Moment maps for non-reductive group actions in Kähler geometry

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Outline of talk:

§1 Classical GIT (geometric invariant theory)

§2 Symplectic reduction

§3 Non-reductive GIT

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§6 Moment map descriptions of non-reductive GIT quotients

§7 Applications to Betti numbers, cohomology rings, integration formulas

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Mumford's ***G****eometric* ***I****nvariant* ***T****heory* (1960s)

G complex **reductive** group
X complex projective variety acted on by G

We require a **linearisation** of the action (i.e. an ample line bundle L on X and a lift of the action to L; e.g. $X \subseteq \mathbb{P}^n$ and the action given by a representation $\rho : G \to GL(n+1)$).

$$\begin{array}{ccccl} X & \rightsquigarrow & A(X) & = & \bigoplus_{k=0}^{\infty} H^0(X, L^{\otimes k}) \\ | & & & & \\ | & & \cup\!| & & \\ \downarrow & & & & \\ X//G & \leftsquigarrow & A(X)^G & & \text{algebra of invariants} \end{array}$$

G reductive implies that $A(X)^G$ is a *finitely generated* graded complex algebra so that $X//G = \text{Proj}(A(X)^G)$ is a projective variety.

The rational map $X - - \rightarrow X//G$ fits into a diagram

$$
\begin{array}{rccccl}
 & X & --\rightarrow & X//G & \text{cx proj variety} \\
 & \bigcup & & \| & \\
\text{semistable} & X^{ss} & \stackrel{\text{onto}}{\longrightarrow} & X//G & \\
 & \bigcup & & \bigcup & \text{open} \\
\text{stable} & X^{s} & \longrightarrow & X^{s}/G &
\end{array}
$$

where the morphism $X^{ss} \rightarrow X//G$ is G-invariant and surjective.

Topologically $\boxed{X//G = X^{ss}/\sim}$ where $x \sim y \Leftrightarrow \overline{Gx} \cap \overline{Gy} \cap X^{ss} \neq \emptyset$.

Every equivalence class for $\sim$ contains a unique *polystable* orbit (that is, a semistable orbit which is closed in X^{ss}), so we get a bijection

$$\{\text{ polystable orbits }\} \rightarrow X//G.$$

Hilbert–Mumford criteria for (semi)stability

Question: Can we describe which points of $X \subseteq \mathbb{P}^n$ are stable and which are semistable for the G-action without having to know what all the G-invariant sections of powers of L are?

Answer: YES

Let T be a maximal torus of G, choose homogeneous coordinates on $\mathbb{P}^n$ so that T acts diagonally with weights $\alpha_0, \ldots, \alpha_n$. Then $x = [x_0 : \cdots : x_n] \in X$ is **semistable (respectively stable) for the action of** T iff 0 lies in (respectively lies in the interior of) the convex hull in $(\mathrm{Lie}(T))^*$ of

$$\{\alpha_j : 0 \leq j \leq n \text{ and } x_j \neq 0\}.$$

Moreover x is **semistable (respectively stable) for the action of** G iff gx is semistable (respectively stable) for the action of T for **every** $g \in G$.

Classical GIT links with SYMPLECTIC REDUCTION .

(X, ω) compact symplectic manifold

K compact Lie group with Lie algebra $\mathfrak{k}$

Hamiltonian action of K on (X, ω)

$\mu : X \to \mathfrak{k}^*$ **moment map** satisfies

$$d\mu_x(\xi).a = \omega_x(\xi, a_x) \qquad \forall x \in X, \xi \in T_xX, a \in \mathfrak{k}$$

and μ is K-equivariant (for the coadjoint action on $\mathfrak{k}^*$) .

Special case: (X, ω) is **Kähler** and K acts holomorphically;
then the action extends to $G = K_{\mathbb{C}}$ = complexification of K.

Let $\zeta \in \mathfrak{k}^*$ be a regular value of $\mu : X \to \mathfrak{k}^*$.
Let K_ζ be its stabiliser for the coadjoint action.
Then the **Marsden-Weinstein reduction at** ζ

$$\mu^{-1}(\zeta)/K_\zeta$$

is a symplectic orbifold. If $\zeta = 0$ we have

$$\mu^{-1}(0)/K \text{ **'symplectic quotient'**}$$

Kähler case: $\mu^{-1}(0)/K =$ (open subset of X)$/G$
inherits a Kähler structure.

N.B. grad$\mu(x).a = i\, a_x \quad \forall a \in \mathfrak{k}$

When 0 is not a regular value of μ, then $\mu^{-1}(0)/K$ has a stratified symplectic/Kähler structure with more serious singularities (Lerman, Sjamaar, Heinzner, Huckleberry, Loose, ...).

Let $G = K_{\mathbb{C}}$ act linearly on $X \subseteq \mathbb{P}^n$ via $\rho : G \to GL(n+1)$.
Assume $\rho(K) \subseteq U(n+1)$ so K preserves the Fubini-Study Kähler form on X.

Then a moment map $\mu : X \to \mathfrak{k}^*$ is given by

$$\mu([x]).a = \frac{\bar{x}^T \rho_*(a) x}{2\pi i |x|^2} \in \mathbb{R} \quad \text{for } a \in \mathfrak{k}.$$

$x \in X$ is semistable iff $\overline{Gx} \cap \mu^{-1}(0) \neq \emptyset$, and $x \in X$ is stable iff $Gx \cap \mu^{-1}(0)_{\text{reg}} \neq \emptyset$.

The inclusion $\mu^{-1}(0) \to X^{ss}$ composed with the quotient morphism $X^{ss} \to X//G$ is K-invariant and the induced map

$$\mu^{-1}(0)/K \to X//G$$

to the GIT quotient is a homeomorphism.

$\rightsquigarrow$ Kähler quotients

What happens to GIT if G is not reductive?
Problem: We can't define a projective variety

$$X//G = \text{Proj}(A(X)^G)$$

because $A(X)^G$ is not necessarily finitely generated.

Question: Can we define a sensible 'quotient' variety $X//G$ when G is not reductive? If so, can we understand it geometrically?

Answer: We can define open subsets X^s ('stable points') and X^{ss} ('semistable points') with a geometric quotient $X^s \to X^s/G$ and an 'enveloping quotient' $X^{ss} \to X//G$. BUT $X//G$ is **not necessarily projective** and $X^{ss} \to X//G$ is **not necessarily onto**.
Also the Hilbert–Mumford criteria for (semi)stability do not generalise, at least not in a very obvious way, in general.

Defn: a unipotent linear alg group U is **graded unipotent** if there is a homomorphism $\lambda : \mathbb{C}^* \to Aut(U)$ with the weights of the $\mathbb{C}^*$ action on $Lie(U)$ all **strictly positive**.

Suppose that $\hat{U} = U \rtimes \mathbb{C}^*$ acts linearly on a projective variety X.
After twisting the linearisation by a suitable rational character, GIT for the $\hat{U}$ action is in some ways as well behaved as for reductive groups.

This can be applied when H is a linear algebraic group with **internally graded unipotent radical** U, in the sense that:
$H = U \rtimes R$ linear algebraic group acting linearly on
X complex projective variety with ample line bundle L;
$\hat{U} = U \rtimes \lambda(\mathbb{C}^*) \leqslant H$ where $\lambda : \mathbb{C}^* \to R$ is a central 1-PS of R which grades the unipotent radical U of H.

Thm: (Berczi, Doran, Hawes, K) Let U be graded unipotent acting linearly on a projective variety X, and suppose that the action extends to $\hat{U} = U \rtimes \lambda(\mathbb{C}^*)$. Suppose also that

$$(*) \quad x \in Z_{\min} \Rightarrow \dim \mathrm{Stab}_U(x) = 0$$

where $Z_{\min}$ is the union of connected components of $X^{\lambda(\mathbb{C}^*)}$ where $\lambda(\mathbb{C}^*)$ acts on the fibres of L^* with minimum weight.

We can twist the action of $\hat{U}$ by a (rational) character so that 0 lies just above the minimum weight for the $\mathbb{C}^*$ action on X, and then

(i) the algebra $A(X)^{\hat{U}}$ of $\hat{U}$-invariants is **finitely generated**, so that $X//\hat{U} = \mathrm{Proj}(A(X)^{\hat{U}})$ is **projective**;

(ii) $X//\hat{U}$ is a **geometric quotient** of $X^{ss,\hat{U}} = X^{s,\hat{U}}$ by $\hat{U}$ and $X^{s,\hat{U}}$ has a simple **Hilbert–Mumford** description.

$\hat{U} = U \rtimes \lambda(\mathbb{C}^*)$ acting linearly on X

Let $\omega_0 < \omega_1 < \ldots < \omega_r$ be the weights of the linear action of $\lambda(\mathbb{C}^*)$ on X. Set

$$Z_{\min} = \left\{x \in X^{\lambda(\mathbb{C}^*)} \,|\, \lambda(\mathbb{C}^*) \text{ acts on } L^*|_x \text{ with weight } \omega_0\right\}$$

$$X^0_{\min} = \left\{x \in X \,|\, p(x) := \lim_{t \to 0} \lambda(t)x \text{ lies in } Z_{min}\right\}.$$

If $\omega_0 < 0 < \omega_1$ then $X^{s,\lambda(\mathbb{C}^*)} = X^{ss,\lambda(\mathbb{C}^*)} = X^0_{\min} \setminus Z_{\min}$.

If $\lambda(\mathbb{C}^*)$ acts nontrivially so $r \geq 1$, the condition $\omega_0 < 0 < \omega_1$ can be achieved by multiplying the linearisation by a suitable (rational) character.

Similarly we make a linearisation *well-adapted* to the action on X of $\hat{U}$ (or of $H = U \rtimes R$ where $\lambda : \mathbb{C}^* \to Z(R)$ grades U) by multiplying it by a rational character so that $0 \in (\omega_0, \omega_1)$ is sufficiently close to ω_0. Then if (*) holds $X^{ss,\hat{U}} = X^{s,\hat{U}} = X^0_{\min} \setminus UZ_{\min}$.

$H = U \rtimes R$ linear algebraic group acting linearly on X;
$\hat{U} = U \rtimes \lambda(\mathbb{C}^*)$ where $\lambda : \mathbb{C}^* \to Z(R)$ grades the unipotent radical U of H;
$\chi : H \to \mathbb{C}^*$ character with $\chi|_{\lambda(\mathbb{C}^*)}$ nontrivial.

Let $Z_{\min}^{s(s),R/\lambda(\mathbb{C}^*)}$ be the (semi)stable locus for the action of $R/\lambda(\mathbb{C}^*)$ on $Z_{\min}$, with the (rational) linearisation induced by twisting by a rational multiple of χ chosen so that $\lambda(\mathbb{C}^*)$ acts trivially on $L|_{Z_{\min}}$.

H-stability=H-semistability means (*) holds and $Z_{\min}^{s,R/\lambda(\mathbb{C}^*)} = Z_{\min}^{ss,R/\lambda(\mathbb{C}^*)} \neq \emptyset$.

Thm Suppose a well-adapted linear action of H on X satisfies H-stability = H-semistability. Then the algebra of invariants $A(X)^H$ is **finitely generated**, and $X//H = \text{Proj}(A(X)^H)$ is a geometric quotient by H of

$$X^{ss,H} = X^{s,H} = p^{-1}(Z_{\min}^{s,R/\lambda(\mathbb{C}^*)}) \setminus UZ_{\min}.$$

$H = U \rtimes R$ linear algebraic group acting linearly on X;
$\hat{U} = U \rtimes \lambda(\mathbb{C}^*)$ where $\lambda : \mathbb{C}^* \to Z(R)$ grades the unipotent radical U of H.
E.g. H parabolic in reductive G, or $H = \mathrm{Aut}(Y)$ where Y is a complete toric variety.

When $H = R$ is reductive $X//H$ can be identified with

$$\mu^{-1}(0)/K$$

where K is a maximal compact subgroup of H and μ is a suitable moment map.
Applications (at least when $X^{ss} = X^s$, achievable by blow-ups of X if $X^s \neq \emptyset$): Betti numbers and cohomology ring of $X//H$, intersection pairings on $X//H$...

Can we find a similar description of $X//H$ more generally (at least when semistability coincides with stability for the action of H)?

Simple example: $U = \mathbb{C}^+$ and $\hat{U} = \mathbb{C}^+ \rtimes \mathbb{C}^*$ acting on $\mathbb{P}^n$. There are coordinates in which $\mathbb{C}^*$ acts diagonally and the generator of $Lie(\mathbb{C}^+)$ has Jordan normal form with blocks of size $k_1 + 1, \dots, k_q + 1$. So the linear $\mathbb{C}^+$ action extends to $G = SL(2)$, where

$$\mathbb{C}^+ = \{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} : a \in \mathbb{C}\} \leqslant G,$$

via $\mathbb{C}^{n+1} \cong \bigoplus_{i=1}^{q} Sym^{k_i}(\mathbb{C}^2)$, and the action of (a cover of) $\hat{U}$ extends to $SL(2)\times\mathbb{C}^*$. In this case the U-invariants are finitely generated (by Weitzenböck's theorem) so we can define

$$\mathbb{P}^n//\mathbb{C}^+ = \text{Proj}((\mathbb{C}[x_0, \dots, x_n])^{\mathbb{C}^+}).$$

Here $G \times_{\mathbb{C}^+} \mathbb{P}^n \cong (G/\mathbb{C}^+) \times \mathbb{P}^n \cong (\mathbb{C}^2 \setminus \{0\}) \times \mathbb{P}^n \subseteq \mathbb{C}^2 \times \mathbb{P}^n \subseteq \mathbb{P}^2 \times \mathbb{P}^n$ via $(g, x) \mapsto (g\mathbb{C}^+, gx)$ and

$$\boxed{\mathbb{P}^n//\mathbb{C}^+ \cong (\mathbb{P}^2 \times \mathbb{P}^n)//SL(2).}$$

Example when $(\mathbb{P}^n)^{ss} \to \mathbb{P}^n/\!/\mathbb{C}^+$ is *not* onto:

$$\mathbb{P}^3 = \mathbb{P}(\mathrm{Sym}^3(\mathbb{C}^2)) = \{\text{ 3 unordered points on } \mathbb{P}^1\}.$$

Then $(\mathbb{P}^3)^{ss} = (\mathbb{P}^3)^s = \{$ 3 points on $\mathbb{P}^1$ with at most one at $\infty\}$ and its image in $\mathbb{P}^3/\!/\mathbb{C}^+ = (\mathbb{P}^3)^s/\mathbb{C}^+ \ \sqcup\ \mathbb{P}^3/\!/SL(2)$ is the open subset $(\mathbb{P}^3)^s/\mathbb{C}^+$ which does not include the 'boundary' points coming from $0 \in \mathbb{C}^2$.

If we quotient *not* just by $U = \mathbb{C}^+$ but by $\hat{U} = \mathbb{C}^+ \rtimes \mathbb{C}^*$, then we can modify the linearisation by multiplying by a rational character of $\hat{U}$. For some such choices of linearisation all the 'boundary' points in the quotient by $\mathbb{C}^+$ coming from $0 \in \mathbb{C}^2$ become unstable for the induced action on $\mathbb{C}^*$, so we do get a surjective morphism

$$(\mathbb{P}^3)^{ss,\hat{U}} \quad \stackrel{\text{onto}}{\longrightarrow} \quad \mathbb{P}^3/\!/\hat{U}.$$

Symplectic implosion (Guillemin, Jeffrey, Sjamaar 2001)

Ingredients: (X, ω) symplectic manifold

Hamiltonian action of compact connected group K

$\mu : X \to \mathfrak{k}^*$ moment map

T maximal torus of K, Lie algebra $\mathfrak{t} \subseteq \mathfrak{k}$

$\mathfrak{t}^*_+$ = positive Weyl chamber

$K_\zeta = \{k \in K | (Ad^* k)\zeta = \zeta\}$, when $\zeta \in \mathfrak{t}^*_+$, with commutator subgroup $[K_\zeta, K_\zeta]$.

$$X_{impl} = \mu^{-1}(\mathfrak{t}^*_+)/ \sim$$

where $x \sim y \Leftrightarrow x = ky$ for some $k \in [K_\zeta, K_\zeta]$ with

$$\zeta = \mu(x) = \mu(y) \in \mathfrak{t}^*_+.$$

Example: $K = SU(2)$ with $\mathfrak{t}^*_+ = (-\infty, 0] = (-\infty, 0) \sqcup \{0\}$, and

$$X_{impl} = \mu^{-1}((-\infty, 0)) \sqcup \frac{\mu^{-1}(0)}{SU(2)}.$$

X_{impl} inherits a **stratified symplectic structure and T-action** with moment map $X_{impl} \to \mathfrak{t}^*_+ \subseteq \mathfrak{t}^*$ induced by the restriction of μ.

$(T^*K)_{impl}$ '**universal imploded cross-section**' is an affine algebraic variety over $\mathbb{C}$. In general

$$X_{impl} \cong (X \times (T^*K)_{impl})//K$$

which is an algebraic variety if X is algebraic.

Link with Kähler/algebraic geometry:
$G = K_{\mathbb{C}}$ complexification of K;
$U_{\max}$ maximal unipotent subgroup of G normalised by T.
Fact: $K_{\mathbb{C}}/U_{\max}$ is a quasi-affine variety whose algebra of regular functions $\mathcal{O}(K_{\mathbb{C}}/U_{\max}) = \mathcal{O}(K_{\mathbb{C}})^{U_{\max}}$ is finitely generated.
So $K_{\mathbb{C}}/U_{\max}$ has a canonical affine completion (usually singular)

$$K_{\mathbb{C}}//U_{\max} = \mathrm{Spec}(\mathcal{O}(K_{\mathbb{C}})^{U_{\max}}).$$

Thm (GJS): $K_{\mathbb{C}}//U_{\max}$ has a K-invariant stratified Kähler structure which is symplectically iso to the universal implosion $(T^*K)_{impl}$.
Cor: X affine or projective variety acted on linearly by $K_{\mathbb{C}} \Rightarrow$

$$X_{impl} \cong (X \times (K_{\mathbb{C}}//U_{\max}))//K_{\mathbb{C}} \cong X//U_{\max}.$$

$\exists$ generalisation replacing $U_{\max}$ with the unipotent radical U_P of any parabolic subgroup P of $G = K_{\mathbb{C}}$. Greb–Miebach study arbitrary unipotent $U \leqslant G$ acting on $Y \subseteq X$ in the Kähler situation.

Back to simple example:

$\mathbb{P}^n//\mathbb{C}^+ \cong \mu_{SU(2)}^{-1}(\mathfrak{t}_+^*)$ / collapsing on the boundary $\mu_{SU(2)}^{-1}(0)$;

$\mathbb{P}^n//(\mathbb{C}^+ \rtimes \mathbb{C}^*) \cong \mu_{SU(2)}^{-1}(\mathfrak{t}_+^*) \cap \mu_{S^1}^{-1}(\xi)$ / $(S^1$ and collapsing).

Suppose $\mu_{SU(2)}^{-1}(\mathfrak{t}) \cap \mu_{S^1}^{-1}(\xi) \subseteq \mu_{SU(2)}^{-1}((\mathfrak{t}_+^*)^o) \cap \mu_{S^1}^{-1}(\xi)$. Then

$$\mathbb{P}^n//(\mathbb{C}^+ \rtimes \mathbb{C}^*) \cong \mu_{SU(2)}^{-1}(\mathfrak{t}) \cap \mu_{S^1}^{-1}(\xi)/S^1 = \mu_{\mathfrak{t}^\perp}^{-1}(0) \cap \mu_{S^1}^{-1}(\xi)/S^1$$

where $\mu_{\mathfrak{t}^\perp} : \mathbb{P}^n \to \mathfrak{t}^\perp \cong \mathrm{Lie}\mathbb{C}^+$ is projection of $\mu_{SU(2)}$ onto $\mathfrak{t}^\perp$. So

$$\mathbb{P}^n//(\mathbb{C}^+ \rtimes \mathbb{C}^*) \cong \mu^{-1}(0)/S^1$$

where $\mu = (\mu_{\mathfrak{t}^\perp}, \mu_{S^1} - \xi) : \mathbb{P}^n \to \mathfrak{t}^\perp \times (\mathrm{Lie}S^1)^*$.

(Y, ω) compact Kähler manifold
holomorphic action on X of complex reductive Lie group G;
K maximal compact subgroup of $G = K_{\mathbb{C}}$ with Lie algebra $\mathfrak{k}$;
$\mathfrak{k}^* = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{k}, \mathbb{R})$ embeds in $\mathfrak{g}^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}, \mathbb{C})$ (where $\mathfrak{g} = \mathfrak{k} \otimes \mathbb{C}$)
as $\mathfrak{k}^* = \{\xi \in \mathfrak{g}^* : \xi(\mathfrak{k}) \subseteq \mathbb{R}\}$.

Traditionally: fix K and assume ω is K-invariant;
then ask for a moment map $\mu : Y \to \mathfrak{k}^*$ for the K-action.
Alternatively: fix a G-orbit Ω in

$$\{(K, \omega) \in \mathcal{K}_G \times \text{Kähler}(Y) : \omega \text{ is } K\text{-invariant}\}$$

where Kähler(Y) is the space of Kähler forms on Y and

$$\mathcal{K}_G = \{K \mid K \text{ is a maximal compact subgroup of } G\}$$

and ask for an Ω-moment map $\mathfrak{m}_{G,Y,\Omega} : \Omega \times Y \to \mathfrak{g}^*$.

Call a G-orbit Ω in $\{(K,\omega) \in \mathcal{K}_G \times \text{Kähler}(Y) : \omega \text{ is } K\text{-invariant}\}$ a *G-equivariant Kähler structure* on Y.

Then an *Ω-moment map* for G acting on Y is a smooth G-equivariant map

$$\mathfrak{m}_{G,Y,\Omega} : \Omega \times Y \to \mathfrak{g}^*$$

such that $\mathfrak{m}_{G,Y,\Omega}(K,\omega,x) = \mu_{(K,\omega)}(x)$ for each $(K,\omega) \in \Omega$ and $x \in Y$, where $\mu_{(K,\omega)} : Y \to \mathfrak{g}^*$ is the composition of a moment map for the K-action on (Y,ω) with the embedding of the dual of the Lie algebra of K in $\mathfrak{g}^*$.

W.l.o.g. $\Omega \cong G/K$.

A smooth G-equivariant map $\mathfrak{m}_{G,Y,\Omega} : \Omega \times Y \to \mathfrak{g}^*$ is an Ω-moment map for a G-action on Y with respect to a G-equivariant Kähler structure Ω on Y if and only if it satisfies the Ω-moment map differential equation:

If $a \in \mathfrak{g}$ and $\xi \in T_x Y$ then the derivative at $(K, \omega, x) \in \Omega \times Y$ of the complex-valued function on $\Omega \times Y$ given by evaluating $\mathfrak{m}_{G,Y,\Omega} : \Omega \times Y \to \mathfrak{g}^*$ at a takes $(0, \xi) \in T_{(K,\omega)}\Omega \times T_x Y$ to

$$\eta_{\omega,x}(\xi, a_x) - \eta_{\omega,x}(\iota_K(a)_x, \xi))/2i,$$

where $\eta_{\omega,x}$ is the value at x of the Hermitian metric on Y whose imaginary part is the K-invariant Kähler form ω, while ι_K is the K-invariant anti-complex involution on $\mathfrak{g}$ with fixed point set $\mathfrak{k}$, and $x \mapsto a_x$ is the holomorphic vector field on Y determined by the infinitesimal action of G.

X compact Kähler manifold with holo action of lin alg gp H;
$H = U \rtimes R$ with internally graded unipotent radical U;
$\lambda : \mathbb{C}^* \to Z(R)$ grades U, with $\hat{U} = U \rtimes \lambda(\mathbb{C}^*) \leqslant H$.

An H-equivariant Kähler structure on X is an H-orbit Ω in

$$\{(K, \omega) \in \Omega_{G,Y} : K \cap H \text{ maximal compact in } H\},$$

where Y is a compact Kähler manifold with a holomorphic action of a complex reductive Lie group G and a G-equivariant Kähler structure $\Omega_{G,Y}$ such that
(i) H is a subgroup of G,
(ii) X is a complex submanifold of Y, and
(iii) the action of G on Y restricts to the action of H on X.

$\Omega \subseteq \Omega_{G,Y}$ an H-equivariant Kähler structure on X

Define an Ω-*moment map* for the H-action on X to be a smooth H-equivariant map

$$\mathfrak{m}_{H,X,\Omega} : \Omega \times X \to \mathfrak{h}^*$$

if the derivative at $(K, \omega, x) \in \Omega \times X$ of the complex-valued function on $\Omega \times X$ given by evaluating $\mathfrak{m}_{H,X,\Omega} : \Omega \times X \to \mathfrak{h}^*$ at $a \in \mathfrak{h}$ takes $(0, \xi) \in T_{(K,\omega)}\Omega \times T_x X$ to

$$(\eta_{\omega,x}(\xi, a_x) - \eta_{\omega,x}(\iota_K(a)_x, \xi))/2i,$$

where $\eta_{\omega,x}$ is the value at x of the Hermitian metric on Y whose imaginary part is the K-invariant Kähler form ω, while ι_K is the K-invariant anti-complex involution on $\mathfrak{g}$ with fixed point set $\mathfrak{k}$, and $x \mapsto b_x$ denotes the infinitesimal action of $b \in \mathfrak{g}$.

Thm (Bérczi–K) [**Moment map description of $X//H$**]
Given: $H = U \rtimes R$ with internally graded unipotent radical U;
well-adapted action of H on $X \subseteq \mathbb{P}^n$ smooth cx proj variety;
H-equivariant Fubini–Study Kähler structure $\Omega \subseteq \Omega_{GL(n+1),\mathbb{P}_n}$;
Ω-moment map $\mathfrak{m}_{H,X,\Omega} : \Omega \times X \to \mathfrak{h}^*$.

If H-stability=H-semistability and $(K, \omega) \in \Omega$ then
(i) 0 is a regular value of $\mu^H_{(K,\omega)} : X \to \mathfrak{h}^*$, $x \mapsto \mathfrak{m}_{H,X,\Omega}(K, \omega, x)$;
(ii) $X^{s,H} = X^{ss,H} = \{x \in X : 0 \in \mathfrak{m}_{H,X,\Omega}(\Omega \times \{x\})\} = H(\mu^H_{(K,\omega)})^{-1}(0)$;
(iii) the embedding $(\mu^H_{(K,\omega)})^{-1}(0) \hookrightarrow X^{s,H}$ induces a surjective local diffeomorphism of orbifolds

$$(\mu^H_{(K,\omega)})^{-1}(0)/(K \cap H) \to X^{s,H}/H = X//H.$$

Thm (Bérczi–K) [**Betti numbers**] Let X be a smooth complex projective variety endowed with a well-adapted action of a linear algebraic group $H = U \rtimes R$ with internally graded unipotent radical U and grading one-parameter subgroup $\lambda : \mathbb{C}^* \to Z(R)$. Assume that semistability coincides with stability for H. Then the stratification $X^0_{\min} = X^{s,\hat{U}} \sqcup UZ_{\min}$ is equivariantly perfect for the actions of $\hat{U} = U \rtimes \lambda(\mathbb{C}^*)$ and H, and the Poincaré series of the GIT quotients $X//\hat{U}$ and $X//H$ satisfy

$$P_t(X//\hat{U}) = P_t(Z_{\min})\frac{1 - t^{2(\dim(X)-\dim(Z_{\min})-\dim(U))}}{1 - t^2}$$

and

$$P_t(X//H) = P_t(Z_{\min}//(R/\lambda(\mathbb{C}^*)))\frac{1 - t^{2(\dim(X)-\dim(Z_{\min})-\dim(U))}}{1 - t^2}.$$

Following Shaun Martin in the reductive case, we can use the moment map description of $X//H$ to express its rational cohomology ring in terms of the rational cohomology ring of the quotient of X by a maximal torus of H.

If $(K,\omega) \in \Omega$ and T^Q is a maximal (compact) torus of a maximal compact subgroup $Q = K \cap H$ of R, with complexification $T^H = (T^Q)_{\mathbb{C}}$:

$$\begin{array}{ccc} (\mu^H_{(K,\omega)})^{-1}(0)/T^Q & \stackrel{i}{\hookrightarrow} & (\mu^{T^H}_{(K,\omega)})^{-1}(0)/T^Q = X//T^H \\ \downarrow \pi & & \\ X//H & & \end{array}.$$

Here π is the composition of the surjective local diffeomorphism

$$(\mu^H_{(K,\omega)})^{-1}(0)/Q \to X//H$$

induced by the embedding of $(\mu^H_{(K,\omega)})^{-1}(0)$ in $X^{ss,H}_{\min +}$ with the natural map

$$(\mu^H_{(K,\omega)})^{-1}(0)/T^Q \to (\mu^H_{(K,\omega)})^{-1}(0)/Q.$$

Thm (Bérczi–K) [**Cohomology rings**]

Let X be a smooth complex projective variety endowed with a well-adapted action of $H = U \rtimes R$ such that H-stability = H-semistability. Then there is a natural ring isomorphism

$$H^*(X//H, \mathbb{Q}) \cong \frac{H^*(X//T^H, \mathbb{Q})^W}{ann(e)}$$

where W denotes the Weyl group of R, which acts naturally on $X//T^H$, while $e = \mathrm{Euler}(V) \in H^*(X//T^H)^W$ is the Euler class of the bundle V associated to the roots of R and weights of the adjoint action of T^H on $\mathrm{Lie}(U)$, and

$$\mathrm{ann}(e) = \{c \in H^*(X//T, \mathbb{Q}) \,|\, c \cup e = 0\} \subseteq H^*(X//T^H, \mathbb{Q}).$$

is the annihilator ideal.

We say that $\tilde{a} \in H^*(X//T^H)$ is a lift of $a \in H^*(X//H)$ if $\pi^* a = i^* \tilde{a}$.

Thm (Bérczi–K) [**Integration formula**] Let X be a smooth projective variety endowed with a well-adapted action of $H = U \rtimes R$ such that H-stability = H-semistability holds. Given a cohomology class $a \in H^*(X//H)$ with a lift $\tilde{a} \in H^*(X//T^H)$, then

$$\int_{X//H} a = \frac{n_0}{|W|} \int_{X//T^H} \tilde{a} \cup e$$

where $n_0 > 0$ is determined by the sizes of the stabilisers in H and T^H of a generic $x \in X$ and the size of a generic fibre of the surjective local diffeomorphism $(\mu^H_{(K,\omega)})^{-1}(0)/Q \to X//H$ induced by the embedding of $(\mu^H_{(K,\omega)})^{-1}(0)$ in $X^{ss,H}$.

A projective variety X is called Brody hyperbolic if any holomorphic map $f : \mathbb{C} \to X$ must be constant.

[**Kobayashi conjecture, 1970**] A generic hypersurface $X \subseteq \mathbb{P}^{n+1}$ of degree d_n is Brody hyperbolic if d_n is sufficiently large.

This is a slightly stronger version of Kobayashi's original conjecture in which 'very general' replaced 'generic'.

The conjectured optimal degree bound is $d_1 = 4, d_n = 2n + 1$ for $n = 2, 3, 4$ and $d_n = 2n$ for $n \geq 5$.

Siu, and later Brotbek, proved Kobayashi hyperbolicity for projective hypersurfaces of sufficiently high (but not effective) degree.
Based on the work of Brotbek, effective degree bounds were worked out by Deng, Demailly, Merker and The-Anh Ta. The best known bound based on these techniques is $(n \log n)^n$.

[**Green-Griffiths-Lang conjecture, 1979/1986**] Any projective variety X of general type contains a proper subvariety $Y \subsetneq X$ such that every nonconstant entire holomorphic curve $f : \mathbb{C} \to X$ satisfies $f(\mathbb{C}) \subseteq Y$.

(A generic projective hypersurface $X \subseteq \mathbb{P}^{n+1}$ is of general type if $\deg(X) \geq n+3$).

McQuillan proved GGL for surfaces assuming $c_1^2 - c_2$ is positive.
Using techniques of Demailly, the first effective lower bound for the degree of a generic hypersurface was given by Diverio, Merker and Rousseau, confirming GGL for generic projective hypersurfaces $X \subseteq \mathbb{P}^{n+1}$ of degree $\deg(X) > 2^{n^5}$.
This was improved to $\deg(X) > (\sqrt{n}\log n)^n$ by Merker and The-Anh Ta.

The two hyperbolicity conjectures are closely related: Riedl and Yang recently showed that if there are integers d_n for all positive n such that GGL holds for hypersurfaces of dimension n for degree at least d_n, then the Kobayashi conjecture is true for hypersurfaces with degree at least d_{2n-1}.

Key object in Demailly's strategy: Jet bundle $\pi_k : J_kX \to X$ of order k
$J_kX = \{$holomorphic k-jets of maps $f : (\mathbb{C}, 0) \to X\}$ with $\pi_k(f) = f(0)$,
open subset $(J_kX)^\circ = \{f : (\mathbb{C}, 0) \to X$ in J_kX with $f'(0) \neq 0\}$.
$\mathrm{Diff}_k = \{$holo k-jets $g : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ with $g'(0) \neq 0\}$
Diff_k acts on J_kX by reparametrisation of jets: $g \in \mathrm{Diff}_k$ sends f to $f \circ g$.
Diff_k is not reductive.

[**Polynomial Green-Griffiths-Lang theorem**] (Bérczi–K) Let $X \subseteq \mathbb{P}^{n+1}$ be a generic smooth projective hypersurface of degree $\deg(X) \geq 16n^5(5n+4)$. Then there is a proper algebraic subvariety $Y \subsetneq X$ containing all nonconstant entire holomorphic curves in X.

Using the recent results of Riedl and Yang this implies

[**Polynomial Kobayashi theorem**] A generic smooth projective hypersurface $X \subseteq \mathbb{P}^{n+1}$ of degree $\deg(X) \geq 16(2n-1)^5(10n-1)$ is Brody hyperbolic.