Higher holonomy

Representations up to homotopy

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Today's talk is about work in progress. I was led to it by a paper by my colleague at Penn, Jonathan Block, and his student, Aaron Smith. I'll tell you more about it after presenting some background.

There are many technical details which I will not discuss. A source I found very helpful is

Picken and Caetano-An axiomatic definition of holonomy.

Classical differential geometry is “a language the muse did not sing at my cradle” The subject developed from a finite dimensional point of view. In particular, classical *parallel transport* is defined in the context of a *connection* on a smooth vector bundle $p : E \to M$.

Connection can refer to a differential 1-form or a set of horizontal subspaces in the tangent bundle $Tp : TE \to TM$ or....

Remark

There is great ambiguity in the literature concerning the terms

- *connection*
- *connection form*
- *parallel transport*
- *holonomy*
- *monodromy*
- *covariant derivative.*

For this talk, connection will usually mean connection form.

There are formal definitions which distinguish these tools. This can be very important when one term appears in a theorem, but another appears in the proof. *Choose the tool to fit the problem.*

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Let $PM = M^I$, the space of smooth paths in M.

Given a connection on a smooth vector bundle $p : E \to M$, the corresponding *parallel transport* $\tau : PM \times_M E \to E$ is constructed by lifting a path $\lambda : I \to M$ in M to a horizontal path in E, *unique!* if a starting point is specified.

Theorem

A vector bundle with typical fiber V and flat connection on M determines a representation $\rho : \pi_1(M) \to GL(V)$, given via the holonomy of a based loop.

According to n-lab:

> *Apparently one of the oldest occurrences of the idea that a principal bundle with connection over a connected base space may be reconstructed from its holonomies around all smooth loops (for any fixed base point)*

appeared in or was implied by Kobayashi in 1954.

Established for principal bundles, there is a corresponding result for associated vector bundles with typical fiber V.

The *holonomy* with respect to a curve is given by the evaluation of the transfer τ on the path in M. It descends to a representation of $\pi_1(M)$ as a result of the flatness.

The *holonomy group* is the image $\tau_* : \pi_1(M) \to GL(V)$ as a subgroup of the structure group of the bundle.

Going from a representation $\pi_1(M) \to GL(V)$ to an ordinary vector bundle with flat connection is achieved by the associated $GL(V)$-bundle construction.

Regarding τ as homotopy lifting (rather than just path lifting), we have that
$\tau : I \times V \to E$ so that

$$\begin{array}{ccc} I \times V & \xrightarrow{\tau} & E \\ \downarrow & & \downarrow p \\ I & \xrightarrow{\lambda} & M \end{array}$$

is commutative.

However, by passing to the homotopy classes of based loops in M, a lot of information is lost. To retrieve some of that information, let us stay on the level of maps and homotopies.

To work at this level, K.T. Chen introduced an idea of *differentiable spaces* and invented differential forms on them. In particular, a based loop space became a differentiable space. Chen's work on the "de Rham" cohomology of loop spaces on a manifold was a breakthrough, giving rise to his idea of *generalized holonomy* using his technology of iterated integrals.

Just as classical holonomy is given by the evaluation of the lifting τ, consider *higher holonomy* by lifting $\sigma : \Delta^n \to M$ to $\tau^n : \Delta^n \times V \to E$, but now with $E \to M$ a graded vector bundle with fiber V, a graded vector space. The higher holonomy corresponds to a *generalized connection* as a set of forms of total degree 0 on the path space $PM = M^I$ of M

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Later, such higher holonomy was related to a notion of *representation up to homotopy*.

Consider a graded vector bundle $p : E = \prod E^k \to M$.

Let $End^p(E)$ denote the degree p part of the endomorphism bundle of E:

$$End^p(E) = \prod Hom(E^k, E^{k+p}).$$

A $\mathbb{Z}$-graded connection *form* is just the analog of a classical connection form, but with careful attention to grading and signs. It corresponds to (a family of) differential forms with values in $End(E)$.

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Let $\Omega^\bullet(M)$ be the graded algebra of smooth differential forms on M and let $\Omega^\bullet(M; E)$ be the graded $\Omega^\bullet(M)$-module of forms with values in $End(E)$. As usual, it is useful to describe graded connections locally (though without local coordinates), so the appropriate covariant derivative can be written: $d + A$, where $A = A_1 + A_2 + \cdots$ with $A_p \in \Omega^p(M; E^{1-p})$

What then corresponds to ‘higher holonomy’?

To capture such 'higher structure', Chen used maps of simplices $\sigma : \Delta^k \to M$. Denote the standard *ordered* n-simplex Δ^n as $< 0, 1, \cdots, n >$ with vertices labelled $0, 1, \cdots, n$. Sub-simplices are denoted $< i_0, i_1, \cdots, i_j >$. The face and degeneracy maps for a simplicial set are:

$$\partial_q < 0, 1, \cdots, n > = < 0, 1, \cdots, q-1, q+1, \cdots, n >$$
$$s_q < 0, 1, \cdots, n > = < 0, 1, \cdots, q, q, \cdots, n >$$

Definition

$Sing(M)$ is the set of (smooth) maps of simplices $\sigma : \Delta^k \to M$. For $\sigma : \Delta^k \to M$, we denote by V_i the fibre over the image of the vertex $i \in \Delta$.

Definition

A *representation up to homotopy* of $Sing(M)$ on a graded vector space V is a collection of maps $\{\theta_k\}_{k\geq 0}$ which assign to any smooth k-simplex $\sigma : \Delta^k \to M$ a map $\theta_k(\sigma) : I^{k-1} \times V_0 \to V_k$ satisfying, for any $v \in V_0$, the relations:

θ_0 is the identity on V_0

$\theta_k(\sigma)(t_1, \cdots, t_{k-1}, -) : V_0 \to V_k$ is an isomorphism for any $(t_1, \cdots, t_{k-1})$.

For any $1 \leq p \leq k-1$ and $e \in V_0$,

$$\theta_k(\sigma)(\cdots, t_p = 0, \cdots, v) = \theta_{k-1}(\partial_p \sigma)(\cdots, \hat{t}_p, \cdots, , v)$$

$$\theta_k(\sigma)(\cdots, t_p = 1, \cdots, v) =$$

$$\theta_p(< 0, \cdots, p >)(t_1, \cdots, t_{p-1}, \theta_q(< p, \cdots, k >)(t_{p+1}, \cdots, t_k, v)).$$

There is a similar but earlier use of that name. They are also called *homotopy coherent representations*.

In this setting, *coherence* refers to the compatibility of the θ_n with respect to the boundary of the cubes and simplices.

Notice the ‘cubical’ nature of the condition, similar to Sugawara’s *strong homotopy multiplicative* for A_∞-maps between associative H-spaces..

Remark

It’s worth pointing out that A_∞-maps between strictly associative dg algebras were studied before my theses by Sugawara as parameterized by cubes and called strong homotopy muliplicative. A_∞-maps of A_∞-spaces are parameterized by more complicated polyhedra.

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There is an alternate definition in terms of forms:
For every $t \in I$, the evaluation map $ev_t : PM \to M$ sends γ to $\gamma(t)$.

Let W_t be the pull back of E to PM along ev_t.

That is, $(W_t)_\gamma = V_{\gamma(t)}$.

For $0 \leq s \leq t \leq 1$, let $Hom^q(W_s, W_t)$ be the space of degree q graded homomorphisms from W_s to W_t. Define

$$\Omega^p(PM, Hom^q(W_s, W_t)),$$

the vector space of smooth p-forms with coefficients in $Hom^q(W_s, W_t)_\gamma$.

Definition

Alternate: Given a graded vector bundle $p : E \to M$, a homotopy coherent representation on E of the smooth singular simplicial set $Sing(M)$ of M consists of a family of forms $\Psi_p(s,t)$ for all $0 \le s \le t \le 1$

$$\Psi_p(s,t) \in \Omega^p(PM, Hom^{-p}(W_s, W_t))$$

satisfying the following at each $\gamma \in PM$:

1. $\Psi_0(s,s)_\gamma$ *is the identity map in* $Hom^0(V_{\gamma(s)}, V_{\gamma(s)})$
2. $\Phi(s,t)_\gamma$ *satisfies a certain first order linear differential equation.*

The differential equation determines $\Psi_p(s,t)$ uniquely as a p-form on PM. Starting with $\Psi_0(s,t)$, one can find $\Psi_p(s,t)$ by induction on p using a version of Chen's iterated integrals.

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Definition

A $\mathbb{Z}$-connection form A on a graded vector bundle $p : E \to M$ is a form of total degree 1, i.e. in $\prod \Omega^p(M; End^{1-p}(E))$.

Theorem

Given a graded vector bundle $p : E \to M$ with typical fiber V and a flat $\mathbb{Z}$-connection form A, there is a homotopy coherent representation on V
of the smooth singular simplicial set $Sing(M)$ of M

A crucial piece of structure is a set of *coherent* maps

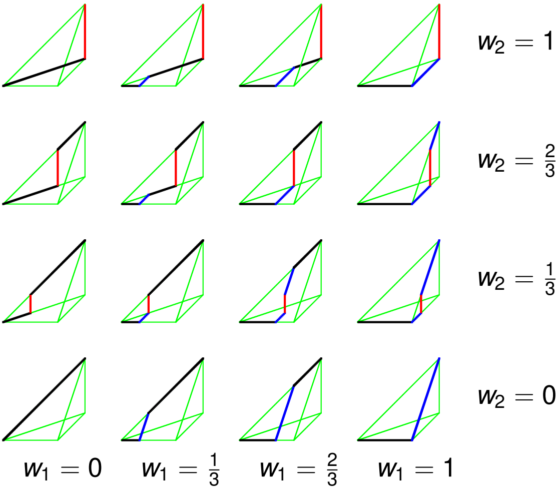
$$\gamma_n : I^{n-1} \to P\Delta^n.$$

These involve realizing any simplex as a family of paths with fixed endpoints 0 and n.

Such maps were first produced by Adams in the topological context by induction using the contractability of Δ^n. Later specific formulas were introduced by Chen and later equivalently but more transparently by Igusa.

For the simplex $< 0, 1, 2 >$, picture the path from 0 to 1 to 2 as gradually moving to the path from 0 to 2 with intermediate paths from 0 to a point in $< 0, 1 >$ and then up to a point on $< 0, 2 >$ and on to 2.

See the front face of the first column in the next figure.


$w_2 = 1$
$w_2 = \frac{2}{3}$
$w_2 = \frac{1}{3}$
$w_2 = 0$
$w_1 = 0$
$w_1 = \frac{1}{3}$
$w_1 = \frac{2}{3}$
$w_1 = 1$

The existence of a representation up to homotopy, i.e. that mappings $\theta_{(n)} : I^{n-1} \to P(\Delta^n, v_n, v_0)$ exist, was shown by Chen in 1977.

The relations those mappings $\theta_{(n)}$ must satisfy follow from two basic lemmas which here are modifications of Chen's.

Lemma

For every $1 \leq i \leq k-1$*, the following diagram commutes.*

$$\begin{array}{ccc} I^{k-2} & \xrightarrow{\theta_{k-1}} & P(\Delta^{k-1}, v_{k-1}, v_0) \\ \downarrow & & \uparrow \\ I^{k-1} & \xrightarrow{\omega_k} & P(\Delta^{k-1}, v_{k-1}, v_0) \end{array}$$

where ω_k *is a reparameterization of* θ_k

This follows from the definitions with careful choice of the reparameterization.

Lemma

For every $1 \leq i \leq k-1$ *and* $j = k - i$*, the following diagram commutes.*

$$
\begin{array}{ccccc}
I^{k-2} & \longrightarrow & I^{k-1} & \xrightarrow{\theta_{(k)}} & P(\Delta^k, v_k, v_0) \\
{\scriptstyle =}\big\downarrow & & & & \big\uparrow {\scriptstyle (f_i, b_j)} \\
I^{i-1} \times I^{j-1} & & \xrightarrow{\theta_{(i)} \times \theta_{(j)}} & & P(\Delta^i, v_i, v_0) \times P(\Delta^j, v_j, v_0)
\end{array}
$$

where $f_i : \Delta^i \to \Delta^k, b_j : \Delta^j \to \Delta^k$ *are the inclusions of the front* i*-face and back* j*-face respectively.*

Remark

These are the two main properties of the mappings $\theta_{(n)} : I^{n-1} \to P(\Delta^n, v_n, v_0)$. Chen used them to pull back simplicial classes from M to give cubical classes for the loop space ΩM.

Let $Flat(M)$ denote the category of vector bundles with given flat connection.

In a variant of the classical Riemann–Hilbert equivalence, a map

$$Flat(M) \to Reps(\pi_1(M))$$

is developed by calculating the holonomy with respect to a flat connection. The holonomy descends to a representation of $\pi_1(M)$ as a result of the flatness.

The functor

$$Reps(\pi_1(M)) \to Flat(M),$$

can be achieved by the associated bundle construction for the structure group $GL(V)$.

Working in categories more appropriate for algebraic geomety, Block and Smith define a *generalized Riemann–Hilbert correspondence* which is an equivalence of categories

$$Flat(M) \to hReps(Sing(M)),$$

given by the generalized holonomy of a flat $\mathbb{Z}$-graded connection. Here *hReps* denotes representations up to homotopy, although Block and Smith refer to them as *infinity-local systems* on M.

Block and Smith remark:

> *It would be an interesting problem in its own right to define an inverse functor which makes use of a kind of associated bundle construction.*

Block and Smith work in algebraic geometry, which led me to seek an analogous result in graded differential geometry, including a change in vocabulary.

The essential problem is to assemble a vector bundle with flat connection over M from the pieces

$$\Delta^k \times I^{k-1} \times V.$$

This is exactly what I am working on now and welcome collaboration.

Speculation

Recall the realization $|Sing(M)|$ of the simplicial space $Sing(M)$ which, for 'nice' M, is homotopy equivalent to M. In fact, assuming M admits partitions of unity, there is an embedding $M \subset |Sing(M)|$.

Now, think of θ_n as analogous to transition functions for a bundle.

For comparison:

Let $E_0 = \{V_i\}$, indexed copies of V.

Define $E_1 := E_0 \cup I \times V$ with the 'obvious' identifications:

The identity I_V for $0 \in I$ and θ_1 for $1 \in I$.