Homotopy fiber product of manifolds

Hsuan-Yi Liao
joint work with Kai Behrend and Ping Xu

國立清華大學
NATIONAL TSING HUA UNIVERSITY

Global Poisson Webinar
Feb 24, 2022

Derived geometry: study spaces with singularities from intersections, zero loci, fiber products, ...

- Algebraic geometric version: appeared around 2004 (Toën–Vezzosi, Lurie, ...)
- Differential geometric version: appeared around 2007 (Spivak, Borisov–Noël, Joyce, Carchedi–Roytenberg, ...)
  Most of the approaches base on C^∞-rings and ∞-categories.
- Our approach to derived differential geometry: we mainly use vector bundles, geodesics, L_∞ algebras

Recall (fiber products of manifolds):

Given smooth maps $X \xrightarrow{f} Z \xleftarrow{g} Y$ between manifolds,

$$X \times_Z Y = \{(x, y) \in X \times Y \mid f(x) = g(y)\}.$$

If f or g is a submersion, then $X \times_Z Y$ is a manifold.

In general, $X \times_Z Y$ is NOT a manifold.

Example

Let X, Y be submanifolds of M. The intersection $X \cap Y$ of X and Y in M can be identified with

$$X \times_M Y = \{(x, y) \in X \times Y \mid \iota_1(x) = \iota_2(y)\},$$

where $\iota_1 : X \hookrightarrow M$, $\iota_2 : Y \hookrightarrow M$ are embeddings of submanifolds.

Example

Let $f \in C^\infty(M, \mathbb{R})$. The zero set $Z(f)$ of f in M can be identified with $M \times_{M \times \mathbb{R}} M$, where the maps are

$$M \to M \times \mathbb{R} : x \mapsto (x, f(x)),$$
$$M \to M \times \mathbb{R} : y \mapsto (y, 0).$$

Purpose of the talk

- Want to study: $X \times_Z Y$.
- Approach: resolve the singular spaces $X \times_Z Y$ by vector bundles and sections.
- Goal of today:
  - introduce quasi-smooth derived manifolds (= vector bundle + a global section)
  - construct homotopy fiber products of manifolds
  - explain the categorical structure behind the construction, i.e., the following theorem:

Theorem (Behrend, L, Xu)

The category of derived manifolds is a category of fibrant objects.

Here, derived manifold = finite-dimensional bundle of curved $L_\infty[1]$ algebras of positive amplitudes.

1. Quasi-smooth derived manifolds

2. Homotopy fiber product of manifolds

3. Categories of fibrant objects

Quasi-smooth derived manifolds

Idea: Resolve $Z(f)$ by $(M \times \mathbb{R}, f)$.

- A quasi-smooth derived manifold $\mathcal{M} = (M, L, \lambda)$ is a vector bundle $L \to M$ together with a global section $\lambda \in \Gamma(M, L)$.
- A morphism $(f, \phi) : \mathcal{M} \to \mathcal{M}'$ is a vector bundle map such that the following diagram commutes:

$$\begin{array}{ccc} L & \xrightarrow{\phi} & L' \\ {\scriptstyle\lambda}\uparrow\downarrow & & \downarrow\uparrow{\scriptstyle\lambda'} \\ M & \xrightarrow[f]{} & M' \end{array}$$

- $(f, \phi) : \mathcal{M} \to \mathcal{M}'$ is called a **fibration / submersion** if f is a submersion and $\phi|_p : L|_p \to L'|_{f(p)}$ is surjective $\forall p \in M$.

Problem: There are too many (M, L, λ) with same zero locus $Z(\lambda)$, so we need a certain notion of equivalence.

Tangent complex and weak equivalence

Assume $p \in M$ is a classical/Maurer-Cartan locus of $\mathcal{M} = (M, L, \lambda)$, i.e., $\lambda(p) = 0_p \in L|_p$. Define $D_p\lambda$ by

$$D_p\lambda : TM|_p \xrightarrow{T\lambda|_p} TL|_{0_p} \cong TM|_p \oplus L|_p \xrightarrow{\text{pr}} L|_p$$

The tangent complex of $\mathcal{M}$ at $p \in Z(\lambda)$ is the two-term complex

$$T\mathcal{M}|_p \quad := \quad TM|_p \xrightarrow{D_p\lambda_0} L|_p$$

The derived dimension $\dim^h(\mathcal{M})$ of $\mathcal{M}$ = the Euler characteristic of $T\mathcal{M}|_p = \dim(M) - \text{rk}(L)$.

A morphism $(f, \phi) : \mathcal{M} \to \mathcal{M}'$ of derived manifolds induces a cochain map

$$T(f, \phi)|_p : T\mathcal{M}|_p \to T\mathcal{M}'|_{f(p)}$$

Definition

A morphism $(f, \phi) : \mathcal{M} \to \mathcal{M}'$ of derived manifolds is called a **weak equivalence** if

- f induces a bijection on classical loci, and
- $T(f, \phi)|_p : T\mathcal{M}|_p \to T\mathcal{M}'|_{f(p)}$ is a quasi-isomorphism at each classical locus $p \in Z(\lambda)$.

In particular, if there is a weak equivalence $\mathcal{M} \to \mathcal{M}'$, then $\mathcal{M}$ and $\mathcal{M}'$ have the same derived dimension.

Example

If $\lambda \in \Gamma(M, L)$ is a regular section (i.e. $D_p\lambda : TM|_p \to L|_p$ is surjective $\forall p \in Z(\lambda)$), then $Z(\lambda)$ is a manifold of dimension $\dim^h \mathcal{M} = \dim M - \operatorname{rk} L$. And the inclusion map $Z(\lambda) = (Z(\lambda), Z(\lambda) \times 0, 0) \to (M, L, \lambda)$ is a weak equivalence.

Algebraic model

The **function algebra** $C^\infty(\mathcal{M})$ of $\mathcal{M} = (M, L, \lambda)$ is the commutative differential graded algebra $(\Gamma(\Lambda^{-\bullet}L^\vee), \iota_\lambda)$:

$$\cdots \xrightarrow{\iota_\lambda} \Gamma(\Lambda^2 L^\vee) \xrightarrow{\iota_\lambda} \Gamma(\Lambda^1 L^\vee) \xrightarrow{\iota_\lambda} C^\infty(M) \to 0$$

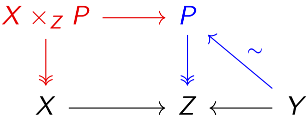
A morphism $(f, \phi) : \mathcal{M} \to \mathcal{M}'$ induces a morphism of cdga's by pullback: $\phi^* : C^\infty(\mathcal{M}') \to C^\infty(\mathcal{M})$.

Proposition

$(f, \phi) : \mathcal{M} \to \mathcal{M}'$ is a weak equivalence iff $\phi^* : C^\infty(\mathcal{M}') \to C^\infty(\mathcal{M})$ is a quasi-isomorphism of cdga's.

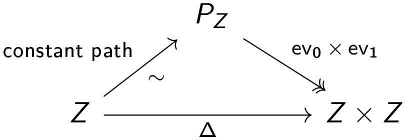
Homotopy fiber product

Idea of homotopy fiber products:



Construction of P:

First construct an important case: diagonal map $\Delta : Z \to Z \times Z$

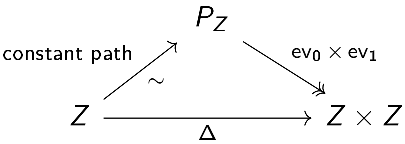


$P_Z =$ (short geodesics, ∇-constant paths in T_Z, derivatives)

- Fix a connection ∇ and fix an open interval $I = (c, d) \supset [0, 1]$.
- $P_Z = (P_g Z, P_{\text{con}} TZ\, dt,\ D)$, where
  - $P_g Z := \{a : I \to Z \mid \nabla_{a'} a' = 0\}$ consists of short geodesics.
  - a fiber $P_{\text{con}} TZ\, dt|_a$ over $a \in P_g Z$ is

$$P_{\text{con}} TZ\, dt|_a = \{\alpha\, dt \mid \alpha \in \Gamma(I, a^* TZ),\ (a^* \nabla)(\alpha) = 0\}$$

  - $D : P_g Z \to P_{\text{con}} TZ\, dt : a \mapsto a'\, dt$ is given by derivatives



Another description of P_Z

$P_Z = (TZ, TZ \times_Z TZ, \delta)$, $\delta(v_p) = (v_p, v_p)$

Base space TZ is actually a neighborhood of zeros where $\exp^\nabla$ is defined.

$$
\begin{array}{ccc}
 & 0_p \in P_Z \ni v_p & \\
 \overset{\sim}{\nearrow} & & \searrow\!\!\searrow \\
p \in Z & \xrightarrow[\Delta]{\qquad\qquad} & Z \times Z \ni (p, \exp^\nabla v_p)
\end{array}
$$

- $Z \xrightarrow{\sim} (P_Z, \delta)$ is a weak equivalence because it induces
  - bijection between zero loci: $Z(0) = Z \to \{0_p \in T_Z\} = Z(\delta)$
  - quasi-isomorphism between tangent complexes:

$$
\begin{array}{ccc}
0 & \longrightarrow & TZ|_p \\
\uparrow & & \uparrow \mathrm{pr}_2 \\
TZ|_p & \xrightarrow[i_1]{} & T(TZ)|_{0_p} = TZ|_p \oplus TZ|_p
\end{array}
$$

- $P_Z \to Z \times Z : (v_p, (v_p, w_p)) \mapsto ((p, \exp^\nabla v_p), 0)$ is a fibration.

General factorization

$$\begin{array}{ccc} & P = P_Z \times_Z Y & \\ \swarrow & & \nwarrow \sim \\ Z & \longleftarrow & Y \end{array}$$

$$P = P_Z \times_Z Y = (TZ \times_Z Y, TZ \times_Z TZ \times_Z Y, \delta), \ \delta(v_p, y) = (v_p, v_p, y)$$

- base space $TZ \times_Z Y$ is a manifold because $TZ \to Z : v_p \mapsto \exp^{\nabla} v_p$ is a submersion.
- $P \to Z$ is a fibration: $TZ \times Y \to Z : (v_p, y) \mapsto p$ is a submersion.
- $Y \to P$ is a weak equivalence: $Z(\delta) = \mathrm{Graph}(Y \to Z) \cong Y$, and its tangent map at y is

$$\begin{array}{ccc} 0 & \longrightarrow & TZ|_p \\ \uparrow & & \uparrow \mathrm{pr}_1 \\ TY|_y & \xrightarrow[i_2]{} & T(TZ \times_Z Y)|_{(0_p, y)} = TZ|_p \oplus TY|_y \end{array}$$

Homotopy fiber product of manifolds

Given smooth maps $X \to Z \leftarrow Y$, the homotopy fiber product $X \times^h_Z Y$ is represented by a quasi-smooth derived manifold

$$X \times_Z P_Z \times_Z Y = (X \times_Z T_Z \times_Z Y, X \times_Z T_Z \times_Z T_Z \times_Z Y, \delta),$$

$\delta(x, v_p, y) = (x, v_p, v_p, y)$.

- the classical locus $Z(\delta) \cong X \times_Z Y$ as sets.
- the derived dimension $\dim^h(X \times_Z P_Z \times_Z Y) = \dim X + \dim Y - \dim Z$.
- if one of $X \to Z \leftarrow Y$ is a submersion, then the map $X \times_Z Y \to X \times_Z P_Z \times_Z Y : (x, y) \mapsto (x, 0_p, y)$ is a weak equivalence.

Derived intersections

Let X, Y be submanifolds of a manifold M. The derived intersection $X \cap^h_M Y$ of X and Y in M is understood as

$$X \cap^h_M Y := X \times^h_M Y$$

which is represented by $X \times_M P_M \times_M Y = (N, E, \tilde{D})$, where

- $N = X \times_M TM \times_M Y = X \times_M P_g M \times_M Y$
  $=$ space of short geodesics which start from a point in X and end at a point in Y
  $=$ an open submanifold of $X \times Y$ consisting of $(x, y) \in X \times Y$ such that x and y are sufficiently close to the set-theoretical intersection $X \cap Y$;

- the fiber $E|_a$ over $a \in N$ is

$$E|_a = \{\alpha \, dt \mid \alpha \in \Gamma(a^* TM),\ (a^*\nabla)(\alpha) = 0\} \cong TM|_{a(0)};$$

- the section

$$\tilde{D} : N \to E : a \mapsto a' \, dt$$

  is given by derivatives.

Furthermore,

- classical locus of $X \cap^h_M Y$ = set-theoretical intersection $X \cap Y$;
- $\dim^h(X \cap^h_M Y) = \dim(X) + \dim(Y) - \dim(M)$;
- if X and Y intersect transversally, then $X \cap Y \to (N, E, \tilde{D})$ is a weak equivalence.

Some problems of quasi-smooth derived manifolds:

- The category of quasi-smooth derived manifolds is NOT closed under homotopy fiber products.
- A weak equivalence is NOT necessarily invertible. To get the expected equivalence relation, we need higher structures.

A solution:

We further extend the category of quasi-smooth derived manifolds to a larger category — the category of derived manifolds — and show that this category is a category of fibrant objects. This structure solves the above problems.

A category of fibrant objects is a category $\mathcal{C}$, together with two subcategories, the category of fibrations and the category of weak equivalences, such that

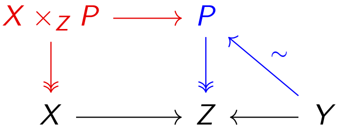
1. weak equivalences satisfy two out of three, all isomorphisms are weak equivalences,
2. all isomorphisms are fibrations,
3. every pullback of a fibration exists, and is again a fibration,
4. every pullback of a trivial fibration (i.e. a fibration which is a weak equivalence) is a trivial fibration,
5. $\mathcal{C}$ has a final object, and all the morphisms ending at the final object are fibrations,
6. there exists a path space object for every object, i.e.

$$\forall X \xrightarrow{\sim} \exists P_X \twoheadrightarrow X \times X, \qquad \forall X \xrightarrow{\Delta} X \times X$$

Factorization Lemma (Brown)

Let $\mathcal{C}$ be a category of fibrant objects. Any morphism $X \to Y$ in $\mathcal{C}$ can be factored $X \xrightarrow{\sim} P \twoheadrightarrow Y$, where $X \xrightarrow{\sim} P$ is a section of a trivial fibration, and $P \twoheadrightarrow Y$ is a fibration.

Idea of homotopy fiber products:



In fact, Brown constructed a homotopy fiber product using path space objects: $X \times^h_Z Y = X \times_Z P_Z \times_Z Y$ which is well-defined in the homotopy category.

Derived manifolds

A derived manifold is a triple $\mathcal{M} = (M, L, \lambda)$, where

- M is a manifold,
- $L = L^1 \oplus \cdots \oplus L^n$ is a finite-dimensional positively graded vector bundle over M,
- $\lambda = (\lambda_k)_{k \geq 0}$ is a smooth family of curved $L_\infty[1]$ structures on L.

That is,

$$\lambda_k : S^k L \to L, \qquad k \geq 0,$$

are degree one vector bundle maps such that

$$Q_\lambda \circ Q_\lambda = 0,$$

where $Q_\lambda \in \mathrm{coDer}^1_{C^\infty(M)}(\Gamma(SL))$ is the coderivation generated by $\lambda : SL = \bigoplus_{k \geq 0} S^k L \to L$.

A morphism of derived manifolds $(f, \phi) : \mathcal{M} \to \mathcal{M}'$ is a smooth map $f : M \to M'$ together with a smooth family of morphisms of curved $L_\infty[1]$ algebras $\phi = (\phi_k)_{k \geq 1} : L \rightsquigarrow f^* L'$.

That is,

$$\phi_k : S^k L \to L', \qquad k \geq 1,$$

are degree zero vector bundle maps such that

$$Q_{\lambda'} \circ F_\phi = F_\phi \circ Q_\lambda,$$

where $F_\phi : \Gamma(SL) \to \Gamma(SL')$ is the coalgbra morphism generated by $\phi = \sum_k \phi_k : SL \to L'$.

Remark

The degree restrictions imply that there are only finite nonzero λ_k and ϕ_k.

Explicit $L_\infty[1]$ equations

First few equations in $L_\infty[1]$:

- $\lambda_1(\lambda_0) = 0$.
- $\lambda_2(\lambda_0, x) = \lambda_1^2(x)$.
- $\lambda_3(\lambda_0, x, y) + \lambda_2(\lambda_1(x), y) + (-1)^{|x||y|}\lambda_2(\lambda_1(y), x) + \lambda_1(\lambda_2(x, y)) = 0$

- $\phi_1(\lambda_0) = \lambda_0'$.
- $\phi_2(\lambda_0, x) + \phi_1(\lambda_1(x)) = \lambda_1'(\phi_1(x))$.

Special cases:

- Manifold case: $L = M \times 0$.
- Quasi-smooth case: $L = L^1$.

Weak equivalences and fibrations

Assume $p \in M$ is a classical/Maurer-Cartan locus of $\mathcal{M}$, i.e., $\lambda_0(p) = 0 \in L^1|_p$. Define $D_p\lambda_0$ by

$$D_p\lambda_0 : TM|_p \xrightarrow{T\lambda_0|_p} TL^1|_{0_p} \cong TM|_p \oplus L^1|_p \xrightarrow{\text{pr}} L^1|_p$$

The tangent complex of $\mathcal{M}$ at $p \in Z(\lambda_0)$ is

$$T\mathcal{M}|_p \quad := \quad TM|_p \xrightarrow{D_p\lambda_0} L^1|_p \xrightarrow{\lambda_1|_p} L^2|_p \xrightarrow{\lambda_1|_p} \cdots$$

The derived dimension $\dim^h(\mathcal{M})$ of $\mathcal{M}$ = the Euler characteristic of $T\mathcal{M}|_p = \dim(M) - \text{rk}(L^1) + \text{rk}(L^2) - \cdots$.

A morphism $(f, \phi) : \mathcal{M} \to \mathcal{M}'$ of derived manifolds induces a cochain map

$$T(f, \phi)|_p : T\mathcal{M}|_p \to T\mathcal{M}'|_{f(p)}$$

Definition

A morphism $(f, \phi) : \mathcal{M} \to \mathcal{M}'$ of derived manifolds is called a **weak equivalence** if

- f induces a bijection on classical loci, and
- $T(f, \phi)|_p : T\mathcal{M}|_p \to T\mathcal{M}'|_{f(p)}$ is a quasi-isomorphism at each classical locus $p \in Z(\lambda_0)$.

In particular, if there is a weak equivalence $\mathcal{M} \to \mathcal{M}'$, then $\mathcal{M}$ and $\mathcal{M}'$ have the same derived dimension.

Definition

A morphism $(f, \phi) : \mathcal{M} \to \mathcal{M}'$ of derived manifolds is called a **fibration** if

- $f : M \to M'$ is a submersion, and
- $\phi_1 : L \to f^*L'$ is surjective.

Theorem (Behrend, L, Xu)

The category of derived manifolds is a category of fibrant objects.

Back to the 2 problems:

- Existence of homotopy fiber products of derived manifolds is guaranteed: $X \times_Z^h Y = X \times_Z P_Z \times_Z Y$.
  For a derived manifold Z, we construct P_Z explicitly by short geodesics.
- By a property of categories of fibrant objects, two derived manifolds X and Y are isomorphic in the homotopy category iff there exists the following diagram of derived manifolds:

$$X \xleftarrow{\sim} \exists\, T \xrightarrow{\sim} Y$$

Thank you!

Kai Behrend, Hsuan-Yi Liao, and Ping Xu, *Derived Differentiable Manifolds*, arXiv e-prints (2020), arXiv:2006.01376.

Kenneth S. Brown, *Abstract homotopy theory and generalized sheaf cohomology*, Trans. Amer. Math. Soc. **186** (1973), 419–458. MR 341469