Nonperturbative analysis of noncritical Ising models

Some applications of the Ornstein–Zernike theory

Yvan Velenik

Université de Genève

— Ornstein–Zernike theory —

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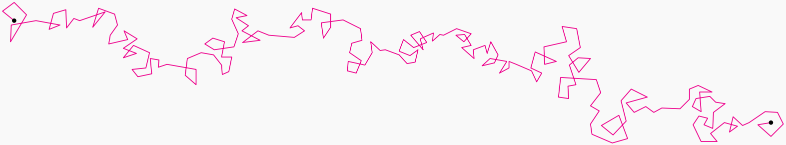

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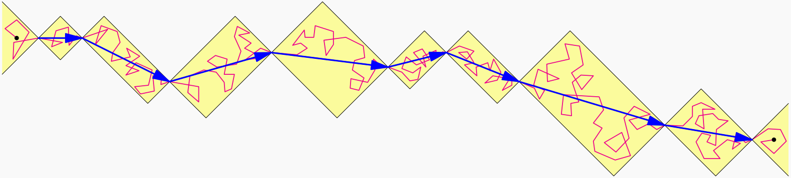

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- Applies, among others, to **interfaces in planar systems** or to paths, clusters, etc., originating from **graphical representations of correlation functions** (high-temperature expansion, FK-percolation, random-current, …).
- Using this coupling, we can in many cases **reduce difficult questions arising in the Ising model to much simpler (and more classical) ones about random walks**.

Some examples of applications of OZ theory

- **A. Asymptotics of correlations**
  - A.1. The 2-point function
  - A.2. General correlation functions
  - A.3 An inhomogeneous system: Ising with a defect line
- **B. Interfaces in the planar Ising model**
  - B.1. Interface in the bulk
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A.1. The 2-point function

Notations

- **Coupling constants:** $(J_x)_{x\in\mathbb{Z}^d}$ such that $J_0 = 0$, $J_x \geq 0$, $J_x = J_{-x}$ and $\sum_x J_x < \infty$.

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$$\langle f\,;g\rangle_{\beta,h} = \langle f\,g\rangle_{\beta,h} - \langle f\rangle_{\beta,h}\langle g\rangle_{\beta,h}.$$

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- For each $\vec{s} \in \mathbb{S}^{d-1}$, the **inverse correlation length** $\nu_{\beta,h}(\vec{s})$ is defined by

$$\langle \sigma_0 \,;\, \sigma_{n\vec{s}} \rangle_{\beta,h} = \mathrm{e}^{-\nu_{\beta,h}(\vec{s})n + \mathrm{o}(n)}.$$

where $\sigma_x = \sigma_{[x]}$, with $[x] \in \mathbb{Z}^d$ the coordinatewise integer part of $x \in \mathbb{R}^d$.

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- Assume that there exist $C, c > 0$ such that $\forall x \in \mathbb{Z}^d,\ J_x \leq Ce^{-c\|x\|}$. Then,

$$\forall (\beta, h) \neq (\beta_c, 0), \qquad \min_{\vec{s}} \nu_{\beta,h}(\vec{s}) > 0.$$

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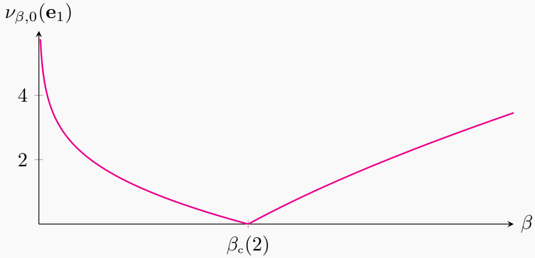

Graph of $\beta \mapsto \nu_{\beta,0}(\mathbf{e}_1)$ for the planar Ising model

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  - $h \neq 0$, $\beta \in \mathbb{R}$: [Lebowitz–Penrose 1968]
  - $h = 0$ and $\beta < \beta_c$: [Aizenman–Barsky–Fernández 1987]
  - (J_x) with finite range, $h = 0$ and $\beta > \beta_c$: [Duminil-Copin–Goswami–Raoufi 2020]

  Open problem: Remove the finite-range assumption when $h = 0$ and $\beta > \beta_c$.

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Assume that $\beta < \beta_c$ and $h = 0$. Let $\vec{s} \in \mathbb{S}^{d-1}$. Then, as $n \to \infty$,

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  - *Ornstein–Zernike 1914, Zernike 1916:* first (non-rigorous) derivation
  - *Wu 1966, Wu et al 1976:* exact computation, planar model, $\beta < \beta_c$
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Theorem [Wu–McCoy–Tracy–Barouch 1976]

Consider the **nearest-neighbor** Ising model on $\mathbb{Z}^2$. Assume that $h = 0$, $\beta > \beta_c$. Let $\vec{s} \in \mathbb{S}^{d-1}$. Then, as $n \to \infty$,

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$$\langle \sigma_0 \,;\, \sigma_{n\vec{s}} \rangle_{\beta,0} = \frac{\Psi_{\beta,0}(\vec{s})}{n^{(d-1)/2}}\, e^{-\nu_{\beta,0}(\vec{s})n}\,(1 + o(1)).$$

- **Open problems** Prove that OZ asymptotics hold for all $\beta > \beta_c$ when $d \geq 3$, but also when $d = 2$ and the graph is not planar.

Sharp asymptotics: $h \neq 0$

- Let us now consider the model in a field $h \neq 0$, assuming **finite-range interactions**.

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- In this case, **Ornstein–Zernike asymptotics apply** (at any temperature):

Theorem [Ott 2020]

Assume that $h \neq 0$ and let $\beta > 0$. Let $\vec{s} \in \mathbb{S}^{d-1}$. Then, as $n \to \infty$,

$$\langle \sigma_0 \,; \sigma_{n\vec{s}} \rangle_{\beta,h} = \frac{\Psi_\beta(\vec{s})}{n^{(d-1)/2}} \, e^{-\nu_{\beta,h}(\vec{s})n} \, (1 + o(1)),$$

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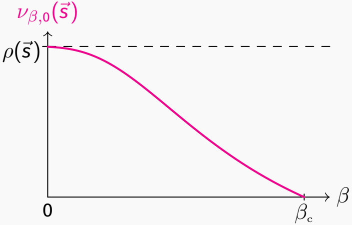
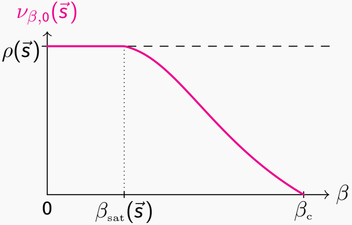
- For simplicity, let us also assume that

$$\forall x \in \mathbb{R}^d, \quad \psi(x) = \psi(\rho(x)) > 0.$$

- One can show that these are the only two possible scenarios:





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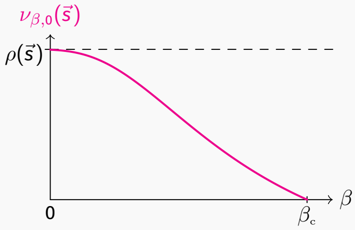

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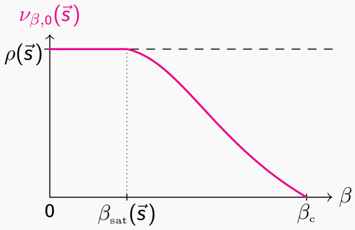

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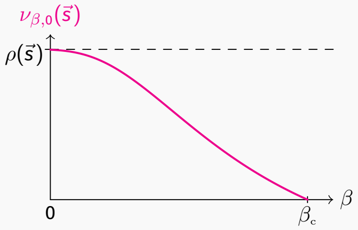

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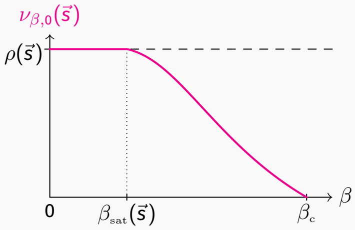

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- Criterion to determine which scenario occurs?

Criterion for the existence of a saturation regime

- Let us introduce the generating function (for $t \in \mathbb{R}^d$)

$$\mathbb{J}(t) = \sum_{x \in \mathbb{Z}^d} e^{t \cdot x} J_x.$$

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$$\mathscr{W} = \{t \in \mathbb{R}^d : \forall \vec{s} \in \mathbb{S}^{d-1},\ t \cdot \vec{s} \leq \rho(\vec{s})\}.$$

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- $t \in \partial\mathscr{W}$ is **dual** to $\vec{s} \in \mathbb{S}^{d-1}$ if

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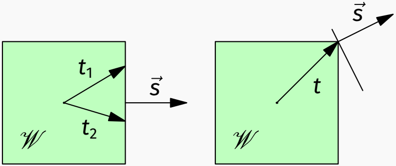

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Theorem [Aoun–Ioffe–Ott-V. 2021, Aoun–Ott–V. 2022]

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▶ What can be said about the **asymptotic behavior of the 2-point function** in the regimes $(0, \beta_{\text{sat}}(\vec{s}))$ and $(\beta_{\text{sat}}(\vec{s}), \beta_c)$?

Sharp asymptotics: $\beta_{\text{sat}}(\vec{s}) < \beta < \beta_c$

- Standard **Ornstein–Zernike asympotics hold** when $\beta \in (\beta_{\text{sat}}(\vec{s}), \beta_c)$:

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Let $\vec{s} \in \mathbb{S}^{d-1}$. For all $\beta \in (\beta_{\text{sat}}(\vec{s}), \beta_c)$, under some (presumably technical) condition, as $n \to \infty$,

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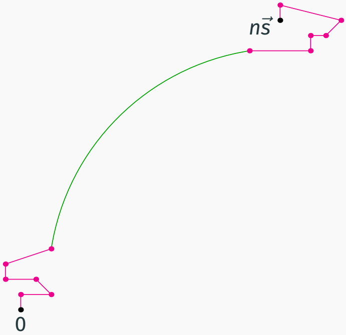
- Very similar asymptotics have been shown to hold in the simpler situation in which the coupling constants decay **subexponentially** [Newman–Spohn 1998].

Behavior of typical paths

▶ These different asymptotics reflect very different behaviors of typical “paths” contributing to graphical representations in both regimes.



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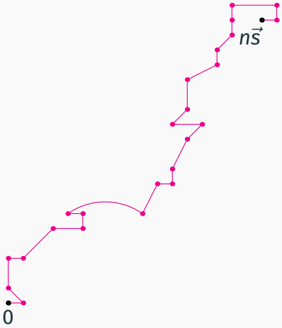

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— A. Asymptotics of correlations —

A.2. General correlation functions

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- Of course, by symmetry, $\langle \sigma_C \rangle_{\beta,0} = 0$ whenever $|C|$ is odd.

$$\rightsquigarrow \langle \sigma_A \,;\, \sigma_B \rangle_{\beta,0} = 0 \text{ whenever } |A| + |B| \text{ is odd.}$$

Decay of correlations

- We assume that $h = 0$, $\beta < \beta_c$ and $(J_x)_{x \in \mathbb{Z}^d}$ has finite range.
- For $A \Subset \mathbb{Z}^d$, let $\sigma_A = \prod_{i \in A} \sigma_i$.
__Remark:__ any local function (that is, depending on finitely many spins) can be expressed as a finite linear combination of such functions.
- Given $A, B \Subset \mathbb{Z}^d$ and $\vec{s} \in \mathbb{S}^{d-1}$, we investigate the asymptotic behavior of

$$\langle \sigma_A \,;\, \sigma_{B+n\vec{s}} \rangle_{\beta,0}$$

as $n \to \infty$.

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- We are thus left with **two cases to consider:**

Odd-odd correlations	Even-even correlations
$\lvert A\rvert, \lvert B\rvert$ both odd	$\lvert A\rvert, \lvert B\rvert$ both even

- Odd-odd correlations always display **Ornstein–Zernike behavior**:

Theorem [Campanino–Ioffe–V. 2004]

Let $\beta < \beta_c$. Let $A, B \Subset \mathbb{Z}^d$ with $|A|$ and $|B|$ odd and let $\vec{s} \in \mathbb{S}^{d-1}$.
Then, there exists a constant $0 < C < \infty$ (depending on $A, B, \vec{s}, \beta$) such that

$$\langle \sigma_A \,;\, \sigma_{B+n\vec{s}} \rangle_{\beta,0} = \frac{C}{n^{(d-1)/2}}\, e^{-\nu_{\beta,0}(\vec{s})n}\, (1 + o(1)),$$

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as $n \to \infty$.

- The first rigorous results of this type were obtained for $\beta \ll 1$ in
  - ▷ *Bricmont–Fröhlich 1985*
  - ▷ *Zhizhina–Minlos 1988*

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- However, concerning the prefactor, two conflicting predictions were put forward:

Polyakov 1969		Camp, Fisher 1971
n^{-2}	$d = 2$	
$(n \log n)^{-2}$	$d = 3$	n^{-d} for all $d \geq 2$
$n^{-(d-1)}$	$d \geq 4$	

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- It turns out that Polyakov was right. This was first shown in
  - *Bricmont–Fröhlich 1985:* $|A| = |B| = 2$ $\beta \ll 1$ $d \geq 4$
  - *Minlos–Zhizhina 1988, 1996:* $|A|, |B|$ even $\beta \ll 1$ $d \geq 2$

- The best nonperturbative result to date is the following:

$$\text{Let} \quad \tau(n) = \begin{cases} n^2 & \text{when } d = 2, \\ (n \log n)^2 & \text{when } d = 3, \\ n^{d-1} & \text{when } d \geq 4. \end{cases}$$

Theorem [Ott–V. 2019]

Let $d \geq 2$ and $\beta < \beta_c$. Let $A, B \Subset \mathbb{Z}^d$ with $|A|$ and $|B|$ even and let $\vec{s} \in \mathbb{S}^{d-1}$. Then, there exist constants $0 < C_- \leq C_+ < \infty$ (depending on $A, B, \vec{s}, \beta$) such that, for all n large enough,

$$\frac{C_-}{\tau(n)} e^{-2\nu_{\beta,0}(\vec{s})n} \leq \langle \sigma_A \,;\, \sigma_{B+n\vec{s}} \rangle \leq \frac{C_+}{\tau(n)} e^{-2\nu_{\beta,0}(\vec{s})n}.$$

— A. Asymptotics of correlations —

A.3. An inhomogeneous system: Ising with a defect line

Settings

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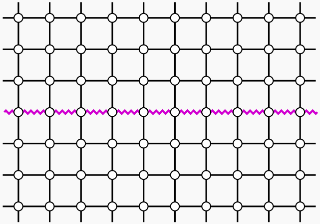

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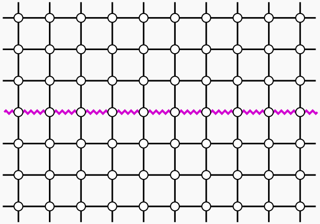

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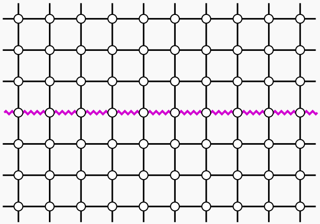

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- Central quantity: **longitudinal inverse correlation length** $\nu_\beta(J)$

$$\langle \sigma_0 \,;\, \sigma_{n\mathbf{e}_1} \rangle_{\beta,J} = \mathrm{e}^{-\nu_\beta(J)n + o(n)}.$$

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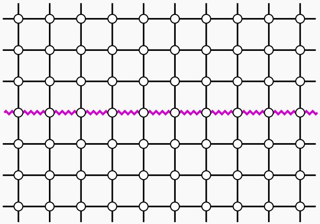

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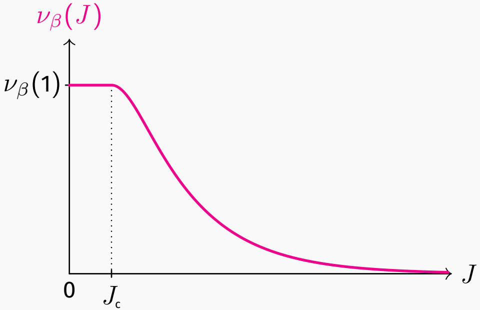
- *McCoy–Perk 1980:* **explicit computation** of $\nu_\beta(J)$ for the planar Ising model.

Properties of the longitudinal correlation length

- The following is proved in [Ott–V. 2018]: For any $d \geq 2$, there exists $J_c \geq 1$ such that



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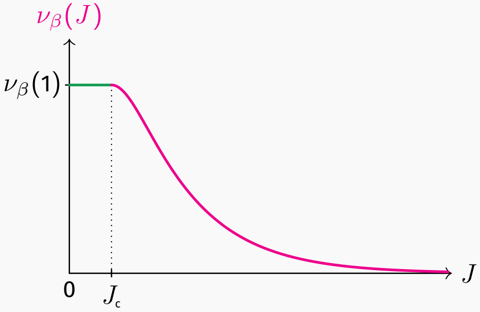

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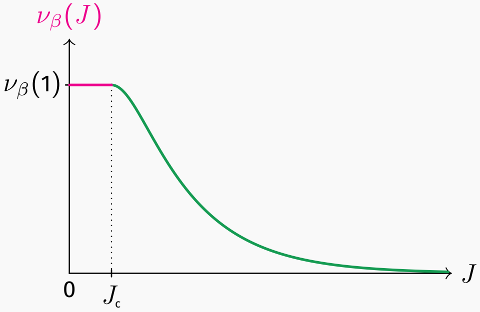

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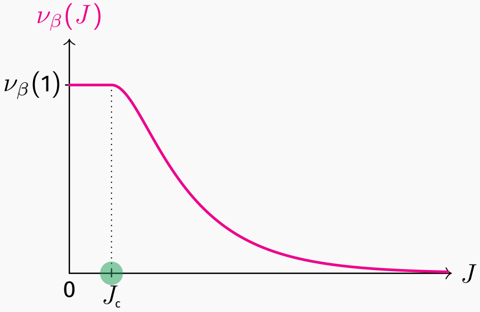

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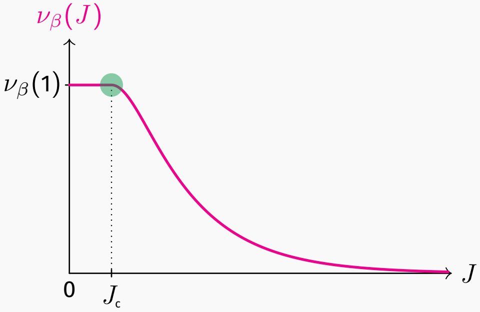
  - $d = 2, 3$: $J_c = 1$, $d \geq 4$: $J_c > 1$.
  - There exist constants $c_2^\pm, c_3^\pm > 0$ such that, as $J \downarrow J_c$,

$$c_2^-(J - J_c)^2 \leq \nu_\beta(J_c) - \nu_\beta(J) \leq c_2^+(J - J_c)^2 \qquad (d = 2)$$

$$e^{-c_3^-/(J-J_c)} \leq \nu_\beta(J_c) - \nu_\beta(J) \leq e^{-c_3^+/(J-J_c)} \qquad (d = 3)$$



- When $J > J_c$, one has **pure exponential decay**:

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Let $d \geq 2$. Assume that $\beta < \beta_c$, $h = 0$ and $J > J_c$. Then, as $n \to \infty$,

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Asymptotics of correlations

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- When $0 \leq J < 1$, one has the following asymptotics:

Theorem [Ott–V. 2019, Ioffe–Ott–V.–Wachtel 2020]

Assume that $\beta < \beta_c$, $h = 0$ and $0 \leq J < 1$. Then, as $n \to \infty$,

$$d = 2: \qquad \langle \sigma_0 \,;\, \sigma_{n\mathbf{e}_1} \rangle_{\beta,J} = \frac{C_{\beta,J}}{n^{3/2}}\, e^{-\nu_\beta(J) n} (1 + o(1)),$$

$$d = 3: \qquad \langle \sigma_0 \,;\, \sigma_{n\mathbf{e}_1} \rangle_{\beta,J} = \frac{C_{\beta,J}}{n(\log n)^2}\, e^{-\nu_\beta(J) n} (1 + o(1)),$$

$$d \geq 4: \qquad \langle \sigma_0 \,;\, \sigma_{n\mathbf{e}_1} \rangle_{\beta,J} = \frac{C_{\beta,J}}{n^{(d-1)/2}}\, e^{-\nu_\beta(J) n} (1 + o(1)).$$

— B. Interfaces in the planar Ising model —

B.1. Interface in the bulk

- We consider the **n.n. Ising model in** $\Lambda_N = \{-N+1, \ldots, N\}^2$ with $\beta > \beta_c$ and $h = 0$.

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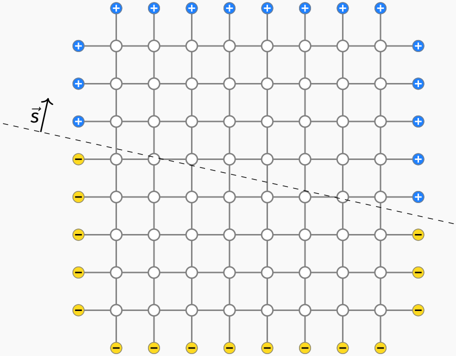
- Let $\vec{s} \in \mathbb{S}^1$. We consider a system with $\vec{s}$**-boundary condition**:



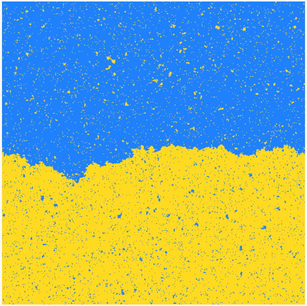
Interface in the bulk: Scaling limit of the interface

Theorem [Greenberg–Ioffe 2005]

Let $\vec{s} \in \mathbb{S}^1$ and $\beta > \beta_c$. The distribution of the centered and diffusively-rescaled interface induced by the $\vec{s}$-boundary condition converges to the distribution of

$$\sqrt{\chi_\beta(\vec{s})}\, \mathfrak{b},$$

where $\mathfrak{b}$ is the standard **Brownian bridge** on $[-1, 1]$ and $\chi_\beta(\vec{s})$ is the curvature of the Wulff shape at the unique point t of its boundary where the normal is $\vec{s}$.



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- Some earlier results:
  - *Abraham–Reed 1976:* Expected magnetization profile ($\vec{s} = \mathbf{e}_2$) (exact computations)
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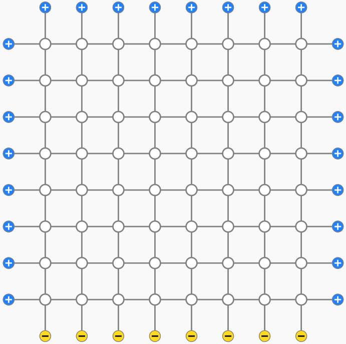
— B. Interfaces in the planar Ising model —

B.2. Interface at a boundary

- We consider a system with boundary condition inducing an interface along the bottom wall:



Interface at a boundary: Scaling limit of the interface

Theorem [Ioffe–Ott–V.–Wachtel 2020]

Let $\beta > \beta_c$. The distribution of the diffusively-rescaled interface converges to the distribution of

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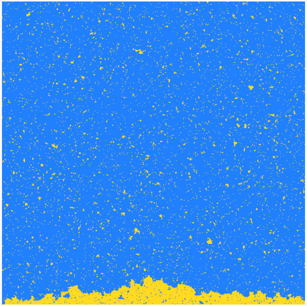

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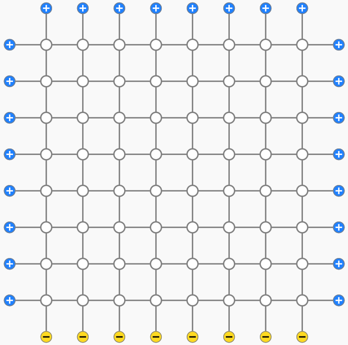
— B. Interfaces in the planar Ising model —

B.3. Interface in a field

Interface in a field: Settings

- We consider again the boundary condition



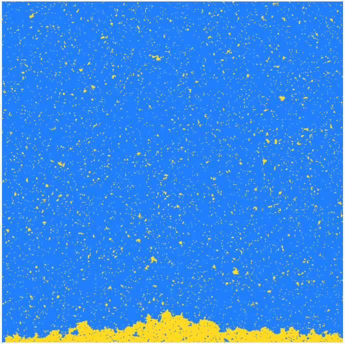
but add to the Hamiltonian a **magnetic field term**

$$-h \sum_{i \in \Lambda_N} \sigma_i$$

with $\boldsymbol{h > 0}$.

Interface in a field: Layer of unstable phase

- Let $\beta > \beta_c$. Since $h > 0$, the layer of $-$ phase becomes **unstable**:



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average width $= O(N^{1/2})$

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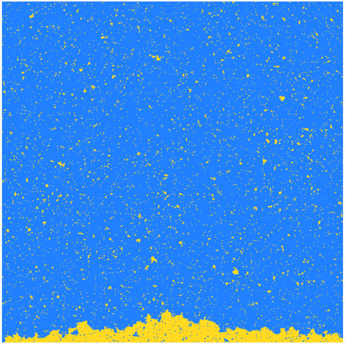

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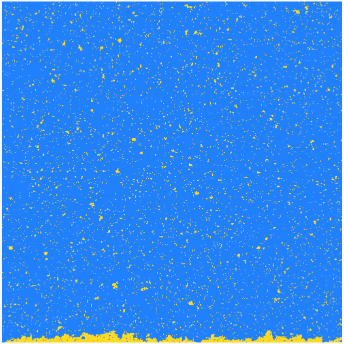
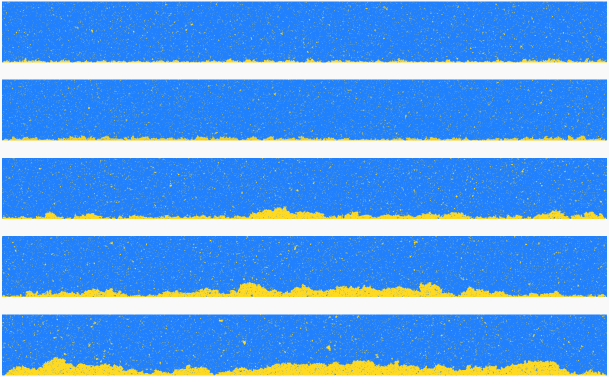


$h > 0$

average width $= O(1)$

Interface in a field: Critical prewetting

- The width of the layer increases as h decreases:



- It turns out to be natural to choose $h = h(N)$ to be of the form

$$h = \frac{\lambda}{N}$$

for some $\lambda > 0$.

Theorem [Ioffe, Ott, Shlosman, V. 2020]

Rescale the interface

- horizontally by $N^{-2/3}$
- vertically by $\chi_\beta^{-1/2} N^{-1/3}$.

Then, as $N \to \infty$, its distribution weakly converges to that of the **Ferrari–Spohn diffusion** introduced in the next slide.

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- vertically by $\chi_\beta^{-1/2} N^{-1/3}$.

Then, as $N \to \infty$, its distribution weakly converges to that of the **Ferrari–Spohn diffusion** introduced in the next slide.

- Some earlier results:
  - *V. 2004:* Layer width $\sim N^{1/3+o(1)}$
  - *Ganguly-Gheissari 2021:* Layer width $\sim N^{1/3}$ and other global estimates

Interface in a field: Scaling limit

Theorem [Ioffe, Ott, Shlosman, V. 2020]

Rescale the interface

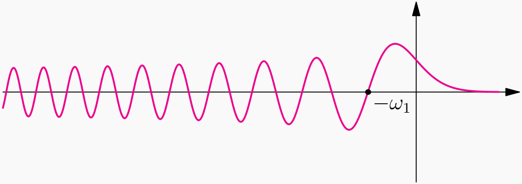
- horizontally by $N^{-2/3}$
- vertically by $\chi_\beta^{-1/2} N^{-1/3}$.

Then, as $N \to \infty$, its distribution weakly converges to that of the **Ferrari–Spohn diffusion** introduced in the next slide.

- Some earlier results:
  - *V. 2004:* Layer width $\sim N^{1/3+o(1)}$
  - *Ganguly-Gheissari 2021:* Layer width $\sim N^{1/3}$ and other global estimates

Interface in a field: Ferrari–Spohn diffusion

- Let us introduce
  - the **spontaneous magnetization**: m^*_β
  - the **curvature of the Wulff shape** (at its apex): χ_β
  - the **Airy function** Ai and its first zero $-\omega_1$



- Set $\varphi_0(r) = \mathrm{Ai}\big((4\lambda m^*_\beta \sqrt{\chi_\beta})^{1/3}\, r - \omega_1\big)$.

- The relevant **Ferrari–Spohn diffusion** in the present context is the diffusion on $(0, \infty)$ with generator

$$L_\beta = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}r^2} + \frac{\varphi_0'}{\varphi_0}\frac{\mathrm{d}}{\mathrm{d}r}$$

and Dirichlet boundary condition at 0.

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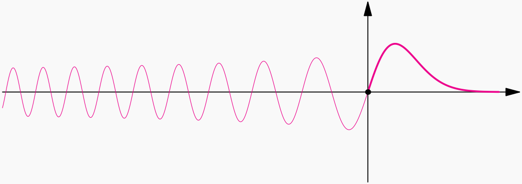

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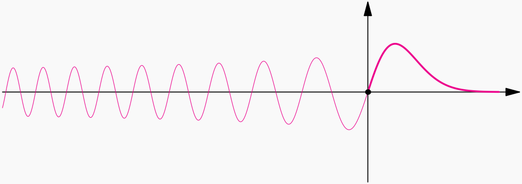

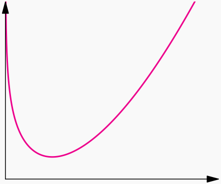

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- It allows, in particular, a detailed understanding of **asymptotics of correlation functions** in any dimension and for general ferromagnetic coupling constants, away from the critical point.
- Combined with planar duality, it enables an in-depth analysis of **interfacial phenomena** in 2D.
- The (modern version of the) Ornstein–Zernike theory was developed in a very large part by Dima Ioffe, to whom I dedicate this talk...

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