Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG
Algebra and Number Theory
Mathematics department

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Hopf algebras
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Sheet 4

In this sheet $\mathbb{K}$ is a field.

Problem 1. Let C a coalgebra and $I \subset C$ a vector subspace.

1. Show that the map
$$\overline{\Delta} : C/I \to C/I \otimes C,$$
$$x + I \mapsto \sum_{(x)} (x_{(1)} + I) \otimes x_{(2)}$$
is a well-defined counital coaction of the coalgebra C on the quotient vector space C/I, iff I is a right coideal.

2. Show that the comultiplication and counit of C define a coalgebra structure on the quotient vector space C/I by the induced maps, iff I is a two-sided coideal.

3. Deduce from the previous question that C is a sum of finite dimensional co-algebra.

Problem 2. Let (C, Δ, ϵ) be a coalgebra and x be an element of C.

1. Prove that for all $n \in \mathbb{N}$ and all i in $[1, n+1]$, we have:
$$\sum_{(x)} x_{(1)} \otimes x_{(2)} \otimes \cdots \otimes x_{(n)} = \sum_{(x)} x_{(1)} \otimes \cdots \otimes x_{(i-1)} \otimes \epsilon(x_{(i)}) \otimes x_{(i+1)} \otimes \cdots \otimes x_{(n+1)}$$

Problem 3 (Frobenius[1] algebra)**.** Let A be a finite dimensional $\mathbb{K}$-algebra. Let $\eta : A \to \mathbb{K}$ be a $\mathbb{K}$-linear map, we suppose that the composition $\eta \circ \mu =: \langle \cdot, \cdot \rangle$ is a non-degenerate[2] bilinear form (A is then called a *Frobenius algebra*).

1. Prove that A is then naturally endowed with a co-algebra structure.
2. Prove $\mathrm{Mat}_{n\times n}(\mathbb{K})$ is a Frobenius algebra.
3. If G is a finite group, prove that $\mathbb{K}G$ is a Frobenius algebra.
4. (A little more difficult) Prove that $\mathbb{K}[X, Y]/(X^2, Y^2, XY)$ is not a Frobenius algebra.

Problem 4. Let $C := \mathbb{K}[X]$ be the vector space of polynomials in one variable and let us consider the following linear maps $\Delta(X^n) = \sum_{p+q=n} X^p \otimes X^q$ and $\epsilon(X^n) = \delta_{n,0}$.

1. Show that (C, Δ, ϵ) is a counital coalgebra.
2. We know that C, with the usual multiplication of polynomials, is an associative algebra. Is C with the comultiplication Δ a bialgebra?
3. Define $\mu(X^p \otimes X^q) := \binom{p+q}{p} X^{p+q}$. Show that this defines an associative multiplication on C. What is the unit?

[1] Georg Frobenius (1849 – 1917), was a german Mathematician
[2] I mean here that for every x, there exists y such that $\langle x, y \rangle \neq 0_{\mathbb{K}}$

4. Show that C is a bialgebra with the product μ and coproduct Δ.

Problem 5. Let C be a $\mathbb{K}$-coalgebra. And let us denote by $C^\star$ the dual of C.

1. (Re)-prove that $C^\star$ is naturally endowed with a structure of algebra.
2. Let M be a comodule-C (I mean here a right C-comodule), (re)-prove that M is naturally endowed with a structure of $C^\star$-module.
3. From now on M will be a $C^\star$-module. Prove that there exists a natural embedding ι of $M \otimes C$ in $\mathrm{Hom}(C^\star, M)$.
4. Prove that from $C^\star$-module structure of M, one can naturally define a map $\rho : M \to \mathrm{Hom}(C^\star, M)$. A module such that $\rho(M) \subseteq \iota M \otimes C$ is called a *rational* module.
5. Prove that if the $C^\star$-module structure of M is obtained by the construction of question 2, then M is rational.
6. Prove that if a $C^\star$-module M is rational, it can be naturally endowed with a comodule-$C^\star$ structure.
7. If M is a rational module, prove that $N \subset M$ is a submodule if and only if $\rho(N) \subseteq N \otimes C$.