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DER FORSCHUNG | DER LEHRE | DER BILDUNG
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Hopf algebras
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Sheet 7

Problem 1 (Adjoint functors). 1. If A is an algebra, we denote by $A^\times$ the set of invertible element in A. Show that this fits in a functor setting, and find a left adjoint functor of $\bullet^\times$.

Solution. *The set $A^\times$ is a naturally endowed with a group structure thank to the multiplication in A, and if $f : A \to B$ is a morphism of unital algebra then $f(A^\times) \subseteq B^\times$, furthermore the restriction $f^\times$ of f to $A^\times$ is a morphism of group.*

It is straightforward to check that $\mathrm{id}^\times = \mathrm{id}_{A^\times}$ and that $(f \circ g)^\times = f^\times \circ g^\times$, this proves that $\bullet^\times$ is a functor. Let us prove that the functor $F :: \mathsf{Grp} \to \mathbb{K} - \mathsf{Alg}$ which associate to a group G its group algebra G is a left adjoint to $\bullet^\times$: Let G be a group and A be an algebra.

We define $\phi_{G,A} : \mathrm{Hom}_{\mathsf{Alg}}(\mathbb{K}G, A) \to \mathrm{Hom}_{\mathsf{Grp}}(G, A^\times)$ by restriction (note that for any $x \in G$, the image of x in A by a morphism of unital algebra is inversible) and its inverse by $\mathbb{K}$-linearization. Given a morphism of groups $f : G_1 \to G_2$ and a moprhism of algebra $g : A_1 \to A_2$, the following diagram obviously commutes:

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathsf{Alg}}(\mathbb{K}G_2, A_1) & \xrightarrow{\phi_{G_2,A_1}} & \mathrm{Hom}_{\mathsf{Grp}}(G_2, A_1^\times) \\
{\scriptstyle "f"}\downarrow & & \downarrow{\scriptstyle "f"} \\
\mathrm{Hom}_{\mathsf{Alg}}(\mathbb{K}G_1, A_1) & \xrightarrow{\phi_{G_1,A_1}} & \mathrm{Hom}_{\mathsf{Grp}}(G_1, A_1^\times) \\
{\scriptstyle "g"}\downarrow & & \downarrow{\scriptstyle "g"} \\
\mathrm{Hom}_{\mathsf{Alg}}(\mathbb{K}G_1, A_1) & \xrightarrow{\phi_{G_1,A_2}} & \mathrm{Hom}_{\mathsf{Grp}}(G_1, A_1^\times)
\end{array}$$

□

2. If $\mathfrak{g}$ is a Lie algebra, we denote by $U(\mathfrak{g})$ the enveloping algebra of $\mathfrak{g}$. Show that this fits in a functor settings, and find a right adjoint functor of $U(\bullet)$.

Solution. *$U(\bullet)$ is obviously a functor. Let us denote by $L : \mathsf{Alg} \to \mathsf{Lie}$ the functor which associate to an algebra A the Lie algebra $L(A)$ which as a vector space is equal to A and whose Lie bracket is the commutator of A.*

We define $\phi_{\mathfrak{g},A} : \mathrm{Hom}_{\mathsf{Alg}}(U(\mathfrak{g}), A) \to \mathrm{Hom}_{\mathsf{Lie}}(\mathfrak{g}, L(A))$ by restriction (note that if a morphism of algebra respect the commutators) and its inverse by the universal property of the enveloping algebra. The diagrams obviously commutes. □

3. If C is a coalgerba, we denote by $G(C)$ the set of group like element of C. Show that this fits in a functor settings, and find a left adjoint functor of $G(\bullet)$.

Solution. *Note that of f is a morphism of coalgebras, it sends a group-like element on a group-like element (why ?). So that $G(\bullet)$ is a functor from* CoAlg *to* Set. *If X is a set one can construct $C(X)$ the ($\mathbb{K}$-) coalgebra which as a vector space is the vector space generated by X, and whose coproduct is given by the fact that the element of X are group-like. We claim that $C(\bullet)$ is a left adjoint functor of $G(\bullet)$. We define $\phi_{X,C}$: $\mathrm{Hom}_{\mathsf{CoAlg}}(C(X), C) \to \mathrm{Hom}_{\mathsf{set}}(X, G(C))$ by restriction and its inverse by $\mathbb{K}$-linearization (note that this gives indeed a morphism of coalgebra). The diagrams obviously commutes.* □

4. If R is a commutative ring without zero divisors, we denote by $\mathfrak{F}(R)$ the field of fractions of R. Show that this fits in a functor settings, and find a right adjoint functors of $\mathfrak{F}(\bullet)$.

Solution. *Given a map $f : R_1 \to R_2$, there is one and only one way to extend it to a map $\mathfrak{F}(f) : \mathfrak{F}(R_1) \to \mathfrak{R}_2$, this describe completely the functor $\mathfrak{F}(\bullet)$. Let us denote by $\mathfrak{R}$, the forgetful functor from the category* Field *to the category $\mathcal{C}$ of commutative ring without zero divisors, I claim that it is a right adjoint to $\mathcal{F}$:*

We define $\phi_{\mathfrak{g},A} : \mathrm{Hom}_{\mathsf{Field}}(\mathfrak{F}(R), k) \to \mathrm{Hom}_{\mathcal{C}}(R, \mathfrak{R}(k))$ by restriction and its inverse thanks to the fact that there is a canonical isomorphism $\mathfrak{F}(k) \simeq k$. The diagrams obviously commutes. □

Problem 2 (The Hopf algebra $U(\mathfrak{sl}_2)$)**.** We consider the Lie algebra $\mathfrak{sl}_2 = \mathfrak{sl}_2(\mathbb{C})$. As a vector space, it consists of all 2×2 matrices with complex coefficient and which have trace equal to 0. The Lie bracket is given by the commutator of the classical matrix product. Choose a base of $\mathfrak{sl}_2$:

1. Prove that $\mathfrak{sl}_2$ is isomorphic to the Lie algebra generated by E, F and H subjected to the relations:

$$[H, E] = -[E, H] = 2E, \quad [H, F] = -[F, H] = -2F \quad \text{and} \quad [E, F] = -[F, E] = H.$$

Solution. *Let us denote by $\mathfrak{g}$, the Lie algebra generated by E, F and H subjected to the given relations. It has dimension at most 3, since it is spanned by E, F and H. Note that $\mathfrak{sl}_2$ is 3 dimensional as a $\mathbb{C}$-vector space. Now if we set:*

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \textit{and} \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

we see (easy computation) that the relations given are satisfied. (To be completely rigorous, one would have to consider a linear map from $\mathfrak{sl}_2$ to $\mathfrak{g}$ and to say that as the relation are satisfied, this is a Lie algebra morphism). □

2. Recall the definition of $U(\mathfrak{sl}_2)$, compute Δ, ϵ and S on the generators.

Solution. *We have*

$$U(\mathfrak{sl}_2) = \left(\bigoplus_{n \geq 0} \mathfrak{sl}_2^{\otimes n} \right) \Big/ \langle x \otimes y - y \otimes x - [x, y] | \, x, y \in \mathfrak{sl}_2 \rangle \, .$$

The comultiplication Δ is determined by the fact that element from $\mathfrak{g}$ are primitive:

$$\Delta(E) = 1 \otimes E + E \otimes 1, \quad \Delta(F) = 1 \otimes F + F \otimes 1 \quad \textit{and} \quad \Delta(H) = 1 \otimes H + H \otimes 1.$$

If $x = x_1 x_2 \cdots x_n$ is a monomial in E, F and H (I mean here that $x_i \in \{E, F, H\}$ and for simplicity I didn't write the $\otimes$), then we have:

$$\Delta(x) = 1 \otimes x + \sum_{i=1}^{n-1} x_1 \cdots x_i \otimes x_{i+1} \cdots x_n + x \otimes n$$

The counity ϵ is determined by $\epsilon(1) = 1$ and $\epsilon(\mathfrak{sl}_2) = \{0\}$. The antipode is determined by:

$$S(E) = -E, \quad S(F) = -F \quad \textit{and} \quad S(H) = -H$$

And by the fact that it is an antimorphism. If $x = x_1 x_2 \cdots x_n$ is a monomial in E, F and H (same notations and abuse of notations as before), we have:

$$S(x_1 \ldots x_n) = (-1)^n x_n \cdots x_1.$$

□

3. Prove that $\mathfrak{sl}_2$ has no non-trivial ideal[1], that is: there is no non-trivial subspace $\mathfrak{i}$ such that $[\mathfrak{i}, \mathfrak{sl}_2] \subseteq \mathfrak{i}$. (Consider an element X in such a subspace and compute $[H, [H, X]]$, then discuss according to the different possible cases).

Solution. *Let X be a non-trivial element of element of $\mathfrak{i}$. We can write $X = aE + bF + cH$. $[[X, H], H] = [-2aE + 2bF, H] = 4aE + 4bF$. If $c \neq 0$, this implies that H, is in $\mathfrak{i}$, and therefor E and F. If $c = 0$, then a or b must be different from 0. Suppose $a \neq 0$. then we have $[X, F] = aH$, so that H is in $\mathfrak{i}$ and we conclude as before.* □

4. A representation of $\mathfrak{sl}_2$ is *irreducible* if it contains no non-trivial sub-representation of $\mathfrak{g}$. Let V be a finite dimensional irreducible representation of $\mathfrak{sl}_2$. Let v be an element of $V \setminus \{0\}$ such that there exists a complex number λ such that $H \cdot v = \lambda v$ (we say that v is an *weight vector*). Prove that if $E \cdot v \neq 0$, it is as well a weight vector.

Solution. *We want to compute $H \cdot (E \cdot v)$. The only information we have is on $H \cdot v$. So that it is reasonable to consider the following equality:*

$$[H, E] \cdot v = H \cdot (E \cdot v) - E \cdot (H \cdot v).$$

On the other hand $[H, E] = 2E$, so that we have:

$$H \cdot (E \cdot v) = E \cdot (\lambda v) + 2E\lambda v,$$

so that $H \cdot (E \cdot v) = (\lambda + 2)(E \cdot v)$. Hence $E \cdot v$ is a weight vector. □

5. Prove that there exists an *highest weight vector* in V, that is a weight vector such that $E \cdot v = 0$.

Solution. *First, observe that as we work over $\mathbb{C}$, we can always find a weight vector in V. The vector space V is supposed to be finite dimensional, this means implies that endomorphism of V induced by the action of H has finitely many eigen-values. If there were no highest weight vector, the set of eigen-value would not be bounded. This is absurd.* □

[1] This property is the *simplicity* of $\mathfrak{sl}_2$.

6. Let v be an highest weight vector in V. Prove that $V =< F^n \cdot v | n \in \mathbb{N} >$.

Solution. *V is suppose to be irreducible, this implies that if $W =< F^n \cdot v|n \in \mathbb{N} >$ is stable by the action of $\mathfrak{sl}_2$ then it is equal to V. W is clearly stable by F. Let us inspect the action of E and H. We will show by induction on n that $E \cdot F^n v$ and $H \cdot F^n v$ is in W. If $n = 0$, $Hv = \lambda v$ and $Ev = 0$, hence it is clear. Let us suppose this holds for n.*

$$E \cdot F^{n+1}v = [E, F]F^n v - FEF^n = HF^n v - FEF^n v \in V,$$
$$H \cdot F^{n+1}v = [H, F]F^n v - FHF^n = -2F^{n+1}v - FEF^n v \in V.$$

□

7. Describe all the finite dimensional representation of $\mathfrak{g}$.

Solution. *Let V be an irreducible representation of $\mathfrak{sl}_2$. By a slight abuse of notation, we identify H and the endomorphism of V it induces. Just like for E, if x is a weight vector of weight λ, Fx is a weight vector of weight $\lambda - 2$. So that for some N, $F^N v = 0$. The description of V in the previous question show that V is spanned as a vector space by the eigen-vectors of H and that all the eigen-values of H are simple. Let v be a highest weight vector of V. Let us denote by λ the weight of v and for every $k \in \mathbb{N}$, $v_k = \frac{1}{k!}F^k v$. One easyl show by induction on k that the following three relation holds:*

$$Hv_k = (\lambda - 2k) \quad \textit{and} \quad Ev_k = (\lambda - k + 1)v_{k-1}.$$

Let now n be the first integer for which $F^{n+1}v = 0$, n is the dimension of V. For every k, we have

$$E^k F^k v = E^k k! v_k = k! \prod_{i=1}^{k} (\lambda - i + 1)v.$$

for $k = n + 1$ the product in the last formula should be equal to 0. This shows that λ is a non-negative integer smaller or equal to n. We claim that $\lambda = n$, If it would be smaller, starting from the vector space $< E^k v_n | k \in \mathbb{N} >$, would be a strict sub-module of V. This is not allowed. This fixes completely the $\mathfrak{sl}_2$-module structures on V. One verifies easily, that for every n we can construct an irreducible $\mathfrak{sl}_2$-module of dimension $n + 1$. □

Problem 3. Let H be a Hopf algebra of dimension n ($< \infty$).

1. Suppose first that as a $\mathbb{K}$-algebra, H is isomorphic to $\mathbb{K} \times \mathbb{K} \times \cdots \times \mathbb{K}$, prove that $G(H^*)$ has order n.

Solution. *Thanks to the proposition 2.6.11 of the script, the order of $G(H^*)$ is at most equal to the dimension of H^*, hence it is smaller than n. If we find n different group like element in H^*, we are done. What is a group-like element of H^*? It is a linear form f on H such that for any $(h_1, h_2) \in H^2$, we have:*

$$\Delta(f)(h_1 \otimes h_2) = f(h_1) \cdot f(h_2).$$

But by definition, we have $\Delta(f)(h_1 \otimes h_2) = f(h_1)f(h_2)$. This means that f is group-like if and only if it is a non-trivial morphism of algebra. On the other hand, the projection $\pi_i : \mathbb{K} \times \cdots \times \mathbb{K} \to \mathbb{K}$ on the ith coordinate is a morphism of algebras. Hence we found n different group-like element in H^*. □

2. Deduce that H is isomorphic as a Hopf algebra to $(\mathbb{K}G)^*$ for some group G (the dual of the group algebra of G).

Solution. *From the previous question we can deduce that $H^* \simeq \mathbb{K}G$ as an Hopf algebra with $G = G(H^*)$. On the other hand, we have: $H \simeq (H^*)^*$ as an algebra and as a coalgebra and hence as a Hopf algebra. Hence we have $H \simeq (\mathbb{K}G)^*$.* □

3. Suppose now that H is isomorphic as a Hopf algebra to $(\mathbb{K}G)^*$, for some finite group G, prove that H is isomorphic to $\mathbb{K} \times \mathbb{K} \times \cdots \times \mathbb{K}$.

Solution. *Let G be a finite group, we want to show $(\mathbb{K}G)^*$ is isomoprhic as an algebra to $\mathbb{K} \times \cdots \times \mathbb{K}$. Note that the only structure which matters on $\mathbb{K}G$ is the coalgebra-structure. The elements of G form a base of $\mathbb{K}G$, we consider the dual base $(g^*)_{g\in G}$ of $(\mathbb{K}G)^*$. And we claim that*

$$\begin{array}{rccc} \phi: & (\mathbb{K}G)^* & \to & \mathbb{K}^{\#G} \\ & \sum_{g\in G} a_g g^* & \mapsto & (a_g)_{g\in G} \end{array}$$

is an isomorphism of algebra. As a linear map, it is clearly injective and surjective, so that we just have to show that it sends 1 to 1 and that it respects the multiplication. The unit of $(\mathbb{K}G)^*$ is $e = \sum_{g\in G} g^*$, indeed for every (h, x) in G^2 and every, we have:

$$(e \cdot h^*)(x) = e(x) \cdot h^*(x) = \sum_{g\in G} g^*(x)h^*(x) = x^*(x)h^*(x) = h^*(x).$$

As, G spans $\mathbb{K}G$, this proves that $e \cdot h^* = h^*$, and as $(h^*)_{h\in G}$ spans $(\mathbb{K}G)^*$, this proves that e is the 1 of $(\mathbb{K}G)^*$. Furthermore, we clearly have $\phi(e) = 1_{\mathbb{K}^{\#G}}$. Let (g, h, x) be an element of G^3, we have:

$$(g^* \cdot h^*)(x) = g^*(x) \cdot h^*(x) = \begin{cases} 1 & \text{if g=h=x,} \\ 0 & \text{else.} \end{cases}$$

This means that $g^* \cdot h^* = 0$ if $g \neq h$ and $g^* \cdot g^* = g^*$. This is now clear that ϕ respects the multiplication. □

Problem 4. We define $\mathcal{O}(M_n(\mathbb{K}))$ as the commutative algebra $\mathbb{K}[X_{i,j} \mid 1 \leq i, j \leq n]$ of polynomials in n^2 indeterminates $\{X_{i,j}\}_{1\leq i,j\leq n}$ together with the maps Δ and ϵ defined by

$$\Delta(X_{i,j}) := \sum_{k=1}^{n} X_{i,k} \otimes X_{k,j} \qquad \text{and} \qquad \epsilon(X_{i,j}) := \delta_{i,j}.$$

1. Show that $\mathcal{O}(M_n(\mathbb{K}))$ is a bialgebra.

Solution. *By the universal property of the polynomial ring in n^2 indeterminates Δ and ϵ are algebra homomorphism, thus one has to check only coassociativity and counitality. And for this it suffice to check on the indeterminates, since they generate $\mathcal{O}(M_n(\mathbb{K}))$ as an algebra: For every $1 \leq i, j \leq n$ we have*

$$(\Delta \otimes \mathrm{id})\Delta(X_{i,j}) = \sum_{k=1}^{n} \Delta(X_{i,k}) \otimes X_{k,j} = \sum_{k,\ell=1}^{n} X_{i,\ell} \otimes X_{\ell,k} \otimes X_{k,j}$$

and

$$(\mathrm{id} \otimes \Delta)\Delta(X_{i,j}) = \sum_{k=1}^{n} X_{i,k} \otimes \Delta(X_{k,j}) = \sum_{k,\ell=1}^{n} X_{i,k} \otimes X_{k,\ell} \otimes X_{\ell,j} \quad .$$

These sums are obviously equal. For counitality consider

$$(\epsilon \otimes \mathrm{id})\Delta(X_{i,j}) = \sum_{k=1}^{n} \epsilon(X_{i,k}) \otimes X_{k,j} = \sum_{k=1}^{n} \delta_{i,k} \cdot X_{k,j} = X_{i,j}$$

and

$$(\mathrm{id} \otimes \epsilon)\Delta(X_{i,j}) = \sum_{k=1}^{n} X_{i,k} \otimes \epsilon(X_{k,j}) = \sum_{k=1}^{n} \delta_{k,j} \cdot X_{i,k} = X_{i,j} \quad .$$

□

2. Consider the $(n \times n)$-matrix $X = (X_{i,j})_{1 \leq i,j \leq n}$ with entries in $\mathbb{K}[X_{i,j}]$. Show that $g := \det X \in \mathcal{O}(M_n(\mathbb{K}))$ is group-like, i.e. $\Delta(g) = g \otimes g$.

Solution. Consider the polynomial algebras $\mathbb{K}[X_{i,j}]$ and $\mathbb{K}[Y_{i,j}]$ in n^2 indeterminates. The tensor product $\mathbb{K}[X_{i,j}] \otimes \mathbb{K}[Y_{i,j}]$ is canonically isomorphic to the polynomial algebra $\mathbb{K}[X_{i,j}, Y_{i,j}]$ in $2n^2$ indeterminates (the algebra isomorphism is given by $\psi : X_{i,j} \otimes 1 \mapsto X_{i,j}, 1 \otimes Y_{i,j} \mapsto Y_{i,j}$).
Consider the matrices $X = (X_{i,j})$ and $Y = (Y_{i,j})$ as matrices with entries in the quotient field F of $\mathbb{K}[X_{i,j}, Y_{i,j}]$, i.e. the field of rational functions. Now we know from linear algebra that the identity

$$\det X \cdot \det Y = \det(X \cdot Y) \tag{1}$$

holds in the field F. But since the entries of X, Y and $X \cdot Y$ are in $\mathbb{K}[X_{i,j}, Y_{i,j}]$ equation (??) also holds in the ring $\mathbb{K}[X_{i,j}, Y_{i,j}]$.
Now consider Δ as a map from $\mathbb{K}[X_{i,j}]$ to $\mathbb{K}[X_{i,j}] \otimes \mathbb{K}[Y_{i,j}]$, i.e. $\Delta(X_{i,j}) = \sum_{k=1}^{n} X_{i,k} \otimes Y_{k,j}$. Observe

$$(\psi\Delta)(X_{i,j}) = \sum_{k=1}^{n} \psi(X_{i,k} \otimes Y_{k,j}) = (X \cdot Y)_{i,j},$$

so we get with the help of Leibniz' formula

$$\begin{aligned}(\psi\Delta)(\det X) &= \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)(\psi\Delta)(X_{1,\sigma(1)}) \cdots (\psi\Delta)(X_{n,\sigma(n)}) \\ &= \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)(X \cdot Y)_{1,\sigma(1)} \cdots (X \cdot Y)_{n,\sigma(n)} \\ &= \det(X \cdot Y) = \det X \cdot \det Y\end{aligned}$$

By application of ψ^{-1} we get the equality $\Delta(\det X) = \det X \otimes \det Y$ which gives us the desired equality if we interpret Δ as map from $\mathcal{O}(M_n(\mathbb{K}))$ to $\mathcal{O}(M_n(\mathbb{K})) \otimes \mathcal{O}(M_n(\mathbb{K}))$. □

3. Show that $\mathcal{O}(M_n(\mathbb{K}))$ is not a Hopf algebra. (Hint: Is $\det X$ multiplicatively invertible?)

Solution. $\det X$ *is a polynomial of degree $n > 0$, thus not invertible. But if there was an antipode S for the bialgebra $\mathcal{O}(M_n(\mathbb{K}))$ then $S(\det X)$ would be an inverse of $\det X$.* □

4. Let I be the two-sided ideal of $\mathcal{O}(M_2(\mathbb{K}))$ generated by $\det X - 1$. Show that $\mathcal{O}(M_2(\mathbb{K}))/I$ is a Hopf algebra where the antipode is given by $S(X_{i,j} + I) = (X^{-1})_{i,j} + I$. Here X^{-1} is the matrix $\begin{pmatrix} X_{2,2} & -X_{1,2} \\ -X_{2,1} & X_{1,1} \end{pmatrix}$. How can one generalize this for larger n?

Solution. *First we show that I is a coideal: Since $g := \det X$ and 1 are group-like, we have*

$$\begin{aligned}\Delta(g-1) &= g\otimes g - 1\otimes 1 = g\otimes g - g\otimes 1 + g\otimes 1 - 1\otimes 1\\ &= \underbrace{g\otimes(g-1)}_{\in\mathcal{O}(M_n(\mathbb{K}))\otimes I} + \underbrace{(g-1)\otimes 1}_{\in I\otimes\mathcal{O}(M_n(\mathbb{K}))}\end{aligned}$$

and

$$\epsilon(g-1) = \epsilon(g) - \epsilon(1) = 1 - 1 = 0.$$

Hence I is a coideal and so Δ and ϵ descend on the quotient $\mathcal{O}(M_n(\mathbb{K}))/I$.
For $n \geq 1$ we define $(X^\sharp)_{i,j} := (-1)^{i+j}\det X'_{j,i}$ where $X'_{j,i}$ is the matrix we obtain by wiping out the j-th row and i-th column of X. Furthermore we define $S(X_{i,j}) := (X^\sharp)_{j,i}$.
Cramer's rule tells us

$$XX^\sharp = \det X\cdot I_n = X^\sharp X \tag{2}$$

where I_n s the $(n\times n)$-unit matrix. We now show that S descends to a well-defined map on the quotient if we extend it to the whole algebra:
With Leibniz' rule one easily sees $S(\det X) = \det X^\sharp$. Now observe

$$\det X\cdot\det X^\sharp = \det(X\cdot X^\sharp) = \det(\det X\cdot I_n) = (\det X)^n$$

Since $\mathbb{K}[X_{i,j}]$ has no zero divisors we conclude $\det X^\sharp = (\det X)^{n-1}$, so

$$S(\det X - 1) = (\det X)^{n-1} - 1 = (\det X - 1)\cdot\sum_{k=0}^{n-2}(\det X)^k \in I,$$

so S gives a map on the quotient.
For the antipode property we compute $p := (\eta\epsilon)(X_{i,j}) = \delta_{i,j}\cdot 1$. The polynomial $q := \mu(S\otimes\mathrm{id})\Delta(X_{i,j})$ is the (i,j)-th entry of $X^\sharp\cdot X$ which is $\delta_{i,j}\cdot\det X$. Modulo I the polynomials p and q are equal. The other equality of the antipode property follows in the same way. □