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Hopf algebras
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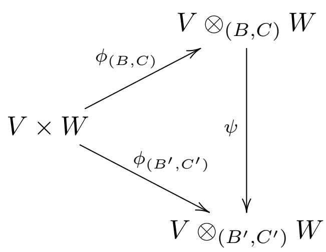
Sheet 1

Problem 1 (Tensor product)**.** Let $\mathbb{K}$ be a field, V and W be two $\mathbb{K}$-vector spaces. We consider $B = (b_i)_{i\in I}$ and $C = (c_j)_{j\in J}$ bases of V and W. Let us denote by $V \otimes_{(B,C)} W$ the $\mathbb{K}$-vector space spanned by the set $(b_i, c_j)_{i\in I, j\in J}$ and by $\phi_{(B,C)} : V \times W \to V \otimes_{(B,C)} W$ the bilinear map defined by $\phi_{(B,C)}(b_i, c_j) = (b_i, c_j)$.

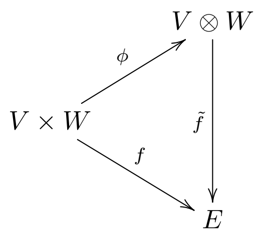
1. Show that if B' and C' are other bases for V and W, there exists a unique isomorphism of vector spaces $\psi : V \otimes_{(B,C)} W \to V \otimes_{(B',C')} W$ such that the following diagram commutes:



   From now on, the symbol $V \otimes W$ denotes the vector space $V \otimes_{(B,C)} W$ for some arbitrary but fixed bases B and C. If (x, y) is an element of $V \times W$, the symbol $x \otimes y$ denotes the image of (x, y) by $\phi_{(B,C)}$ and is called an *elementary tensor*. In the following we write ϕ instead of $\phi_{(B,C)}$,. If we want to emphasize the ground field, we might write $V \otimes_{\mathbb{K}} W$ and $x \otimes_{\mathbb{K}} y$.

2. If V and W are finite dimensional, what is the dimension of $V \otimes W$?

3. Prove that, for every $\mathbb{K}$-vector space E and for every bilinear map f from $V \times W$, there exists a unique linear map $\tilde{f}$ such that the following diagram commutes:



4. Prove that the property given in the previous question determines the pair $(V \otimes W, \phi)$ up to a unique isomorphism (meaning that if a pair (U, ρ) satisfies the property, then there exists a unique isomorphism π from $V \otimes W$ to U such that $\phi = \pi \circ \rho$.

5. Generalizing the previous questions, define the tensor product of a finite collection of vector spaces.

6. Suppose that V and W are finite dimensional, prove that $W^\star \otimes V$ is "canonically" isomorphic to $\mathrm{Hom}(W, V)$. This means that every linear map from W to V can be expressed as a finite linear combination of elementary tensors.

7. If V is finite dimensional and if g is an endomorphism of V, write a formula for the trace of g tanks to the identification of $\mathrm{End}(V)$ with $V^\star \otimes V$.

8. Let V_1, V_2, W_1 and W_2 be four $\mathbb{K}$-vector spaces, let $f_1 : V_1 \to W_1$ and $f_2 : V_2 \to W_2$ two linear maps. Use the question 3 to define a "natural" linear map $f_1 \otimes f_2 : V_1 \otimes W_1 \to V_2 \otimes W_2$. If M_1 and M_2 are matrices of f_1 and f_2 in some bases, describe a matrix representing $f_1 \otimes f_2$ in some appropriate bases.

Problem 2 (Group algebra)**.** Let $\mathbb{K}$ be a field and G a group. Let $\mathbb{K}[G]$ denote the $\mathbb{K}$-vector space with basis G.

1. Show that the multiplication of the group G induces a multiplication on $\mathbb{K}[G]$ making this vector space an (associative) $\mathbb{K}$-algebra. It is called the group algebra of G. Is $\mathbb{K}[G]$ unital? For which groups G is the algebra $\mathbb{K}[G]$ comutative?

2. Let n be a positive integer and let us denote by A_n the set of matrices with shape

$$\begin{pmatrix} a_1 & a_2 & \dots & a_{n-1} & a_n \\ a_n & a_1 & a_2 & \dots & a_{n-1} \\ a_{n-1} & a_n & a_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_2 & \dots & \dots & a_n & a_1 \end{pmatrix}$$

   for $a_1, \dots a_n$ elements of $\mathbb{K}$. Prove that A_n is an algebra isomorphic to a group algebra.

3. We denote by $\mathbb{K}[X^{\pm 1}]$ the set of Laurent polynomials over $\mathbb{K}$. It is defined by the following formula:

$$\mathbb{K}[X^{\pm 1}] = \{f(X) \in \mathbb{K}(X) | \exists l \in \mathbb{N} \text{ such that } X^l f(X) \in \mathbb{K}[X]\}$$

   Prove that $\mathbb{K}[X^{\pm}]$ is isomorphic to a group algebra.

4. Suppose that G is finite of order n and $\mathbb{K}$ is of characteristic 0. Show that $\mathbb{K}[G]$ decomposes as a direct sum of an ideal of dimension $n-1$ and an ideal of dimension 1.

Problem 3 (If you have never heard about categories, do not work on this problem)**.** A functor $F : \mathcal{C} \to \mathcal{D}$ is *essentially surjective* if for every object W of $\mathcal{D}$, there exists an object U of $\mathcal{C}$ such that $F(U) \simeq W$.

A functor $F : \mathcal{C} \to \mathcal{D}$ is *faithful (resp. fully faithful)* if for every pair of objects (U_1, U_2) of C, the map $F : \mathrm{Hom}(U_1, U_2) \to \mathrm{Hom}(F(U_1), F(U_2))$ is injective (resp. bijective).

A functor $F : \mathcal{C} \to \mathcal{D}$ is an *equivalence of categories* if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ and two natural isomorphisms $\eta : \mathrm{id}_{\mathcal{D}} \to F \circ G$ and $\theta : G \circ F \to \mathrm{id}_{\mathcal{C}}$.

In this problem, we intend to prove the following theorem:

Theorem 1. *A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if and only if it is fully faithful and essentially surjective.*

1. We first suppose that F is an equivalence of categories. Prove that F is essentially surjective.

2. Let U_1 and U_2 be two objects of $\mathcal{C}$. Show that θ (we use the notations introduced in the definitions) induces a bijection between $\mathrm{Hom}(G \circ F(U_1), G \circ F(U_2))$ and $\mathrm{Hom}(U_1, U_2)$. Prove that F is faithful. Prove that G is faithful.

3. Let U_1 and U_2 be two objects of $\mathcal{C}$ and $g : F(U_1) \to F(U_2)$ a morphism of $\mathcal{C}$. Compute $F(\theta(U_2) \circ G(g) \circ \theta(U_1)^{-1})$. Prove that F is fully faithful.

4. We now suppose that F is essentially surjective and fully faithful. We want to define a functor $G : \mathcal{D} \to \mathcal{C}$ and two natural isomorphisms $\eta : \mathrm{id}_{\mathcal{D}} \to F \circ G$ and $\theta : G \circ F \to \mathrm{id}_{\mathcal{C}}$. For every object W of $\mathcal{D}$ we choose[1] an object $G(W)$ of $\mathcal{C}$ such that $F(G(W))$ is isomorphic to W and we choose[2] an isomorphism $\eta(W) : W \to F(G(W))$. If g is a morphism in the category $\mathcal{D}$, what is the "natural" definition of $G(g)$? Prove that with this definition, G is indeed a functor and $\eta : \mathrm{id}_{\mathcal{D}} \to F \circ G$ a natural transformation.

5. What is the "natural" definition of $\theta : G \circ F \to \mathrm{id}_{\mathcal{C}}$? Prove that F is an equivalence of category.

[1] We use the axiom of choice.
[2] We use it again.