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DER FORSCHUNG | DER LEHRE | DER BILDUNG
Algebra and Number Theory
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Hopf algebras
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Sheet 8

Problem 1. Let H be a finite dimensional Hopf algebra. In this case we know that H^* is also a Hopf algebra. We consider the multiplication in H^* given by

$$\langle f \cdot g, h \rangle = \sum_{(h)} f(h_{(1)}) \cdot g(h_{(2)}) \quad f, g \in H^*, h \in H.$$

1. Show that the following defines a left resp. right action of the algebra H^* on the vector space H.

$$f \rightharpoonup h := \sum_{(h)} f(h_{(2)}) \cdot h_{(1)}, \qquad h \leftharpoonup f := \sum_{(h)} f(h_{(1)}) \cdot h_{(2)} \qquad f \in H^*, h \in H.$$

   Show that H is an H^*-bimodule with the above actions, i.e. $(f \rightharpoonup h) \leftharpoonup g = f \rightharpoonup (h \leftharpoonup g)$ for all $f, g \in H^*$ and $h \in H$.

Solution. *The unit of H^* is ϵ, so we have to prove for every $h \in H$ the equality*

$$\epsilon \rightharpoonup h = h = h \leftharpoonup \epsilon$$

which holds by counitality of Δ. In the following let $h \in H$ and $f, g \in H^*$. We show $f \rightharpoonup (g \rightharpoonup h) = (f \cdot g) \rightharpoonup h$:

$$\begin{aligned} f \rightharpoonup (g \rightharpoonup h) &= \sum_{(h)} g(h_{(2)}) \cdot (f \rightharpoonup h_{(1)}) \\ &= \sum_{(h)} g(h_{(3)}) \cdot f(h_{(2)}) \cdot h_{(1)} \\ &= \sum_{(h)} (f \cdot g)(h_{(2)}) \cdot h_{(1)} \\ &= (f \cdot g) \rightharpoonup h \end{aligned}$$

Now we check $(h \leftharpoonup f) \leftharpoonup g = h \leftharpoonup (f \cdot g)$:

$$\begin{aligned} (h \leftharpoonup f) \leftharpoonup g &= \sum_{(h)} f(h_{(1)})(h_{(2)} \leftharpoonup g) \\ &= \sum_{(h)} f(h_{(1)}) \cdot g(h_{(2)}) \cdot h_{(3)} \\ &= \sum_{(h)} (f \cdot g)(h_{(1)}) \cdot h_{(2)} \\ &= h \leftharpoonup (f \cdot g) \end{aligned}$$

At last we prove $(f \rightharpoonup h) \leftharpoonup g = f \rightharpoonup (h \leftharpoonup g)$*:*

$$\begin{aligned}(f \rightharpoonup h) \leftharpoonup g &= \sum_{(h)} f(h_{(2)} \cdot (h_{(1)} \leftharpoonup g) \\ &= \sum_{(h)} f(h_{(3)} \cdot g(h_{(1)})h_{(2)} \\ &= \sum_{(h)} g(h_{(1)}) \cdot (f \rightharpoonup h_{(2)}) \\ &= f \rightharpoonup (h \leftharpoonup g)\end{aligned}$$

□

2. Show that the following defines a left resp. right action of H on the vector space H^*

$$h \rightharpoonup f := (k \mapsto \langle f, kh\rangle), \qquad f \leftharpoonup h := (k \mapsto \langle f, hk\rangle) \qquad f \in H^*, h \in H.$$

Show that H^* is an H-bimodule with the above actions, i.e. $(h \rightharpoonup f) \leftharpoonup k = h \rightharpoonup (f \leftharpoonup k)$ for all $f \in H^*$ and $h, k \in H$.

Solution. We have to show $1 \rightharpoonup f = f = f \leftharpoonup 1$*. This follows since for all* $\ell \in H$ *we have*

$$(1 \rightharpoonup f)(\ell) = f(\ell \cdot 1) = f(\ell) = f(1 \cdot \ell) = (f \leftharpoonup 1)(\ell).$$

In the following assume $h, k, \ell \in H$ *and* $f \in H^*$*. We prove* $h \rightharpoonup (k \rightharpoonup f) = (hk) \rightharpoonup f$*:*

$$(h \rightharpoonup (k \rightharpoonup f))(\ell) = (k \rightharpoonup f)(\ell h) = f((\ell h)k) = f(\ell(hk)) = ((hk) \rightharpoonup f)(\ell).$$

Next we show $(f \leftharpoonup h) \leftharpoonup k = f \leftharpoonup (hk)$*:*

$$((f \leftharpoonup h) \leftharpoonup k)(\ell) = (f \leftharpoonup h)(k\ell) = f(h(k\ell)) = f((hk)\ell) = (f \leftharpoonup (hk))(\ell).$$

The last thing to show is $(h \rightharpoonup f) \leftharpoonup k = h \rightharpoonup (f \leftharpoonup k)$*:*

$$\begin{aligned}((h \rightharpoonup f) \leftharpoonup k)(\ell) &= (h \rightharpoonup f)(k\ell) = f((k\ell)h) = f(k(\ell h)) \\ &= (f \leftharpoonup k)(\ell h) = (h \rightharpoonup (f \leftharpoonup k))(\ell).\end{aligned}$$

□

3. Show that H^* becomes a left H-module with

$$h.f := \sum_{(h)} h_{(1)} \rightharpoonup f \leftharpoonup S(h_{(2)}) \qquad f \in H^*, h \in H$$

This action is called coadjoint (left) action of H on H^*.

Solution. We show a more general result: Let M *be an* H*-bimodule, i.e. there are an* H*-left action (denoted by* $h \triangleright m$*) and an* H*-right action (denoted by* $m \triangleleft k$*) on* M*, such that the* bimodule property *holds:*

$$(h \triangleright m) \triangleleft k = h \triangleright (m \triangleleft k) \quad \text{for all } h, k \in H \text{ and } m \in M.$$

The vector space M has the structure of an H left-module by

$$h.m := \sum_{(h)} h_{(1)} \triangleright m \triangleleft S(h_{(2)}) \quad .$$

Note that the bimodule property allows us to omit parentheses. The equality $1.m = m$ for $m \in M$ follows by $\Delta(1) = 1 \otimes 1$, $S(1) = 1$ and $1 \triangleright m = m = m \triangleleft 1$. Now we prove associativity of the action: Let $h, k \in H$ and $m \in M$

$$\begin{aligned}
h.(k.m) &= \sum_{(h)} h_{(1)} \triangleright (k.m) \triangleleft S(h_{(2)}) \\
&= \sum_{(h),(k)} h_{(1)} \triangleright (k_{(1)} \triangleright m \triangleleft S(k_{(2)})) \triangleleft S(h_{(2)}) \\
&= \sum_{(h),(k)} (h_{(1)} \cdot k_{(1)}) \triangleright m \triangleleft (S(k_{(2)}) \cdot S(h_{(2)})) && \textit{associativity of } \triangleright \textit{ and } \triangleleft \\
&= \sum_{(h),(k)} (h_{(1)} \cdot k_{(1)}) \triangleright m \triangleleft (S(h_{(2)} \cdot k_{(2)})) && S \textit{ is anti-homomorphism} \\
&= \sum_{(hk)} (hk)_{(1)} \triangleright m \triangleleft S((hk)_{(2)}) && \Delta \textit{ is algebra homomorphism} \\
&= (hk).m
\end{aligned}$$

□

4. How do the actions above look in the graphical notation introduced in the lecture?

Problem 2. Let H be a finite-dimensional Hopf algebra over a field $\mathbb{K}$. Assume there is a left integral $\lambda \in \mathcal{I}_\ell(H)$, such that $\epsilon(\lambda) = 1$. Further let M be a left H-module and $N \subset M$ a submodule.

1. Choose a $\mathbb{K}$-linear $\pi : M \to M$, with $\pi^2 = \pi$ and $\mathrm{im}\pi = N$. Show that

$$\Pi : M \to M, \quad m \mapsto \sum_{(\lambda)} \lambda_{(1)}.\pi(S(\lambda_{(2)}).m)$$

is H-linear, $\Pi^2 = \Pi$ and $\mathrm{im}\Pi = N$.

Solution. *We first show $\mathrm{im}\Pi = N$. Let $n \in N$, i.e. there is is an $m \in M$ with $n = \pi(m)$. Note*

$$\pi(n) = \pi^2(m) = \pi(m) = n. \tag{1}$$

Now we compute

$$\Pi(\pi(m)) = \sum_{(\lambda)} \lambda_{(1)}.\pi(S(\lambda_{(2)}).n) \overset{(1)}{=} \sum_{(\lambda)} (\lambda_{(1)} \cdot S(\lambda_{(2)})).n = \underbrace{\epsilon(\lambda)}_{=1 \textit{ by assumption}} .n = n$$

So $\mathrm{im}\Pi = N$. Now we show $\Pi^2 = \Pi$. Since $\mathrm{im}\Pi = \mathrm{im}\pi$ for every $m \in M$ there is a $M' \in M$ such that $\Pi(m) = \pi(m')$ and we get

$$\Pi(\Pi(m)) = \Pi(\pi(m')) = \pi(m') = \Pi(m).$$

We still have to check H-linearity: Note for every $h \in H$

$$\begin{aligned}
\Delta(\lambda) \otimes h &\overset{\textit{couni.}}{=} \sum_{(h)} \Delta(\epsilon(h_{(1)}) \cdot \lambda) \otimes h_{(2)} \overset{\textit{left-int.}}{=} \sum_{(h)} \Delta(h_{(1)} \cdot \lambda) \otimes h_{(2)} \\
&= \sum_{(h)(\lambda)} h_{(1)}\lambda_{(1)} \otimes h_{(2)}\lambda_{(2)} \otimes h_{(3)}
\end{aligned} \tag{2}$$

So we get

$$
\begin{aligned}
\Pi(h.m) &= \sum_{(\lambda)} \lambda_{(1)}.\pi(S(\lambda_{(2)}) \cdot h.m) \\
&= \sum_{(h)(\lambda)} h_{(1)}\lambda_{(1)}\pi(S(h_{(2)}\lambda_{(2)})h_{(3)}m) && \textit{by (2)} \\
&= \sum_{(h)(\lambda)} h_{(1)}\lambda_{(1)}\pi(S(\lambda_{(2)})S(h_{(2)})h_{(3)}m) && S \textit{ anti-hom.} \\
&= \sum_{(h)(\lambda)} h_{(1)}\lambda_{(1)}\pi(S(\lambda_{(2)})\epsilon(h_{(2)})m) && S \textit{ antipode} \\
&= \sum_{(\lambda)} h\lambda_{(1)}\pi(S(\lambda_{(2)})m) && \textit{couni.} \\
&= h\Pi(m)
\end{aligned}
$$

□

2. Show that there is a complement for every H-submodule $N \subset M$, i.e. there exists an H-submodule P of M, such that $M = N \oplus P$.

Solution.

Take the kernel $P := \ker \Pi$*, it is an* H*-submodule of* M *since* Π *is* H*-linear. We have to show* $M = \ker \Pi \oplus \mathrm{im}\Pi$*. Write* $m = \Pi(m) + m - \Pi(m)$*. From* $\Pi^2 = \Pi$ *we see* $\Pi(m) \in \mathrm{im}\Pi = N$ *and* $m - \Pi(m) \in \ker \Pi = P$*. Now let* $m \in \ker \Pi \cap \mathrm{im}\Pi$*, then there is* $n \in M$ *such that* $m = \Pi(n)$ *and we have*

$$0 = \Pi(m) = \Pi(\Pi n) = \Pi(n) = m.$$

□

Problem 3. Let $\mathbb{K}$ be a field of characteristic 2, and let $\mathfrak{g}$ be the following 2-dimensional Lie algebra over k: as a k-vector space, it is spanned by x and y, and $[x, y] = x$.

1. Show that $\mathfrak{g}$ can be endowed with a structure of restricted Lie algebra.

Solution. We have $(\mathrm{ad}\, x)^2 = 0$ *and* $(\mathrm{ad}\, y)^2 = \mathrm{ad}\, y$*, this suggests to set* $x^{[2]} = 0$ *and* $y^{[2]} = y$*. So that we set for all* $(\lambda, \mu) \in \mathbb{K}^2$ *the following:*

$$(\lambda x + \mu y)^{[2]} = \mu^2 y + \mu\lambda x.$$

With this definition we have indeed for all $a, b \in \mathfrak{g}$ *and* $\lambda \in \mathbb{K}$*:*

$$(\lambda a)^{[2]} = \lambda^2 a^{[2]}, \quad ad(a^{[2]}) = (ad(a))^2 \quad (a + b)^{[2]} = a^{[2]} + b^{[2]} + [b, a].$$

□

2. Recall the structure of restricted Lie algebra on $\mathcal{U}(\mathfrak{g}) = U(\mathfrak{g})/I$ where I is the ideal of $U(\mathfrak{g})$ generated by $a^{[2]} - a^2$, for all $a \in \mathfrak{g}$.

Solution. *We just need to give the application* $\bullet^{[2]}$*: we set* $x^{[2]} = x^2 (:= x \otimes x)$ *for all* $x \in \mathcal{U}(\mathfrak{g})$*. This is consistent with the definition on* $\mathfrak{g}$ *and all the relations* $\bullet^{[2]}$ *should satisfied are satisfied.* □

3. Give a basis of $\mathcal{U}(\mathfrak{g})$.

Solution. *A base is given by* $(1, x, y, xy)$*: The PBW theorem tells us that a base of* $U(\mathfrak{g})$ *is given by* $(x^i y^j)_{(i,j)\in\mathbb{N}^2}$*. With the relations* $y^2 = y^{[2]} = y$ *and* $x^{[2]} = x^2 = 0$*, we have that* $(1, x, y, xy)$ *spans* $\mathcal{U}(\mathfrak{g})$*.* □

4. Recall the structure of Hopf Algebra on $\mathcal{U}(\mathfrak{g})$.

Solution. *This is given by*

$$\begin{aligned}&\epsilon(1) = 1,\ \epsilon(x) = \epsilon(y) = 0,\\ &\Delta(1) = 1 \otimes 1,\ \Delta(x) = x \otimes 1 + 1 \otimes x,\ \Delta(y) = y \otimes 1 + 1 \otimes y, \qquad S(1) = 1,\ S(x) = x,\ S(y) = y.\end{aligned}$$

□

5. Compute the left and right integrals of H.

Solution. *We easily check that* $\int_H^l = \mathbb{K}xy$ *and that* $\int_H^r = \mathbb{K}yx$*. It means that they are not equal even if the Hopf algebra was cocommutative.* □

Problem 4. 1. Let H be a Hopf algebra, prove the following equality for all $f \in H^\star$ and all $x, y \in H$:

$$(f \rightharpoonup x)y = \sum_{(y)} (f \leftharpoonup y_{(2)}) \rightharpoonup (xy_{(1)})$$

Solution. *We can definitely do this graphically, but here is a "classical" proof:*

$$\begin{aligned}\sum_{(y)}(f \leftharpoonup y_{(2)}) \rightharpoonup (xy_{(1)}) &= \sum_{(y)}(S(y_{(2)}) \rightharpoonup f) \rightharpoonup (xy_{(1)}) \\ &= \sum_{(y),\,(f)} f_{(2)}(S(y_{(2)}))f_{(1)}) \rightharpoonup (xy_{(1)}) \\ &= \sum_{(x),\,(y),\,(f)} f_{(2)}(S(y_{(3)}))f_{(1)}(x_{(2)}y_{(2)})x_{(1)}y_{(1)} \\ &= \sum_{(x),\,(y),\,(f)} f_{(2)}(S(y_{(3)}))f_{(1)}(x_{(2)}y_{(2)})x_{(1)}y_{(1)} \\ &= \sum_{(x),\,(y),\,(f)} f_{(1)}(x_{(2)}y_{(2)})(f_{(2)}S(y_{(3)}))x_{(1)}y_{(1)} \\ &= \sum_{(x),\,(y)} f(x_{(2)}y_{(2)}S(y_{(3)}))x_{(1)}y_{(1)} \\ &= \sum_{(x),\,(y)} f(x_{(2)}\epsilon(y_2))x_{(1)}y_{(1)} \\ &= \sum_{(x),\,(y)} f(x_{(2)})x_{(1)}y_{(1)}\epsilon(y_2) \\ &= \sum_{(x)} f(x_{(2)})x_{(1)}y \\ &= (f \rightharpoonup x)y\end{aligned}$$

□

2. Show that if J is a right ideal in H, then the right coideal (or equivalently the left rational H^*-module) generated by J is still a right ideal.

Solution. *This is the interpretation of the previous statement: H is a aright co-module over itself, hence it is a rational $H*$-module (via the $\rightharpoonup$-action), hence being a right co-ideal of H is the same as being a left H^* module.* □

3. Show that it $K \subset H$ is a right ideal and a right coideal, then K is an H-Hopf module, and prove that $K = H$.

Solution. *We just need to prove that $\Delta : K \to K \otimes H$ is a map of H-modules. Let $h \in H$ and $k \in k$, we want to show that the following equality holds: $\Delta(kh) = h\Delta(k)$. This is trivial since this equality holds for $\Delta : H \to H \otimes H$. Hence K is a H-Hopf module. Using theorem 3.1.5 of the script, we have that: $K \simeq K^{coH} \otimes H$. This proves that $H = K$.* □

4. Prove that of an Hopf algebra H contains a non-zero finite dimensional right ideal, it is itself finite dimensional.

Solution. *Thank to the previous question, we just need to show that, if J is a non-zero finite dimensional ideal of H, then H contains a finite dimensional tight ideal and right co-ideal. We claim that the co-ideal K generated by J satisfies this.*

We already did this once: let (k_i) be a base of J, and let us write $\Delta(k_i) = \sum_j k_{ij} \otimes h_{ij}$. I claim that the co-ideal generated by J is included in the vector space spanned by the k_{ij}'s. The fact that it contains K follows from the property of the co-unit: for all i, $k_i = \sum_j k_{ij}\epsilon(h_{ij})$. Furthermore, it is indeed a finite dimensional. We need to see that it contains a co-ideal containing K. We use once more the fact that a right co-ideal is a left H^*-module. Hence the right co-ideal we are looking for is nothing but $H^* \rightharpoonup J$, and if f is an element of H^*, we have for all i: $f \rightharpoonup k_i = \sum_j f(h_{ij})k_{ij}$.

This proves that K is a finite dimensional right ideal and right co-ideal, as we have $K \simeq H$, H is finite dimensional. □

5. Prove that if H is a Hopf algebra and $\{0\} \neq J$ is a right coideal, then $JH = H$.

Solution. *Le us prove that JH is a right still co-ideal of H. Just as before, we will prove instead that JH is a left H^*-module. for all $f f \in H^*$, x in J and $h \in H$, we have the following:*

$$\begin{aligned} f \leftharpoonup (xh) &= \sum_{(xh)} (xh)_{(1)} f((xh)_{(2)}) \\ &= \sum_{(x),\,(h)} x_{(1)}h_{(1)}f(x_{(2)}h_{(2)}) \\ &= \sum_{(x),\,(h),\,(f)} x_{(1)}h_{(1)}f_{(1)}(x_{(2)})f_{(2)}(h_{(2)}) \\ &= \sum_{(x),\,(h),\,(f)} x_{(1)}f_{(1)}(x_{(2)})h_{(1)}f_{(2)}(h_{(2)}) \\ &= \sum_{(f)} (f_{(1)} \rightharpoonup x) \cdot (f_{(2)} \rightharpoonup h). \end{aligned}$$

This shows that JH is a sub H^*-module of H, hence JH is a non-zero right co-ideal and right ideal of H, and it is equal to H, thanks to question 3. □