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# Wonderful Compactification of a Cartan Subalgebra of a Semisimple Lie Algebra

Li, Yu

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Joint work w/ S. Evans.

Related work: T. Braden, J. Huh, J. Matherne, N. Proudfoot, B. Wang.

## 1. Definition and motivation

IC

$G$ : semisimple gp.  $\ni H$ : Cartan subgp.  
adjoint type

$\mathfrak{g} = \text{Lie } G$ ,  $\mathfrak{h} = \text{Lie } H$ .  $n := \dim \mathfrak{g}$ ,  $\mathfrak{g}^*$ : dual space of  $\mathfrak{g}$ .

$\text{Gr}(n, \mathfrak{g} \times \mathfrak{g}^*)$ : Grassmannian of  $n$ -dimensional vect. subsp. of  $\mathfrak{g} \times \mathfrak{g}^*$ .

Have an embedding  $\mathfrak{g}^* \hookrightarrow \text{Gr}(n, \mathfrak{g} \times \mathfrak{g}^*)$ .  
 $\alpha \mapsto \{(y, -\text{adj}_y \alpha) : y \in \mathfrak{g}\}$

Def The wonderful compactification  $\bar{\mathfrak{g}}^*$  of  $\mathfrak{g}^*$  is the closure of the image of  $\mathfrak{g}^*$  in  $\text{Gr}(n, \mathfrak{g} \times \mathfrak{g}^*)$ .

$\kappa$ : Killing form.  $\hookrightarrow \mathfrak{g} \hookrightarrow \mathfrak{g}^*$ .  
 $x \mapsto \kappa(x, -)$

Def The wonderful compactification  $\bar{\mathfrak{t}}_g$  of  $\mathfrak{t}_g$  is the closure of the image of  $\mathfrak{t}_g$  in  $\text{Gr}(n, \mathfrak{g} \times \mathfrak{g}^*)$  under the composition  $\mathfrak{t}_g \hookrightarrow \mathfrak{g}^* \hookrightarrow \text{Gr}(n, \mathfrak{g} \times \mathfrak{g}^*)$ .

$(U, \pi_U)$ : Poisson Lie gp.  $\rightsquigarrow (\mathfrak{u} \times \mathfrak{u}^*, \mathfrak{u}, \mathfrak{u}^*, \langle -, - \rangle)$ : Manin triple.

- $\mathfrak{u} \times \mathfrak{u}^*$ : Drinfeld double Lie alg.  
 $\mathfrak{u}$  (resp.  $\mathfrak{u}^*$ ) acts on  $\mathfrak{u}^*$  (resp.  $\mathfrak{u}$ ) by the coadj. action
- $\langle -, - \rangle$ : symmetric, nondegenerate, invariant bilinear form on  $\mathfrak{u} \times \mathfrak{u}^*$
- $\mathfrak{u}, \mathfrak{u}^*$ : Lie subalgebras of  $\mathfrak{u} \times \mathfrak{u}^*$

Lagrangian subspaces w.r.t.  $\langle -, - \rangle$   
 $q \otimes q^* \cong q \oplus q^*$  as vect. spaces.

$\mathcal{L} = \mathcal{L}(q \otimes q^*) := \{ \text{Lie subalg.s of } q \otimes q^* \text{ that are Lagrangian subspaces w.r.t. } \langle -, - \rangle \}$

- variety of Lagrangian subalgebras
- projective variety
- closed subvariety of  $\text{Gr}(\dim q, q \otimes q^*)$ .

$(M, \pi_M)$ : Poisson variety.  $U \curvearrowright M$ : transitive action.

We say that  $(M, \pi_M)$  is Poisson homogeneous space for  $(U, \pi_U)$  if the action map  $U \times M \xrightarrow{\text{act}} M$  is Poisson.

Assume  $(M, \pi_M)$  is a Poisson homogeneous space for  $(U, \pi_U)$ .

Drinfeld map:  $M \rightarrow \mathcal{L}(q \otimes q^*)$ :  $m \mapsto l_m := \{(x, \alpha) \in q \otimes q^*: \alpha|_{qm} = 0,$   
 (Drinfeld, Evens-Lu)  
 $\alpha \cup \pi_M(m) = x \bmod q^m\}$ .

- $U$ -equivariant

- Poisson map : there exists a Poisson structure on  $\mathcal{L}(u \otimes u^*)$  s.t the Drinfeld map is Poisson

- Almost an embedding : an embedding up to a covering map.

Therefore, it is reasonable to regard  $\mathcal{L}(u \otimes u^*)$  as the "universal Poisson homogeneous space" for the Poisson Lie gp.  $(U, \pi_U)$ .

So it is important to understand algebraic and Poisson geometric properties of  $\mathcal{L}(u \otimes u^*)$ . (E.g.  $U$ -orbits, symplectic leaves, etc.)

### Important case A (Evens-Lu)

$(G, \pi_{st})$ : standard Poisson Lie gp.  $\rightsquigarrow (\mathfrak{g} \oplus \mathfrak{g}, \mathfrak{g}_\Delta^\pm, \mathfrak{g}_{st}^\pm, \langle \cdot, \cdot \rangle)$ , where  $\mathfrak{g}_\Delta^\pm$  = diagonal Lie subalg.,  $\mathfrak{g}_{st}^\pm = \mathfrak{g}_\Delta^\pm + \mathfrak{g}_+ \oplus \mathfrak{g}_-$   $\rightsquigarrow \mathcal{L} = \mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$  has been studied in great detail by Evens-Lu.

$$[\mathfrak{g}_\Delta^\pm] \in \mathcal{L}(\mathfrak{g} \oplus \mathfrak{g}) \rightsquigarrow \text{Stab}_{G \times G}([\mathfrak{g}_\Delta^\pm]) = G_\Delta \rightsquigarrow (G \times G).[\mathfrak{g}_\Delta^\pm] \cong (G \times G)/G_\Delta \cong G.$$

$$[L^{(e,g)}] \sim q$$

Facts • The closure of  $(G \times G) \cdot [g_\Delta]$  in  $\mathcal{Z}(g \otimes g)$  (i.e. in  $\text{Gr}(n, g \otimes g)$ ) is the De Concini-Procesi wonderful compactification  $\bar{G}$  of  $G$ .

• The closure of every  $(G \times G)$ -orbit in  $\mathcal{Z}(g \otimes g)$  is smooth; have a filtration  $\bar{G}_S \hookrightarrow \overline{(G \times G)\text{-orbit in } \mathcal{Z}(g \otimes g)}$ , where the fiber is the



$$(G/P_S) \times (Q/P_T^-)$$

wonderful compactification of a semisimple subgp.  $G_S$  of  $Q$ .

$$H \hookrightarrow G \cong (G \times G) \cdot [g_\Delta] \hookrightarrow \mathcal{Z}(g \otimes g) (\hookrightarrow \text{Gr}(n, g \otimes g)).$$

• The closure  $\bar{H}$  of the image of  $H$  under this composition is the wonderful compactification of  $H$

•  $\bar{H}$  is a smooth toric variety

• The combinatorial data corresponding to  $\bar{H}$  is the Weyl chambers.

Important case B

$(\mathfrak{g}^*, \cdot)$ : additive gp. .  $\pi_{KK}$ : Kirillov - Kostant Poisson structure on  $\mathfrak{g}^*$ .

$(\mathfrak{g}^*, \pi_{KK})$ : Poisson Lie gp.  $\hookrightarrow (\mathfrak{g} \times \mathfrak{g}^*, \mathfrak{g}, \mathfrak{g}^*, \langle \cdot, \cdot \rangle)$ : Manin triple  $\hookrightarrow$   
 $\mathcal{L} = \mathcal{L}(\mathfrak{g} \times \mathfrak{g}^*)$ .

Facts (Evens-L.) •  $\{$  closed  $(G \times \mathfrak{g}^*)$ -orbits in  $\mathcal{L}(\mathfrak{g} \times \mathfrak{g}^*)\}_{G \times G} \cong \{\text{abelian ideals of}$   
a fixed Borel subalg. of  $\mathfrak{g}\}$ .  
 $\mathcal{L}(\mathfrak{g} \otimes \mathfrak{g})$

• "classification of  $G$ -orbits in  $\mathcal{L}(\mathfrak{g} \times \mathfrak{g}^*)$ "  $\approx$  "classification  
of finite dimensional Lie algebras".  $G_\Delta$   $\mathcal{L}(\mathfrak{g} \otimes \mathfrak{g})$

$$[\mathfrak{g}] \in \mathcal{L}(\mathfrak{g} \times \mathfrak{g}^*) \hookrightarrow \text{Stab}_{G \times \mathfrak{g}^*}([\mathfrak{g}]) = G \hookrightarrow (G \times \mathfrak{g}^*).[\mathfrak{g}] \cong (G \times \mathfrak{g}^*)/G \cong \mathfrak{g}^* \dots$$

$[\mathfrak{g}] \in \mathcal{L}(\mathfrak{g} \otimes \mathfrak{g})$        $\text{Stab}_{G \times G}([\mathfrak{g}_\Delta]) = G_\Delta$        $\text{Ad}_{(e,\alpha)} \mathfrak{g} \hookrightarrow [(e,\alpha)] \hookrightarrow \alpha$   
 $= \{(y, -\alpha y^* \alpha) : y \in \mathfrak{g}\}$

$\bar{\mathfrak{g}}^*$  defined above is the closure of  $(G \times \mathfrak{g}^*).[\mathfrak{g}]$  (analogous to  $\bar{G}$  in important  
case A)  $\hookrightarrow \bar{\mathfrak{g}}$  defined above is analogous to  $\bar{H}$  in important case A.

$\Phi = \Phi^+ \sqcup \Phi^-$ : roots.  $\Pi$ : simple roots.

For  $\lambda \in \bar{\Phi}$ , choose a root vector  $e_\lambda$ .

Identify  $g^*$  with  $g$  via the Killing form  $K$ .

$$\mathfrak{t}_y \hookrightarrow g^* \hookrightarrow \text{Gr}(n, g \times g^*)$$

$$x \mapsto K(x, -) \mapsto \{ (y, -\text{adj}^* K(x, -)) : y \in g \}$$

$$\cong \{ (y, [x, y]) : y \in g \}$$

$$= \mathfrak{t}_y \oplus 0 + \text{Span}((e_\lambda, \lambda(x)e_\lambda) : \lambda \in \bar{\Phi})$$

$$= \mathfrak{t}_y \oplus 0 + \text{Span}((e_\lambda, \underline{\lambda(x)} e_\lambda), (e_{-\lambda}, -\lambda(x) e_{-\lambda}) : \lambda \in \Phi^+).$$

Consider the embedding  $\mathbb{C}^{\bar{\Phi}^+} \hookrightarrow \text{Gr}(n, g \times g^*)$ .

$$(x_\lambda)_{\lambda \in \bar{\Phi}^+} \mapsto \mathfrak{t}_y \oplus 0 + \text{Span}((e_\lambda, \underline{x_\lambda} e_\lambda), (e_{-\lambda}, -\underline{x_\lambda} e_{-\lambda}) : \lambda \in \bar{\Phi}^+)$$

In fact, have a closed embedding  $(\mathbb{P}^1)^{\bar{\Phi}} \hookrightarrow \text{Gr}(n, g \times g^*)$ .

$$([\underline{x}_{\lambda,0} : \underline{x}_{\lambda,1}])_{\lambda \in \bar{\Phi}^+} \mapsto \mathfrak{t}_y \oplus 0 + \text{Span}((\underline{x}_{\lambda,0} e_\lambda, \underline{x}_{\lambda,1} e_\lambda), (\underline{x}_{\lambda,0} e_{-\lambda}, -\underline{x}_{\lambda,1} e_{-\lambda}) : \lambda \in \bar{\Phi}^+)$$

Get  $\mathfrak{t}_y \hookrightarrow \text{Gr}(n, g \otimes g^*)$ . Consequently,  $\mathbb{T}_y$  is the closure of  $\mathfrak{t}_y$  in  $(\mathbb{P}^1)^{\Phi^+}$ .

Ex 1 Type A<sub>1</sub>.

$$\bar{g}^* = \bar{\mathfrak{sl}}_2^* \cong \mathbb{P}^3.$$

$$\mathfrak{t}_y \cong \mathbb{C} \hookrightarrow \mathbb{P}^1 \implies \mathbb{T}_y \cong \mathbb{P}^1 = \mathfrak{t}_y \amalg \overbrace{\{ \infty \}}^{\text{irreducible component of } \partial \mathfrak{t}_y}.$$

↓  
irreducible component of  $\partial \mathfrak{t}_y$

aff. stratification

Ex 2 Type A<sub>2</sub>.

$$\Phi^+ = \{\alpha, \beta, \alpha+\beta\}.$$

$$\mathfrak{t}_y \hookrightarrow \mathbb{C}^3 \hookrightarrow (\mathbb{P}^1)^3 \implies \text{Can show that } \mathbb{T}_y \text{ is cut out by the multi-} \\ x \mapsto (\alpha(x), \beta(x), (\alpha+\beta)(x))$$

homogenization of  $x_\alpha + x_\beta - x_{\alpha+\beta}$ , where  $x_\alpha, x_\beta, x_{\alpha+\beta}$  are coordinates on  $\mathbb{C}^3$ .

$\alpha_0 \beta_0 \alpha+\beta_0 \alpha_0 \beta_1 \alpha+\beta_0 \alpha_0 \beta_0 \wedge \alpha+\beta_1 \rightarrow \gamma$  looks as follows:

$x_\alpha$	$x_\beta$	$x_{\alpha+\beta}$	
fin.	fin.	fin.	$\mathbb{P}^1$
arb.	$\infty$	$\infty$	$\mathbb{P}'$
$\infty$	arb.	$\infty$	$\mathbb{P}'$
$\infty$	$\infty$	arb.	$\mathbb{P}'$

$$\hookrightarrow T_y = \mathbb{P}_y \amalg \underbrace{(\mathbb{P}' \cup \mathbb{P}' \cup \mathbb{P}')}_{\Phi} = \mathbb{P}_y \amalg (\mathbb{C} \times \{0\} \times \{00\}) \amalg (\{00\} \times \mathbb{C} \times \{00\}) \amalg (\{00\} \times \{00\} \times \mathbb{C}) \amalg (\{00\} \times \{00\} \times \{00\}).$$

irreducible components of  $\mathbb{P}_y$   
corresponding to  $\alpha, \beta, \alpha+\beta$

aff. stratification

### 3. Basic properties of $T_y$

#### 3.1. Defining ideal of $T_y$

Naively, I would expect the following:

for each  $\lambda \in \Phi^+ - \Pi$ , write  $\lambda = \sum_{\alpha \in \Pi} c_\alpha(\lambda) \alpha$ ,  $c_\alpha(\lambda) \in \mathbb{Z}$ . Then the linear

polynomial  $x_\lambda - \sum c_\alpha(\lambda) x_\alpha$  on  $\mathbb{C}^{\Phi^+}$  vanishes on  $\mathbb{P}_y$ . The naive guess

is that the multi-homogenization of these polynomials cut out  $\mathbb{P}^1$  in  $(\mathbb{P}^1)^{\Phi^+}$ . But this is NOT the case.

Ex Type  $A_3$ .

$$\Phi^+ = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}.$$

$x \ y \ z \ u \ v \quad s$  : coordinates on  $\mathbb{C}^{\Phi^+} \cong \mathbb{C}^6$ .

Relations above:  $\alpha_1 + \alpha_2 = 1 \cdot \alpha_1 + 1 \cdot \alpha_2 \rightarrow$  linear polynomials

$$\alpha_2 + \alpha_3 = 1 \cdot \alpha_2 + 1 \cdot \alpha_3$$

$$\alpha_1 + \alpha_2 + \alpha_3 = 1 \cdot \alpha_1 + 1 \cdot \alpha_2 + 1 \cdot \alpha_3$$

above:  $u - x - y$

$v - y - z$

$s - x - y - z$ .

Consider, for example, the "infinity chart"  $(\mathbb{P}^1 - \{0\})^6$  of  $(\mathbb{P}^1)^6$ .

On this chart, the defining polynomials above become  $\begin{cases} xy - xu - yu = 0 \\ yz - yv - zv = 0 \end{cases}$

$$1xyz - xys - xzs - yzs = 0.$$

These are satisfied when  $x=y=z=0$  and  $u,v,s$  are arbitrary. But this is a 3-dimensional variety, too large to be in the boundary of the 3-dim. variety  $\mathbb{J}_y$ .

For each rel.  $\sum_{\lambda \in \Phi^+} c_\lambda \lambda = 0$  among the pos. roots, consider the polynomial  $\sum_{\lambda \in \Phi^+} c_\lambda x_\lambda$  in  $\mathbb{C}[\Phi^+]$ .

Def The ideal  $I(\Phi)$  is the multi-homogeneous ideal of the multi-graded alg.  $\mathbb{C}[x_{\lambda,0}, x_{\lambda,1} : \lambda \in \Phi^+]$  generated by the multi-homogenization of the polynomials  $\sum_{\lambda \in \Phi^+} c_\lambda x_\lambda$  above.

Thm!  $V(I(\Phi)) \cong \mathbb{J}_y$ .

The naive guess above can be remedied as follows, which is also a convenient way to find a finite generating set of  $I(\Phi)$ .

Let  $A$  be the  $(\Phi^+ - \Pi) \times \Phi^+$  matrix whose rows are the coefficients of

the relations  $\lambda - \sum_{\alpha \in \Pi} c_\alpha(\lambda) \alpha, \lambda \in \Phi^+ - \Pi$ .

Def A cocircuit is a set of columns of the matrix  $A$  (i.e. a subset of  $\Phi^+$ ) which intersects each set of columns of  $A$  which is a basis of  $\text{Span}_F$  and is minimal, w.r.t. inclusion, among sets of columns of  $A$  with this property.

Fact Let  $S \subseteq \Phi^+$  be a cocircuit. Then, up to a nonzero scalar, there is a unique relation  $\sum_{\lambda \in \Phi^+} c_\lambda \lambda = 0$  among the pos. roots whose support is  $S$ .

Let  $S, \sum_{\lambda \in \Phi^+} c_\lambda \lambda$  be as above. Write  $p_S$  for the multi-homogenization of  $\sum_{\lambda \in \Phi^+} c_\lambda \chi_\lambda$ .

Fact  $I(\Phi) = (p_S : S \text{ a cocircuit})$ .

Ex Type  $A_2$ .

$\Phi^+ = \{\alpha, \beta, \alpha+\beta\}$ . Relation:  $(\alpha+\beta) - \alpha - \beta = 0 \Rightarrow A = \begin{pmatrix} \alpha & \beta & \alpha+\beta \\ 1 & 1 & 1 \end{pmatrix}, \alpha + \beta$

Each column of  $A$  is a basis of  $\text{Span}_F A$  so a cocircuit must

meet  $\{\alpha\}$ ,  $\{\beta\}$ ,  $\{\alpha+\beta\}$ . Hence the only cocircuit is  $\{\alpha, \beta, \alpha+\beta\}$ . The uniq. rel. whose support is  $\{\alpha, \beta, \alpha+\beta\}$  is  $(\alpha+\beta)-\alpha-\beta=0$ . So  $P_{\{\alpha, \beta, \alpha+\beta\}}$  is the multihomogenization of  $\chi_{\alpha+\beta} - \chi_\alpha - \chi_\beta$ .

Ex Type A<sub>3</sub>.

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \begin{matrix} \alpha_1 + \alpha_2 \\ \alpha_2 + \alpha_3 \\ \alpha_1 + \alpha_2 + \alpha_3 \end{matrix} \rightsquigarrow \text{Cocircuits are } \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$$

$\{\alpha_2, \alpha_3, \alpha_2 + \alpha_3\}$ ,  $\{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$ ,  $\{\alpha_1, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$ ,  $\{\alpha_1 + \alpha_2, \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$ ,  
 $\{\alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_2, \alpha_1 + \alpha_2 + \alpha_3\}$ ,  $\{\alpha_1, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2, \alpha_3\} \rightsquigarrow U-X-Y, V-Y-Z, S-X-Y-Z$ ,  
 $S-X-V, S-Z-U, U+V-Y-S, X+V-U-Z$ .

### 3.2 Irreducible components of $\partial_\Phi^k$

Def Let  $\Phi'$  be a root subsys. of  $\Phi$ .

We say that  $\Phi'$  is closed if  $\lambda.u \in \Phi'$  and  $\lambda+u \in \Phi$  implies.

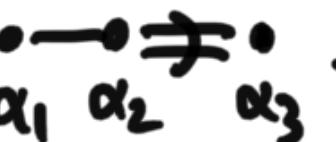
$\lambda + \mu \in \Phi'$ .

We say that  $\Phi'$  is good if  $\Phi'$  is closed, of rank  $\text{rk } \Phi - 1$  and is maximal, w.r.t. inclusion, among root subsys.s of  $\Phi$  with these properties.

Ex  $(x_\lambda)_{\lambda \in \Phi^+} \in (\mathbb{P}^1)^{\Phi^+}$ . Assume  $(x_\lambda)$  is in  $\mathcal{J}_y$ . Then  $F_m := \{ \lambda \in \Phi^+ : x_\lambda \neq \infty \}$  is the set of pos. roots of a closed root subsys. of  $\Phi$ . Since for  $(x_\lambda)$  to be a generic point of an irreducible component of  $\partial_y$ , we want  $F_m$  to be as large as possible, it is reasonable to expect that irreducible components of  $\partial_y$  have to do with good root subsys.s.

Ex  $\Phi$  of type  $A_1$ .  $\Phi' = \emptyset$ .

$\Phi$  of type  $A_2$ .  $\Phi' = \{ \pm \alpha \}, \{ \pm \beta \}, \{ \pm (\alpha + \beta) \}$ .

Ex  $\Phi$  of type  $B_3$ . 

$\Phi' = \{ \alpha_1, \alpha_2 \} \cong \Phi''$  (i.e.  $\Phi' \subset \Phi''$ )

$\Phi := \langle \alpha_2, \alpha_3 \rangle$  s.t.  $\Phi^\vee = \{ \text{short roots in } \Phi \}.$

good, type $B_2$	closed, not good, type $A_2$
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For each good root subsys.  $\Phi'$  of  $\Phi$ , choose a s.s. Lie subalg.  $g'$  of  $g$  and a Cartan subalg.  $t_{g'}$  of  $g'$  s.t. the root sys. of  $(g', t_{g'})$  is  $\Phi'$ . Then  $\overline{t_{g'}} = V(I(\Phi')) \subseteq (\mathbb{P}')^{(\Phi')^+}$ .

Define an embedding  $(\mathbb{P}')^{(\Phi')^+} \hookrightarrow (\mathbb{P}')^{\Phi^+}$  by  $y_\lambda = \begin{cases} x_\lambda & \lambda \in (\Phi')^+ \\ \infty & \lambda \notin (\Phi')^+. \end{cases}$  Via  
 $(x_\lambda)_{\lambda \in (\Phi')^+} \mapsto (y_\lambda)_{\lambda \in \Phi^+}$

this embedding,  $\overline{t_{g'}}$  becomes a closed subvariety of  $(\mathbb{P}')^{\Phi^+}$ .

Theorem 2  $\overline{t_{g'}} = t_g \amalg \bigcup_{\Phi': \text{good}} \overline{t_{g'}}$ .

In particular, there is a bijection:

$\{ \text{good root subsystems of } \Phi \} \rightarrow \{ \text{irreducible components of } \overline{t_{g'}} \}$

$$\Phi' \longmapsto \overline{t_{g'}}.$$

Thm 3  $I_y \amalg \cup_{\Phi: \text{good}} \mathfrak{t}_y' \subseteq (\mathfrak{t}_y)^{\text{reg}}$ .

In particular,  $\mathfrak{t}_y$  is regular in codimension one. (R1).

Thm 4 (Ardila-Booscher)  $\mathfrak{t}_y$  is Cohen-Macaulay.

In particular,  $\mathfrak{t}_y$  satisfies Serre's condition (S2).

Cor  $\mathfrak{t}_y$  is a normal variety.

Recall  $X$ : normal variety,  $H \curvearrowright X$ .

$X$  is called a toric variety if there exists an open dense  $H$ -orbit in  $X$  which is  $H$ -equivariantly iso. to  $H$ .

$(\mathfrak{t}_y, +)$ : additive gp.

The  $\mathfrak{t}_y$ -action on itself by translation extends to an  $\mathfrak{t}_y$ -action on  $\mathfrak{t}_y$ .

So  $\mathfrak{t}_y$  becomes a normal variety with an action of the additive gp.  $(\mathfrak{t}_y, +)$  s.t. there exists an open dense  $\mathfrak{t}_y$ -orbit which is  $\mathfrak{t}_y$ -equivariantly iso. to  $\mathfrak{t}_y$ .

Reasonable to regard  $\tilde{\mathfrak{t}}_y$  as an "additive toxic variety". Will come back to this later.

### 3.4. Borel-de Siebenthal algorithm

How to find good root subsys.s of  $\tilde{\Phi}$ ?

W. Weyl gp.

Thm 5 Up to the action of  $W$ , all good root subsys.s of  $\tilde{\Phi}$  can be obtained as follows:

- ① Take a connected component  $D$  of the Dynkin diagram of  $\tilde{\Phi}$ ;
- ② Affinize  $D$  to get an affine Dynkin diagram  $\hat{D}$ ;
- ③ Remove two nodes whose Dynkin labels are coprime from  $\hat{D}$ ;
- ④ Take the root sys. generated by roots corresponding to the remaining nodes.

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### 4. Cohomology of $\tilde{\mathfrak{t}}_y$ and the Coxeter arrangement

#### 4.1. Betti numbers

By Thm. 2,  $\overline{\mathcal{I}}_g = \overline{\mathcal{I}}_g \amalg \bigcup_{\overline{\Phi}' \text{ good}} \overline{\mathcal{I}}'_g$ . Apply this theorem to  $\overline{\mathcal{I}}'_g$ . We get  $\overline{\mathcal{I}}'_g = \overline{\mathcal{I}}'_g \amalg \bigcup_{\substack{\overline{\Phi}'' \text{ good, s.t.} \\ \text{subsys. of } \overline{\Phi}'}} \overline{\mathcal{I}}''_g$ . Iterate this procedure, we get an affine stratification of  $\overline{\mathcal{I}}_g$ . Consequently, to determine the Betti numbers of  $\overline{\mathcal{I}}_g$ , we need to find the number of good root subsys.s, the number of good root subsys.s of good wot subsys.s, etc.

When  $\mathfrak{g}$  (or  $\overline{\Phi}$ ) is of classical type, the Betti numbers of  $\overline{\mathcal{I}}_g$  have combinatorial significance.

$f(\mathfrak{g}, k) = f(\overline{\Phi}, k) := *$  codim.  $k$  cells in  $\overline{\mathcal{I}}_g$ .

Def The Stirling number of the second kind  $S(n, k)$  is the number of partitions of  $\{1, \dots, n\}$  into  $k$  nonempty subsets.

Ex  $n=2, k=1 \rightarrow \{1, 2\} \rightarrow S(2, 1) = 1$ .

$n=2, k=2 \rightarrow \{\{1\}, \{2\}\} \rightarrow S(2,2)=1.$

$n=3, k=1 \rightarrow \{\{1,2,3\}\} \rightarrow S(3,1)=1.$

$n=3, k=2 \rightarrow \{\{1\}, \{2,3\}\}; \{\{2\}, \{1,3\}\}; \{\{3\}, \{1,2\}\} \rightarrow S(3,2)=3.$

$n=3, k=3 \rightarrow \{\{1\}, \{2\}, \{3\}\} \rightarrow S(3,3)=1.$

Thm 6  $f(A_n, k) = S(n+1, k+1).$

Fact  $S(n, k) = \frac{1}{k!} \sum_{i=0}^k (-1)^i \binom{k}{i} (k-i)^n.$

$F_A(q, t) := \sum_{n, k=0}^{\infty} f(A_n, k) q^k \frac{t^n}{(n+1)!}$  : two variable generating fcn.

Cor  $F_A(q, t) = \frac{e^{q(t-1)}}{qt} - 1.$

$K$ : finite gp. of order  $m$ .

$\Omega_n(K)$ : Dowling lattice of  $K$

- "colored partitions of  $\{1, \dots, n\}$  with colors in  $K$ "
- geometric lattice: ranked poset  
has  $\wedge$  and  $\vee$

semi-modular:  $rk(x) + rk(y) \geq rk(xy) + rk(x \vee y)$

atomic: every elem. is the join of atoms

- depends only on  $m$
- $\mathcal{Q}_n(\{1\}) = \{\text{partitions of } \{1, \dots, n\}\}$ .

Def The Whitney number of the second kind  $W_k(\mathcal{Q}_n(k))$  of the Dowling lattice is the number of  $k$  elements of  $\mathcal{Q}_n(k)$ .

Ex  $W_k(\mathcal{Q}_n(\{1\})) = S(n, k)$ .

$W_k(\mathcal{Q}_n(\{1, q\})) = *$  "type B partitions" of  $\{1, \dots, n\}$  into  $k$  blocks.

Thm 7  $f(B_n, k) = f(C_n, k) = W_k(\mathcal{Q}_n(\{1, q\}))$ .

Cor  $f(B_n, k) = f(C_n, k) = \sum_{i=0}^{m-k} \binom{n}{i} S(m-i, k) 2^{m-k-i}$ .

$F_B(q, t) := \sum_{n,k=0}^{\infty} f(B_n, k) q^k \frac{t^n}{n!}$ .

Cor  $F_B(q, t) = \exp(t + \frac{q(e^{2t}-1)}{2})$ .

Thm 8  $f(D_n, k) = \sum_{i=k}^n \binom{n}{i} 2^{i-k} S(i, k) - n 2^{m-k} S(m, k)$ .

$$\text{Or } F_D(q,t) = (e^t - t) \exp\left(\frac{t}{2}(e^{2t} - 1)\right).$$

## 4.2. Coxeter arrangement and Weyl gp. representation

$\mathcal{A} := \{\lambda^\perp : \lambda \in \Phi^+\}$  : hyperplane arrangement in  $\mathfrak{t}_y^*$   
 Coxeter arrangement

$L(\mathcal{A}) := \left\{ \bigcap_{\lambda \in S} \lambda^\perp : S \subseteq \Phi^+ \right\}$ :

- intersection lattice (geometric lattice)
- partial order :  $X \in Y \Leftrightarrow X \supseteq Y$
- $\text{rk}(X) := \text{codim } X$
- $X \vee Y = X \cap Y$
- $W$ -action induced from the  $W$ -action on  $\mathfrak{t}_y^*$ .

{cells in  $\mathfrak{t}_y^*$ }:

- partial order :  $C \in C' \Leftrightarrow C \subseteq \overline{C'}$

- $\text{rk}(C) := \dim C$

- $W$ -action :  $W$ -action on  $\mathfrak{t}_y^*$  extends to a  $W$ -action

on  $\mathfrak{t}_y^*$ , hence a  $W$ -action on  $H^*(\mathfrak{t}_y^*)$ .

Have maps  $L(G) \xrightarrow{G} \{\text{cells in } \mathbb{F}_q\}$ .

$$\begin{array}{ccc} X & \mapsto & X^\perp \cap \bar{\Psi} \\ \bar{\Psi}^\perp & \longleftarrow & \bar{\Psi} \end{array}$$

Thm 9 The maps  $F$  and  $G$  are mutually inverse iso.s of graded posets equipped with a  $\mathbb{W}$ -action.

Def Let  $L$  be a finite graded poset w/ a uniq. min. elem.  $\hat{0}$ .

The Möbius function  $\mu$  of  $L$  is defined as follows:

$$\mu(X, Y) = 1, \quad \mu(X, Y) = 0 \text{ if } X \neq Y$$

$$\mu(X, Y) = - \sum_{X \leq Z \leq Y} \mu(X, Z) \text{ if } X \leq Y.$$

The characteristic polynomial  $P_L$  of  $L$  is  $P_L(t) := \sum_{X \in L} \mu(\hat{0}, X) t^{\text{rk}(X)}$ .

The generating function  $q_L$  of  $L$  is  $q_L(t) := \sum_{X \in L} t^{\text{rk}(X)}$ .

The Whitney number of the first kind  $w_F(L)$  of  $L$  is the coeff.

$$1 + \text{rk}(L) - k \text{ in } P_L(t)$$

The Whitney number of the second kind  $W_k(L)$  of  $L$  is the coeff.

of  $t^{rk(L)-k}$  in  $g_L(t)$ . (= # cark.  $k$  elem.s of  $L$ ).

Cor  $f(\Phi, k) = W_k(L(s\Phi))$ , where  $s\Phi = \{\lambda^\perp : \lambda \in \Phi^+\}$ .

$M := \bigcup_{\lambda \in \Phi^+} \lambda^\perp$ : complement of the Coxeter arrangement.

$f'(Φ, k) := \dim H^{dim M - k}(M)$ : Betti numbers of  $M$ .

Fact (Orlik-Solomon)  $f'(\Phi, k) = |W_k(L(s\Phi))|$ .

$\mathbb{C}.L(s\Phi)$ :  $\mathbb{C}$ -vector space with  $L(s\Phi)$  as a basis.  $\rightarrow$   $W$ -representation because  $W$  acts on  $L(s\Phi)$ , a permutation representation.

Cor The maps  $F$  and  $G$  induce iso.s of graded  $W$ -representations

$$\mathbb{C}.L(s\Phi) \xrightarrow[G]{F} H^*(\mathbb{P}_y).$$

In particular,  $H^*(\mathbb{P}_y)$  is a permutation representation of  $W$ .

$\lambda \in L(\mathfrak{g})$   $\Rightarrow$   $W_\lambda :=$  pointwise stabilizer of  $\lambda$  in  $W$   
 $\lambda$  parahoric subgp. of  $W$ .

Fact (Orlik-Solomon) For  $X, Y \in L(\mathfrak{g})$ ,  $W.X = W.Y$  iff  $W_X$  and  $W_Y$  are conjugate in  $W$ .

$\mathcal{P} :=$  a set of representatives of conjugacy classes of parabolic subgps of  $W$

(can choose this to be a subset of the set of standard parabolic subgps; so  $|\mathcal{P}| \leq 2^{rk\Phi}$ )

Gr There is an iso. of  $W$ -representations

$$H^*(\overline{G}) \cong \bigoplus_{P \in \mathcal{P}} \text{Ind}_P^W \mathbf{1}.$$

Moreover, if we place  $\text{Ind}_P^W \mathbf{1}$  in degree  $rk P$ , then the above is an iso. of graded  $W$ -representations.

Fact (Stembridge)  $H^*(\overline{H})$  is a permutation representation of  $W$ .

Let  $S$  be a set with a  $W$ -action where  $b_{\alpha, \beta} = 1$

is  $H^*(\bar{A})$ . Then there are  $2^{rk \bar{W}}$   $\bar{W}$ -orbits in  $S$ .

#### 4.3. Cup product

$X \in L(s)$   $\mapsto \beta_X$ : cohomology class on  $\bar{T}_j$  dual to the cell  $F(X)$ .

By Thm. 9,  $\{\beta_X : X \in L(s)\}$  form a basis of  $H^*(\bar{T}_j)$ . It would be nice if  $H^*(\bar{T}_j)$  is iso. as an algebra to the monoid algebra of the semilattice  $(L(s), v)$ . But this too good to be true.

Thm 10 For  $X, Y \in L(s)$ , we have

$$\beta_X \cup \beta_Y = \begin{cases} \beta_{X \vee Y} & rk(X \vee Y) = rk X + rk Y \\ 0 & \text{else.} \end{cases}$$

In particular,  $H^*(\bar{T}_j)$  is generated by its degree 2 component.

Ex Type A<sub>2</sub>.  $H^0(\bar{T}_j) \cong \mathbb{C}$ ,  $H^2(\bar{T}_j) \cong \mathbb{C}^3$ ,  $H^4(\bar{T}_j) \cong \mathbb{C}$ .

The cup product of any two generators of degree 2 is equal to the generator of degree 4.

In this case,  $H^{\frac{1}{2}}(\mathbb{F}_q)$  is a quadratic algebra.

#### 4.4. Concluding remarks

1. For any central, essential hyperplane arrangement in  $\mathbb{C}^r$ , one can associate to it an additive toric variety for the additive gp.  $(\mathbb{C}^r, +)$ .

Does this lead to a general theory of additive toric varieties?

$$w_k(L(A)) \xleftarrow[\text{"M\"obius inversion"}]{\text{Betti number}} M = \mathbb{F}_q - \bigcup_{X \in \mathfrak{A}^+} X^\perp \quad | \quad \begin{array}{l} \text{Koszul dual in type A of } H^*(M) \\ \text{is } \mathcal{U} \text{ (Drinfeld-Kohno Lie alg.)} \\ \text{Is } H^*(\mathbb{F}_q) \text{ related to Drinfeld-Kohno?} \end{array}$$

???

$$w_k(L(d)) \xleftarrow[\text{Betti number}]{\text{Betti number}} \mathbb{F}_q$$

Is there a geometric duality between  $M$  and  $\mathbb{F}_q$  that corresponds to the combinatorial duality between the two kinds of Whitney numbers?

3. The multiplicative structure on  $H^*(\mathbb{F}_q)$  is quite "degenerate":  $\mathbb{F}_X \cup \mathbb{F}_Y$  is nonzero only when  $\text{rk}(X \cup Y) = \text{rk}X + \text{rk}Y$ , i.e.  $X$  and  $Y$  intersect transversally.

Is there a graded ring whose underlying vector space is the same as that of  $H^*(\mathbb{P}_Y)$ , but is more closely related to the monoid algebra of  $(L(\text{sd}), \wedge)$ ?

4. What is the geometric meaning of the monoid algebra of the semilattice  $(L(\text{sd}), \wedge)$ ?

Since  $\mathbb{P}_Y$  admits an affine stratification, its Chow groups (tensored with  $\mathbb{C}$ ) as a vector space is iso. to  $\mathbb{C}.L(\text{sd})$ . It would be nice if the "Chow ring" of  $\mathbb{P}_Y$  is somehow related to the monoid algebra of  $(L(\text{sd}), \wedge)$ , in the same sense as in 3 (Thm. (o)).