\big), \quad \forall u, v \in V.$$

A relative Rota-Baxter Lie algebra is a triple $((\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}), \rho, T)$, where $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g})$ is a Lie algebra, $\rho: \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ is a representation of $\mathfrak{g}$ on a vector space V and $T: V \longrightarrow \mathfrak{g}$ is a relative Rota-Baxter operator

Graph characterization of relative Rota-Baxter operators

Consider the semidirect product Lie algebra $(\mathfrak{g}\oplus V,[\cdot,\cdot]_{\ltimes})$:

$$[x+u,y+v]_{\ltimes}=[x,y]_{\mathfrak{g}}+\rho(x)v-\rho(y)u.$$

Proposition

A linear map $T:V\longrightarrow\mathfrak{g}$ is a relative Rota-Baxter operator if and only if the graph of T,

$$G_T:=\{Tu+u|\ \forall u\in V\}$$

is a subalgebra of the semidirect product Lie algebra $(\mathfrak{g}\oplus V,[\cdot,\cdot]_{\ltimes})$.

Examples

Example

Let $r \in \wedge^2\mathfrak{g}$ be a triangular r-matrix. Define $r^\sharp : \mathfrak{g}^* \to \mathfrak{g}$ by $\langle r^\sharp(\xi), \eta\rangle = r(\xi, \eta)$. Then $r^\sharp$ is a relative RB operator on $\mathfrak{g}$ with respect to the coadjoint representation.

It is well known that a triangular r-matrix gives rise to a triangular Lie bialgebra, which we denote by $(\mathfrak{g}, r)$. We will see that deformations and cohomologies of triangular Lie bialgebras can be studied using the general framework of relative Rota-Baxter Lie algebras.

Example

Let $\omega \in \wedge^2\mathfrak{g}^*$ be a symplectic structure on a Lie algebra $\mathfrak{g}$. Then $(\omega^\sharp)^{-1}$ is a relative Rota-Baxter operator on the Lie algebra $\mathfrak{g}$ with respect to the coadjoint representation, where $\omega^\sharp : \mathfrak{g} \to \mathfrak{g}^*$ is defined by $\langle \omega^\sharp(x), y\rangle = \omega(x, y)$ for $x, y \in \mathfrak{g}$.

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Rota-Baxter operators $\rightsquigarrow$ splitting of algebras

$$\begin{array}{ccccc}
\text{zinbiel alg.} & \longrightarrow & \text{dendriform alg.} & \longrightarrow & \text{pre-Lie alg.} \\
\uparrow \text{splitting} & & \uparrow \text{splitting} & & \uparrow \text{splitting} \\
\text{comm. alg.} & \longrightarrow & \text{ass. alg.} & \longrightarrow & \text{Lie alg.} \\
\downarrow \text{duplication} & & \downarrow \text{duplication} & & \downarrow \text{duplication} \\
\text{perm. alg.} & \longrightarrow & \text{dialg.} & \longrightarrow & \text{Leibniz alg.}
\end{array}$$

Averaging operators (embedding tensors) $\rightsquigarrow$ duplication of algebras

Main results

Our approach to study deformations, cohomologies and homotopy of RB Lie algebras is using the 'controlling algebra'.

More precisely, we construct an L_∞-algebra whose Maurer-Cartan elements are relative Rota-Baxter Lie algebras.

- A relative Rota-Baxter Lie algebra gives rise to a twisted L_∞-algebra that controls its deformations;
- The l_1 in the above twisted L_∞-algebra defines a cohomology of the relative Rota-Baxter Lie algebra;
- Replace vector spaces by graded vector spaces, we get homotopy relative Rota-Baxter Lie algebras.

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- Replace vector spaces by graded vector spaces, we get homotopy relative Rota-Baxter Lie algebras.

Controlling algebra for Lie algebras

Let $\mathfrak{g}$ be a vector space. The graded vector space

$$C^*(\mathfrak{g},\mathfrak{g}) = \oplus_{n=0}^{+\infty} C^n(\mathfrak{g},\mathfrak{g}) = \oplus_{n=0}^{+\infty} \mathrm{Hom}\,(\wedge^{n+1}\mathfrak{g},\mathfrak{g}).$$

equipped with the Nijenhuis-Richardson bracket

$$[P,Q]_{\mathsf{NR}} = P\bar{\circ}Q - (-1)^{pq} Q\bar{\circ}P, \quad \forall P \in C^p(\mathfrak{g},\mathfrak{g}), Q \in C^q(\mathfrak{g},\mathfrak{g}),$$

is a graded Lie algebra, where $P\bar{\circ}Q \in C^{p+q}(\mathfrak{g},\mathfrak{g})$ is defined by

$$(P\bar{\circ}Q)(x_1,\cdots,x_{p+q+1}) = \sum_{\sigma\in\mathbb{S}_{(q+1,p)}} P(Q(x_{\sigma(1)},\cdots,x_{\sigma(q+1)}),x_{\sigma(q+2)},\cdots,x_{\sigma(p+q+1)})$$

Lemma

$\mu \in \mathrm{Hom}\,(\wedge^2\mathfrak{g},\mathfrak{g})$ defines a Lie algebra structure on $\mathfrak{g}$ if and only if

$$[\mu,\mu]_{\mathsf{NR}} = 0.$$

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Lemma

$\mu \in \mathrm{Hom}\,(\wedge^2\mathfrak{g},\mathfrak{g})$ *defines a Lie algebra structure on* $\mathfrak{g}$ *if and only if*

$$[\mu,\mu]_{\mathsf{NR}} = 0.$$

Applications to deformations of Lie algebras

Define d_μ by $d_\mu = [\mu, \cdot]_{\mathsf{NR}}$. Then $(C^*(\mathfrak{g}, \mathfrak{g}), [\cdot, \cdot]_{\mathsf{NR}}, d_\mu)$ is a dgLa, that controls deformations of the Lie algebra $(\mathfrak{g}, \mu)$.

Theorem

Let $(\mathfrak{g}, \mu)$ be a Lie algebra and $\mu' \in \mathrm{Hom}\,(\wedge^2\mathfrak{g}, \mathfrak{g})$. Then $(\mathfrak{g}, \mu + \mu')$ is a Lie algebra if and only if μ' is a Maurer-Cartan element of the dgLa $(C^*(\mathfrak{g}, \mathfrak{g}), [\cdot, \cdot]_{\mathsf{NR}}, d_\mu)$.

Applications to cohomologies of Lie algebras

It is well known that the Chevalley-Eilenberg coboundary operator $\mathrm{d}_{\mathsf{CE}} : \mathrm{Hom}\,(\wedge^k \mathfrak{g}, \mathfrak{g}) \to \mathrm{Hom}\,(\wedge^{k+1}\mathfrak{g}, \mathfrak{g})$ for a Lie algebra $\mathfrak{g}$ is given by

$$\mathrm{d}_{\mathsf{CE}} f(x_1, \cdots x_{k+1}) = \sum_{i=1}^{k} (-1)^{i+1} [x_i, f(x_1, \cdots, \hat{x_i}, \cdots, x_{k+1})] + \sum_{i<j} (-1)^{i+j} f([x_i, x_j], x_1, \cdots, \hat{x_i}, \cdots, \hat{x_j}, \cdots x_{k+1}).$$

Theorem-Definition

Let $(\mathfrak{g}, \mu)$ *be a Lie algebra. Then*

$$\mathrm{d}_{\mathsf{CE}} f = (-1)^{k+1} [\mu, f]_{\mathsf{NR}}, \quad \forall f \in \mathrm{Hom}\,(\wedge^k \mathfrak{g}, \mathfrak{g}).$$

Applications to homotopy Lie algebras

Let V be a $\mathbb{Z}$-graded vector space. Denote by $\mathrm{Hom}^n(\mathsf{Sym}(V), V)$ the space of degree n linear maps from the graded vector space $\mathsf{Sym}(V) = \oplus_{i=1}^{+\infty}\mathsf{Sym}^i(V)$ to the $\mathbb{Z}$-graded vector space V. Set $C^n(V,V) := \mathrm{Hom}^n(\mathsf{Sym}(V), V)$ and $C^*(V,V) := \oplus_{n\in\mathbb{Z}}C^n(V,V)$. Then we have the **graded Nijenhuis-Richardson bracket** $[\cdot,\cdot]_{\mathsf{NR}}$ on the graded vector space $C^*(V,V)$, which is a graded Lie algebra.

Theorem-Definition

The Maurer-Cartan elements $\sum_{k=1}^{+\infty} l_k$ of the graded Lie algebra $(C^*(V,V),[\cdot,\cdot]_{\mathsf{NR}})$ are the L_∞-algebra structures on V.

The controlling algebras for other algebras

- The gLa structure for associative algebras is given by the Gerstenhaber bracket on $\oplus \mathrm{Hom}(\otimes^k V, V)$. The cohomology groups for the deformation theories of associative algebras are the Hochschild cohomology groups;
- The gLa structure for pre-Lie algebras (left-symmetric algebras) is given by the Matsushima-Nijenhuis bracket on $\oplus \mathrm{Hom}(\wedge^{k-1} V \otimes V, V)$. The cohomology groups for the deformation theories of pre-Lie algebras are the Dzhumadil′daev cohomology groups;
- The controlling algebra for Leibniz algebras is a gLa;
- The controlling algebra for n-Lie algebras is a gLa.

L_∞-algebra

Definition

An L_∞-algebra *is a* $\mathbb{Z}$*-graded vector space* $\mathfrak{g} = \oplus_{k\in\mathbb{Z}}\mathfrak{g}^k$ *equipped with a collection* $(k \geq 1)$ *of linear maps* $l_k : \otimes^k\mathfrak{g} \longrightarrow \mathfrak{g}$ *of degree* 1 *with the property that, for any homogeneous elements* $x_1, \cdots, x_n \in \mathfrak{g}$*, we have*

(i) (graded symmetry) *for every* $\sigma \in \mathbb{S}_n$,

$$l_n(x_{\sigma(1)}, \cdots, x_{\sigma(n-1)}, x_{\sigma(n)}) = \varepsilon(\sigma)l_n(x_1, \cdots, x_{n-1}, x_n),$$

(ii) (generalized Jacobi identity) *for all* $n \geq 1$,

$$\sum_{i=1}^{n} \sum_{\sigma\in\mathbb{S}_{(i,n-i)}} \varepsilon(\sigma)l_{n-i+1}(l_i(x_{\sigma(1)}, \cdots, x_{\sigma(i)}), x_{\sigma(i+1)}, \cdots, x_{\sigma(n)}) = 0.$$

Maurer-Cartan elements

Definition (Getzler)

The set of Maurer-Cartan elements of an L_∞-algebra $\mathfrak{g}$ is the set of those $\alpha \in \mathfrak{g}^0$ satisfying the Maurer-Cartan equation

$$\sum_{k=1}^{+\infty} \frac{1}{k!} l_k(\alpha, \cdots, \alpha) = 0.$$

E. Getzler, Lie theory for nilpotent L_∞-algebras. *Ann. Math. (2)* **170** (2009), 271-301.

Main results: the controlling algebra of relative Rota-Baxter Lie algebra

Let $\mathfrak{g}$ and V be vector spaces. Denote by L' the following graded vector space

$$L' = \oplus_{n=0}^{+\infty} \mathrm{Hom}\,(\wedge^{n+1}\mathfrak{g}, \mathfrak{g}) \oplus \mathrm{Hom}\,(\wedge^{n}\mathfrak{g} \otimes V, V),$$

and denote by $\mathfrak{h} = \oplus_{n=0}^{+\infty} \mathrm{Hom}\,(\wedge^{n+1} V, \mathfrak{g})$.

Theorem (Lazarev-S-Tang)

There is an L_∞-algebra structure $\{l_k\}_{k=1}^{+\infty}$ on the graded vector space $L'[1] \oplus \mathfrak{h}$, such that its Maurer-Cartan elements are exactly relative Rota-Baxter Lie algebra structures on $\mathfrak{g}$ and V.

Maurer-Cartan elements contains

$$\mu \in \mathrm{Hom}\,(\wedge^2\mathfrak{g}, \mathfrak{g}), \quad \rho \in \mathrm{Hom}\,(\mathfrak{g} \otimes V, V), \quad T \in \mathrm{Hom}\,(V, \mathfrak{g}).$$

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Deformations of relative Rota-Baxter Lie algebras

Let (μ, ρ, T) be a relative Rota-Baxter Lie algebra structure on $\mathfrak{g}$ and V. Define $l_k^{\mu,\rho,T}$ by

$$l_k^{\mu,\rho,T}(x_1, \cdots, x_k) = \sum_{n=0}^{+\infty} \frac{1}{n!} l_{k+n}(\underbrace{\mu+\rho+T, \cdots, \mu+\rho+T}_{n}, x_1, \cdots, x_k).$$

According to Getzler's results, $\left(L'[1] \oplus \mathfrak{h}, \{l_k^{\mu,\rho,T}\}_{k=1}^{+\infty}\right)$ is an L_∞-algebra, called the twisted L_∞-algebra.

Theorem (Lazarev-S-Tang)

The triple $(\mu+\mu', \rho+\rho', T+T')$ is again a relative Rota-Baxter Lie algebra for

$$\mu' \in \mathrm{Hom}\,(\wedge^2\mathfrak{g}, \mathfrak{g}), \quad \rho' \in \mathrm{Hom}\,(\mathfrak{g}, \mathfrak{gl}(V)), \quad T' \in \mathrm{Hom}\,(V, \mathfrak{g}),$$

if and only if $\mu'+\rho'+T'$ is a Maurer-Cartan element of the L_∞-algebra $\left(L'[1] \oplus \mathfrak{h}, \{l_k^{\mu,\rho,T}\}_{k=1}^{+\infty}\right)$.

Slogan

There is a well known slogan, often attributed to Deligne, Drinfeld and Kontsevich: *every reasonable deformation theory is controlled by a differential graded Lie algebra (an L_∞-algebra), determined up to quasi-isomorphism*.

Related works in Poisson geometry

- **F. Schatz and M. Zambon**, Deformations of pre-symplectic structures and the Koszul L_∞-algebra. **IMRN** 2020.
- **M. Gualtieri, M. Matviichuk and G. Scott**, Deformation of Dirac structures via L_∞-algebras. **IMRN** 2020.
- **Y. Frégier and M. Zambon**, Simultaneous deformations and Poisson geometry. **Compos. Math.** (2015).
- **H. Le, Y. Oh, A. Tortorella and L. Vitagliano**, Deformations of coisotropic submanifolds in Jacobi manifolds. *J. Symplectic Geom.* (2018).
- **Y. Frégier and M. Zambon**, Simultaneous deformations of algebras and morphisms via derived brackets. *JPAA* (2015).
- **Y. Frégier, M. Markl and D. Yau**, The L_∞-deformation complex of diagrams of algebras. *New York J. Math.* (2009).

Extensions of L_∞-algberas

Recall that $\mathfrak{h} = \oplus_{k=1}^{+\infty}\mathrm{Hom}\,(\wedge^{n+1}V, \mathfrak{g})$ and

$$L' = \oplus_{n=0}^{+\infty}\mathrm{Hom}\,(\wedge^{n+1}\mathfrak{g}, \mathfrak{g}) \oplus \mathrm{Hom}\,(\wedge^{n}\mathfrak{g} \otimes V, V).$$

Theorem (Lazarev-S.-Tang)

Let $((\mathfrak{g}, \mu), \rho, T)$ be a relative Rota-Baxter Lie algebra. Then the L_∞-algebra $(L'[1] \oplus \mathfrak{h}, \{l_k^{\mu,\rho,T}\}_{k=1}^{+\infty})$ is a strict extension of the L_∞-algebra (dgLa) $L'[1]$ by the L_∞-algebra (dgLa) $\mathfrak{h}$, that is, we have the following short exact sequence of L_∞-algebras:

$$0 \longrightarrow \mathfrak{h} \xrightarrow{\iota} L'[1] \oplus \mathfrak{h} \xrightarrow{p} L'[1] \longrightarrow 0,$$

where $\iota(\theta) = (0, \theta)$ and $p(f, \theta) = f$.

Let (μ, ρ, T) be a relative Rota-Baxter Lie algebra structure on $\mathfrak{g}$ and V. Define the space of n-cochains $\mathfrak{C}^n(\mu, \rho, T)$ by

$$\Big(\mathrm{Hom}\,(\wedge^n \mathfrak{g}, \mathfrak{g}) \oplus \mathrm{Hom}\,(\wedge^{n-1}\mathfrak{g} \otimes V, V)\Big) \oplus \mathrm{Hom}\,(\wedge^{n-1} V, \mathfrak{g}).$$

Define the **coboundary operator** $\mathcal{D} : \mathfrak{C}^n(\mu, \rho, T) \to \mathfrak{C}^{n+1}(\mu, \rho, T)$ by

$$\mathcal{D}(f, \theta) = (-1)^n l_1^{\mu,\rho,T}(f, \theta).$$

We obtain that $(\oplus_{n=0}^{+\infty}\mathfrak{C}^n(\mu, \rho, T), \mathcal{D})$ is a cochain complex.

Definition

The cohomology of the cochain complex $(\oplus_{n=0}^{+\infty}\mathfrak{C}^n(\mu, \rho, T), \mathcal{D})$ is called the **cohomology of the relative Rota-Baxter Lie algebra**. *We denote its n-th cohomology group by $\mathcal{H}^n(\mu, \rho, T)$*

Consider infinitesimal deformations of a relative Rota-Baxter Lie algebra structure (μ, ρ, T) on $\mathfrak{g}$ and V:

$$\begin{aligned} \mu_t &= \mu + t\mu' \\ \rho_t &= \rho + t\rho' \\ T_t &= T + tT'. \end{aligned}$$

Theorem

There is a one-to-one correspondence between equivalence classes of infinitesimal deformations of the relative Rota-Baxter Lie algebra $(\mathfrak{g}, \mu, \rho, T)$ *and the second cohomology group* $\mathcal{H}^2(\mu, \rho, T)$.

Relations among cohomologies

Denote by

$$
\begin{aligned}
\mathfrak{C}^n(\mu,\rho) &= \mathrm{Hom}\,(\wedge^n\mathfrak{g},\mathfrak{g}) \oplus \mathrm{Hom}\,(\wedge^{n-1}\mathfrak{g}\otimes V, V),\\
\mathfrak{C}^n(T) &= \mathrm{Hom}\,(\wedge^{n-1}V,\mathfrak{g}).
\end{aligned}
$$

The coboundary operator $\mathcal{D}:\mathfrak{C}^n(\mu,\rho,T)\ \to\ \mathfrak{C}^{n+1}(\mu,\rho,T)$ is given by

$$
\begin{aligned}
\mathcal{D}(f,\theta) &= (-1)^n\Big(-[\pi,f]_{\mathsf{NR}},\ [[\pi,T]_{\mathsf{NR}},\theta]_{\mathsf{NR}} + \frac{1}{n!}\underbrace{[\cdots[[}_{n} f,T]_{\mathsf{NR}},T]_{\mathsf{NR}},\cdots,T]_{\mathsf{NR}}\Big)\\
&= (\partial f,\ \delta\theta + h_T f).
\end{aligned}
$$

$\mathcal{D}$ can be well-explained by the following diagram:

$$
\begin{array}{ccccccccc}
\cdots \longrightarrow & \mathfrak{C}^n(\mu,\rho) & \xrightarrow{\ \partial\ } & \mathfrak{C}^{n+1}(\mu,\rho) & \xrightarrow{\ \partial\ } & \mathfrak{C}^{n+2}(\mu,\rho) & \longrightarrow \cdots & & \\
 & & \searrow^{h_T} & & \searrow^{h_T} & & & & \\
\cdots \longrightarrow & \mathfrak{C}^n(T) & \xrightarrow{\ \delta\ } & \mathfrak{C}^{n+1}(T) & \xrightarrow{\ \delta\ } & \mathfrak{C}^{n+2}(T) & \longrightarrow \cdots . & &
\end{array}
$$

Relations among cohomologies

Theorem (Lazarev-S.-Tang)

There is a short exact sequence of the cochain complexes:

$$0 \longrightarrow (\oplus_{n=0}^{+\infty}\mathfrak{C}^n(T),\delta) \stackrel{\iota}{\longrightarrow} (\oplus_{n=0}^{+\infty}\mathfrak{C}^n(\mu,\rho,T),\mathcal{D}) \stackrel{p}{\longrightarrow} (\oplus_{n=0}^{+\infty}\mathfrak{C}^n(\mu,\rho),\partial) \longrightarrow 0,$$

and there is a long exact sequence of the cohomology groups:

$$\cdots \longrightarrow \mathcal{H}^n(T) \stackrel{\mathcal{H}^n(\iota)}{\longrightarrow} \mathcal{H}^n(\mu,\rho,T) \stackrel{\mathcal{H}^n(p)}{\longrightarrow} \mathcal{H}^n(\mu,\rho) \stackrel{c^n}{\longrightarrow} \mathcal{H}^{n+1}(T) \longrightarrow \cdots,$$

where the connecting map c^n is defined by $c^n([\alpha]) = [h_T\alpha]$, for all $[\alpha] \in \mathcal{H}^n(\mathfrak{g},\rho)$.

Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, T)$ be a Rota-Baxter Lie algebra. Define the space of n-cochains $\mathfrak{C}^n_{\mathrm{RB}}(\mathfrak{g}, T)$ by

$$\mathfrak{C}^n_{\mathrm{RB}}(\mathfrak{g}, T) := \mathfrak{C}^n_{\mathrm{Lie}}(\mathfrak{g};\mathfrak{g}) \oplus \mathfrak{C}^n(T) = \mathrm{Hom}\,(\wedge^n\mathfrak{g}, \mathfrak{g}) \oplus \mathrm{Hom}\,(\wedge^{n-1}\mathfrak{g}, \mathfrak{g}).$$

Define $\mathcal{D}_{\mathrm{RB}} : \mathfrak{C}^n_{\mathrm{RB}}(\mathfrak{g}, T) \longrightarrow \mathfrak{C}^{n+1}_{\mathrm{RB}}(\mathfrak{g}, T)$ by

$$\mathcal{D}_{\mathrm{RB}}(f, \theta) = \Big(\mathrm{d}_{\mathsf{CE}} f, \delta\theta + \Omega f\Big), \quad \forall f \in \mathrm{Hom}\,(\wedge^n\mathfrak{g}, \mathfrak{g}),\ \theta \in \mathrm{Hom}\,(\wedge^{n-1}\mathfrak{g}, \mathfrak{g}),$$

where $\Omega : \mathrm{Hom}\,(\wedge^n\mathfrak{g}, \mathfrak{g}) \longrightarrow \mathrm{Hom}\,(\wedge^n\mathfrak{g}, \mathfrak{g})$ is defined by

$$\begin{aligned}(\Omega f)(x_1, \cdots, x_n) &= (-1)^n\Big(f(Tx_1, \cdots, Tx_n) \\ &\quad - \sum_{i=1}^n Tf(Tx_1, \cdots, Tx_{i-1}, x_i, Tx_{i+1}, \cdots, Tx_n)\Big).\end{aligned}$$

Cohomology of RB Lie algebras

Theorem (Lazarev-S.-Tang)

The map $\mathcal{D}_{\mathrm{RB}}$ *is a coboundary operator, i.e.* $\mathcal{D}_{\mathrm{RB}} \circ \mathcal{D}_{\mathrm{RB}} = 0$.

Definition (Lazarev-S.-Tang)

Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, T)$ *be a Rota-Baxter Lie algebra. The cohomology of the cochain complex* $(\oplus_{n=0}^{+\infty} \mathfrak{C}^n_{\mathrm{RB}}(\mathfrak{g}, T), \mathcal{D}_{\mathrm{RB}})$ *is taken to be the* cohomology of the Rota-Baxter Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, T)$. *Denote the* n*-th cohomology group by* $\mathcal{H}^n_{\mathrm{RB}}(\mathfrak{g}, T)$.

Let $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},r)$ be a triangular Lie bialgebra. Define the space of n-cochains $\mathfrak{C}^n_{\mathrm{TLB}}(\mathfrak{g},r)$ by

$$\mathfrak{C}^n_{\mathrm{TLB}}(\mathfrak{g},r) := \mathrm{Hom}\,(\wedge^n\mathfrak{g},\mathfrak{g}) \oplus \wedge^n\mathfrak{g}.$$

Define the *coboundary operator* $\mathcal{D}_{\mathrm{TLB}}:\mathfrak{C}^n_{\mathrm{TLB}}(\mathfrak{g},r)\longrightarrow\mathfrak{C}^{n+1}_{\mathrm{TLB}}(\mathfrak{g},r)$ by

$$\mathcal{D}_{\mathrm{TLB}}(f,\chi)=\Big(\mathrm{d}_{\mathsf{CE}}f,\ \mathrm{d}_r\chi+\Theta f\Big),\quad \forall f\in\mathrm{Hom}\,(\wedge^n\mathfrak{g},\mathfrak{g}),\ \chi\in\wedge^n\mathfrak{g},$$

where $\mathrm{d}_r:\wedge^n\mathfrak{g}\longrightarrow\wedge^{n+1}\mathfrak{g}$ is given by $\mathrm{d}_r\chi=[r,\chi]$ and $\Theta:\mathrm{Hom}\,(\wedge^n\mathfrak{g},\mathfrak{g})\longrightarrow\wedge^{n+1}\mathfrak{g}$ is defined by

$$\langle\Theta f,\xi_1\wedge\cdots\wedge\xi_{n+1}\rangle=\sum_{i=1}^{n+1}(-1)^{i+1}\langle\xi_i,f(r^\sharp(\xi_1),\cdots,r^\sharp(\xi_{i-1}),r^\sharp(\xi_{i+1}),\cdots,r^\sharp(\xi_{n+1}))\rangle.$$

Theorem (Lazarev-S.-Tang)

The map $\mathcal{D}_{\mathrm{TLB}}$ *is a coboundary operator, i.e.* $\mathcal{D}_{\mathrm{TLB}} \circ \mathcal{D}_{\mathrm{TLB}} = 0$.

Definition (Lazarev-S.-Tang)

Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, r)$ *be a triangular Lie bialgebra. The cohomology of the cochain complex* $(\oplus_{n=0}^{+\infty}\mathfrak{C}^n_{\mathrm{TLB}}(\mathfrak{g}, r), \mathcal{D}_{\mathrm{TLB}})$ *is called the* cohomology of the triangular Lie bialgebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, r)$. *Denote the* n*-th cohomology group by* $\mathcal{H}^n_{\mathrm{TLB}}(\mathfrak{g}, r)$.

Let $(\mathfrak{g},\{l_k\}_{k=1}^{+\infty})$ be an L_∞-algebra and $(V,\{\rho_k\}_{k=1}^{+\infty})$ a representation. Then $(\mathfrak{h}:=\oplus_{n\in\mathbb{Z}}\mathrm{Hom}^n(\mathsf{Sym}(V),\mathfrak{g}),\{\mathfrak{l}_k\}_{k=1}^{+\infty})$ is an L_∞-algebra, where $\mathfrak{l}_k$ is given by

$$\mathfrak{l}_k(a_1,\cdots,a_k)=P\underbrace{[\cdots[[}_{k}\sum_{k=1}^{+\infty}(l_k+\rho_k),a_1]_{\mathsf{NR}},a_2]_{\mathsf{NR}},\cdots,a_k]_{\mathsf{NR}}.$$

Theorem-Definition (Lazarev-S.-Tang)

A degree 0 element $T=\sum_{k=1}^{+\infty}T_k\in\mathrm{Hom}(\mathsf{Sym}(V),\mathfrak{g})$ is a homotopy relative Rota-Baxter operator on $(\mathfrak{g},\{l_k\}_{k=1}^{+\infty})$ with respect to the representation $(V,\{\rho_k\}_{k=1}^{+\infty})$ if and only if $T=\sum_{k=1}^{+\infty}T_k$ is a Maurer-Cartan element of the L_∞-algebra $(\mathfrak{h},\{\mathfrak{l}_k\}_{k=1}^{+\infty})$.

Homotopy relative Rota-Baxter Lie algebras

Let $(\mathfrak{g}, \{l_k\}_{k=1}^{+\infty})$ be an L_∞-algebra and $(V, \{\rho_k\}_{k=1}^{+\infty})$ a representation. Then $(\mathfrak{h} := \oplus_{n\in\mathbb{Z}}\mathrm{Hom}^n(\mathsf{Sym}(V), \mathfrak{g}), \{\mathfrak{l}_k\}_{k=1}^{+\infty})$ is an L_∞-algebra, where $\mathfrak{l}_k$ is given by

$$\mathfrak{l}_k(a_1, \cdots, a_k) = P\underbrace{[\cdots[[}_{k}\sum_{k=1}^{+\infty}(l_k + \rho_k), a_1]_{\mathsf{NR}}, a_2]_{\mathsf{NR}}, \cdots, a_k]_{\mathsf{NR}}.$$

Theorem-Definition (Lazarev-S.-Tang)

A degree 0 element $T = \sum_{k=1}^{+\infty} T_k \in \mathrm{Hom}(\mathsf{Sym}(V), \mathfrak{g})$ is a homotopy relative Rota-Baxter operator on $(\mathfrak{g}, \{l_k\}_{k=1}^{+\infty})$ with respect to the representation $(V, \{\rho_k\}_{k=1}^{+\infty})$ if and only if $T = \sum_{k=1}^{+\infty} T_k$ is a Maurer-Cartan element of the L_∞-algebra $(\mathfrak{h}, \{\mathfrak{l}_k\}_{k=1}^{+\infty})$.

Remark

Dotsenko and Khoroshkin studied the homotopy of Rota-Baxter operators on associative algebras in

V. Dotsenko and A. Khoroshkin, Quillen homology for operads via Gröbner bases. Doc. Math. **18** *(2013), 707-747.*

and noted that "in general compact formulas are yet to be found". For Rota-Baxter Lie algebras, one encounters a similarly challenging situation.

We use the approach of 'controlling algebras' and their MC elements to formulate the notion of a homotopy version of a relative Rota-Baxter Lie algebra. Hopefully our research will provide some useful information for the operadic research of the homotopy of relative Rota-Baxter Lie algebras.

Explicit formula for a homotopy RB operator

A degree 0 element $T=\sum_{k=1}^{+\infty}T_k\in\mathrm{Hom}\,(\mathsf{Sym}(V),\mathfrak{g})$ with $T_k\in\mathrm{Hom}\,(\mathsf{Sym}^k(V),\mathfrak{g})$ is a homotopy relative Rota-Baxter operator on an L_∞-algebra $(\mathfrak{g},\{l_k\}_{k=1}^{+\infty})$ with respect to the representation $(V,\{\rho_k\}_{k=1}^{+\infty})$ if the following equalities hold for all $p\geq 1$ and all homogeneous elements $v_1,\cdots,v_p\in V$,

$$
\begin{aligned}
&\sum_{\substack{k_1+\cdots+k_m=t\\ 1\leq t\leq p-1}}\ \sum_{\sigma\in\mathbb{S}_{(k_1,\cdots,k_m,1,p-1-t)}}\frac{\varepsilon(\sigma)}{m!}\cdot\\
&T_{p-t}\Big(\rho_{m+1}\Big(T_{k_1}\big(v_{\sigma(1)},\cdots,v_{\sigma(k_1)}\big),\cdots,T_{k_m}\big(v_{\sigma(k_1+\cdots+k_{m-1}+1)},\cdots,v_{\sigma(t)}\big),v_{\sigma(t+1)}\Big),\\
&v_{\sigma(t+2)},\cdots,v_{\sigma(p)}\Big)\\
=\;&\sum_{k_1+\cdots+k_n=p}\ \sum_{\sigma\in\mathbb{S}_{(k_1,\cdots,k_n)}}\\
&\frac{\varepsilon(\sigma)}{n!}l_n\Big(T_{k_1}\big(v_{\sigma(1)},\cdots,v_{\sigma(k_1)}\big),\cdots,T_{k_n}\big(v_{\sigma(k_1+\cdots+k_{n-1}+1)},\cdots,v_{\sigma(p)}\big)\Big).
\end{aligned}
$$

F. Chapoton and **M. Livernet**, Pre-Lie algebras and the rooted trees operad. **IMRN 8** (2001), 395-408.

Definition

A pre-Lie$_\infty$-algebra structure on a graded vector space V consists of degree 1 linear map $\{\theta_k\}_{k=1}^{\infty}$ from the graded vector space $\mathsf{Sym}(V)\otimes V$ to V such that $\sum_{k=1}^{+\infty}\theta_k$ is a Maurer-Cartan element of the graded Lie algebra $(C^*(V,V),[\cdot,\cdot]_{\mathsf{MN}})$.

From pre-Lie$_\infty$-algebras to homotopy RB operators

Let $(\mathfrak{g}, \{\theta_k\}_{k=1}^{+\infty})$ be a pre-Lie$_\infty$-algebra. Define l_k by

$$l_k(x_1, \cdots, x_k) = \sum_{i=1}^{k} (-1)^{x_i(x_{i+1}+\cdots+x_k)} \theta_k(x_1, \cdots, \hat{x}_i, \cdots, x_k, x_i).$$

Then $(\mathfrak{g}, \{l_k\}_{k=1}^{+\infty})$ is an L_∞-algebra, denoted by $\mathfrak{g}^C$.
For all $k \geq 1$, we define L_k : Sym$^{k-1}(\mathfrak{g}) \to \mathfrak{gl}(\mathfrak{g})$ by

$$L_k(x_1, \cdots, x_{k-1})(x_k) = \theta_k(x_1, \cdots, x_{k-1}, x_k).$$

Proposition

With the above notation, $(\mathfrak{g}, \{L_k\}_{k=1}^{+\infty})$ is a representation of the sub-adjacent L_∞-algebra $\mathfrak{g}^C$. Moreover, the identity map $\mathrm{Id} : \mathfrak{g} \to \mathfrak{g}$ is a strict homotopy relative Rota-Baxter operator on the L_∞-algebra $\mathfrak{g}^C$ with respect to the representation $(\mathfrak{g}, \{L_k\}_{k=1}^{+\infty})$.

From homotopy RB operators to pre-Lie$_\infty$-algebras

Theorem (Lazarev-S.-Tang)

Let $T \in \mathrm{Hom}(V, \mathfrak{g})$ be a strict homotopy relative Rota-Baxter operator on an L_∞-algebra $(\mathfrak{g}, \{l_k\}_{k=1}^{+\infty})$ with respect to the representation $(V, \{\rho_k\}_{k=1}^{+\infty})$. Then $(V, \{\theta_k\}_{k=1}^{+\infty})$ is a pre-Lie$_\infty$-algebra, where $\theta_k : \otimes^k V \to V$ $(k \geq 1)$ are linear maps of degree 1 defined by

$$\theta_k(v_1, \cdots, v_k) := \rho_k(Tv_1, \cdots, Tv_{k-1}, v_k), \quad \forall v_1 \cdots, v_k \in V.$$

The notion of an r_∞-matrix was introduced by A. Voronov and his collaborators in

D. Bashkirov and **A. Voronov**, r_∞-Matrices, triangular L_∞-bialgebras, and quantum$_\infty$ groups, 2015.

As expected, an r_∞-matrix gives rise to a homotopy relative Rota-Baxter operator with respect to the coadjoint representation.

V-structure

Definition

A V*-structure consists of a quadruple* $(L, \mathfrak{h}, P, \Delta)$ *where*

- $(L, [\cdot,\cdot])$ *is a graded Lie algebra,*
- $\mathfrak{h}$ *is an abelian graded Lie subalgebra of* $(L, [\cdot,\cdot])$*,*
- $P : L \longrightarrow L$ *is a projection, that is* $P \circ P = P$*, whose image is* $\mathfrak{h}$ *and kernel is a graded Lie subalgebra of* $(L, [\cdot,\cdot])$*,*
- Δ *is an element in* $\ker(P)^1$ *such that* $[\Delta, \Delta] = 0$*.*

Theorem (T. Voronov)

Let $(L, \mathfrak{h}, P, \Delta)$ be a V-structure. Then the graded vector space $L[1] \oplus \mathfrak{h}$ is an L_∞-algebra where

$$\begin{aligned} l_1(x,a) &= (-[\Delta,x], P(x+[\Delta,a])), \\ l_2(x,y) &= (-1)^x[x,y], \\ l_k(x,a_1,\cdots,a_{k-1}) &= P[\cdots[[x,a_1],a_2]\cdots,a_{k-1}], \quad k\geq 2, \\ l_k(a_1,\cdots,a_{k-1},a_k) &= P[\cdots[[\Delta,a_1],a_2]\cdots,a_k], \quad k\geq 2. \end{aligned}$$

Here $a, a_1, \cdots, a_k \in \mathfrak{h}$ and $x, y \in L$.

Remark

Let L' be a graded Lie subalgebra of L that satisfies $[\Delta, L'] \subset L'$. Then $L'[1] \oplus \mathfrak{h}$ is an L_∞-subalgebra of the above L_∞-algebra $(L[1] \oplus \mathfrak{h}, \{l_k\}_{k=1}^{+\infty})$.

A V-structure

Let $\mathfrak{g}$ and V be two vector spaces.

Proposition (Lazarev-S.-Tang)

We have a V-structure $(L, \mathfrak{h}, P, \Delta)$ as follows:

- *the graded Lie algebra $(L, [\cdot, \cdot])$ is given by*
$$\big(\oplus_{n=0}^{+\infty} C^{n+1}(\mathfrak{g} \oplus V, \mathfrak{g} \oplus V), [\cdot, \cdot]_{\mathsf{NR}}\big);$$
- *the abelian graded Lie subalgebra $\mathfrak{h}$ is given by*
$$\mathfrak{h} := \oplus_{n=0}^{+\infty} \mathrm{Hom}\,(\wedge^{n+1} V, \mathfrak{g});$$
- *$P : L \longrightarrow L$ is the projection onto the subspace $\mathfrak{h}$, and $\Delta = 0$.*

Consequently, we obtain an L_∞-algebra $(L[1] \oplus \mathfrak{h}, \{l_k\}_{k=1}^{+\infty})$,

$$\begin{aligned} l_1(Q, \theta) &= P(Q), \\ l_2(Q, Q') &= (-1)^{Q}[Q, Q']_{\mathsf{NR}}, \\ l_k(Q, \theta_1, \cdots, \theta_{k-1}) &= P[\cdots[Q, \theta_1]_{\mathsf{NR}}, \cdots, \theta_{k-1}]_{\mathsf{NR}}, \end{aligned}$$

for $\theta, \theta_1, \cdots, \theta_{k-1} \in \mathfrak{h}$, $Q, Q' \in L$.

Main Theorem

Note that $L' = \oplus_{k=0}^{+\infty}(\mathrm{Hom}\,(\wedge^{n+1}\mathfrak{g},\mathfrak{g}) \oplus \mathrm{Hom}\,(\wedge^{n}\mathfrak{g}\otimes V, V))$ is a subalgebra of L.

Theorem (Lazarev-S.-Tang)

With above notation, $(L'[1]\oplus\mathfrak{h}, \{l_k\}_{k=1}^{+\infty})$ is an L_∞-algebra, where l_k are given by

$$
\begin{aligned}
l_2(Q,Q') &= (-1)^{Q}[Q,Q']_{\mathsf{NR}},\\
l_k(Q,\theta_1,\cdots,\theta_{k-1}) &= P[\cdots[Q,\theta_1]_{\mathsf{NR}},\cdots,\theta_{k-1}]_{\mathsf{NR}},
\end{aligned}
$$

for $\theta_1,\cdots,\theta_{k-1}\in\mathfrak{h}$, $Q,Q'\in L'$.

Let $\mu\in\mathrm{Hom}\,(\wedge^2\mathfrak{g},\mathfrak{g})$, $\rho\in\mathrm{Hom}\,(\mathfrak{g}\otimes V,V)$, $T\in\mathrm{Hom}\,(V,\mathfrak{g})$. Then $((\mathfrak{g},\mu),\rho,T)$ is a relative Rota-Baxter Lie algebra if and only if $\mu+\rho+T$ is a Maurer-Cartan element of the L_∞-algebra $(L'[1]\oplus\mathfrak{h},\{l_i\}_{i=1}^{+\infty})$.

Proof

Let $\mu+\rho+T$ be a Maurer-Cartan element. Then we have

$$\begin{aligned} & \sum_{k=1}^{+\infty}\frac{1}{k!}l_k\Big(\mu+\rho+T,\cdots,\mu+\rho+T\Big) \\ = & \frac{1}{2!}l_2\Big(\mu+\rho+T,\mu+\rho+T\Big) \\ & +\frac{1}{3!}l_3\Big(\mu+\rho+T,\mu+\rho+T,\mu+\rho+T\Big) \\ = & \Big(-\frac{1}{2}[\mu+\rho,\mu+\rho]_{\mathsf{NR}},\frac{1}{2}[[\mu+\rho,T]_{\mathsf{NR}},T]_{\mathsf{NR}}\Big) \\ = & (0,0). \end{aligned}$$

Thus, we obtain $[\mu+\rho,\mu+\rho]_{\mathsf{NR}}=0$ and $[[\mu+\rho,T]_{\mathsf{NR}},T]_{\mathsf{NR}}=0$, which implies that $(\mathfrak{g},\mu)$ is a Lie algebra, $(V;\rho)$ is its representation and T is a relative Rota-Baxter operator on the Lie algebra $(\mathfrak{g},\mu)$ with respect to the representation $(V;\rho)$.

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Thanks for your attention!