Almost complex structures, transverse complex structures, and $(p, 0)$ Dolbeault cohomology

Simone Gutt

Université Libre de Bruxelles, Institut Elie Cartan de Lorraine

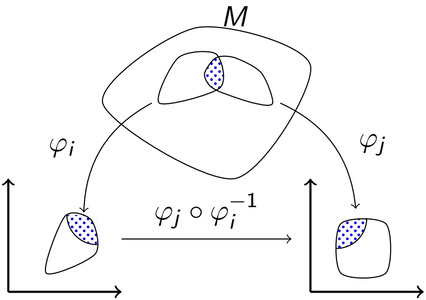
joint work with Michel Cahen and Jean Gutt

Global Poisson, March 10, 2022

Plan of the talk

- Almost complex structures.
- Distributions associated to an almost complex structure.
- Transverse complex structure defined by an almost complex structure.
- Transverse Dolbeault cohomology.
- Dolbeault cohomology for an almost complex structure.
- Interpretation of $H^{(p,0)}_{Dol}$.

Complex structure on a manifold



A smooth (real) manifold is a topological space that is locally modeled on open subsets of $\mathbb{R}^m$ with smooth coordinate changes.

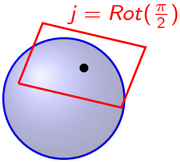
It has *a complex structure* if it can be viewed as a complex manifold, i.e. a topological space locally modeled on open subsets of $\mathbb{C}^m$ with holomorphic coordinate changes.

Almost complex structures

An *almost complex structure* j on M (real) is a smooth section of $\mathrm{End}(TM)$ such that $j^2 = -\mathrm{Id}$.

If M is a complex manifold, it has an *associated almost complex structure*:
$\{z^1 = x^1 + iy^1, \ldots, z^n = x^n + iy^n\}$ being holomorphic coordinates, j reads

$$j\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial}{\partial y^j} \quad j\left(\frac{\partial}{\partial y^j}\right) = -\frac{\partial}{\partial x^j}$$



Integrability - Nijenhuis tensor

An almost complex structure is *integrable* if it comes from a complex structure on M.

The Nijenhuis tensor N^j associated to an almost complex structure j is defined by

$$N^j(X,Y) := [jX, jY] - j[jX, Y] - j[X, jY] - [X, Y]$$

for $X, Y \in \Gamma^\infty(TM)$.

Theorem(Newlander, Nirenberg) : An almost complex structure j is integrable iff its Nijenhuis tensor (Nijenhuis torsion) N^j vanishes.

Irreducibility of the Nijenhuis tensor

Properties of the Nijenhuis tensor: $N^j(X,Y) := [jX, jY] - j[jX, Y] - j[X, jY] - [X, Y]$

$$N^j(X,Y) = -N^j(Y,X); \qquad N^j(jX,Y) = -jN^j(X,Y);$$

Theorem • Let (V, J) be a vector space endowed with a complex structure. The space of tensors $T : V \times V \to V$ satisfying

$$T(X,Y) = -T(Y,X); \qquad T(JX,Y) = -JT(X,Y)$$

is irreducible under the action of the complex linear group $Gl(V,J)$.

Image Distribution associated to the Nijenhuis tensor

$$(\mathrm{Im}\, N^j)_p = \mathrm{span}\{\, N^j_p(X, Y) \,|\, X, Y \in T_pM \,\}$$

Properties : • If $\dim(M) = 4, \dim(\mathrm{Im}\, N^j)_p = 2$ or 0 $\forall p \in M$

• Examples with $\dim(M) > 4$, where $\dim(\mathrm{Im}\, N^j)_p$ can be any $2q$, $0 \leq q \leq n$, the image distribution can be involutive, or not.

Definition An almost complex structure j on a manifold M is *maximally non integrable* if

$$\mathrm{Im}\, N^j = TM.$$

Theorem : (R. Coelho, G. Placini, and J. Stelzig ,arXiv 2021) If there is an almost complex structure on M and if $\dim M \geq 10$, then there exists a maximally non integrable one.

Natural examples On twistor spaces and on some 4-symmetric spaces.

Derived distribution $\left(\mathcal{T}_j^{1,0}\right)^{(1)}$

Decompose $TM^{\mathbb{C}} = T_j^{1,0} \oplus T_j^{0,1}$ into $\pm i$ eigenspaces for j; $\mathcal{T}_j^{1,0}, \mathcal{T}_j^{0,1}$ the spaces of sections

$$A^+ : \mathfrak{X}(M) \to \mathcal{T}_j^{1,0} : X \mapsto \tfrac{1}{2}(X - ijX) \qquad A^- : \mathfrak{X}(M) \to \mathcal{T}_j^{0,1} : X \mapsto \tfrac{1}{2}(X + ijX),$$

$$X + iW = A^+(X + jW) + A^-(X - jW).$$

An almost complex structure j is integrable iff N^j vanishes. This says that the bracket of two sections of $T_j^{1,0}$ is a section of $T_j^{1,0}$.

Given a smooth (real or complex) distribution D whose sections are denoted $\mathcal{D}$, the *derived flag of the distribution* is the sequence of distributions

$$\mathcal{D}^{(0)} = \mathcal{D} \qquad \mathcal{D}^{(1)} = \mathcal{D} + [\mathcal{D}, \mathcal{D}] \qquad \mathcal{D}^{(i+1)} = \mathcal{D}^{(i)} + [\mathcal{D}^{(i)}, \mathcal{D}^{(i)}].$$

Proposition : The first derived distribution of $\mathcal{T}_j^{1,0}$ is given by

$$\left(\mathcal{T}_j^{1,0}\right)^{(1)} = \mathcal{T}_j^{1,0} + [\mathcal{T}_j^{1,0}, \mathcal{T}_j^{1,0}] = \mathcal{T}_j^{1,0} \oplus A^-(\mathcal{I}m\, N^j) = \mathcal{T}_j^{1,0} + \left(\mathcal{I}m\, N^j\right)^{\mathbb{C}}.$$

Derived distribution $\left(\mathcal{T}_j^{1,0}\right)^{(k)}$

The k-th derived distribution of $\mathcal{T}_j^{1,0}$ writes

$$\left(\mathcal{T}_j^{1,0}\right)^{(k)} =: \mathcal{T}_j^{1,0} \oplus A^-(\mathcal{D}_j^{(k)}) = \mathcal{T}_j^{1,0} + \left(\mathcal{D}_j^{(k)}\right)^{\mathbb{C}}.$$

where $\mathcal{D}_j^{(1)} = \mathcal{I}m\, N^j$ and

$$\mathcal{D}_j^{(k+1)} = \mathcal{D}_j^{(k)} + \sum_{M \in \mathcal{D}_j^{(k)}} (\mathcal{I}m\, \mathcal{L}_M j) + [\mathcal{D}_j^{(k)}, \mathcal{D}_j^{(k)}].$$

Each of them is stable under j.

Involutivity of derived distributions

Proposition : The k-th derived distribution $\left(\mathcal{T}_j^{1,0}\right)^{(k)} =: \mathcal{T}_j^{1,0} \oplus A^-(\mathcal{D}_j^{(k)})$ is involutive iff $\mathcal{D}_j^{(k)}$ is involutive and has the property that $\mathcal{I}m\,\mathcal{L}_M j \subset \mathcal{D}_j^{(k)}$ for each $M \in \mathcal{D}_j^{(k)}$.

$\left(\mathcal{T}_j^{1,0}\right)^{\infty} := \lim_k \left(\mathcal{T}_j^{1,0}\right)^{(k)} = \mathcal{T}_j^{1,0} \oplus A^-(\lim_k \mathcal{D}_j^{(k)})$ is involutive and

$\mathcal{D}_j^{\infty} := \lim_k \mathcal{D}_j^{(k)}$ is involutive and has the property that $\mathcal{I}m\,\mathcal{L}_M j \subset \mathcal{D}_j^{\infty}$ for each $M \in \mathcal{D}_j^{\infty}$.

The first derived distribution of $\mathcal{T}_j^{1,0}$, $\left(\mathcal{T}_j^{1,0}\right)^{(1)} = \mathcal{T}_j^{1,0} + \left(\mathcal{I}m\,N^j\right)^{\mathbb{C}}$, is involutive iff

$$[N, jX] - j[N, X] = (\mathcal{L}_N j)X \in \mathcal{I}m\,N^j \qquad \forall X \in \mathfrak{X}(M), \forall N \in \mathcal{I}m\,N^j.$$

$\mathcal{D}$-Transverse structures

Let $\mathcal{D} = \Gamma^\infty(D)$ for D a (real) smooth involutive distribution (not necessarily of constant rank) on a manifold M, or a limit of such, as for $\mathcal{D}_j^\infty$.

If D is regular and M/D has a manifold structure s.t. $p : M \to M/D$ is a submersion, then transverse objects for $\mathcal{D}$ are corresponding objects on M/D.

• $p^* T(M/D) \simeq TM/D =: Q$ the *normal bundle* Q. Let $\Pi : TM \to Q$ denote the canonical projection.

Action of $\mathcal{D}$ on sections of Q: for $F \in \mathcal{D}$ and $u \in \Gamma^\infty(M, Q)$,

$$L_F^Q u := \Pi([F, U]) \quad \text{with } U \in \mathfrak{X}(M), \Pi \circ U = u.$$

A vector field on M/D can be viewed as a section u of Q such that $L_F^Q u = 0$ for $F \in \mathcal{D}$.

A vector field $U \in \mathfrak{X}(M)$ is *$\mathcal{D}$-foliated* iff

$$[F, U] \in \mathcal{D} \quad \text{for any} \quad F \in \mathcal{D}.$$

$\mathfrak{X}_D(M) = \{\text{foliated vector fields}\}$. A *$\mathcal{D}$-transverse vector field* is an element of $\mathfrak{X}_D(M)/\mathcal{D}$.

$\mathcal{D}$-Transverse almost complex structures

- An almost complex structure $\hat{j}$ on M/D, gives a section $\tilde{j}$ of $\mathrm{End}(Q)$ which squares to $-\mathrm{Id}$ and such that $L_F^{\mathrm{End}(Q)}\tilde{j} = 0$ for any $F \in \mathcal{D}$, where $L_F^{\mathrm{End}(Q)}s = L_F^Q \circ s - s \circ L_F^Q$,

One can consider a lift $j \in \mathrm{End}(TM)$ of $\tilde{j}$ (i.e. $\tilde{j}(\Pi(U)) = \Pi(jU)$).

A *$\mathcal{D}$-transverse almost complex structure* is the equivalence class $[j]$ of a section j of $\mathrm{End}(TM)$ such that $j(\mathcal{D}) \subset \mathcal{D}$, $j^2U + U \in \mathcal{D}$, $\forall U \in \mathfrak{X}(M)$,

$$[F, jU] - j[F, U] \in \mathcal{D} \quad \text{for all } F \in \mathcal{D}, U \in \mathfrak{X}(M),$$

the equivalence being given by $j \sim j'$ iff $\mathcal{I}m\,(j - j') \subset \mathcal{D}$.

$\mathcal{D}$-Transverse complex structures

• $\hat{j}$ on M/D is integrable iff its Nijenhuis tensor vanishes, iff $N^{\tilde{j}}(u,v) := [\tilde{j}u, \tilde{j}v] - \tilde{j}[\tilde{j}u, v] - \tilde{j}[u, \tilde{j}v] - [u,v] = 0$ for any u, v sections u of Q such that $L_F^Q u = 0$ for $F \in \mathcal{D}$, hence iff

$$\Pi\left([jU, jV] - j[jU, V] - j[U, jV] - [U,V]\right) = 0$$

for j a lift of $\tilde{j}$ and any U, V foliated vector fields.

A *$\mathcal{D}$-transverse complex structure* is a $\mathcal{D}$-transverse almost complex structure $[j]$ (j of $\mathrm{End}(TM)$, $j(\mathcal{D}) \subset \mathcal{D}$, $j^2U + U \in \mathcal{D}$, $[F, jU] - j[F, U] \in \mathcal{D}$ for all $F \in \mathcal{D}$, $U \in \mathfrak{X}(M)$) such that

$$N^j(U,V) := [jU, jV] - j[jU, V] - j[U, jV] - [U,V] \in \mathcal{D}, \text{ for all } U, V \in \mathfrak{X}(M)$$

$\mathcal{D}$-Transverse forms

A *$\mathcal{D}$-transverse -or basic- p-form* is a p-form ω on M such that

$$\iota(F)\omega = 0 \qquad \text{and} \qquad \mathcal{L}_F\omega = 0 \qquad \forall F \in \mathcal{D}.$$

The differential of a $\mathcal{D}$-transverse form is again $\mathcal{D}$-transverse.

Let j of $\mathrm{End}(TM)$ define a $\mathcal{D}$-transverse complex structure
$(j(\mathcal{D}) \subset \mathcal{D}, j^2U + U \in \mathcal{D}, (\Pi([F, jU] - j[F, U]) = 0$, for $F \in \mathcal{D}, U \in \mathfrak{X}(M)$ and $\Pi(N^j(U, V)) = 0)$.

A $\mathcal{D}$-transverse 1-form ω is of type $(1, 0)$ (resp. $(0, 1)$) if $\omega(U + ijU) = 0, \forall U \in \mathfrak{X}(M)$ (resp. $\omega(U - ijU) = 0)$.

We have the splitting of $\mathcal{D}$-transverse p-forms

$$\Omega^k_{\mathcal{D}}(M) = \oplus_{p+q=k}\Omega^{p,q}_{\mathcal{D},[j]}(M).$$

$\mathcal{D}$-Transverse Dolbeault cohomology

Since $N^j(U, V)) \in \mathcal{D}$, we have

$$d\Omega^{p,q}_{\mathcal{D},[j]}(M) \subset \Omega^{p+1,q}_{\mathcal{D},[j]}(M) \oplus \Omega^{p,q+1}_{\mathcal{D},[j]}(M)$$

and denote by ∂ and $\bar\partial$ the corresponding projections.

The $\mathcal{D}$-transverse $\bar\partial$-cohomology is $H^{p,q}_{\mathcal{D},\bar\partial} = \operatorname{Ker} \bar\partial|_{\Omega^{p,q}_{\mathcal{D},[j]}(M)} / \operatorname{Im} \bar\partial|_{\Omega^{p,q-1}_{\mathcal{D},[j]}(M)}$.

Almost complex foliations with transverse complex structures induced by an almost complex structure

Let j be an almost complex structure, and let $\mathcal{D}$ be an involutive distribution, stable under j.

This j yields a $\mathcal{D}$-transverse almost complex structure iff

$$[F, jU] - j[F, U] \in \mathcal{D} \quad \forall F \in \mathcal{D}, \quad \forall U \in \mathfrak{X}(M),$$

and a complex $\mathcal{D}$-transverse structure iff, furthermore, $\mathcal{D}$ contains the image of N^j.

j defines a complex $\mathcal{D}$-transverse structure iff

$$\mathcal{T}_j^{1,0} + \mathcal{D} = \mathcal{T}_j^{1,0} \oplus A^-(\mathcal{D}) = \mathcal{T}_j^{1,0} + \mathcal{D}^{\mathbb{C}} \quad \text{is involutive} \ .$$

Then $\mathcal{D}^{\mathbb{C}} \supset \mathcal{D}_j^{\infty} = \lim_k \mathcal{D}_j^{(k)} \supset \operatorname{Im} N^j$.

j defines a complex structure transverse to $\mathcal{D}_j^{\infty}$ and a corresponding $\mathcal{D}_j^{\infty}$-transverse Dolbeault cohomology $H^{p,q}_{\mathcal{D}_j^{\infty},\bar{\partial}}$, called the j-transverse Dolbeault cohomology.

j-transverse Dolbeault cohomology: $H^{p,0}_{\mathcal{D}_j^\infty,\bar\partial}$

$$H^{p,0}_{\mathcal{D}_j^\infty,\bar\partial} = \left\{\omega \in \Omega^{p,0}_{\mathcal{D}_j^\infty,[j]}(M) \mid \bar\partial\omega = 0\right\}$$

$$= \left\{\omega \in \Omega^p(M,\mathbb{C}) \mid \iota(Z)\omega = 0,\ \mathcal{L}_Z\omega = 0\,\forall Z \in \left(\mathcal{T}_j^{0,1}\right)^\infty = \lim_k \left(\mathcal{T}_j^{0,1}\right)^{(k)}\right\}$$

Dolbeault cohomology of an almost complex structure (J.Cirici and S Wilson, Adv. in Math., 2021)

j almost complex structure, $TM^{\mathbb{C}} = T_j^{1,0} \oplus T_j^{0,1}$, $T^*M^{\mathbb{C}} = (T_j^*)^{1,0} \oplus (T_j^*)^{0,1}$, so

$$\Omega^k(M,\mathbb{C}) = \oplus_{p+q=k}\Omega_j^{p,q}.$$

$$d\Omega_j^{p,q} \subset \Omega_j^{p-1,q+2} \oplus \Omega_j^{p,q+1} \oplus \Omega_j^{p+1,q} \oplus \Omega_j^{p+2,q-1}$$

and

$$d = \bar{\mu} \oplus \bar{\partial} \oplus \partial \oplus \mu.$$

$$\begin{aligned} H_{\bar{\mu}}^{(p,q)} &= \operatorname{Ker} \bar{\mu}|_{\Omega_j^{p,q}} / \operatorname{Im} \bar{\mu}|_{\Omega_j^{p+1,q-2}} \\ \widetilde{(\bar{\partial})} &: H_{\bar{\mu}}^{(p,q)} \to H_{\bar{\mu}}^{(p,q+1)} : \omega + \operatorname{Im}\bar{\mu} \mapsto \bar{\partial}\omega + \operatorname{Im}\bar{\mu} \\ H_{Dol}^{(p,q)} &= \operatorname{Ker} \widetilde{(\bar{\partial})}|_{H_{\bar{\mu}}^{(p,q)}} / \operatorname{Im} \widetilde{(\bar{\partial})}|_{H_{\bar{\mu}}^{(p,q-1)}} \end{aligned}$$

$$H_{Dol}^{(p,0)} = \left\{ \omega \in \Omega_j^{p,0} \mid \bar{\mu}\omega = 0, \bar{\partial}\omega = 0 \right\}.$$

$H^{(p,0)}_{Dol}$ and transverse cohomology.

$$H^{(p,0)}_{Dol} = \left\{ \omega \in \Omega^{p,0}_j \,|\, \bar{\mu}\omega = 0, \bar{\partial}\omega = 0 \right\}.$$

$$H^{(p,0)}_{\bar{\mu}} = \left\{ \omega \in \Omega^p(M,\mathbb{C}) \,|\, \iota(W)\omega = 0 \,\forall W \in \left(\mathcal{T}^{0,1}_j\right)^{(1)} \right\}.$$

$$\begin{aligned} H^{(p,0)}_{Dol} &= \left\{ \omega \in \Omega^p(M,\mathbb{C}) \,|\, \iota(W)\omega = 0 \; \mathcal{L}_Z\omega = 0 \,\forall Z \in \mathcal{T}^{0,1}_j, \forall W \in \left(\mathcal{T}^{0,1}_j\right)^{(1)} \right\} \\ &= \left\{ \omega \in \Omega^p(M,\mathbb{C}) \,|\, \iota(Z)\omega = 0 \; \mathcal{L}_Z\omega = 0 \,\forall Z \in \left(\mathcal{T}^{0,1}_j\right)^{\infty} = \lim_k \left(\mathcal{T}^{0,1}_j\right)^{(k)} \right\} \\ &= H^{p,0}_{\mathcal{D}^\infty_j, \bar{\partial}} \end{aligned}$$

Minimally non integrable almost complex structures.

$$[N, jX] - j[N, X] \in \mathcal{I}mN^j, \forall X \in \mathfrak{X}(M), N \in \mathcal{I}mN^j$$
$$\Downarrow$$
maximal transverse complex structure.

Definition : An almost complex structure j on M is minimally non integrable if (dim Im N^j is constant $= 2$ and)

$$[N, jX] - j[N, X] \in \mathcal{I}mN^j, \forall X \in \mathfrak{X}(M), N \in \mathcal{I}mN^j.$$

Examples : Complex line bundles over a Kähler manifold, whose first Chern Class can not be represented by a $(1, 1)$-form.

Maximal transverse complex structures induced by an invariant almost complex structure on a Lie group.

j left invariant almost complex structure on a Lie group G; $J = j_e : \mathfrak{g} \to \mathfrak{g}$.
N^j is left invariant; its value at e is

$$N^J(X,Y) := [JX,JY] - J[JX,Y] - J[X,JY] - [X,Y] \qquad X,Y \in \mathfrak{g}.$$

$\operatorname{Im} N^j$ is a a left invariant distribution; $(\operatorname{Im} N^j)_e = \operatorname{Im} N^J$

$$[N,JX] - J[N,X] \in \operatorname{Im} N^J, \quad \forall X \in \mathfrak{g},\ N \in \operatorname{Im} N^J. \tag{1}$$

$$\Downarrow$$

$\operatorname{Im} N^j$ defines a foliation, the leaves carry an induced almost complex structure, and j induces a transverse complex structure.

(1) implies that $\operatorname{Im} N^J$ is a subalgebra of $\mathfrak{g}$ and is satisfied if $\operatorname{Im} N^J$ is an ideal in $\mathfrak{g}$.

Maximal transverse complex structures on a homogeneous space

$$G \times M \to M : (g, x) \mapsto g \cdot x =: \rho(g)x$$

transitive action of a Lie group G on a manifold M; x_0 a base point,
$H = \{g \in G \,|\, g \cdot x_0 = x_0\}$ its stabilizer;

j a G-invariant almost complex structure: $\rho(g)_{*x} \circ j_x = j_{g\cdot x} \circ \rho(g)_{*x} \; \forall g \in G$.

A^* the fundamental vector field defined by $A \in \mathfrak{g}$ ($A^*_x = \frac{d}{dt} \exp -tA \cdot x|_{t=0}$;)

Choose a linear map $J : \mathfrak{g} \to \mathfrak{g}$ such that $j_{x_0} A^* = (JA)^*_{x_0}$ for all $A \in \mathfrak{g}$. Then

$$N^j_{x_0}(A^*, B^*) = \Big(-N^J(A, B) \Big)^*_{x_0} \tag{2}$$

where $N^J(A, B) := [JA, JB] - J[JA, B] - J[A, JB] - [A, B] \quad \forall A, B \in \mathfrak{g}$.

Im N^j is involutive and gives a foliation with transverse complex structure if and only if

$$[JA, N^J(B, C)] - J[A, N^J(B, C)] \in \mathrm{Im}\, N^J + \mathfrak{h} \qquad \forall A, B, C \in \mathfrak{g},$$

Michel Cahen, Maxime Gérard, Simone Gutt and Manar Hayyani, " Distributions associated to almost complex structures on symplectic manifolds", Journal of Symplectic Geometry 19,5 (2022).

Michel Cahen, Jean Gutt and Simone Gutt , "Almost complex structures, transverse complex structures, and $(p, 0)$ Dolbeault cohomology", Preprint (2022).

J.Cirici and S Wilson, Dolbeault cohomology for almost complex manifolds, Advances in Mathematics, 391 (2021).

R. Coelho, G. Placini, and J. Stelzig, Maximally non-integrable almost complex structures: an h-principle and cohomological properties, arXiv 2105.12113.

Thank you for your attention!