Deformations, cohomologies and homotopy of relative Rota-Baxter Lie algebras

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The notion of Rota-Baxter operators on Lie algebras was introduced as the operator form of the classical Yang-Baxter equation.

**Definition**

A linear operator \( T : \mathfrak{g} \rightarrow \mathfrak{g} \) on a Lie algebra \( \mathfrak{g} \) is called a **Rota-Baxter operator** (of weight \( \lambda \)) if the following condition is satisfied:

\[
[T(x), T(y)]_\mathfrak{g} = T([T(x), y]_\mathfrak{g} + [x, T(y)]_\mathfrak{g} + \lambda [x, y]_\mathfrak{g}).
\]

Semonov-Tian-Shansky proved that if there is an \( \text{ad} \)-invariant, non-degenerate, symmetric bilinear form on \( \mathfrak{g} \), then a Rota-Baxter operator and a **triangular \( r \)-matrix** \( ([r, r] = 0) \) are equivalent.
Example

Let the Lie algebra $\mathfrak{g}$ be the direct sum of two subalgebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$. Then

$$T(x_1, x_2) = -x_1, \quad \forall x_1 \in \mathfrak{g}_1, x_2 \in \mathfrak{g}_2$$

is a Rota-Baxter operator of weight 1.
Let $R \in \text{End}(\mathfrak{g})$ be a solution of the modified Yang-Baxter equation:

$$[R(u), R(v)]_\mathfrak{g} = R([R(u), v]_\mathfrak{g}) + R([u, R(v)]_\mathfrak{g}) - [u, v]_\mathfrak{g},$$

from which Semenov-Tian-Shansky obtained an Infinitesimal Factorization Theorem for the Lie algebra $\mathfrak{g}$ with important applications to integrable systems. Under the transformation

$$R = \text{Id} + 2T,$$

the operator $R$ satisfies the modified Yang-Baxter equation if and only if the operator $T$ is a Rota-Baxter operator of weight 1.
Kupershmidt introduced the notion of a relative Rota-Baxter operator (also called an $O$-operator) on a Lie algebra $\mathfrak{g}$ with respect to arbitrary representation.

**Definition**

A *relative Rota-Baxter operator* on a Lie algebra $(\mathfrak{g}, [-, -]_\mathfrak{g})$ with respect to a representation $(V; \rho)$ is a linear map $T : V \rightarrow \mathfrak{g}$ satisfying the following quadratic constraint:

$$[Tu, Tv]_\mathfrak{g} = T(\rho(Tu)(v) - \rho(Tv)(u)), \quad \forall u, v \in V.$$ 

A *relative Rota-Baxter Lie algebra* is a triple $((\mathfrak{g}, [\cdot, \cdot]_\mathfrak{g}), \rho, T)$, where $(\mathfrak{g}, [\cdot, \cdot]_\mathfrak{g})$ is a Lie algebra, $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ is a representation of $\mathfrak{g}$ on a vector space $V$ and $T : V \rightarrow \mathfrak{g}$ is a relative Rota-Baxter operator.
Consider the semidirect product Lie algebra \((\mathfrak{g} \oplus V, [\cdot, \cdot]_{\times})\):

\[
[x + u, y + v]_{\times} = [x, y]_{\mathfrak{g}} + \rho(x)v - \rho(y)u.
\]

**Proposition**

A linear map \(T : V \to \mathfrak{g}\) is a relative Rota-Baxter operator if and only if the graph of \(T\),

\[
G_T := \{Tu + u \mid \forall u \in V\}
\]

is a subalgebra of the semidirect product Lie algebra \((\mathfrak{g} \oplus V, [\cdot, \cdot]_{\times})\).
Example

Let \( r \in \wedge^2 g \) be a **triangular \( r \)-matrix**. Define \( r^\#: g^* \rightarrow g \) by
\[
\langle r^\#(\xi), \eta \rangle = r(\xi, \eta).
\]
Then \( r^\# \) is a relative Rota-Baxter operator on \( g \) with respect to the coadjoint representation.

It is well known that a triangular \( r \)-matrix gives rise to a triangular Lie bialgebra, which we denote by \((g, r)\). We will see that deformations and cohomologies of triangular Lie bialgebras can be studied using the general framework of relative Rota-Baxter Lie algebras.

Example

Let \( \omega \in \wedge^2 g^* \) be a **symplectic structure** on a Lie algebra \( g \). Then \( (\omega^\#)^{-1} \) is a relative Rota-Baxter operator on the Lie algebra \( g \) with respect to the coadjoint representation, where \( \omega^\# : g \rightarrow g^* \) is defined by
\[
\langle \omega^\#(x), y \rangle = \omega(x, y)
\]
for \( x, y \in g \).
Examples

Example

Let $r \in \wedge^2 g$ be a triangular $r$-matrix. Define $r^\# : g^* \rightarrow g$ by 
\[
\langle r^\#(\xi), \eta \rangle = r(\xi, \eta).
\]
Then $r^\#$ is a relative RB operator on $g$ with respect to the coadjoint representation.

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Let \( r \in \wedge^2 \mathfrak{g} \) be a **triangular \( r \)-matrix.** Define \( r^\# : \mathfrak{g}^* \to \mathfrak{g} \) by
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\[
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\]
for \( x, y \in \mathfrak{g} \).
Rota-Baxter operators $\rightsquigarrow$ splitting of algebras

zinbiel alg. $\longrightarrow$ dendriform alg. $\longrightarrow$ pre-Lie alg.

comm. alg. $\longrightarrow$ ass. alg. $\longrightarrow$ Lie alg.

perm. alg. $\longrightarrow$ dialg. $\longrightarrow$ Leibniz alg.

Averaging operators (embedding tensors) $\rightsquigarrow$ duplication of algebras
Main results

Our approach to study deformations, cohomologies and homotopy of RB Lie algebras is using the ‘controlling algebra’.

More precisely, we construct an $L_\infty$-algebra whose Maurer-Cartan elements are relative Rota-Baxter Lie algebras.

- A relative Rota-Baxter Lie algebra gives rise to a twisted $L_\infty$-algebra that controls its deformations;
- The $l_1$ in the above twisted $L_\infty$-algebra defines a cohomology of the relative Rota-Baxter Lie algebra;
- Replace vector spaces by graded vector spaces, we get homotopy relative Rota-Baxter Lie algebras.
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- The $l_1$ in the above twisted $L_\infty$-algebra defines a cohomology of the relative Rota-Baxter Lie algebra;
- Replace vector spaces by graded vector spaces, we get homotopy relative Rota-Baxter Lie algebras.
Let $\mathfrak{g}$ be a vector space. The graded vector space
\[ C^*(\mathfrak{g}, \mathfrak{g}) = \bigoplus_{n=0}^{+\infty} C^n(\mathfrak{g}, \mathfrak{g}) = \bigoplus_{n=0}^{+\infty} \text{Hom}(\wedge^{n+1} \mathfrak{g}, \mathfrak{g}). \]
equipped with the Nijenhuis-Richardson bracket
\[ [P, Q]_{\text{NR}} = P \bar{\circ} Q - (-1)^{pq} Q \bar{\circ} P, \quad \forall P \in C^p(\mathfrak{g}, \mathfrak{g}), Q \in C^q(\mathfrak{g}, \mathfrak{g}), \]
is a graded Lie algebra, where $P \bar{\circ} Q \in C^{p+q}(\mathfrak{g}, \mathfrak{g})$ is defined by
\[ (P \bar{\circ} Q)(x_1, \cdots, x_{p+q+1}) = \sum_{\sigma \in S_{(q+1), p}} P(Q(x_{\sigma(1)}, \cdots, x_{\sigma(q+1)}), x_{\sigma(q+2)}, \cdots, x_{\sigma(p+q+1)}) \]

**Lemma**

$\mu \in \text{Hom}(\wedge^2 \mathfrak{g}, \mathfrak{g})$ defines a Lie algebra structure on $\mathfrak{g}$ if and only if
\[ [\mu, \mu]_{\text{NR}} = 0. \]
Let $\mathfrak{g}$ be a vector space. The graded vector space

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**Lemma**

$\mu \in \text{Hom}(\wedge^2\mathfrak{g}, \mathfrak{g})$ defines a Lie algebra structure on $\mathfrak{g}$ if and only if

$$[\mu, \mu]_{\text{NR}} = 0.$$
Define $d_\mu$ by $d_\mu = [\mu, \cdot]_{\text{NR}}$. Then $(C^*(\mathfrak{g}, \mathfrak{g}), [\cdot, \cdot]_{\text{NR}}, d_\mu)$ is a dgLa, that controls deformations of the Lie algebra $(\mathfrak{g}, \mu)$.

**Theorem**

Let $(\mathfrak{g}, \mu)$ be a Lie algebra and $\mu' \in \text{Hom}(\wedge^2 \mathfrak{g}, \mathfrak{g})$. Then $(\mathfrak{g}, \mu + \mu')$ is a Lie algebra if and only if $\mu'$ is a Maurer-Cartan element of the dgLa $(C^*(\mathfrak{g}, \mathfrak{g}), [\cdot, \cdot]_{\text{NR}}, d_\mu)$. 
It is well known that the Chevalley-Eilenberg coboundary operator $d_{CE} : \text{Hom}(\wedge^k g, g) \rightarrow \text{Hom}(\wedge^{k+1} g, g)$ for a Lie algebra $g$ is given by

$$d_{CE}f(x_1, \cdots x_{k+1}) = \sum_{i=1}^{k} (-1)^{i+1} [x_i, f(x_1, \cdots, \hat{x}_i, \cdots, x_{k+1})] + \sum_{i<j} (-1)^{i+j} f([x_i, x_j], x_1, \cdots, \hat{x}_i, \cdots, \hat{x}_j, \cdots x_{k+1}).$$

**Theorem-Definition**

*Let $(g, \mu)$ be a Lie algebra. Then*

$$d_{CE}f = (-1)^{k+1} [\mu, f]_{NR}, \quad \forall f \in \text{Hom}(\wedge^k g, g).$$
Let $V$ be a $\mathbb{Z}$-graded vector space. Denote by $\text{Hom}^n(\text{Sym}(V), V)$ the space of degree $n$ linear maps from the graded vector space $\text{Sym}(V) = \bigoplus_{i=1}^{+\infty} \text{Sym}^i(V)$ to the $\mathbb{Z}$-graded vector space $V$. Set $C^m(V, V) := \text{Hom}^n(\text{Sym}(V), V)$ and $C^*(V, V) := \bigoplus_{n \in \mathbb{Z}} C^m(V, V)$. Then we have the graded Nijenhuis-Richardson bracket $[\cdot, \cdot]_{NR}$ on the graded vector space $C^*(V, V)$, which is a graded Lie algebra.

Theorem-Definition

The Maurer-Cartan elements $\sum_{k=1}^{+\infty} l_k$ of the graded Lie algebra $(C^*(V, V), [\cdot, \cdot]_{NR})$ are the $L_\infty$-algebra structures on $V$. 
The gLa structure for associative algebras is given by the Gerstenhaber bracket on $\bigoplus \text{Hom} (\otimes^k V, V)$. The cohomology groups for the deformation theories of associative algebras are the Hochschild cohomology groups;

The gLa structure for pre-Lie algebras (left-symmetric algebras) is given by the Matsushima-Nijenhuis bracket on $\bigoplus \text{Hom} (\wedge^{k-1} V \otimes V, V)$. The cohomology groups for the deformation theories of pre-Lie algebras are the Dzhumadil’daev cohomology groups;

The controlling algebra for Leibniz algebras is a gLa;

The controlling algebra for $n$-Lie algebras is a gLa.
An $L_\infty$-algebra is a $\mathbb{Z}$-graded vector space $\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}^k$ equipped with a collection $(k \geq 1)$ of linear maps $l_k : \otimes^k \mathfrak{g} \to \mathfrak{g}$ of degree 1 with the property that, for any homogeneous elements $x_1, \cdots, x_n \in \mathfrak{g}$, we have

(i) (graded symmetry) for every $\sigma \in \mathfrak{S}_n$,

$$l_n(x_{\sigma(1)}, \cdots, x_{\sigma(n-1)}, x_{\sigma(n)}) = \varepsilon(\sigma)l_n(x_1, \cdots, x_{n-1}, x_n),$$

(ii) (generalized Jacobi identity) for all $n \geq 1$,

$$\sum_{i=1}^{n} \sum_{\sigma \in \mathfrak{S}_{(i, n-i)}} \varepsilon(\sigma)l_{n-i+1}(l_i(x_{\sigma(1)}, \cdots, x_{\sigma(i)}), x_{\sigma(i+1)}, \cdots, x_{\sigma(n)}) = 0.$$
Maurer-Cartan elements

Definition (Getzler)

The set of **Maurer-Cartan elements** of an $L_\infty$-algebra $\mathfrak{g}$ is the set of those $\alpha \in \mathfrak{g}^0$ satisfying the Maurer-Cartan equation

\[
\sum_{k=1}^{+\infty} \frac{1}{k!} l_k(\alpha, \cdots, \alpha) = 0.
\]

Main results: the controlling algebra of relative Rota-Baxter Lie algebra

Let $\mathfrak{g}$ and $V$ be vector spaces. Denote by $L'$ the following graded vector space

$$L' = \bigoplus_{n=0}^{+\infty} \text{Hom} (\wedge^{n+1} \mathfrak{g}, \mathfrak{g}) \oplus \text{Hom} (\wedge^n \mathfrak{g} \otimes V, V),$$

and denote by $\mathfrak{h} = \bigoplus_{n=0}^{+\infty} \text{Hom} (\wedge^{n+1} V, \mathfrak{g})$.

Theorem (Lazarev-S-Tang)

There is an $L_\infty$-algebra structure $\{l_k\}_{k=1}^{+\infty}$ on the graded vector space $L'[1] \oplus \mathfrak{h}$, such that its Maurer-Cartan elements are exactly relative Rota-Baxter Lie algebra structures on $\mathfrak{g}$ and $V$.

Maurer-Cartan elements contains

$$\mu \in \text{Hom} (\wedge^2 \mathfrak{g}, \mathfrak{g}), \quad \rho \in \text{Hom} (\mathfrak{g} \otimes V, V), \quad T \in \text{Hom} (V, \mathfrak{g}).$$
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Maurer-Cartan elements contains

$$\mu \in \text{Hom}(\wedge^2 \mathfrak{g}, \mathfrak{g}), \quad \rho \in \text{Hom}(\mathfrak{g} \otimes V, V), \quad T \in \text{Hom}(V, \mathfrak{g}).$$
Let $(\mu, \rho, T)$ be a relative Rota-Baxter Lie algebra structure on $g$ and $V$. Define $l_{k}^{\mu, \rho, T}$ by

$$l_{k}^{\mu, \rho, T}(x_1, \cdots, x_k) = \sum_{n=0}^{+\infty} \frac{1}{n!} l_{k+n}(\underbrace{\mu + \rho + T, \cdots, \mu + \rho + T}_{n}, x_1, \cdots, x_k).$$

According to Getzler’s results, $(L'[1] \oplus \mathfrak{h}, \{l_{k}^{\mu, \rho, T}\}_{k=1}^{+\infty})$ is an $L_{\infty}$-algebra, called the twisted $L_{\infty}$-algebra.

**Theorem (Lazarev-S-Tang)**

The triple $(\mu + \mu', \rho + \rho', T + T')$ is again a relative Rota-Baxter Lie algebra for

$$\mu' \in \text{Hom}(\wedge^2 g, g), \quad \rho' \in \text{Hom}(g, \mathfrak{gl}(V)), \quad T' \in \text{Hom}(V, g),$$

if and only if $\mu' + \rho' + T'$ is a Maurer-Cartan element of the $L_{\infty}$-algebra $(L'[1] \oplus \mathfrak{h}, \{l_{k}^{\mu, \rho, T}\}_{k=1}^{+\infty})$. 
There is a well known slogan, often attributed to Deligne, Drinfeld and Kontsevich: *every reasonable deformation theory is controlled by a differential graded Lie algebra (an $L_\infty$-algebra), determined up to quasi-isomorphism.*
Related works in Poisson geometry


Recall that $\mathfrak{h} = \bigoplus_{k=1}^{+\infty} \text{Hom} (\wedge^{n+1} V, \mathfrak{g})$ and

$$L' = \bigoplus_{n=0}^{+\infty} \text{Hom} (\wedge^{n+1} \mathfrak{g}, \mathfrak{g}) \oplus \text{Hom} (\wedge^n \mathfrak{g} \otimes V, V).$$

**Theorem (Lazarev-S.-Tang)**

Let $((\mathfrak{g}, \mu), \rho, T)$ be a relative Rota-Baxter Lie algebra. Then the $L_\infty$-algebra $(L'[1] \oplus \mathfrak{h}, \{\mathfrak{l}_k^{\mu, \rho, T}\}_{k=1}^{+\infty})$ is a strict extension of the $L_\infty$-algebra $(\text{dgLa}) L'[1]$ by the $L_\infty$-algebra $(\text{dgLa}) \mathfrak{h}$, that is, we have the following short exact sequence of $L_\infty$-algebras:

$$0 \longrightarrow \mathfrak{h} \overset{\iota}{\longrightarrow} L'[1] \oplus \mathfrak{h} \overset{p}{\longrightarrow} L'[1] \longrightarrow 0,$$

where $\iota(\theta) = (0, \theta)$ and $p(f, \theta) = f$. 
Let $(\mu, \rho, T)$ be a relative Rota-Baxter Lie algebra structure on $g$ and $V$. Define the space of $n$-cochains $\mathcal{C}^n(\mu, \rho, T)$ by

$$\left( \text{Hom}(\wedge^n g, g) \oplus \text{Hom}(\wedge^{n-1} g \otimes V, V) \right) \oplus \text{Hom}(\wedge^{n-1} V, g).$$

Define the 	extbf{coboundary operator} $\mathcal{D} : \mathcal{C}^n(\mu, \rho, T) \to \mathcal{C}^{n+1}(\mu, \rho, T)$ by

$$\mathcal{D}(f, \theta) = (-1)^n \lambda_{1}^{\mu, \rho, T}(f, \theta).$$

We obtain that $(\bigoplus_{n=0}^{+\infty} \mathcal{C}^n(\mu, \rho, T), \mathcal{D})$ is a cochain complex.

**Definition**

The cohomology of the cochain complex $(\bigoplus_{n=0}^{+\infty} \mathcal{C}^n(\mu, \rho, T), \mathcal{D})$ is called the 	extbf{cohomology of the relative Rota-Baxter Lie algebra}. We denote its $n$-th cohomology group by $\mathcal{H}^n(\mu, \rho, T)$.
Consider infinitesimal deformations of a relative Rota-Baxter Lie algebra structure \((\mu, \rho, T)\) on \(g\) and \(V\):

\[
\begin{align*}
\mu_t &= \mu + t\mu' \\
\rho_t &= \rho + t\rho' \\
T_t &= T + tT' .
\end{align*}
\]

**Theorem**

There is a one-to-one correspondence between equivalence classes of infinitesimal deformations of the relative Rota-Baxter Lie algebra \((g, \mu, \rho, T)\) and the second cohomology group \(H^2(\mu, \rho, T)\).
Relations among cohomologies

Denote by

\[ C^n(\mu, \rho) = \text{Hom}(\wedge^n g, g) \oplus \text{Hom}(\wedge^{n-1} g \otimes V, V), \]
\[ C^n(T) = \text{Hom}(\wedge^{n-1} V, g). \]

The coboundary operator \( D : C^n(\mu, \rho, T) \rightarrow C^{n+1}(\mu, \rho, T) \) is given by

\[ D(f, \theta) = (-1)^n (-[\pi, f]_{NR}, [[\pi, T]_{NR}, \theta]_{NR} + \frac{1}{n!} \underbrace{\cdots [[f, T]_{NR}, T]_{NR}, \cdots, T]_{NR}}_{n}) \]
\[ = (\partial f, \delta \theta + h_T f). \]

\( D \) can be well-explained by the following diagram:

\[ \cdots \rightarrow C^n(\mu, \rho) \xrightarrow{\partial} C^{n+1}(\mu, \rho) \xrightarrow{\partial} C^{n+2}(\mu, \rho) \rightarrow \cdots \]
\[ \xrightarrow{h_T} \]
\[ \cdots \rightarrow C^n(T) \xrightarrow{\delta} C^{n+1}(T) \xrightarrow{\delta} C^{n+2}(T) \rightarrow \cdots. \]
Theorem (Lazarev-S.-Tang)

There is a short exact sequence of the cochain complexes:

\[ 0 \longrightarrow (\bigoplus_{n=0}^{+\infty} C^n(T), \delta) \overset{\iota}{\longrightarrow} (\bigoplus_{n=0}^{+\infty} C^n(\mu, \rho, T), D) \overset{p}{\longrightarrow} (\bigoplus_{n=0}^{+\infty} C^n(\mu, \rho), \partial) \longrightarrow 0, \]

and there is a long exact sequence of the cohomology groups:

\[ \cdots \longrightarrow H^n(T) \overset{H^n(\iota)}{\longrightarrow} H^n(\mu, \rho, T) \overset{H^n(p)}{\longrightarrow} H^n(\mu, \rho) \overset{c^n}{\longrightarrow} H^{n+1}(T) \longrightarrow \cdots, \]

where the connecting map \( c^n \) is defined by \( c^n([\alpha]) = [h_T \alpha] \), for all \( [\alpha] \in H^n(\mathfrak{g}, \rho) \).
Let \((g, [\cdot, \cdot]_g, T)\) be a Rota-Baxter Lie algebra. Define the space of \(n\)-cochains \(C^n_{RB}(g, T)\) by

\[
C^n_{RB}(g, T) := C^n_{Lie}(g; g) \oplus C^n(T) = \text{Hom}(\wedge^n g, g) \oplus \text{Hom}(\wedge^{n-1} g, g).
\]

Define \(D_{RB}: C^n_{RB}(g, T) \rightarrow C^{n+1}_{RB}(g, T)\) by

\[
D_{RB}(f, \theta) = \left( d_{CE}f, \delta \theta + \Omega f \right), \quad \forall f \in \text{Hom}(\wedge^n g, g), \quad \theta \in \text{Hom}(\wedge^{n-1} g, g),
\]

where \(\Omega : \text{Hom}(\wedge^n g, g) \rightarrow \text{Hom}(\wedge^n g, g)\) is defined by

\[
(\Omega f)(x_1, \cdots, x_n) = (-1)^n \left( f(Tx_1, \cdots, Tx_n) - \sum_{i=1}^{n} Tf(Tx_1, \cdots, Tx_{i-1}, x_i, Tx_{i+1}, \cdots, Tx_n) \right).
\]
Theorem (Lazarev-S.-Tang)

The map $D_{RB}$ is a coboundary operator, i.e. $D_{RB} \circ D_{RB} = 0$.

Definition (Lazarev-S.-Tang)

Let $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, T)$ be a Rota-Baxter Lie algebra. The cohomology of the cochain complex $(\bigoplus_{n=0}^{+\infty} \mathfrak{c}^n_{RB}(\mathfrak{g}, T), D_{RB})$ is taken to be the cohomology of the Rota-Baxter Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, T)$. Denote the $n$-th cohomology group by $\mathcal{H}^n_{RB}(\mathfrak{g}, T)$. 
Let \((g, [\cdot, \cdot]_g, r)\) be a triangular Lie bialgebra. Define the space of \(n\)-cochains \(C^n_{\text{TLB}}(g, r)\) by

\[
C^n_{\text{TLB}}(g, r) := \text{Hom} (\wedge^n g, g) \oplus \wedge^n g.
\]

Define the \textit{coboundary operator} \(D_{\text{TLB}} : C^n_{\text{TLB}}(g, r) \to C^{n+1}_{\text{TLB}}(g, r)\) by

\[
D_{\text{TLB}}(f, \chi) = \left( d_{CE} f, d_r \chi + \Theta f \right), \quad \forall f \in \text{Hom} (\wedge^n g, g), \ \chi \in \wedge^n g,
\]

where \(d_r : \wedge^n g \to \wedge^{n+1} g\) is given by \(d_r \chi = [r, \chi]\) and \(\Theta : \text{Hom} (\wedge^n g, g) \to \wedge^{n+1} g\) is defined by

\[
\langle \Theta f, \xi_1 \wedge \cdots \wedge \xi_{n+1} \rangle = \sum_{i=1}^{n+1} (-1)^{i+1} \langle \xi_i, f (r^\# (\xi_1), \cdots, r^\# (\xi_{i-1}), r^\# (\xi_{i+1}), \cdots, r^\# (\xi_{n+1})) \rangle.
\]
Theorem (Lazarev-S.-Tang)

The map $\mathcal{D}_{TLB}$ is a coboundary operator, i.e. $\mathcal{D}_{TLB} \circ \mathcal{D}_{TLB} = 0$.

Definition (Lazarev-S.-Tang)

Let $(\mathfrak{g}, [\cdot, \cdot]_g, r)$ be a triangular Lie bialgebra. The cohomology of the cochain complex $(\bigoplus_{n=0}^{\infty} \mathfrak{c}_n^{TLB}(\mathfrak{g}, r), \mathcal{D}_{TLB})$ is called the cohomology of the triangular Lie bialgebra $(\mathfrak{g}, [\cdot, \cdot]_g, r)$. Denote the $n$-th cohomology group by $\mathcal{H}_{n}^{TLB}(\mathfrak{g}, r)$. 
Let \((g, \{l_k\}_{k=1}^{+\infty})\) be an \(L_\infty\)-algebra and \((V, \{\rho_k\}_{k=1}^{+\infty})\) a representation. Then \((\mathfrak{h} := \bigoplus_{n\in\mathbb{Z}} \text{Hom}^n(\text{Sym}(V), g), \{l_k\}_{k=1}^{+\infty})\) is an \(L_\infty\)-algebra, where \(l_k\) is given by

\[
l_k(a_1, \cdots, a_k) = P \left[ \cdots \left[ \sum_{k=1}^{+\infty} (l_k + \rho_k), a_1 \right]_{\text{NR}}, a_2 \right]_{\text{NR}}, \cdots, a_k \right]_{\text{NR}}.\]

**Theorem-Definition (Lazarev-S.-Tang)**

A degree 0 element \(T = \sum_{k=1}^{+\infty} T_k \in \text{Hom}(\text{Sym}(V), g)\) is a homotopy relative Rota-Baxter operator on \((g, \{l_k\}_{k=1}^{+\infty})\) with respect to the representation \((V, \{\rho_k\}_{k=1}^{+\infty})\) if and only if \(T = \sum_{k=1}^{+\infty} T_k\) is a Maurer-Cartan element of the \(L_\infty\)-algebra \((\mathfrak{h}, \{l_k\}_{k=1}^{+\infty})\).
Let \((\mathfrak{g}, \{l_k\}_{k=1}^{+\infty})\) be an \(L_\infty\)-algebra and \((V, \{\rho_k\}_{k=1}^{+\infty})\) a representation. Then \((\mathfrak{h} := \bigoplus_{n \in \mathbb{Z}} \text{Hom}^n(\text{Sym}(V), \mathfrak{g}), \{l_k\}_{k=1}^{+\infty})\) is an \(L_\infty\)-algebra, where \(l_k\) is given by

\[
l_k(a_1, \cdots, a_k) = P \left[ \cdots \left[ \sum_{k=1}^{+\infty} (l_k + \rho_k), a_1 \right]_{\text{NR}}, a_2 \right]_{\text{NR}}, \cdots, a_k \right]_{\text{NR}}.
\]

Theorem-Definition (Lazarev-S.-Tang)

A degree 0 element \(T = \sum_{k=1}^{+\infty} T_k \in \text{Hom}(\text{Sym}(V), \mathfrak{g})\) is a homotopy relative Rota-Baxter operator on \((\mathfrak{g}, \{l_k\}_{k=1}^{+\infty})\) with respect to the representation \((V, \{\rho_k\}_{k=1}^{+\infty})\) if and only if \(T = \sum_{k=1}^{+\infty} T_k\) is a Maurer-Cartan element of the \(L_\infty\)-algebra \((\mathfrak{h}, \{l_k\}_{k=1}^{+\infty})\).
Remark

Dotsenko and Khoroshkin studied the homotopy of Rota-Baxter operators on associative algebras in


and noted that “in general compact formulas are yet to be found”. For Rota-Baxter Lie algebras, one encounters a similarly challenging situation.

We use the approach of ‘controlling algebras’ and their MC elements to formulate the notion of a homotopy version of a relative Rota-Baxter Lie algebra. Hopefully our research will provide some useful information for the operadic research of the homotopy of relative Rota-Baxter Lie algebras.
A degree 0 element $T = \sum_{k=1}^{+\infty} T_k \in \text{Hom} \left( \text{Sym}(V), \mathfrak{g} \right)$ with $T_k \in \text{Hom} \left( \text{Sym}^k(V), \mathfrak{g} \right)$ is a homotopy relative Rota-Baxter operator on an $L_\infty$-algebra $(\mathfrak{g}, \{l_k\}_{k=1}^{+\infty})$ with respect to the representation $(V, \{\rho_k\}_{k=1}^{+\infty})$ if the following equalities hold for all $p \geq 1$ and all homogeneous elements $v_1, \cdots, v_p \in V$,

$$\sum_{1 \leq t \leq p-1} \sum_{k_1 + \cdots + k_m = t} \sum_{\sigma \in S(k_1, \cdots, k_m, 1, p-1-t)} \frac{\varepsilon(\sigma)}{m!} \cdot T_{p-t} \left( \rho_{m+1} \left( T_{k_1} (v_{\sigma(1)}, \cdots, v_{\sigma(k_1)}), \cdots, T_{k_m} (v_{\sigma(k_1+\cdots+k_m-1+1)}, \cdots, v_{\sigma(t)}, v_{\sigma(t+1)}) \right), v_{\sigma(t+2)}, \cdots, v_{\sigma(p)} \right)$$

$$= \sum_{k_1 + \cdots + k_n = p} \sum_{\sigma \in S(k_1, \cdots, k_n)} \frac{\varepsilon(\sigma)}{n!} \cdot l_n \left( T_{k_1} (v_{\sigma(1)}, \cdots, v_{\sigma(k_1)}), \cdots, T_{k_n} (v_{\sigma(k_1+\cdots+k_n-1+1)}, \cdots, v_{\sigma(p)}) \right).$$
A **pre-Lie**$_\infty$-**algebra** structure on a graded vector space $V$ consists of degree 1 linear map $\{\theta_k\}_{k=1}^\infty$ from the graded vector space $\text{Sym}(V) \otimes V$ to $V$ such that $\sum_{k=1}^{+\infty} \theta_k$ is a Maurer-Cartan element of the graded Lie algebra $(C^*(V, V), [\cdot, \cdot]_{MN})$. 

---

Let $(g, \{\theta_k\}_{k=1}^{\infty})$ be a pre-$\operatorname{Lie}_\infty$-algebra. Define $l_k$ by

$$l_k(x_1, \ldots, x_k) = \sum_{i=1}^{k} (-1)^{x_i(x_{i+1} + \cdots + x_k)} \theta_k(x_1, \ldots, \hat{x_i}, \ldots, x_k, x_i).$$

Then $(g, \{l_k\}_{k=1}^{\infty})$ is an $L_\infty$-algebra, denoted by $g^C$. For all $k \geq 1$, we define $L_k : \operatorname{Sym}^{k-1}(g) \to \operatorname{gl}(g)$ by

$$L_k (x_1, \ldots, x_{k-1})(x_k) = \theta_k(x_1, \ldots, x_{k-1}, x_k).$$

**Proposition**

*With the above notation, $(g, \{L_k\}_{k=1}^{\infty})$ is a representation of the sub-adjacent $L_\infty$-algebra $g^C$. Moreover, the identity map $\operatorname{Id} : g \to g$ is a strict homotopy relative Rota-Baxter operator on the $L_\infty$-algebra $g^C$ with respect to the representation $(g, \{L_k\}_{k=1}^{\infty})$.***
Theorem (Lazarev-S.-Tang)

Let $T \in \text{Hom} (V, \mathfrak{g})$ be a strict homotopy relative Rota-Baxter operator on an $L_\infty$-algebra $(\mathfrak{g}, \{l_k\}_{k=1}^{+\infty})$ with respect to the representation $(V, \{\rho_k\}_{k=1}^{+\infty})$. Then $(V, \{\theta_k\}_{k=1}^{+\infty})$ is a pre-Lie$_\infty$-algebra, where $\theta_k : \otimes^k V \to V \ (k \geq 1)$ are linear maps of degree 1 defined by

$$\theta_k(v_1, \cdots, v_k) := \rho_k(Tv_1, \cdots, Tv_{k-1}, v_k), \quad \forall v_1 \cdots, v_k \in V.$$
The notion of an $r_\infty$-matrix was introduced by A. Voronov and his collaborators in

D. Bashkirov and A. Voronov, $r_\infty$-Matrices, triangular $L_\infty$-bialgebras, and quantum$_\infty$ groups, 2015.

As expected, an $r_\infty$-matrix gives rise to a homotopy relative Rota-Baxter operator with respect to the coadjoint representation.
Definition

A $V$-structure consists of a quadruple $(L, \mathfrak{h}, P, \Delta)$ where

- $(L, [\cdot, \cdot])$ is a graded Lie algebra,
- $\mathfrak{h}$ is an abelian graded Lie subalgebra of $(L, [\cdot, \cdot])$,
- $P : L \rightarrow L$ is a projection, that is $P \circ P = P$, whose image is $\mathfrak{h}$ and kernel is a graded Lie subalgebra of $(L, [\cdot, \cdot])$,
- $\Delta$ is an element in $\ker(P)$ such that $[\Delta, \Delta] = 0$. 
Theorem (T. Voronov)

Let \((L, \mathfrak{h}, P, \Delta)\) be a \(V\)-structure. Then the graded vector space \(L[1] \oplus \mathfrak{h}\) is an \(L_\infty\)-algebra where

\[
l_1(x, a) = (-[\Delta, x], P(x + [\Delta, a])),
\]
\[
l_2(x, y) = (-1)^x [x, y],
\]
\[
l_k(x, a_1, \cdots, a_{k-1}) = P[\cdots [[x, a_1], a_2] \cdots , a_{k-1}], \quad k \geq 2,
\]
\[
l_k(a_1, \cdots, a_{k-1}, a_k) = P[\cdots [[\Delta, a_1], a_2] \cdots , a_k], \quad k \geq 2.
\]

Here \(a, a_1, \cdots, a_k \in \mathfrak{h}\) and \(x, y \in L\).

Remark

Let \(L'\) be a graded Lie subalgebra of \(L\) that satisfies \([\Delta, L'] \subset L'\). Then \(L'[1] \oplus \mathfrak{h}\) is an \(L_\infty\)-subalgebra of the above \(L_\infty\)-algebra \((L[1] \oplus \mathfrak{h}, \{l_k\}_{k=1}^{+\infty})\).
A V-structure

Let $\mathfrak{g}$ and $V$ be two vector spaces.

Proposition (Lazarev-S.-Tang)

We have a $V$-structure $(L, \mathfrak{h}, P, \Delta)$ as follows:

- the graded Lie algebra $(L, [\cdot, \cdot])$ is given by
  
  $$( \oplus_{n=0}^{+\infty} C^{n+1}(\mathfrak{g} \oplus V, \mathfrak{g} \oplus V), [\cdot, \cdot]_{\text{NR}} );$$

- the abelian graded Lie subalgebra $\mathfrak{h}$ is given by
  
  $$\mathfrak{h} := \oplus_{n=0}^{+\infty} \text{Hom}(\wedge^{n+1} V, \mathfrak{g});$$

- $P : L \longrightarrow L$ is the projection onto the subspace $\mathfrak{h}$, and $\Delta = 0$.

Consequently, we obtain an $L_\infty$-algebra $(L[1] \oplus \mathfrak{h}, \{l_k\}_{k=1}^{+\infty}),$

$$l_1(Q, \theta) = P(Q),$$

$$l_2(Q, Q') = (-1)^Q [Q, Q']_{\text{NR}},$$

$$l_k(Q, \theta_1, \cdots, \theta_{k-1}) = P[\cdots [Q, \theta_1]_{\text{NR}}, \cdots, \theta_{k-1}]_{\text{NR}},$$

for $\theta, \theta_1, \cdots, \theta_{k-1} \in \mathfrak{h}, Q, Q' \in L.$
Note that $L' = \bigoplus_{k=0}^{+\infty}(\text{Hom} (\wedge^{n+1}g, g) \oplus \text{Hom} (\wedge^n g \otimes V, V))$ is a subalgebra of $L$.

**Theorem (Lazarev-S.-Tang)**

With above notation, $(L'[1] \oplus \mathfrak{h}, \{l_k\}_{k=1}^{+\infty})$ is an $L_\infty$-algebra, where $l_k$ are given by

\[
\begin{align*}
l_2(Q, Q') &= (-1)^Q [Q, Q']_{\text{NR}}, \\
l_k(Q, \theta_1, \cdots, \theta_{k-1}) &= P[\cdots [Q, \theta_1]_{\text{NR}}, \cdots, \theta_{k-1}]_{\text{NR}},
\end{align*}
\]

for $\theta_1, \cdots, \theta_{k-1} \in \mathfrak{h}$, $Q, Q' \in L'$.

Let $\mu \in \text{Hom} (\wedge^2 g, g)$, $\rho \in \text{Hom} (g \otimes V, V)$, $T \in \text{Hom} (V, g)$. Then $((g, \mu), \rho, T)$ is a **relative Rota-Baxter Lie algebra** if and only if $\mu + \rho + T$ is a **Maurer-Cartan element of the** $L_\infty$-**algebra** $(L'[1] \oplus \mathfrak{h}, \{l_i\}_{i=1}^{+\infty})$.  

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Proof

Let $\mu + \rho + T$ be a Maurer-Cartan element. Then we have

$$\sum_{k=1}^{+\infty} \frac{1}{k!} l_k \left( \mu + \rho + T, \cdots, \mu + \rho + T \right)$$

$$= \frac{1}{2!} l_2 \left( \mu + \rho + T, \mu + \rho + T \right)$$

$$+ \frac{1}{3!} l_3 \left( \mu + \rho + T, \mu + \rho + T, \mu + \rho + T \right)$$

$$= \left( -\frac{1}{2} [\mu + \rho, \mu + \rho]_{NR}, \frac{1}{2} [[\mu + \rho, T]_{NR}, T]_{NR} \right)$$

$$= (0, 0).$$

Thus, we obtain $[\mu + \rho, \mu + \rho]_{NR} = 0$ and $[[\mu + \rho, T]_{NR}, T]_{NR} = 0$, which implies that $(g, \mu)$ is a Lie algebra, $(V; \rho)$ is its representation and $T$ is a relative Rota-Baxter operator on the Lie algebra $(g, \mu)$ with respect to the representation $(V; \rho)$. 


Thanks for your attention!