Vector bundles over Lie groupoids
and related structures

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To the memory of Alfonso Gracia-Saz (U. Toronto), who died of Covid-19 last May 6 at age 45.

Contents

1. Introduction

Poisson structures as Lie algebroids

Given (M, π) Poisson mfd, the **induced Lie algebroid** A_π over M has $T^*M \to M$ as vector bundle and bracket and anchor given by

$$[df, dg] = d\{f, g\} \qquad \rho = \pi^\# : T^*M \to TM$$

- For $(M, 0)$ and $(\mathfrak{g}^*, \{,\})$ the algebroid A_π recovers π.
- For $(M, \{\})$ symplectic it does not! $A_\pi \cong TM$.

If $A \Rightarrow M$ Lie algebroid, then $TA \Rightarrow TM$, $T^*A \Rightarrow A^*$ are **VB-algebroids**, and $\omega \in \Omega^2(A)$ is **IM** iff $\omega^\flat : TA \to T^*A$ is a VB-algebroid morphism.

Example: $\omega_{can} \in \Omega^2(T^*M)$ is IM for the induced Lie algebroid A_π.

Theorem (Roytenberg 2002; Bursztyn, Cabrera, Ortiz 2009)

- *A Poisson manifold is the same as a Lie algebroid coupled with closed, nondegenerate IM 2-form (A, ω).*
- *A (twisted) Dirac manifold is the same as a Lie algebroid coupled with a* **homotopy** *closed nondegenerate IM 2-form (A, ω).*

Motivating questions

- Understand VB-algebroids and their global analogue **VB-groupoids**
- Understand morphisms of VB-algebroids and VB-groupoids
- Elucidate the Lie theory (differentiation / integration) relating them

It turns out that a VB-groupoid / VB-algebroid has an underlying 2-term complex of vector bundles:

$$TG \rightrightarrows TM \quad \rightsquigarrow \quad A_G \xrightarrow{\rho} TM$$

- **Homotopies** between the VB-algebroids and VB-groupoids
- Tensor products of VB-groupoids and VB-algebroids
- Classify homtopy classes of VB-groupoids

Vector bundles

Lie algebroids $\xleftarrow{\text{differentiation}}$ / $\dashrightarrow$ Lie groupoids $\xrightarrow{\text{Morita equivalence}}$ / $\dashleftarrow$ Diff. stacks

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Lie algebroids		Lie groupoids		Diff. stacks
flat connections ∇ on vector bundles $E \to M$		representations of fundamental groupoid $\pi_1(M) \rightrightarrows M$		usual group representations $\pi_1(M, p) \curvearrowright V$
Lie algebroid representations		Lie groupoid representations		(naive) stacky vector bundles
VB-algebroids	$\leftrightarrow$ [1,4]	**VB-groupoids** [2]		
		homotopy VB-groupoids	$\leftrightarrow$ [3]	stacky vector bundles [7]
		higher VB-groupoids [5,6]		

My collaborations on the subject

1. H. Bursztyn, A. Cabrera, M del Hoyo; Vector bundles over Lie groupoids and algebroids; Adv. in Math. 290 (2016), 163-207
2. M. del Hoyo, D. Stefani; The general linear 2-groupoid; Pacific J. of Math. 298 (2019), 33-57
3. M. del Hoyo, C. Ortiz; Morita equivalences of vector bundles; IMRN (2020), 4395-4432
4. H. Bursztyn, A. Cabrera, M. del Hoyo; Poisson double structures; J. of Geom. Mech. online first (2021)
5. M. del Hoyo, G. Trentinaglia; Simplicial vector bundles and representations up to homotopy; (arxiv preprint)
6. M. del Hoyo, C. Ortiz, F. Studzinski; On the cohomology of differentiable stacks; (in progress)
7. M. del Hoyo, J. Desimoni; The classification of stacky vector bundles; (in progress)

2. Lie theory of vector bundles

VB-groupoids

A **VB-groupoid** is a compatible diagram of Lie groupoids and vector bundles

$$\begin{array}{ccc} \Gamma & \rightrightarrows & E \\ \downarrow & C & \downarrow \\ G & \rightrightarrows & M \end{array}$$

Hidden piece of data: the **core** $C = \ker(s : \Gamma|_M \to E)$:

$$\Gamma|_M \rightrightarrows E \quad \underset{Dold-Kan}{\rightleftharpoons} \quad C \xrightarrow{\partial} E$$

Core sequence $0 \to t^*C \to \Gamma \to s^*E \to 0$ plays a key role in duality.

Examples

- **Tangent** and **cotangent** gpds $\begin{array}{ccc} TG & \rightrightarrows & TM \\ \downarrow & A_G & \downarrow \\ G & \rightrightarrows & M \end{array}$, $\begin{array}{ccc} T^*G & \rightrightarrows & A_G^* \\ \downarrow & T^*M & \downarrow \\ G & \rightrightarrows & M \end{array}$
- A **representation** $G \curvearrowright E$ yields a VB-gpd $\begin{array}{ccc} G \times_M E & \rightrightarrows & E \\ \downarrow & 0_M & \downarrow \\ G & \rightrightarrows & M \end{array}$

Representations up to homotopy

Given $G \rightrightarrows M$ a Lie gpd and $C \oplus E \to M$ a graded vector bundle, a **representation up to homotopy** $G \curvearrowright C \oplus E$ consists of

- for each $x \in M$ a differential $\partial_x : C_x \to E_x$
- for each $g \in G$ a chain map $\rho_g : \partial_x \to \partial_y$
- for each pair $(h, g) \in G_2$ a chain homotopy $\gamma_{(h,g)} : \rho_{hg} \Rightarrow \rho_h \rho_g$

Theorem (GraciaSaz-Mehta 2017)

Correspondence between RUTHs $(G \rightrightarrows M) \curvearrowright (C \oplus E \to M)$ and VB-groupoids $(\Gamma \rightrightarrows E) \to (G \rightrightarrows M)$.

From RUTH to VB-groupoids: Given $G \curvearrowright (C \oplus E)$ define a **semi-direct product** VB-groupoid $\Gamma = t^*C \oplus s^*E \rightrightarrows E$ with

$$(c_1, g_1, e_1) \cdot (c_2, g_2, e_2) = (c_1 + R_1^{g_1}(c_2) + R_2^{g_1,g_2}(e_2), g_1 g_2, e_2)$$

From VB-groupoids to RUTH: Given $\Gamma \rightrightarrows E$, pick a **cleavage** $\Sigma : s^*E \to \Gamma$ to break the fibration

Smooth linear version of Grothendieck correspondence

$$\begin{array}{ccc} w & \overset{\Sigma(g,v)}{\dashleftarrow} & v \\ & & \\ y & \overset{g}{\longleftarrow} & x \end{array}$$

VB-algebroids

A **double vector bundle** is a compatible diagram of vector bundles

$$\begin{array}{ccc} D & \to & B \\ \downarrow & C & \downarrow \\ A & \to & M \end{array} \qquad C = \ker(s : \Gamma|_M \to E)$$

A DVB has a **vertical dual** and a **horizontal dual** and there is a canonical pairing between them

$$\begin{array}{ccc} D^* & \to & C^* \\ \downarrow & B^* & \downarrow \\ A & \to & M \end{array} \qquad \begin{array}{ccc} D^\bullet & \to & B \\ \downarrow & A^* & \downarrow \\ C^* & \to & M \end{array}$$

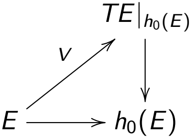
A **VB-algebroid** is a compatible diagram of Lie algebroids and vector bundles, in the sense that $(\Omega^\bullet, \pi)$ must be double linear Poisson

$$\begin{array}{ccc} \Omega & \Rightarrow & E \\ \downarrow & & \downarrow \\ A & \Rightarrow & M \end{array}$$

Examples: TA, T^*A, representations, etc.
Rmks: Characterizations via RUTH and via Weil algebra $W(D)$ of a DVB [Meinrenken, Pike 2021]

Lie theory of vector bundles

Prop: A vector bundle over a manifold $E \to M$ is the same as an action $h : (\mathbb{R}, \cdot) \curvearrowright E$ that is **regular** meaning $V(x) = 0 \iff h_0(x) = x$



$$V(x) = \frac{\partial}{\partial t}|_{t=0} h_t(x)$$

Theorem (Bursztyn, Cabrera, dH 2016)

- *VB-groupoid is the same as a regular action* $(\mathbb{R}, \cdot) \curvearrowright (\Gamma \rightrightarrows E)$
- *VB-algebroid is the same as a regular action* $(\mathbb{R}, \cdot) \curvearrowright (\Omega \Rightarrow E)$
- *A VB-groupoid differentiates to a VB-algebroid, and if the top of a VB-algebroid is integrable, then integrates to a VB-groupoid.*

Some applications

Key role in the differentiation and integration of the following structures:

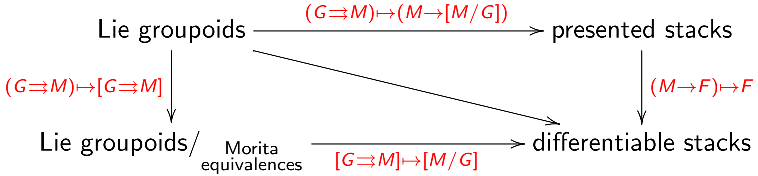
- Dirac structures and foliations on Lie groupoids
  [Ortiz 2013; Ortiz, Jotz 2014]
- 2-term representations up to homotopy
  [GraciaSaz, Mehta 2010, 2017; Bursztyn, Cabrera, dH 2016]
- double Lie groupoids and double Lie algebroids
  [Mackenzie 1992, 2000, 2011; Bursztyn, Cabrera, dH 2016]
- Poisson double groupoids and Poisson double algebroids
  [Mackenzie 1999; Bursztyn, Cabrera, dH 2021]

3. Morita equivalence of vector bundles

Differentiable stacks

A **stack** is a locally determined functor $\mathrm{Manifolds}^{\circ} \xrightarrow{F} \mathrm{Groupoids}$
A **presentation** is a surjective submersion $M \to F$.
If F admits a presentation then it is a **differentiable stack**.

A Lie groupoid $G \rightrightarrows M$ induces an **orbit stack** $[M/G]$ covered by M



Theorem (Moerdijk 1988, Behrend-Xu 2006, etc)

A Lie groupoid is a stack endowed with a presentation.
A differentiable stack is the **Morita class** *of a Lie groupoid.*

Two approaches: **principal bibundles** or **Morita maps**.

Morita morphisms

A morphism $\phi : (G \rightrightarrows M) \xrightarrow{\sim} (G' \rightrightarrows M')$ is **weak equivalence** or **Morita** if it is smoothly fully faitful and essentially surjective:

$$\begin{array}{ccc} G & \xrightarrow{\phi} & G' \\ \downarrow & & \downarrow \\ M \times M & \xrightarrow{\phi\times\phi} & M' \times M' \end{array} \qquad \begin{array}{c} t\pi_1 : G' \times_{M'} M \to M' \\ (x' \xleftarrow{g'} \phi(x), x) \mapsto x' \end{array}$$

Example

Given $\pi : M \to N$ submersion and $\phi : (M \times_N M \rightrightarrows M) \to (N \rightrightarrows N)$,

- ϕ isomorphism iff π diffeomorphism;
- ϕ equivalence iff π admits a global section;
- ϕ Morita iff π surjective.

Theorem (dH 2013)

$$\phi : (G \rightrightarrows M) \xrightarrow{\sim} (G' \rightrightarrows M') \iff \begin{cases} \bar{\phi} : M/G \xrightarrow{\cong} M'/G' \\ \phi_x : G_x \xrightarrow{\cong} G'_{\phi(x)} \\ d_x\phi : N_x O \xrightarrow{\cong} N_{\phi(x)} O' \end{cases}$$

Morita equivalence

Two Lie gpds $G \rightrightarrows M$, $G' \rightrightarrows M'$ are **Morita equivalent** and define the same **differentiable stack** if they are linked by morphisms as below:

$$(G \rightrightarrows M) \xleftarrow{\sim} (\tilde{G} \rightrightarrows \tilde{M}) \xrightarrow{\sim} (G' \rightrightarrows M') \qquad [M/G] \cong [M'/G']$$

Example

- Unit groupoids $M \rightrightarrows M$ regard manifolds as stacks: $M = [M/M]$
- A Lie group $G \rightrightarrows *$ yields the classifying stack $BG = [*/G]$
- An action groupoid $K \times M$ yields finite dimensional model for Borel's recipe on equivariant cohomology $[M/K \times M]$
- Proper groupoids with finite isotropy yield orbifolds
- an holonomy groupoid $\mathrm{Hol}(F) \rightrightarrows M$ gives a leaf space for foliation F

Geometry preserved by Morita morphisms = Geometry of stacks
Example: properness, representations, cohomology, etc

The derived category $VB[G]$

Theorem (dH, Ortiz 2018)

$\phi : (\Gamma \rightrightarrows E) \to (\Gamma' \rightrightarrows E')$ *linear Morita iff it is so in base and in fibers:*

$$\begin{array}{ccc} \Gamma \rightrightarrows E & \xrightarrow{\phi} & \Gamma' \rightrightarrows E' \\ \downarrow & & \downarrow \\ G \rightrightarrows M & \xrightarrow[\varphi]{} & G' \rightrightarrows M' \end{array} \qquad (\Gamma_x \rightrightarrows E_x) \xrightarrow{\phi_x} (\Gamma'_x \rightrightarrows E'_x)\ \forall x$$

In $VB(G)$, every Morita morphism admits a quasi-inverse.
We define the **derived category** as $VB[G] = VB(G)[\text{Morita}^{-1}]$.

$$\Gamma_1 \sim \Gamma_2 \text{ in } VB[G] \iff \exists \text{ acyclic } \Omega_1, \Omega_2 \text{ such that } \Gamma_1 \oplus \Omega_1 \cong \Gamma_2 \oplus \Omega_2$$

Theorem (dH-Ortiz 2018)

$\phi : G' \to G$ *Morita then* $\phi^* : VB[G] \to VB[G']$ *equivalence.*
We can talk about **stacky vector bundles**.

Some applications

- Morita invariance of 2-term ruth
  [AriasAbad, Crainic 2013; GraciaSaz, Mehta 2017; dH, Ortiz 2018]
- Morita invariance of deformation cohomology of Lie groupoids
  [Crainic, Mestre, Struchiner 2020]
- Simpler approach to integration of Dirac structures
  [Bursztyn, Crainic, Weinstein, Zhu 2004; Xu 2004; del Hoyo, Ortiz 2018]
- Notion of tangent and cotangent stack
  [Behrend, Ginot, Noohi, Xu 2007; dH, Ortiz 2018]

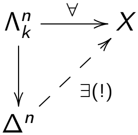
  [Bonechi, Ciccoli, Laurent-Gengoux, Xu 2020]

4. Higher groupoids and vector bundles

4.1 Higher Lie groupoids

Characterizing nerves

A simplicial set X is (the nerve of) a groupoid iff every horn $\Lambda^n_k \to X$ has a filling $\Delta^n \to X$ $\forall n$ and is unique if $n > 1$.



A **weak Lie** ***m*-groupoid** is a simplicial manifold X such that the map $X_n \to \hom(\Lambda^n_k, X)$ is a surjective submersion $\forall n$ and a diffeo if $n > m$.

A filling yields a composition!

Theorem

dH-Stefani 2019 Strict 2-groupoids G (defined as double diagrams) yield weak 2-groupoids via the **lax nerve** *$NG_k = \{$lax functors $[k] \dashrightarrow G\}$.*

General linear 2groupoid

Given $C \oplus E \to M$ a 2-term graded vector bundle, we constructed a **genearl linear 2-groupoid** $GL(C \oplus E)$ with:

- objects: differential $\partial^x : C^x \to E^x$ on fibers;
- arrows: chain maps between fibers (commutative squares);
- 2-cells: chain homotopies.

Example: $\mathbb{R}^p \oplus \mathbb{R}^q \to *$ leads to $GL_{p,q} = GL(\mathbb{R}^p \oplus \mathbb{R}^q)$

Theorem (dH, Stefani 2019)

- *$GL(C \oplus E \to M)$ is a Lie 2-groupoid*
- *A ruth $(G \rightrightarrows M) \curvearrowright (C \oplus E \to M)$ is the same as a pseudo-functor*

$$(G \rightrightarrows M) \dashrightarrow GL(C \oplus E \to M)$$

Classification of stacky vector bundles

A lax functor $\phi : G \to G'$ between Lie 2-gpds is **Morita** if ϕ_0 meets transversely every orbit and is locally Morita $G(y,x) \to G'(\phi_0(y), \phi_0(x))$

Theorem (dH-Desimoni)

- *When ϕ_0 surj. submersion, then ϕ Morita iff $N\phi : NG \to NG'$* **hypercover** [Zhu 2009].
- *When ϕ_0 surj. submersion and ϕ strict, then ϕ Morita iff $\phi : (G_2 \rightrightarrows G_1) \to (\phi_0^* G_2' \rightrightarrows \phi_0^* G_0')$ is Morita* [Ginot, Stiénon 2015]

A **differentiable 2-stack** $[G]$ is the Morita class of a Lie 2-groupoid G.
A **map of 2-stacks** $[G] \to [G']$ is (the class of) a fraction $G \xleftarrow{\sim} H \to G'$.

Theorem (dH-Desimoni)

Stacky vector bundles are classified by the **categorified Grassmanians**

$$Maps_{2stack}([G],[GL_{p,q}]) \rightleftharpoons VB_{pq}[G]$$

When $p = 0$ this gives classic classification thm

Def: Bott-Shulman cohomology $H^\bullet([GL_{p,q}])$ is the universal algebra of **characteristic classes** of stacky vector bundles

Higher VB-groupoids

A **higher VB-groupoid** is a simplicial vector bundle $V \to G$ where the base is (the nerve of) a Lie groupoid G, V is a higher Lie groupoid (horn-filling), and the projection is a fibration.

Given $R : G_\bullet \curvearrowright E_\bullet$ a RUTH we build a higher VB-groupoid $V_\bullet \to G_\bullet$ by

$$V_n = \bigoplus_{\substack{[k] \xrightarrow{\alpha} [n] \\ \alpha \text{ inj}, \alpha(0)=0}} x^*_{\alpha(k)} E_k \qquad (d_0)_{\alpha\beta} = (-1)^l R_{l-k+1}^{g \circ \bar{\beta}'|_{[k,l+1]}} \text{ if } \alpha' = \bar{\beta}' |_{[0,k]}$$

- When $G = M$ this gives the inverse for Dold-Kan (new formulas!!)
- When $E = E_1 \oplus E_0$ our $G \ltimes E$ is isomorphic to the nerve of the associated VB-groupoid via Grothendieck correspondence.

Theorem (dH-Trentinaglia)

This gives a well-defined higher VB-groupoid $G \ltimes E$ over G. And every higher VB-groupoid arises in this way.

Cohomology of stacks

We still do not know if higher VB-groupoids are a Morita invariant.
The same for RUTH (question posed by [Abad, Crainic 2013]).

What we know is a cohomological version:

Theorem (dH, Ortiz, Studzinsky)

If $\phi : G \to G'$ is Morita and $G \curvearrowright E$ is a RUTH then $\phi^* : H^\bullet(G', E) \to H^\bullet(G, \phi^* E)$ is an isomorphism.

- Main result of F. Studzinski's Thesis (USP, Nov 2019)
- It uses the semi-direct product construction
- We hope similar techniques allow us to prove the general result.

Thanks!