Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG
Algebra and Number Theory
Mathematics department

PD Dr. Ralf Holtkamp
Prof. Dr. C. Schweigert
Hopf algebras
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Sheet 6

Problem 1. 1. Prove that the left (or right) dual of an object is essentially unique. (It is unique up to a unique isomorphism).

Solution. *We do it for the right dual: Let V be a right dualizable and let (V_1, b_1, d_1) and (V_2, b_2, d_2) two right dual of V. The morphism $(d_1 \otimes \otimes \mathrm{id}_{V_2}) \circ a^{-1}_{V_1,V,V_2} \circ (id_{V_1} \otimes b_{V_2})$ and $(d_2 \otimes \otimes \mathrm{id}_{V_1}) \circ a^{-1}_{V_2,V,V_1} \circ (id_{V_2} \otimes b_{V_1})$ are mutually inverse and the diagrams which should commute indeed commute. One can compute everything diagrammatically. On the other hand if f is an isomorphism between V_1 and V_2, one can once more compute diagramtically f and show that it is equal to: $(d_1 \otimes \otimes \mathrm{id}_{V_2}) \circ a^{-1}_{V_1,V,V_2} \circ (id_{V_1} \otimes b_{V_2})$.* □

2. Suppose $\mathcal{C}$ is autonomous and V is an object of $\mathcal{C}$. Show that there are canonical isomorphisms: ${}^{\vee}(V^{\vee}) \simeq V \simeq ({}^{\vee}V)^{\vee}$.

Solution. *Like for the previous question we can do everything diagrammatically.* □

3. Let us consider the category $\mathbb{K}$-Vect of $\mathbb{K}$-vector spaces, with its usual tensor structure. Prove that an object of $\mathbb{K}$-Vect is left (or right) dualizable if and only if it is finite dimensional.

Solution. *If V is finite dimensional, it is clear that the (usual) of V is a right dual (and left) dual of V, in this case d is the evaluation. And if $e_1, \dots e_n$ is a base of V and $e_1^\star, \dots, e_n^\star$, the dual base of $V^\star$, then $b(1) = \sum_i e_i \otimes e_i^\star$. If V is not finite dimensional, suppose, that it is right dualizable. We might right $d(1) = \sum_i e_i \otimes \lambda_i \in V \otimes V^\vee$. Then, the image of the map $(\mathrm{id}_V \otimes d) \circ a_{V,V^\vee,V} \circ (b \otimes \mathrm{id}_V)$ is included the span of the e_i and is therefor finite dimensional. This is absurd, since this map is suppose to be the identity of V.* □

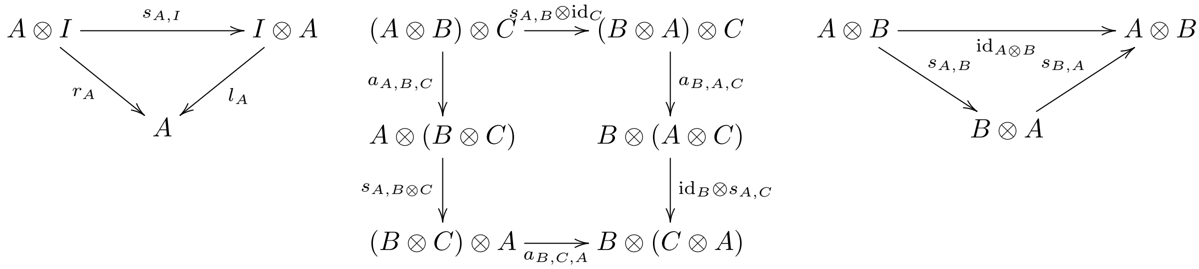
4. Let us consider the category $\mathsf{Cob}(k+1)$ whose objects are oriented k-dimensional manifold and whose morphisms are $(k+1)$-dimensional cobordism. Prove that the disjoint union turns this category into a tensor category.

Solution. *It is even a strict tensor cateogry. The unit is of course the empty set, the tensor product at the level of morphism is given by the disjoint union as well.* □

5. Prove that every object in $\mathsf{Cob}(k+1)$ is left (or right) dualizable.

Solution. *Let M be a oriented k-manifold, then $(-M, b_M, d_M)$ is a right (and left) dual for M where $-M$ is the manifold M with the opposite orientation, b_M is the cylinder $M \times [0,1]$ thought as a cobordism from the empty set to $M \sqcup -M$ and b_M is the same cylinder thought as a morphism from $-M \sqcup M$ to the empty set.* ☐

6. A tensor catogory $(\mathcal{C}, I, a, r, l)$ is *symmetric* if there exists a natural isomorphism between s between $\bullet \otimes \bullet$ and $\bullet \overset{\tau}{\otimes} \bullet$ (wehre $A \overset{\tau}{\otimes} B := B \otimes A$, and similarly for morphisms) such that for any triple of objects (A, B, C) of $\mathcal{C}$ such that the following diagrams commute:



Prove that in a symmetric tensor category every the notion of left dual and of right dual coincide.

Solution. *Thanks to the first question and of the symmetry between right and left, we only have to show that in a symmetric category, a right dual of an object is, also a left dual. If $(V^\vee, b, d)$ is a right dual of V, then one can check that $(V^\vee, s_{V \otimes V^\vee} \circ b, d \circ s_{V^\vee \otimes V})$ is a left dual (diagrammatically for example).* ☐

7. Prove that in a symmetric right autonomous tensor category there is a good notion of trace.

Solution. *The real question is: what is a "good" notion of trace. A possible answer is: for every object V a map from $\mathrm{End}(V)$ to $\mathrm{End}(I)$ such that if $f \in \mathrm{Hom}(V, W)$ and $g \in \mathrm{Hom}(W, V)$, then $\mathrm{tr}(f \circ g) = \mathrm{tr}(g \circ f)$. Let $f \in \mathrm{End}(V)$, we define the trace of f by the following composition:*

$$\mathrm{tr}(f) = b_V \circ s_{V^\vee \otimes V} \circ (f \otimes \mathrm{id}_{V^\vee}) \circ b_V.$$

One can check diagrammatically that $\mathrm{tr}(f \circ g) = \mathrm{tr}(g \circ f)$. ☐

8. What is the trace in $\mathsf{Cob}(n)$?

Solution. *The trace of a cobordism $W : M \to M$. is the manifold W/id_M where the two boundary component are identified. This is indeed a closed manifold and can be therefor thought as a morphism from the emptyset to the empty set.* ☐

Problem 2. Let H be a Hopf algebra over a field $\mathbb{K}$. Let $a \in H$ and define

$$\begin{aligned} &\mathrm{ad}_a : H \to H, \\ &\mathrm{ad}_a(x) := \sum_{(a)} a_{(1)} \cdot x \cdot S(a_{(2)}). \end{aligned}$$

1. Show that $\mathrm{ad} : H \otimes H \to H, a \otimes x \mapsto \mathrm{ad}_a(x)$ defines the structure of a left H-module $H_{\mathrm{ad}} = (H, \mathrm{ad})$ on H. H_{ad} is called the *adjoint module of* H.

Solution. *The map* ad_a *is linear, since the multiplication in* H *is bilinear. For associativity take* $a, b, x \in H$

$$\begin{aligned}
\mathrm{ad}_a(\mathrm{ad}_b(x)) &= \sum_{(b)} \mathrm{ad}_a b_{(1)} x S(b_{(2)}) \\
&= \sum_{(a)} \sum_{(b)} a_{(1)} b_{(1)} x S(b_{(2)}) S(a_{(2)}) \\
&= \sum_{(a)} \sum_{(b)} a_{(1)} b_{(1)} x S(a_{(2)} b_{(2)}) && S \textit{ is an anti-algebra hom.} \\
&= \sum_{(ab)} (ab)_{(1)} x S((ab)_{(2)}) && \Delta \textit{ is an algebra-hom.} \\
&= \mathrm{ad}_{ab}(x)
\end{aligned}$$

We still have to see $\mathrm{ad}_1(x) = x$*:*

$$\mathrm{ad}_1(x) = 1 \cdot x \cdot S(1) = x \cdot 1 = x$$

□

2. Show that the multiplication $\mu : H_{\mathrm{ad}} \otimes H_{\mathrm{ad}} \to H_{\mathrm{ad}}$ is a homomorphism of H-modules.

Solution.

$$\begin{aligned}
\sum_{(a)} \mathrm{ad}_{a_{(1)}}(x) \mathrm{ad}_{a_{(2)}}(y) &= \sum_{(a)} a_{(1)} \cdot x \cdot S(a_{(2)}) \cdot a_{(3)} \cdot y \cdot S(a_{(4)}) \\
&= \sum_{(a)} a_{(1)} \cdot x \cdot \epsilon(a_{(2)}) \cdot 1 \cdot y \cdot S(a_{(3)}) && \textit{antipode axiom} \\
&= \sum_{(a)} a_{(1)} \cdot x \cdot y \cdot S(a_{(2)}) && \textit{counit and unit axiom} \\
&= \mathrm{ad}_a(xy)
\end{aligned}$$

□

3. Show that if $\epsilon(a) = 1$, ad_a preserves the counit and the unit.

Solution. *Unit:*

$$\begin{aligned}
\mathrm{ad}_a(1) &= \sum_{(a)} a_{(1)} \cdot 1 \cdot S(a_{(2)}) \\
&= \sum_{(a)} a_{(1)} \cdot S(a_{(2)}) && \textit{unit axiom} \\
&= \epsilon(a) \cdot 1 \overset{\epsilon(a)=1}{=} 1 && \textit{antipode axiom}
\end{aligned}$$

Counit:

$$\begin{aligned}
\epsilon(\mathrm{ad}_a(x)) &= \sum_{(a)} \epsilon(a_{(1)}) \cdot \epsilon(x) \cdot \epsilon(S(a_{(2)})) && \textit{ϵ algebra morphism} \\
&= \sum_{(a)} \epsilon(a_{(1)}) \cdot \epsilon(x) \cdot \epsilon(a_{(2)}) && \textit{S coalgebra homomorphism} \\
&= \epsilon(a) \cdot \epsilon(x) \overset{\epsilon(a)=1}{=} \epsilon(x) && \textit{counit axiom}
\end{aligned}$$

□

4. Suppose a is group-like. Show that ad_a preserves the comultiplication, i.e.

$$(\mathrm{ad}_a \otimes \mathrm{ad}_a) \circ \Delta = \Delta \circ \mathrm{ad}_a.$$

Solution.

$$\begin{aligned}
&\sum_{(\mathrm{ad}_a x)} (\mathrm{ad}_a x)_{(1)} \otimes (\mathrm{ad}_a x)_{(2)} \\
&= \sum_{(axS(a))} (axS(a))_{(1)} \otimes (axS(a))_{(2)} && \Delta(a) = a \otimes a \\
&= \sum_{(a)(x)(S(a))} a_{(1)} \cdot x_{(1)} \cdot (S(a))_{(1)} \otimes a_{(2)} \cdot x_{(2)} \cdot (S(a))_{(2)} && \textit{Δ is alg. hom.} \\
&= \sum_{(x)(S(a))} a \cdot x_{(1)} \cdot (S(a))_{(1)} \otimes a \cdot x_{(2)} \cdot (S(a))_{(2)} && \Delta(a) = a \otimes a \\
&= \sum_{(x)(a)} a \cdot x_{(1)} \cdot S(a_{(2)}) \otimes a \cdot x_{(2)} \cdot S(a_{(1)}) && \textit{S anti-coalg. hom.} \\
&= \sum_{(x)} ax_{(1)}S(a) \otimes ax_{(2)}S(a) && \Delta(a) = a \otimes a \\
&= \sum_{(x)} \mathrm{ad}_a(x_{(1)}) \otimes \mathrm{ad}_a(x_{(2)})
\end{aligned}$$

□

Problem 3. Let H be a bialgebra and V a sub-space of H. Let us denote by I_l, I_r and I_2 respectively the left, right and bi-sided ideal generated by V.

1. Prove that if $\Delta(V) \subset I_\bullet \otimes H$ then $\Delta(I_\bullet) \subset I_\bullet \otimes H$ for $\bullet = l$, or 2.
2. Prove that if $\Delta(V) \subset H \otimes I_\bullet$ then $\Delta(I_\bullet) \subset H \otimes I_\bullet$ for $\bullet = l$, or 2.
3. Prove that if $\Delta(V) \subset H \otimes I_\bullet + I_\bullet \otimes H$ then $\Delta(I_\bullet) \subset H \otimes I_\bullet + I_\bullet \otimes H$ for $\bullet = l$, or 2.
4. Prove that if $\epsilon(V) = \{0\}$, then $\epsilon(I_\bullet) = \{0\}$.
5. From now on we suppose that H is a Hopf algebra with antipode S. Prove that if $S(V) \subset I_l$ then $S(I_r) \subset I_l$.
6. Prove that if $S(V) \subset I_r$ then $S(I_l) \subset I_r$.

7. Prove that if $S(V) \subset I_2$ then $S(I_2) \subset I_2$.

Problem 4. In this problem we will construct Hopf algebra with antipode of any even order. Let F be the free non-commutative algebra with on three variable X, Y and Z.

1. Prove that the following data yields a well defined bi-alegra:

$$\begin{aligned} \Delta(X) &= X \otimes X, & \epsilon(X) &= 1, \\ \Delta(Y) &= Y \otimes Y, & \epsilon(Y) &= 1, \\ \Delta(Z) &= 1 \otimes Z + Z \otimes X, & \epsilon(Z) &= 0. \end{aligned}$$

2. Prove that the two sided ideal I generated by $XY - 1$ and $YX - 1$ is a bi-ideal. We write $H = F/I$.
3. Prove that H is a Hopf algebra (find the antipode S).
4. Prove that S has infinite order.
5. Let, n be a natural number. Starting from H construct a Hopf algebra with antipode of order $2n$.