

Poisson Geometry of Big Quantum Supergroups at Roots of Unity

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joint work with Nicolás Andruskiewitsch and Iván Angiono
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Big quantum groups I

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The big quantum group $U_\epsilon(\mathfrak{g})$ is the Hopf algebra with the same generators as $U_q(\mathfrak{g})$ and relations in which q is replaced by ϵ .

Generators $K_i^{\pm 1}, E_i, F_i, 1 \leq i \leq n$. Relations: e.g. for $U_\epsilon(\mathfrak{sl}(2))$:

$$KEK^{-1} = \epsilon^2 E, \quad KFK^{-1} = \epsilon^{-2} F, \quad EF - FE = \frac{K - K^{-1}}{\epsilon - \epsilon^{-1}}.$$

From the DKP work to the supercase

Super case data: Given a **contragredient finite dimensional Lie superalgebra \mathfrak{g}** (types \mathfrak{sl} , \mathfrak{osp} , $D(2, 1; \alpha)$, $F(4)$, $G(3)$ in Kac's classification) and a root of unity ϵ , one can define the **big quantum supergroup $U_\epsilon(\mathfrak{g})$** .

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Goals of the talk: Describe an extension of the works of De Concini, Kac and Procesi to the super case. **Difficulty:** DKP rely on concrete arguments and computations with reductions to low rank case: **2 kinds of quantum Serre relations** $\text{ad}_{E_i, \epsilon}^2(E_j) = 0$ and $\text{ad}_{E_i, \epsilon}^3(E_j) = 0$.

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The supercase has **13 kinds of additional Serre relations** on up to **4 generators!** Repeating DKP is not an option.

Our approach uses **Yetter–Drinfeld modules**, **Nichols algebras**, **pairings between integral forms**, and no Poisson bracket computations. An axiomatically defined class of algebras that includes **the DKP big quantum groups**, **big quantum supergroups** and families of **root of unity quantizations of simple Lie algebras over fields of characteristic 2 and 3**.

Representation theoretic side I

Ingredients of DKP work:

- The big quantum group $U_\epsilon(\mathfrak{g})$ has the **central subalgebra**

$$Z_\epsilon(\mathfrak{g}) := \mathbb{C}[E_{\beta_1}^N, \dots, E_{\beta_\ell}^N, F_{\beta_1}^N, \dots, F_{\beta_\ell}^N][K_1^{\pm N}, \dots, K_n^{\pm N}],$$

where $E_{\beta_1}, \dots, E_{\beta_\ell}, F_{\beta_1}, \dots, F_{\beta_\ell}$ are the root vectors of $U_\epsilon(\mathfrak{g})$, and

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The **small quantum group of Lusztig** is

$$u_\epsilon(\mathfrak{g}) = U_\epsilon(\mathfrak{g})/\mathfrak{m}_0 U_\epsilon(\mathfrak{g}),$$

where $\mathfrak{m}_0 := (E_{\beta_j}^N, F_{\beta_j}^N, K_i^N - 1, \forall i, j) \in \text{MaxSpec} Z_\epsilon(\mathfrak{g}) \cong \mathbb{C}^{2\ell} \times (\mathbb{C}^*)^n$.

Bundle of algebras over $\text{MaxSpec} Z_\epsilon(\mathfrak{g})$ with fibers $U_\epsilon(\mathfrak{g})/\mathfrak{m} U_\epsilon(\mathfrak{g})$ for maximal ideals \mathfrak{m} of $Z_\epsilon(\mathfrak{g})$. **Lusztig's small quantum group is one fiber.**

Representation theoretic side II

General picture (Procesi, Artin, Kaplansky, Posner): A is a fin gen \mathbb{C} -algebra, which is **module-finite over a central subalgebra** Z .

Irr(A)=set of irreducible representations of A (must be fin. dim.)

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By **Schur's lemma**, Z acts by a character on every irrep of A . A map:

$$\psi : \text{Irr}(A) \rightarrow \text{MaxSpec}(Z), \quad V \mapsto \text{Ann}_Z(V).$$

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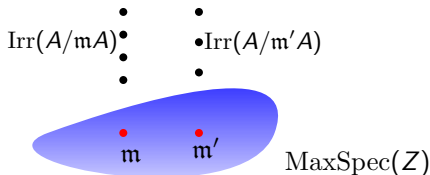
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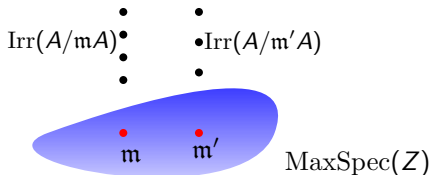
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If A is a **free Z -module**, we have a **bundle of algebras over $\text{MaxSpec}(Z)$** with fibers $A/\mathfrak{m}A$.

Poisson geometric side

Definition. A \mathbb{C} -algebra A , which is module-finite over a central subalgebra Z is called a **Poisson order** if

- 1 Z is equipped with the structure of a **Poisson algebra** and
- 2 \exists a **linear map** $\partial : Z \rightarrow \text{Der}_{\mathbb{C}}(A)$ such that ∂_z extends the **Hamiltonian derivation** $\{z, -\}$ of Z for all $z \in Z$.

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Theorem [Brown–Gordon, De Concini–Kac–Procesi]

Let (A, Z) be a Poisson order.

For every $\mathfrak{m}, \mathfrak{m}'$ in the same symplectic leaf of $\text{MaxSpec} Z$,

$$A/\mathfrak{m}A \cong A/\mathfrak{m}'A.$$

In particular, there is a **dimension-preserving bijection** between the corresponding fibers of ψ :

$$\text{Irr}(A/\mathfrak{m}A) \leftrightarrow \text{Irr}(A/\mathfrak{m}'A).$$

Specialization

An algebra A is obtained by **specialization**, if there is a \mathbb{C} -algebra R and a central element $\hbar \in R$ that is not a zero divisor, such that

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Then $(A, \mathcal{Z}(A))$ has a canonical structure of a **Poisson order**. In terms of the projection $\pi : R \rightarrow R/\hbar R \cong A$ and a linear section $\iota : A \rightarrow R$:

- The Poisson structure on Z is $\{z_1, z_2\} = \pi \left(\frac{[\iota(z_1), \iota(z_2)]}{\hbar} \right)$
- The linear map $\partial : Z \rightarrow \text{Der}_{\mathbb{C}}(A)$ is $\partial_z(a) = \pi \left(\frac{[\iota(z_1), \tilde{a}]}{\hbar} \right)$, where $\tilde{a} \in \pi^{-1}(a)$.

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In our applications $\hbar = q - \epsilon$ for some $\epsilon \in \mathbb{C}$.

However, $\mathcal{Z}(A)$ is usually **singular** and we need to construct a **large smooth Poisson subalgebra** $Z \subset \mathcal{Z}(A)$ turning (A, Z) into a Poisson order.

The De Concini–Kac–Procesi work

De Concini–Kac–Procesi proved the following:

- 1 $U_\epsilon(\mathfrak{g})$ is a specialization, so $(U_\epsilon(\mathfrak{g}), \mathcal{Z}(U_\epsilon(\mathfrak{g})))$ is a Poisson order.
- 2 $Z_\epsilon(\mathfrak{g})$ is a Poisson subalgebra of $\mathcal{Z}(U_\epsilon(\mathfrak{g}))$.
- 3 $Z_\epsilon(\mathfrak{g})$ is isomorphic to the coordinate ring of the dual Poisson algebraic group of a simple algebraic group G with the standard Poisson structure \Rightarrow Invariance of $U_\epsilon(\mathfrak{g})/\mathfrak{m}U_\epsilon(\mathfrak{g})$ along conjugacy classes of G .

Goals of the talk

- 1 In the late 1990s **Andruskiewitsch and Schneider** initiated a program for classifying **pointed Hopf algebras**.
Intuitive introduction to the underlying notions.
- 2 In that program a large family of (infinite dimensional) Hopf algebras arises (**intrinsically defined**), which contains as special cases:
 - (a) all **DKP big quantum groups at roots of unity**,
 - (b) all **big quantum cotangent supergroups at roots of unity**
 - (c) **exceptional families** that can be viewed as quantizations of **finite dimensional simple Lie algebras in prime characteristic**.
- 3 We will describe extensions of the DKP results to all of these algebras.
- 4 DKP rely on **concrete calculations of Poisson brackets and reductions to rank 2 cases**.
- 5 We use methods based on **perfect pairings of restricted and non-restricted integral forms** and **avoid any ad hoc calculations of Poisson brackets**. Sketch.

Yetter–Drinfeld modules

All pointed Hopf algebras are deformations of semidirect products

$$\mathfrak{B}(V) \rtimes \mathbb{C}[G]$$

(called bosonizations). Here

- G is a group,
- V is a Yetter–Drinfeld module of G ,
- $\mathfrak{B}(V)$ is the Nichols algebra of V .

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Best understood is the case of abelian groups G . For an abelian group G , a Yetter–Drinfeld module is a **G -graded G -module V** :

$$V = \bigoplus_{g \in G} V_g \quad \text{where } V_g \text{ are } G\text{-modules.}$$

V is of diagonal type if G acts by characters on V_g , i.e., “diagonally.”

Examples

Take $G := \mathbb{Z}^n = \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_n$, so $\mathbb{C}[\mathbb{Z}^n] = \mathbb{C}[K_1^{\pm 1}, \dots, K_n^{\pm 1}]$. For

$$\mathfrak{q} = (q_{ij}) \in M_{n \times n}(\mathbb{C}^*)$$

consider the Yetter-Drinfeld module $V = \mathbb{C}E_1 \oplus \cdots \oplus \mathbb{C}E_n$ such that

$$K_i.E_j = q_{ij}E_j \quad \text{and} \quad V_{\alpha_j} = \mathbb{C}E_j.$$

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Example 2. $\mathfrak{sl}(m|n)$ type:

$$q_{ii} = \begin{cases} \epsilon^{-2}, & i < m \\ -1, & i = m \\ \epsilon^2, & i > m \end{cases} \quad q_{i-1,i} = q_{i,i-1} = \begin{cases} \epsilon, & i \leq m \\ \epsilon^{-1}, & i > m \end{cases} \quad \text{other } q_{ij} = 1.$$

Nichols algebras

A Yetter–Drinfeld module V is a **braided vector space** with braiding

$$\sigma(E_i \otimes E_j) = q_{ij} E_j \otimes E_i.$$

The tensor algebra $T(V)$ is a **Hopf algebra in the category of braided vector spaces**: as usual, $\Delta T(V) \rightarrow T(V) \otimes T(V)$ must be a homomorphism of algebras but in the RHS:

$$(E_i \otimes E_j)(E_k \otimes E_l) = q_{jk} E_i E_k \otimes E_j E_l, \quad \text{etc.}$$

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The **Nichols algebra** is the quotient

$$\mathfrak{B}(V) := T(V)/\mathfrak{J}(V) \quad \text{for the maximal Hopf ideal } \mathfrak{J}(V) \subseteq V^{\otimes 2} \oplus V^{\otimes 3} \oplus \dots$$

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Example 1 $\rightsquigarrow u_{\epsilon}^{+}(\mathfrak{g})$,

Example 2 $\rightsquigarrow u_{\epsilon}^{+}(\mathfrak{sl}(m|n))$.

Root systems etc

Braiding matrices q whose Nichols algebra $\mathfrak{B}(V)$ is fin. dim were classified by Heckenberger following earlier work of Andruskiewitsch–Schneider.

Heckenberger and Angiono: Root systems $\Delta_+^q \supseteq$ Cartan roots \mathfrak{D}_+^q (like even/odd roots for Lie superalgebras), e.g. $\mathfrak{sl}(m|n)$ -case α_m non-Cartan simple root, the other simple roots are Cartan, a generalized Lusztig action of a Weyl groupoid on a collection of

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which are **vast generalizations** of small quantum groups.

Angiono: (1) Explicit relations of $\mathfrak{B}(V)$: (complicated) Serre relations and **power relations** $E_\alpha^{N_\alpha} = 0$ for $\alpha \in \mathfrak{D}_+^q$ and $E_i^{N_i} = 0$.

(2) Defined distinguished pre-Nichols algebras

$\tilde{\mathfrak{B}}(V) := T(V)/(\text{complicated Serre relations and } E_i^{N_i}, i \text{ non-Cartan vertex}).$

Dictionary I

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$$\begin{aligned} U_q &:= D(\tilde{\mathfrak{B}}(V) \rtimes \mathbb{C}[\mathbb{Z}^n]) \rightsquigarrow U_\epsilon(\mathfrak{g}) && \text{large quantum group,} \\ U_q^\geq &:= \tilde{\mathfrak{B}}(V) \rtimes \mathbb{C}[\mathbb{Z}^n] \rightsquigarrow U_\epsilon^\geq(\mathfrak{g}) && \text{large quantum Borel subalg,} \\ U_q^+ &:= \tilde{\mathfrak{B}}(V) \rightsquigarrow U_\epsilon^+(\mathfrak{g}) && \text{large quantum unipotent subalg.} \end{aligned}$$

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What do we get?

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What do we get? Answer: All **big quantum groups of DKP**, all **big quantum supergroups (contragredient Lie superalgebras)**, and root of unity quantizations of the 34 dim simple Lie algebra in char 2 of Kac–Wesfeiler and the 10 dim simple Lie algebra in char 3 of Brown.

Dictionary II

Generators of the large quantum group U_q^{\geq} :

$$E_i, F_i, K_i^{\pm 1}, L_i^{\pm 1}, \quad 1 \leq i \leq n.$$

Relations:

$$\begin{aligned} K_i E_j &= q_{ij} E_j K_i, & L_i E_j &= q_{ji}^{-1} E_j L_i, \\ K_i F_j &= q_{ij}^{-1} F_j K_i, & L_i F_j &= q_{ji} F_j L_i, \\ E_i F_j - F_j E_i &= \delta_{ij} (K_i - L_i^{-1}), & K_i L_j &= L_j K_i \end{aligned}$$

together with **quantum Serre relations** on E's and F's and $E_i^{N_i} = F_i^{N_i} = 0$ for non-Cartan simple roots.

Construction of Poisson orders

Theorem 1 [Andruskiewitsch–Angiono–Y]

- 1 Each large quantum group U_q is a specialization of a non-restricted integral form of U_q . This makes $(U_q, \mathcal{Z}(U_q))$ a Poisson order.
- 2 The central Hopf subalgebra

$$Z_q := \mathbb{C}[E_\beta^{N_\beta}, F_\beta^{N_\beta}, K_i^{\pm N_i}, L_i^{\pm N_i}, \text{Cartan roots } \beta, 1 \leq i \leq n]$$

of U_q is invariant under the generalized Lusztig action of a Weyl groupoid and U_q is module-finite over it.

- 3 Z_q is a Poisson subalgebra of $\mathcal{Z}(U_q)$, so (U_q, Z_q) a Poisson order.

Construction of Poisson orders

Theorem 1 [Andruskiewitsch–Angiono–Y]

- 1 Each large quantum group U_q is a specialization of a non-restricted integral form of U_q . This makes $(U_q, \mathcal{Z}(U_q))$ a Poisson order.
- 2 The central Hopf subalgebra

$$Z_q := \mathbb{C}[E_\beta^{N_\beta}, F_\beta^{N_\beta}, K_i^{\pm N_i}, L_i^{\pm N_i}, \text{Cartan roots } \beta, 1 \leq i \leq n]$$

of U_q is invariant under the generalized Lusztig action of a Weyl groupoid and U_q is module-finite over it.

- 3 Z_q is a Poisson subalgebra of $\mathcal{Z}(U_q)$, so (U_q, Z_q) a Poisson order.

Step 1: Z_q^+ is the fixed subalgebra of the center $\mathcal{Z}(U_q^+)$ under a set of skew-derivations, which descend from U_q^+ .

Step 2: Prove a general theorem that all such are Poisson subalgebras of specializations.

Step 3: Use that the specialization maps $U_q \rightarrow U_q$ are equivariant with respect to the Weyl groupoid action.

Description of Poisson orders

Poisson orders:

- (U_q, Z_q) (large quantum groups), (U_q^{\geq}, Z_q^{\geq}) (large quantum Borels).

Problem: Here Z_q and Z_q^{\geq} are Poisson–Hopf algebras. What are the (connected) Poisson algebraic groups

$$M_q := \text{MaxSpec} Z_q \text{ and } M_q^{\geq} := \text{MaxSpec} Z_q^{\geq}?$$

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Theorem 2 [Andruskiewitsch–Angiono–Y]

- 1 For each large quantum group U_q ,

$$M_q \cong B_q^+ \times B_q^-, \quad M_q^{\geq} \cong B_q^+,$$

where B_q^{\pm} are opposite Borel subgroups of a semisimple algebraic group G_q of adjoint type associated to the Cartan roots (supercase: the adjoint group of the derived subgroup of the even part).

- 2 They are the dual Poisson algebraic groups of G_q and B_q^- with (nonstandard) Poisson structures with empty Belavin–Drinfeld triples.

Key argument

We construct a restricted and a non-restricted integral forms $U_{\mathfrak{q},\mathcal{A}}^{\text{res}\pm}$ and $U_{\mathfrak{q},\mathcal{A}}^{\pm}$ of $U_{\mathfrak{q}}^{\pm}$ and an \mathcal{A} -linear perfect pairing

$$U_{\mathfrak{q},\mathcal{A}}^{\text{res}+} \times U_{\mathfrak{q},\mathcal{A}}^{-} \rightarrow \mathcal{A} := \mathbb{C}[q][f(q)^{-1}]$$

- 1 From the pairing we obtain a description of the dual tangent Lie bialgebra of $M_{\mathfrak{q}}^{\geq}$ in terms of $U_{\mathfrak{q}}$ and then prove that it is isomorphic to $\text{Lie } B_{\mathfrak{q}}^{-}$.
- 2 Then we use Hopf algebra arguments to show that $M_{\mathfrak{q}}$ and $M_{\mathfrak{q}}^{\geq}$ are isomorphic to Borel subgroups in semisimple algebraic groups of adjoint type.
- 3 Then we use Lie theory to prove that the dual tangent Lie bialgebra of $M_{\mathfrak{q}}$ is isomorphic to $\text{Lie } G_{\mathfrak{q}}$.

Symplectic leaves and irreps

$B_q^+ \times B_q^-$ projects to an open dense subset of

$G_q \times$ (a quotient of the max torus of G_q).

Theorem 2 [Andruskiewitsch–Angiono–Y]

- 1 (Large quantum groups) The symplectic leaves of $M_q \cong B_q^+ \times B_q^-$ are in bijection with the conjugacy classes of G_q times the points of that extra torus. The algebras $U_q/\mathfrak{m}U_q$ are isomorphic to each other across conjugacy classes.
- 2 (Large quantum Borels) The torus-orbits of symplectic leaves of $M_q^{\geq} \cong B_q^+$ are in bijection with the double Bruhat cells $B_q^+ \cup B_q^- w B_q^-$ for Weyl group elements w . The algebras $U_q^{\geq}/\mathfrak{m}U_q^{\geq}$ are isomorphic to each other across double Bruhat cells.

Thank you!