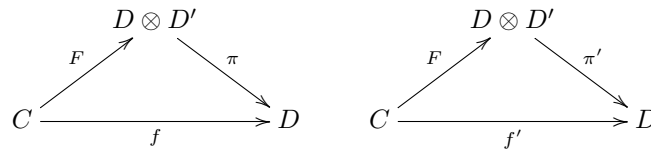




Sheet 5

In this sheet \mathbb{K} is a field, and all algebras, co-algebras and bialgebras are over \mathbb{K}

- Problem 1.**
1. Let H be a bialgebra and C be a sub-coalgebra of H , prove that the sub-algebra of H generated by C is a sub-bialgebra.
 2. Show that a coalgebra C is cocommutative if and only if Δ is a coalgebra map.
 3. Let C, D and D' be coalgebras such that C is cocommutative, and $f : C \rightarrow D$ and $f' : C \rightarrow D'$ two coalgebra maps. Define the canonical coalgebra maps $\pi : D \otimes D' \rightarrow D$ and $\pi' : D \otimes D' \rightarrow D'$, and prove that there exists a unique map $F : C \rightarrow D \otimes D'$ such that the two following diagrams are commutatives.



Problem 2 (Rational modules, second part, see sheet 4). Let C be a coalgebra, and M and M' be two rational C^* -modules and L a C^* -module.

1. Prove that if N is a cyclic¹ sub-module of M then it is finite dimensional
2. Prove that every finitely generated rational module is finite dimensional.
3. Prove that if a C^* -module N is a quotient of M , then it is rational.
4. Prove that $L^{\text{rat}} = \rho^{-1}(L \otimes C)$ is a rational sub-module. Prove that it contains all rational sub-modules of L .
5. Prove that $f : M \rightarrow M'$ is a C^* -module map iff f is a comodule- C map.

Problem 3 (Fundamental theorem on Coalgebras). In this exercise C is a coalgebra and c an element of C .

1. Let $(C_i)_{i \in I}$ a collection of subcoalgebras of C . Prove that $\bigcap_{i \in I} C_i$ is a coalgebra. Let S be a subset of C , define the notion of *coalgebra generated by S* .
2. Recall how C is naturally endowed with a structure of C^* -module.
3. Show that the sub- C^* -module N generated by c is finite dimensional (one should use the exercise about rational modules).
4. (A little more difficult) Let $J = \{a \in C^* | a \cdot N = \{0\}\}$. Prove that J^\perp is a sub-coalgebra of C and that it is finite dimensional.
5. Prove the following theorem:

Theorem 1. *Every coalgebra is a sum² of finite dimensional coalgebras.*

¹This a module generated by one element

²Just a sum, not a direct sum

Problem 4. Let $(\mathcal{C}, \otimes, I, a, l, r)$ a (non-strict) monoidal category. In this problem we want to construct a strict monoidal category \mathcal{D} such that \mathcal{C} and \mathcal{D} are tensor equivalent.

1. We start with a (useful) example. Let \mathcal{V} be the monoidal category whose objects are non-negative integers and whose morphisms from m to n are matrices of size $n \times m$ with coefficient in a field \mathbb{K} , tensor products being given by the sum of integers. Prove that this category is tensor equivalent to \mathbb{K} -vect, the category of finite dimensional vector spaces over \mathbb{K} .
2. The objects of \mathcal{D} are finite sequences (the empty sequence is allowed) of objects of \mathcal{C} . We construct at the same time a (tensor) functor $F : \mathcal{D} \rightarrow \mathcal{C}$ even if \mathcal{D} is not completely defined. If $S = (V_1, V_2, \dots, V_l)$ is an object of \mathcal{D} , we set $F(S) = (\dots((V_1 \otimes V_2) \otimes V_3) \otimes \dots) \otimes V_l$ (what should be $F(\emptyset)$?). Define the hom-spaces of \mathcal{D} and the tensor product on objects of \mathcal{D} , denoted by \star .
3. Finish the definition of F and prove that it is fully faithful and essentially surjective (see the script or sheet 1, for the definitions). This proves that F is an equivalence of category which admit $G : \mathcal{D} \rightarrow \mathcal{C}, G(V) = (V)$ as an inverse.
4. For S and S' two objects of \mathcal{D} , let us define $\phi(S, S') : F(S \otimes S') \rightarrow F(S \star S')$ inductively on the length of S' by:

$$\begin{aligned} \phi(\emptyset, S') &= l_{S'}, & \phi(S, \emptyset) &= r_S, & \phi(S, (V_1)) &= \text{id}_{F(S) \otimes V_1} \quad \text{and} \\ \phi(S, (V_1, \dots, V_{l+1})) &= (\phi(S, (V_1, \dots, V_l)) \otimes \text{id}_{V_{l+1}}) \circ a_{F(S), F((V_1, \dots, V_l)), V_{l+1}}^{-1}. \end{aligned}$$

Prove that if S, S' and S'' are objects of \mathcal{D} , we have the following equality:

$$\phi(S, S' \star S'') \circ (\text{id}_S \otimes \phi(S', S'')) \circ a_{F(S), F(S'), F(S'')} = \phi(S, S' \star S'') \circ (\phi(S, S') \otimes S'').$$

5. Define the tensor product of two morphisms in \mathcal{D} and prove that with this structure \mathcal{C}^* is a strict monoidal category.
6. Prove that F and G are tensor functors. Conclude.