

$(G, \pi_G)$  complex semisimple Lie group with standard Poisson Lie group structure

$\mathfrak{g} = \text{Lie}(G)$  is a Lie bialgebra, so dual  $\mathfrak{g}^*$  is a Lie algebra.

For  $\mathfrak{c} \subset \mathfrak{g}$ , let  $\mathfrak{c}^\circ = \{ \lambda \in \mathfrak{g}^* \mid \lambda(x) = 0 \}$   
= annihilator of  $\mathfrak{c}$  in  $\mathfrak{g}^*$ .

Defn: If  $\mathfrak{c}$  is a Lie subalgebra of  $\mathfrak{g}$ , we say  $\mathfrak{c}^\circ$  is coisotropic if  $\mathfrak{c}^\circ$  is a Lie subalgebra of  $\mathfrak{g}^*$ .

I learned about this notion from a paper by Marco Zambrini, JPAA, 2011.

Today: Present some ideas/results on this problem due to N. Krömer (PAMS, 2016) and Joan Leites in Notre Dame PhD thesis 2021. Main idea is to restate the problem in terms of Lagrangian subalgebras.

Recall, the double  $\mathfrak{g}$  of  $\mathfrak{g}_1$  is  $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_1^*$   
 and  $(\mathfrak{g}, \mathfrak{g}_1, \mathfrak{g}_1^*)$  is a Manin triple corresponding  
 to the Lie bracket.

Here  $\mathfrak{g}_1^* = \{(x, x) \mid x \in \mathfrak{g}_1\}$

Let  $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{t} \oplus \mathfrak{n}$  be decomposition, with  
 $\mathfrak{t}$  Cartan subalgebra,  $\mathfrak{n} = \bigoplus_{\alpha \in \mathfrak{R}^+} \mathfrak{g}_\alpha$ ,  $\mathfrak{n}_- = \bigoplus_{\alpha \in \mathfrak{R}^-} \mathfrak{g}_\alpha$

Then  $\mathfrak{g}_1^* = \mathfrak{t} \oplus (\mathfrak{n} \oplus \mathfrak{n}_-)$   
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$\{(x, -x) \mid x \in \mathfrak{t}\}$

Consider the <sup>invariant nondegenerate</sup> SBLF  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{g}$  given by

$$\langle (x_1, x_2), (y_1, y_2) \rangle = k(x_1, y_1) - k(x_2, y_2)$$

where  $k(\cdot, \cdot)$  is the Killing form  
 and  $(x_1, x_2), (y_1, y_2) \in \mathfrak{g}$

$$\mathcal{L}(\mathfrak{g}) = \left\{ \mathfrak{l} \in \text{Gr}(\dim \mathfrak{g}, \mathfrak{g}) \mid \mathfrak{l} \text{ is isotropic w.r.t } \langle \cdot, \cdot \rangle \text{ and } \mathfrak{l} \text{ is a Lie subalgebra} \right\}$$

= variety of Lagrangian subalgebras  
 [E-Lu, Ann ENS 2001 and 2006]

Prop. (Kroeger)

(i)  $\mathfrak{g}$  is coisotropic  $\Rightarrow \mathfrak{g} + \mathfrak{g}^\perp \in \mathcal{L}(\mathfrak{g})$

For  $\mathfrak{g}$  coisotropic, let  $l(\mathfrak{g}) = \mathfrak{g} + \mathfrak{g}^\perp$

(ii) If  $l \in \mathcal{L}(\mathfrak{g})$ , then  $l = \mathfrak{g}(l \cap \mathfrak{g}^\perp)$

$\Leftrightarrow l \cap \mathfrak{g}^\perp$  is coisotropic

Proof is very easy

Defn  $\mathcal{C}\mathcal{L}(\mathfrak{g}) = \{l(\mathfrak{g}) \mid \mathfrak{g} \text{ is coisotropic}\}$

The map:  $\{\text{coisotropic subalgebras}\} \rightarrow \mathcal{L}(\mathfrak{g})$   
 $\mathfrak{g} \longmapsto l(\mathfrak{g})$

is an isomorphism.

Assume  $Z(G)$  is trivial.

Let  $T$  be max torus of  $G$  with Lie algebra

$\mathfrak{t}$ .  $T_\Delta = \{(a, a) \mid a \in \mathfrak{t}\}$ .  $T_\Delta = N_{\mathfrak{g}}(\mathfrak{g}) \cap N_{\mathfrak{g}}(\mathfrak{g}^\perp)$

-  $\mathcal{C}\mathcal{L}(\mathfrak{g})$  is a closed  $T_\Delta$ -stable subvariety of  $\mathcal{L}(\mathfrak{g})$

Approach to understanding coisotropic subalgebras of  $\mathfrak{g}$ :

the variety  $\mathcal{L}(\mathfrak{g})$  is a well-understood Poisson variety. Use this to understand  $\mathcal{C}\mathcal{L}(\mathfrak{g})$

Some motivation: There is a degeneration  $I_0$  of  $I(S)$ . If we give  $G$  the  $O$ -Poisson structure,  $(G, \omega)$  is a Poisson Lie group. Its double is  $\mathfrak{g} := \mathfrak{g}_+ \oplus_{\text{ad}} \mathfrak{g}_-^*$  semi-direct sum of Lie algebras, where  $\mathfrak{g}_+^*$  is abelian.

Understanding  $I(S_0)$  is a wild problem, but  $\lim_{t \rightarrow 0} I(S_t)$  should be more tractable.

$\mathfrak{g}_t =$  Lie algebra associated to  $(G_t, \omega_t)$

If we identify  $\mathfrak{g} = \mathfrak{g}_0$  as vector spaces

in the obvious way (both are  $\mathfrak{g}_+ \oplus \mathfrak{g}_-^*$  as vector spaces), then  $\mathcal{L}I(S) =$  points

in  $I(S)$  that are stable under degeneration.

Example<sup>(Lo)</sup>;  $G = \text{PSL}(2, \mathbb{C})$ .

$\mathcal{Y}(G)$  has 2 connected components  $Z_1$  and  $Z_2$ .

$$Z_1 \cong \mathbb{P}(M_2(\mathbb{C})) = \{ [A] \mid A \in M_2(\mathbb{C}) \} = \mathbb{P}^3$$
$$Z_2 \cong \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\text{separé}} \mathbb{P}(M_2(\mathbb{C})), \quad Z_2 \cong \{ [A] \mid \det(A) = 0 \}$$

$$\mathcal{O}_{\mathcal{Y}}(Z_1) := \mathcal{O}_{\mathcal{Y}}(\mathcal{Y}(G)) \cap Z_1$$

$$\mathcal{O}_{\mathcal{Y}}(Z_1) = \left\{ \left[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right] \right\} \cup \{ [\text{Id}] \}$$

$\mathbb{P}^2$  point

$$\mathcal{O}_{\mathcal{Y}}(Z_2) = \mathbb{P}^1 \sqcup \mathbb{P}^2 \sqcup \left[ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right]$$

$$\mathbb{P}^1 = \left\{ \left[ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \right] \mid \det = 0 \right\}$$

$$\Rightarrow \mathcal{O}_{\mathcal{Y}}(\mathcal{Y}(G)) = \mathbb{P}^3 \sqcup \mathbb{P}^1 \sqcup 2 \text{ points}$$

## Poisson structure on $\mathcal{I}(\mathfrak{g})$

Let  $\mathcal{P} = \mathfrak{g} \ltimes \mathfrak{g}_{\text{iso}}$   $\text{Lie}(\mathcal{P}) = \mathfrak{g}$ . Then  $\mathcal{P}$  acts on  $\mathcal{I}(\mathfrak{g})$  by  $(g, \ell) \rightarrow \text{Ad}(g)(\ell)$

$$\Rightarrow \theta: \mathfrak{g} \rightarrow T(\mathcal{I}(\mathfrak{g}))$$

$$\Rightarrow \Lambda^2 \theta: \Lambda^2 \mathfrak{g} \rightarrow T^* \Lambda^2 T(\mathcal{I}(\mathfrak{g}))$$

Identify  $\mathfrak{g} = \mathfrak{g}^*$  using  $\langle \cdot, \cdot \rangle$

$\Lambda^2 \mathfrak{g}$  has a canonical bivector  $R$

$$\text{given by } R(x_1 + \lambda_1, x_2 + \lambda_2) = \lambda_2(x_1) - \lambda_1(x_2)$$

$$\text{Let } \Pi = \Lambda^2 \theta \left( \frac{1}{2} R \right)$$

(E-Lu, 2001)

Prop.  $(\mathcal{I}(\mathfrak{g}), \Pi)$  is a Poisson variety

Let  $G_A$  and  $G^*$  be the connected subgroups corresponding to  $\mathfrak{g}_A$  and  $\mathfrak{g}^*$

Then  $G$  and  $G^*$ -orbits on  $\mathcal{I}(\mathfrak{g})$  are

Poisson submanifolds. By a theorem

of Drinfel'd, many Poisson homogeneous spaces of  $G$  and  $G^*$  occur as  $G$ -orbits

in  $\mathcal{I}(\mathfrak{g})$ . Part of our initial interest

in  $\mathcal{I}(\mathfrak{g})$  was that it allows the study

of Poisson homogeneous spaces in

families

Prop (Kroeger): If  $l \in \mathcal{C}\mathcal{I}(S)$ , then  $\pi(l) = \emptyset$ .

In fact, there is a simple formula for the anchor map  $\pi: T_l^*(\mathcal{P}\cdot l) \rightarrow T_l(\mathcal{P}\cdot l)$

The converse is false in general.

Lemma (Le) If  $N_S(l) = l$ , then  $\pi(l) = \emptyset \Rightarrow$   
normalizer  
of  $l$  in  $S$

$$l \in \mathcal{C}\mathcal{I}(S)$$

Let  $\alpha: \mathfrak{g} \rightarrow \mathfrak{g}$  be an automorphism.

Let  $l_\alpha = \{(\alpha(x), x) \mid x \in \mathfrak{g}\}$ . Then  $l_\alpha \in \mathcal{I}(S)$   
and  $N_S(l_\alpha) = l_\alpha$ .

Consequence: For  $\alpha \in \text{Aut}(\mathfrak{g})$ ,  $l_\alpha \in \mathcal{C}\mathcal{I}(S)$   
 $\Leftrightarrow \pi(l_\alpha) = \emptyset$ .

Note:  $\text{Aut}(\mathfrak{g}) \longrightarrow \mathcal{I}(S)$  is an open embedding  
 $\alpha \longrightarrow l_\alpha$

Work of Lu (see also Cantini-Carnaro-Costantini, Chau, To); *Indagationes Math*, 2014

Want to know  $\{ \varphi \in \text{Aut}(\mathfrak{g}) \mid \Pi(\ell_\varphi) = 0 \}$

A conjugacy class  $C$  in  $G$  is called spherical if it has an open orbit for  $B$ .

Given  $C$ ,  $\exists!$  element  $m_C \in W$  such that  $C \cap B m_C B$  is dense in  $C$ .

Thm (Lu)  $\{ g \in G \mid \Pi(\ell_{\text{Ad } g}) = 0 \} = \bigsqcup_{C \text{ spherical}} C \cap (B m_C B)$

Furthermore, each  $C \cap (B m_C B)$  is a T-conjugacy class.

- The spherical conjugacy classes are known for each simple  $G$ . For  $SL(n, \mathbb{C})$ , they are either conjugacy classes through  $\left\{ \begin{pmatrix} \lambda \text{Id}_k & 0 \\ 0 & \mu \text{Id}_{n-k} \end{pmatrix} \mid \lambda \neq \mu \right\}$

or conjugacy classes  $\{ \lambda \cdot \text{Id}_n - \mu \mid \mu \text{ is unipotent with Jordan type } \left( \begin{smallmatrix} m-2 & 2 \\ 1 & 2 \end{smallmatrix} \right) \}$

- Lu also precisely determines when  $\Pi(\ell_\varphi) = 0$  for any  $\varphi$ .

## Partial Results on problem in general:

$\mathcal{I}(S)$  has  $T \times T$ -action, and  $T_\Delta$ -action preserves  $\mathcal{C}\mathcal{I}(S)$ :

Idea: (1) Understand  $\mathcal{I}(S)^{T_\Delta}$  and describe the points in  $\mathcal{C}\mathcal{I}(S)$  fixed by  $T_\Delta$ .

(2)  $T_\Delta$ -action on  $\mathcal{I}(S)$  gives Białynicki-Birula decomposition of  $\mathcal{I}(S)$ . Roughly, we decompose  $\mathcal{I}(S)$  into attracting sets for each connected component  $I_i$  of  $\mathcal{I}(S)^{T_\Delta}$ .

The attracting set  $I_i^+$  is a vector bundle over  $I_i$ , and the  $T_\Delta$ -action lets us decompose  $I_i^+$  into sub-bundles.

$\mathcal{C}\mathcal{I}$ -behavior is constant along sub-bundles.

Remark:  $\mathcal{I}(S)$  is singular, but its irreducible components are smooth (E-LW) so we can apply Białynicki-Birula component by component.

Progress on this:  $\mathcal{L}(\mathcal{G})$  has an irreducible component  $\bar{G}$ ; the wonderful compactification of  $G$  due to De Concini-Procesi.

$$\bar{G}^{TxT} = \bigcup_{(y,w) \in W \times W} z_{y,w} \quad , \quad W = \text{Weyl group}$$

$$z_{y,w} = (y,w) \left( \frac{1}{2} + (n_D) + (D, \pi) \right)$$

$$\text{let } \Phi_y = \{ \alpha \in \Phi^+ \mid y^{-1}(\alpha) \in -\Phi^+ \}$$

$$\text{Thm (Kroeger)} \quad z_{y,w} \in \mathcal{E}\mathcal{L}(\mathcal{G}) \Leftrightarrow \Phi_y \cap \Phi_w = \emptyset \\ \text{and } (y^{-1}w)^2 = \text{Id}$$

Remarks (1) kroeger's result is a bit more general than this, and should determine which points in  $\mathcal{L}(\mathcal{G})^{TxT}$  are in  $\mathcal{E}\mathcal{L}(\mathcal{G})$   
 (2) kroeger's result generalizes a construction of coisotropic subalgebras using particular long roots

- Note:  $\mathcal{D}$  has a finite number of orbits on  $\bar{G}_S$  indexed by subsets  $S$  of simple roots
- (1) Open orbit: coisotropic set understood using Lu's result
  - (2) Closed orbit  $\cong G/B \times G/B$   
Coisotropic set in  $(G/B \times G/B)^{T_\Delta}$  is due to Kroeger
  - (3) Le has <sup>some</sup> results on coisotropic set in the other orbits. This is expressed in terms of root system data.

Recently, my student Song Gao computed  $\bar{G}^{T_\Delta}$ . It is a union of toric varieties

Goal: compute coisotropic locus in  $\bar{G}^{T_\Delta}$  and generalize to rest of  $\mathcal{D}(S)$

Thank YOU