Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG
Algebra and Number Theory
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Hopf algebras
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Sheet 5

In this sheet $\mathbb{K}$ is a field, and all algebras, co-algebras and bialgebras are over $\mathbb{K}$

Problem 1.

1. Let H be a bialgebra and C be a sub-coalgebra of H, prove that the sub-algebra of H generated by C is a sub-bialgebra.

2. Show that a coalgebra C is cocommutative if and only if Δ is a coalgebra map.

3. Let C, D and D' be coalgebras such that C is cocommutative, and $f : C \to D$ and $f' : C \to D'$ two coalgebra maps. Define the canonical coalgebra maps $\pi : D \otimes D' \to D$ and $\pi' : D \otimes D' \to D$, and prove that there exists a unique map $F : C \to D \otimes D'$ such that the two following diagrams are commutatives.

$$\begin{tikzcd} & D \otimes D' \arrow[dr, "\pi"] & \\ C \arrow[ur, "F"] \arrow[rr, "f"'] & & D \end{tikzcd} \qquad \begin{tikzcd} & D \otimes D' \arrow[dr, "\pi'"] & \\ C \arrow[ur, "F"] \arrow[rr, "f'"'] & & D \end{tikzcd}$$

Problem 2 (Rational modules, second part, see sheet 4)**.** Let be C a coalgebra, and M and M' be two rational $C^\star$-modules and L a $C^\star$-module.

1. Prove that if N is a cyclic[1] sub-module of M then it is finite dimensional

2. Prove that every finitely generated rational module is finite dimensional.

3. Prove that if a $C^\star$-module N is a quotient of M, then it is rational.

4. Prove that $L^{\mathrm{rat}} = \rho^{-1}(L \otimes C)$ is a rational sub-module. Prove that it contains all rational sub-modules of L.

5. Prove that $f : M \to M'$ is a $C^\star$-module map iff f is a comodule-C map.

Problem 3 (Fundamental theorem on Coalgebras)**.** In this exercise C is a coalgebra and c an element of C.

1. Let $(C_i)_{i \in I}$ a collection of subcoalgeras of C. Prove that $\bigcap_{i \in I} C_i$ is a coalgebra. Let S be a subset of C, define the notion of *coalgebra generated by S*.

2. Recall how C is naturally endowed with a structure of $C^\star$-module.

3. Show that the sub-$C^\star$-module N generated by c is finite dimensional (one should use the exercise about rational modules).

4. (A little more difficult) Let $J = \{a \in C^\star | a \cdot N = \{0\}\}$. Prove that $J^\perp$ is a sub-coalgebra of C and that it is finite dimensional.

5. Prove the following theorem:

   Theorem 1. *Every coalgebra is a sum[2] of finite dimensional coalgebras.*

[1] This a module generated by one element
[2] Just a sum, not a direct sum

Problem 4. Let $(\mathcal{C}, \otimes, I, a, l, r)$ a (non-strict) monoidal category. In this problem we want to construct a strict monoidal category $\mathcal{D}$ such that $\mathcal{C}$ and $\mathcal{D}$ are tensor equivalent.

1. We start with a (useful) example. Let $\mathcal{V}$ be the monoidal category whose objects are non-negative integers and whose morphisms from m to n are matrices of size $n \times m$ with coefficient in a field $\mathbb{K}$, tensor products being given by the sum of integers. Prove that this category is tensor equivalent to $\mathbb{K}$-vect, the category of finite dimensional vector spaces over $\mathbb{K}$.

2. The objects of $\mathcal{D}$ are finite sequences (the empty sequence is allowed) of objects of $\mathcal{C}$. We construct at the same time a (tensor) functor $F : \mathcal{D} \to \mathcal{C}$ even if $\mathcal{D}$ is not completely defined. If $S = (V_1, V_2, \ldots, V_l)$ is an object of $\mathcal{D}$, we set $F(S) = (\cdots((V_1 \otimes V_2) \otimes V_3) \otimes \cdots) \otimes V_l$ (what should be $F(\emptyset)$?). Define the hom-spaces of $\mathcal{D}$ and the tensor product on objects of $\mathcal{D}$, denoted by $\star$.

3. Finish the definition of F and prove that it is fully faithful and essentially surjective (see the script or sheet 1, for the definitions). This proves that F is an equivalence of category which admit $G : \mathcal{D} \to \mathcal{C}, G(V) = (V)$ as an inverse.

4. For S and S' two objects of $\mathcal{D}$, let us define $\phi(S, S') : F(S \otimes S') \to F(S \star S')$ inductively on the length of S' by:

$$\phi(\emptyset, S') = l_{S'}, \qquad \phi(S, \emptyset) = r_S, \qquad \phi(S, (V_1)) = \mathrm{id}_{F(S) \otimes V_1} \quad \text{and}$$
$$\phi(S, (V_1, \ldots V_{l+1})) = (\phi(S, (V_1, \ldots, V_l)) \otimes \mathrm{id}_{V_{l+1}}) \circ a^{-1}_{F(S), F((V_1, \ldots, V_l)), V_{l+1}}.$$

   Prove that if S, S' and S'' are objects of $\mathcal{D}$, we have the following equality:

$$\phi(S, S' \star S'') \circ (\mathrm{id}_S \otimes \phi(S', S'')) \circ a_{F(S), F(S'), F(S'')} = \phi(S, S' \star S'') \circ (\phi(S, S') \otimes S'').$$

5. Define the tensor product of two morphisms in $\mathcal{D}$ and prove that with this structure $C^\star$ is a strict monoidal category.

6. Prove that F and G are tensor functors. Conclude.