RANDOM PERMUTATIONS WITH CYCLE WEIGHTS

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Abstract. We study the distribution of cycle lengths in models of nonuniform random permutations with cycle weights. We identify several regimes. Depending on the weights, the length of typical cycles grows like the total number n of elements, or a fraction of n, or a logarithmic power of n.

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1. Introduction

We study the cycle distributions in models of weighted random permutations. The probability of a permutation π of n elements is defined by

$$P(\pi) = \frac{1}{h_n n!} \prod_{j \geqslant 1} \theta_j^{r_j(\pi)}, \tag{1.1}$$

where $(\theta_1, \theta_2, \dots) \equiv \boldsymbol{\theta}$ are real nonnegative numbers, $r_j(\pi)$ denotes the number of j-cycles in π (we always have $\sum_j j r_j(\pi) = n$), and h_n is the normalization. We are mainly interested in the distribution of cycle lengths in the limit $n \to \infty$, and in how these lengths depend on the set of parameters $\boldsymbol{\theta}$.

The probability P is really a probability on sequences $\boldsymbol{r} = (r_1, r_2, \dots)$ that satisfy $\sum_j j r_j = n$. It is well-known that $\boldsymbol{r}$ are the "occupation numbers" of a partition $\boldsymbol{\lambda}$ of n. That is, if $\boldsymbol{\lambda}$ denotes the partition $\lambda_1 \geqslant \lambda_2 \geqslant \dots$ with $\sum_i \lambda_i = n$, then r_j is the number of λ_i that satisfy $\lambda_i = j$. Thus we are really dealing with random partitions. The number of permutations that are compatible with occupation numbers $\boldsymbol{r}$ is equal to

$$\frac{n!}{\prod_{j \geqslant 1} j^{r_j} r_j!}.$$

It follows that the marginal of (1.1) on partitions is given by

$$P(\boldsymbol{\lambda}) = \frac{1}{h_n} \prod_{j \geqslant 1} \frac{1}{r_j!} \bigl(\tfrac{1}{j}\theta_j\bigr)^{r_j}. \tag{1.2}$$

The formulæ look simpler and more elegant for permutations than for partitions, and this is why we consider the former.

Random permutations with the uniform distribution have a compelling history [13, 7, 1, 6]. They are a special case of the present setting, with $\theta_j \equiv 1$. The uniform distribution of random partitions have been studied e.g. in [8, 16, 12, 18]. They do not fit the present setting because there are no parameters $\boldsymbol{\theta}$ that make the right side of (1.2) constant. Another distribution for random partitions is the Plancherel measure, where the probability of $\boldsymbol{\lambda}$ is proportional to $\frac{1}{n!}(\dim \boldsymbol{\lambda})^2$; the "dimension" $\dim \boldsymbol{\lambda}$ of a partition is defined as the number of Young tableaux in Young diagrams and it does not seem to have an easy expression in terms of $\boldsymbol{r}$. Here again, we do not know of any direct relation between weighted random permutations and the Plancherel measure.

The present model was introduced in [4] but variants of it have been studied previously. The case of constant $\theta_j \equiv \theta$ is known as the Ewens distribution. It appears in the study of population dynamics in mathematical biology [9]; detailed results about the number of cycles were obtained by Hansen [14] and by Feng and Hoppe [10]. The distribution of cycle lengths was considered by

Lugo [15]. Another variant of this model involves parameters $\theta_j \in \{0,1\}$, with finitely many 1's [17, 2], or with parity dependence [15].

Weighted random permutations also appear in the study of large systems of quantum bosonic particles [3, 5], where the parameters $\boldsymbol{\theta}$ depend on such quantities as the temperature, the density, and the particle interactions. The θ_j's are thus forced upon us and they do not necessarily take a simple form. This motivates the present study, where we only fix the asymptotic behavior of θ_j as $j \to \infty$.

The relevant random variables in our analysis are the lengths $\ell_i = \ell_i(\pi)$ of the cycle containing the index $i = 1, \dots, n$. These random variables are always identically distributed, and obviously not independent. Another relevant random variable is the number of indices belonging to cycles of length between a and b, $N_{a,b}(\pi) = \#\{i = 1, \dots, n : a \leqslant \ell_i(\pi) \leqslant b\}$. It follows from the exchangeability of $\ell_1, \dots, \ell_n$ that

$$\frac{1}{n} E(N_{a,b}) = P(\ell_1 \in [a,b]). \tag{1.3}$$

The properties of the distribution of ℓ_1 that we derive below can then be translated into properties of the expectation of $N_{a,b}$.

From a statistical mechanics point of view, it is natural to introduce the sequence $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots)$ of parameters such that $\mathrm{e}^{-\alpha_j} = \theta_j$. The model has an important symmetry, which is also a source of confusion. Namely, the probability of the permutation π is left invariant under the transformation

$$\alpha_j \mapsto \alpha_j + cj, \qquad h_n \mapsto \mathrm{e}^{-cn} h_n, \tag{1.4}$$

for any constant $c \in \mathbb{R}$. In particular, the case $\alpha_j = cj$ is identical to $\alpha_j \equiv 0$, the case of uniform random permutations.

The general results which we prove in this article rely on various technical assumptions. To keep this introduction simple, we only describe the results in the particular but interesting case $\alpha_j = j^\gamma$.

- The case $\gamma < 0$ is a special case of the model studied in [4], which is close to the uniform distribution.
- In the case $\gamma = 0$, i.e. when $\theta_j \to \theta$ (the Ewens case, asymptotically), we find that $P(\ell_1 > sn) \to (1-s)^\theta$. Thus, almost all indices belong to cycles whose length is a fraction of n. Precise statements and proofs can be found in Section 2.
- The case $0 < \gamma < 1$ is surprising. At first glance we might expect smaller cycles than in the uniform case $\alpha_j \equiv 0$. However, we find that almost all indices belong to a single giant cycle! The symmetry (1.4) is playing tricks upon us indeed. In addition, we prove that the probability of the occurrence of a single cycle of length n is strictly positive, and strictly less than 1. This is explained in details in Section 3.
- The case $\gamma = 1$ corresponds to uniform permutations because of the symmetry (1.4).
- When $\gamma > 1$ the cycles become shorter, and ℓ_1 behaves asymptotically as $(\frac{1}{\gamma-1} \log n)^{1/\gamma}$; see Section 4.

Weighted random permutations clearly show a rich behavior and only a little part has been uncovered so far. The case of negative parameters, $\alpha_j \asymp -j^\gamma$ remains to be explored, and the future will hopefully bring more results regarding concentration properties.

In the case of uniform permutations, it is known that the random variables r_k converge to independent Poisson random variables with parameter $1/k$ in the limit $n \to \infty$ [13, 1]. An open problem is to understand how this generalizes to weighted random permutations.

2. Asymptotic Ewens distribution

In the case of the uniform distribution, it is an easy exercise to show that $P(\ell_1 = a) = 1/n$ for any $a = 1, \dots, n$. It follows that $P(\ell_1 > sn) \to 1 - s$ for any $0 \leqslant s \leqslant 1$. This result was extended to the case of small weights in [4]. We consider here parameters that are close to Ewens weights. A result similar to (a) below has been recently derived by Lugo [15].

Theorem 2.1. *Let $\theta \in \mathbb{R}_+$. We suppose that $\sum_{j=1}^\infty \frac{1}{j}|\theta_j - \theta| < \infty$ if $\theta \geqslant 1$, or that $\sum_{j=1}^\infty |\theta_j - \theta| < \infty$ if $\theta < 1$.*

(a) *The distribution of ℓ_1 satisfies, for $0 \leqslant s \leqslant 1$,*

$$\lim_{n\to\infty} P(\ell_1 > sn) = (1-s)^\theta. \tag{2.1}$$

(b) *The joint distribution of ℓ_1 and ℓ_2 satisfies, for $0 \leqslant s,t \leqslant 1$,*

$$\lim_{n\to\infty} P(\ell_1 > sn, \ell_2 > tn) = \frac{\theta}{1+\theta}(1-s-t)_+^{\theta+1} + \frac{1+\theta(s\vee t)}{1+\theta}(1-s\vee t)^\theta, \tag{2.2}$$

where f_+ denotes the positive part of a function f.

Let us recall a few properties that are satisfied by the normalization factors h_n. Summing over the length j of the cycle that contains 1 we find the useful relation

$$P(\ell_1 \in [a,b]) = \frac{1}{n!h_n}\sum_{j=a}^b \frac{(n-1)!}{(n-j)!}\theta_j (n-j)! h_{n-j} = \frac{1}{n}\sum_{j=a}^b \theta_j \frac{h_{n-j}}{h_n}. \tag{2.3}$$

Choosing $[a,b] = [1,n]$, we get

$$h_n = \frac{1}{n}\sum_{j=1}^n \theta_j h_{n-j}, \qquad h_0 = 1. \tag{2.4}$$

Next, let $G_h(s) = \sum_{n\geqslant 0} h_n s^n$ be the generating function of the sequence (h_n). One can view a permutation as a combinatorial structure made of cycles. It follows from standard combinatorics results that $G_h(s) = \exp\sum_{j\geqslant 1}\frac{1}{j}\theta_j s^j$. We also refer to [4] for a direct proof of this formula. The first step in the proof of Theorem 2.1 is to control the normalization h_n. Here, $(\theta)_n = \theta(\theta+1)\dots(\theta+n-1)$ denotes the ascending factorial.

Proposition 2.2. *Under the assumptions of Theorem 2.1, we have*

$$h_n = C(\boldsymbol{\theta})\frac{(\theta)_n}{n!}(1+o(1)) \qquad \text{with} \qquad C(\boldsymbol{\theta}) = \exp\sum_{j\geqslant 1}\tfrac{1}{j}(\theta_j - \theta).$$

Proof. We have

$$G_h(s) = \exp\Big\{\theta\sum_j \tfrac{1}{j}s^j + \sum_j \tfrac{1}{j}(\theta_j-\theta)s^j\Big\} = (1-s)^{-\theta}\,\mathrm{e}^{u(s)}, \tag{2.5}$$

with

$$u(s) = \sum_{j\geqslant 1}\tfrac{1}{j}(\theta_j-\theta)s^j. \tag{2.6}$$

Notice that $u(1) = \lim_{s\nearrow 1} u(s)$ exists. Let c_j be the Taylor coefficients of $\mathrm{e}^{u(s)}$, i.e. $\mathrm{e}^{u(s)} = \sum c_j s^j$. Then, by Leibniz' rule,

$$h_n = \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}s^n}G_h(s)\Big|_{z=0} = \frac{(\theta)_n}{n!}\sum_{k\geqslant 0} d_{n,k}c_k, \tag{2.7}$$

with

$$d_{n,k} = \begin{cases} \frac{n(n-1)\dots(n-k+1)}{(\theta+n-1)\dots(\theta+n-k)} & \text{if } k \leqslant n, \\ 0 & \text{otherwise.}\end{cases} \tag{2.8}$$

It is not hard to check that

$$d_{n,k} \leqslant \begin{cases} 1 & \text{if } \theta \geqslant 1, \\ \theta^{-1}k+1 & \text{if } \theta > 0.\end{cases} \tag{2.9}$$

Let $U(s) = \sum\frac{1}{j}|\theta_j-\theta|s^j$ and C_j the Taylor coefficients of $\mathrm{e}^{U(s)}$. It is clear that $|c_j| \leqslant C_j$ for all j. When $\theta \geqslant 1$, the first bound of (2.9) and the dominated convergence theorem imply

$$\lim_{n\to\infty}\sum_{k\geqslant 0} d_{n,k}c_k = \sum_{k\geqslant 0} c_k = \mathrm{e}^{u(1)} = C(\boldsymbol{\theta}). \tag{2.10}$$

When $\theta < 1$, the second bound of (2.9) gives $d_{n,k}|c_k| \leqslant (\theta^{-1}k+1)C_k$. The sequence (kC_k) is absolutely convergent:

$$\sum kC_k = \frac{\mathrm{d}}{\mathrm{d}s}\,\mathrm{e}^{U(s)}\Big|_{s=1} = \mathrm{e}^{U(1)}\,U'(1) = \mathrm{e}^{\sum \frac{1}{j}|\theta_j-\theta|}\sum|\theta_j-\theta| < \infty. \tag{2.11}$$

We again obtain (2.10) by the dominated convergence theorem. □

Proof of Theorem 2.1. We show that, for any $0 < s < t < 1$, we have

$$\lim_{n\to\infty} P(\ell_1 \in [sn,tn]) = (1-s)^\theta - (1-t)^\theta. \tag{2.12}$$

Using Proposition 2.2, we have

$$P(\ell_1 \in [sn,tn]) = \frac{1}{n}\sum_{j=sn}^{tn}\theta_j\frac{h_{n-j}}{h_n} = \frac{\theta}{n}\sum_{j=sn}^{tn}\frac{(\theta)_{n-j}}{(n-j)!}\frac{n!}{(\theta)_n}(1+o(1)). \tag{2.13}$$

Here and throughout this article, when a and b are not integer we use the convention

$$\sum_{j=a}^{b} f(j) = \sum_{j\in[a,b]\cap\mathbb{N}} f(j) = \sum_{j=\lceil a\rceil}^{\lfloor b\rfloor} f(j). \tag{2.14}$$

We now use the identity

$$(\theta)_n = \frac{\Gamma(n+\theta)}{\Gamma(\theta)} \tag{2.15}$$

and the asymptotic

$$\frac{\Gamma(n+\theta)}{n!} = n^{\theta-1}(1+o(1)). \tag{2.16}$$

We get

$$P(\ell_1 \in [sn,tn]) = \frac{\theta}{n}\sum_{j=sn}^{tn}(1-\tfrac{j}{n})^{\theta-1}(1+o(1)). \tag{2.17}$$

As $n\to\infty$, the right side converges to the Riemann integral $\theta\int_s^t(1-\xi)^{\theta-1}\mathrm{d}\xi$, and we obtain the first claim of Theorem 2.1.

Let us now turn to the second claim. Let $1 \leqslant a \leqslant b \leqslant n$ and $1 \leqslant c \leqslant d \leqslant n$. We get an expression for the joint probability of ℓ_1 and ℓ_2 in a similar fashion as for (2.3). When both indices belong to different cycles (noted $1 \not\sim 2$), we have

$$P\big(\ell_1 \in [a,b], \ell_2 \in [c,d], 1 \not\sim 2\big) = \frac{1}{n!h_n}\sum_{\substack{j\in[a,b]\\ k\in[c,d]\\ j+k\leqslant n}}\sum_{\substack{|c_1|=j\\ |c_2|=k}}\theta_j\theta_k\sum_{\pi'}\prod_{\ell\geqslant 1}\theta_\ell^{r_\ell(\pi')}. \tag{2.18}$$

Here, c_1 and c_2 denote the cycles that contain 1 and 2, respectively, and π' denotes a permutation of the $n-j-k$ indices that do not belong to c_1 or c_2. The number of cycles of length j that contain 1 but not 2 is $\frac{(n-2)!}{(n-1-j)!}$; given c_1, the number of cycles of length k that contain 2 is $\frac{(n-j-1)!}{(n-j-k)!}$. Since the sum over π' gives $(n-j-k)!h_{n-j-k}$, we get

$$P\big(\ell_1 \in [a,b], \ell_2 \in [c,d], 1 \not\sim 2\big) = \frac{1}{n(n-1)}\sum_{\substack{j\in[a,b]\\ k\in[c,d]\\ j+k\leqslant n}}\theta_j\theta_k\frac{h_{n-j-k}}{h_n}. \tag{2.19}$$

When both indices belong to the same cycle, one can first sum over the length j of the common cycle, then over $j-2$ indices other than 1,2, and then over $j-1$ locations for 2. This gives

$\frac{(n-2)!}{(n-j)!}(j-1)$ possibilities. The sum over permutations on remaining indices gives $(n-j)!h_{n-j}$. The result is

$$P\big(\ell_1 \in [a,b], \ell_2 \in [c,d]\big) = \frac{1}{n(n-1)} \sum_{\substack{j\in[a,b]\\ k\in[c,d]\\ j+k\,\leqslant\, n}} \theta_j\theta_k \frac{h_{n-j-k}}{h_n} + \frac{1}{n(n-1)} \sum_{j\in[a,b]\cap[c,d]} (j-1)\theta_j \frac{h_{n-j}}{h_n}. \tag{2.20}$$

Let $\epsilon > 0$ and set $a = sn$, $c = tn$ and $b = d = n$. We assume, without loss of generality, that $1 \geqslant s \geqslant t \geqslant 0$. Using the above expression, Proposition 2.2 and Eqs (2.15)–(2.16), we deduce that, for n large,

$$\begin{aligned} P\big(\ell_1 \geqslant sn, \ell_2 \geqslant tn\big) = \frac{\theta^2}{n^2} &\sum_{\substack{j\,\geqslant\, sn, k\,\geqslant\, tn\\ j+k\,\leqslant\,(1-\epsilon)n}} \Big(1-\frac{j+k}{n}\Big)^{\theta-1}(1+o_\epsilon(1)) \\ &+ \frac{\theta}{n^2} \sum_{sn\,\leqslant\, j\,\leqslant\,(1-\epsilon)n} (j-1)\Big(1-\frac{j}{n}\Big)^{\theta-1}(1+o_\epsilon(1)) \; + O(\epsilon). \end{aligned} \tag{2.21}$$

Taking first the limit $n \to \infty$ and then the limit $\epsilon \to 0$, the right side of the latter expression is seen to converge to

$$1_{\{s+t\,\leqslant\,1\}}\theta^2 \int_{s+t}^1 (\xi - s - t)(1-\xi)^{\theta-1}\mathrm{d}\xi + \theta\int_s^1 \xi(1-\xi)^{\theta-1}\mathrm{d}\xi, \tag{2.22}$$

and the second claim of Theorem 2.1 follows. □

3. Slowly diverging parameters

This section is devoted to parameters α_j that grow slowly to $+\infty$. The typical case is $\alpha_j = j^\gamma$ with $0 < \gamma < 1$, but our conditions allow more general sequences. As mentioned in the introduction, the system displays a surprising behavior: almost all indices belong to a single giant cycle.

Theorem 3.1. *We assume that $0 < \frac{\theta_{n-j}\theta_j}{\theta_n} \leqslant c_j$ for all n and for $j = 1, \dots, \frac{n}{2}$, with constants c_j that satisfy $\sum_{j\,\geqslant\,1} \frac{c_j}{j} < \infty$. Then*

$$\lim_{m\to\infty}\lim_{n\to\infty} P(\ell_1 > n-m) = 1.$$

It may be worth recalling that in this article n always denotes the number of elements and that P depends on n. The proof of this theorem can be found later in this section. In the case $\alpha_j = j^\gamma$, we have

$$\frac{\theta_{n-j}\theta_j}{\theta_n} = \mathrm{e}^{-n^\gamma[(1-\frac{j}{n})^\gamma + (\frac{j}{n})^\gamma - 1]} \approx \begin{cases} \mathrm{e}^{-j^\gamma} & \text{if } j \ll n, \\ \mathrm{e}^{-cn^\gamma} & \text{if } j = sn, \end{cases} \tag{3.1}$$

where the constant in the last equation is $c = (1-s)^\gamma + s^\gamma - 1$. It is positive for $0 < \gamma < 1$, and the condition of the theorem is fulfilled. Another interesting example is $\theta_j = j^{-\gamma}$ with $\gamma > 0$, where we can choose $c_j = 2j^{-\gamma}$.

Let us understand why parameters $\alpha_j = j^\gamma$ favor longer and longer cycles. The heuristics is actually provided by statistical mechanics. Namely, we can write the probability $P(\pi)$ as a Gibbs distribution $\frac{1}{Z}\,\mathrm{e}^{-H(\pi)}$ with "Hamiltonian" $H(\pi) = \sum_{i=1}^n \frac{\alpha_{\ell_i(\pi)}}{\ell_i(\pi)}$. Thus, to each index i that belongs to a cycle of length j is associated an "energy" $\frac{\alpha_j}{j} = j^{\gamma-1}$. Indices in longer cycles have lower energy, so they are favored. This discussion also provides an illustration for the symmetry (1.4); it amounts to shifting the Hamiltonian by a constant and this does not affect the Gibbs distribution.

We can state a more precise result than Theorem 3.1 if we make the additional assumption that $\frac{\theta_{n+1}}{\theta_n}$ converges to 1 as $n \to \infty$. This condition is easy to check when $\alpha_j = j^\gamma$, $0 < \gamma < 1$, or when $\alpha_j = \gamma \log j$, $\gamma > 0$.

Theorem 3.2. *Suppose that the assumptions of Theorem 3.1 hold true. In addition, we suppose that $\frac{\theta_{n+1}}{\theta_n}$ converges to 1 as $n \to \infty$. Then $\sum_j h_j < \infty$, and for any fixed m,*

$$\lim_{n\to\infty} P(\ell_1 = n-m) = \frac{h_m}{\sum_{j \geqslant 1} h_j}.$$

Theorem 3.2 shows in particular that a single cycle of length n occurs with probability $1/\sum_j h_j$, but that finite cycles may be present as well.

This theorem is proved at the end of the section. We first obtain estimates for h_n.

Proposition 3.3. *Under the assumptions of Theorem 3.1 there exists a constant B such that, for all n,*

$$1 \leqslant \frac{nh_n}{\theta_n} \leqslant B.$$

The constant B depends on $\{c_j\}$ only.

Proof. The lower bound follows obviously from (2.4), but the upper bound requires some work. Let $a_n = \frac{nh_n}{\theta_n}$. The relation (2.4) can be written as

$$a_n = 1 + \sum_{j=1}^{n-1} \frac{1}{j} \frac{\theta_{n-j}\theta_j}{\theta_n} a_j. \tag{3.2}$$

We can rewrite this relation as

$$a_n = \begin{cases} 1 + \sum_{j=1}^{\frac{n-1}{2}} \frac{\theta_{n-j}\theta_j}{\theta_n} \Big(\frac{a_j}{j} + \frac{a_{n-j}}{n-j} \Big) & \text{if } n \text{ is odd,} \\ 1 + \sum_{j=1}^{\frac{n}{2}-1} \frac{\theta_{n-j}\theta_j}{\theta_n} \Big(\frac{a_j}{j} + \frac{a_{n-j}}{n-j} \Big) + \frac{2\theta_{n/2}^2}{n\theta_n} a_{n/2} & \text{if } n \text{ is even.} \end{cases} \tag{3.3}$$

We define the sequence (b_n) by the recursion equation

$$b_n = 1 + \sum_{j=1}^{n/2} c_j \Big(\frac{b_j}{j} + \frac{b_{n-j}}{n-j} \Big). \tag{3.4}$$

It is clear that $a_n \leqslant b_n$ for all n. Next, let m be a number such that

$$\frac{2}{n} \sum_{j=1}^{n/2} c_j + \sum_{j>m/2} \frac{c_j}{j} \leqslant \frac{1}{2} \tag{3.5}$$

for all $n \geqslant m$. Such an m exists because (c_j/j) is summable, and the first term of the above equation is less than $\frac{2}{\sqrt{n}} \sum_{j=1}^{\sqrt{n}} \frac{c_j}{j} + \sum_{j>\sqrt{n}} \frac{c_j}{j} = O(n^{-1/2})$. We set

$$B = 2 \max_{1 \leqslant j \leqslant m} b_j. \tag{3.6}$$

Notice that B depends on the c_j's but not on the θ_j's. Finally, we introduce another sequence (b'_n) defined by

$$b'_n = \begin{cases} b_n & \text{if } n \leqslant m, \\ 1 + \sum_{j=1}^{n/2} c_j \Big(\frac{b'_j}{j} + \frac{2B}{n} \Big) & \text{if } n > m. \end{cases} \tag{3.7}$$

It is clear that $b'_n \leqslant \frac{1}{2}B$ for $n \leqslant m$; we now show by induction that $b'_n \leqslant B$ for all n. We have

$$\begin{aligned} b'_n - b'_m &= \frac{2B}{n} \sum_{j=1}^{n/2} c_j - \sum_{j=1}^{m/2} c_j \frac{b'_{m-j}}{m-j} + \sum_{j=\frac{m}{2}+1}^{n/2} c_j \frac{b'_j}{j} \\ &\leqslant \Big(\frac{2}{n} \sum_{j=1}^{n/2} c_j + \sum_{j>m/2} \frac{c_j}{j} \Big) B. \end{aligned} \tag{3.8}$$

This is less than $\frac12 B$ by the definition (3.5) of m. Since $b'_m \leqslant \frac12 B$, we find that $b'_n \leqslant B$ for all n. The final step is to see that $b_n \leqslant b'_n$. This is clear when $n \leqslant m$, and we get it by induction when $n > m$:

$$b_{n+1} = 1 + \sum_{j=1}^{n/2} c_j \Big(\frac{b_j}{j} + \frac{b_{n-j+1}}{n-j+1} \Big) \leqslant 1 + \sum_{j=1}^{n/2} c_j \Big(\frac{b'_j}{j} + \frac{2B}{n+1} \Big) = b'_{n+1}. \tag{3.9}$$

We have shown that $a_n \leqslant b_n \leqslant b'_n \leqslant B$ for all n. □

Proof of Theorem 3.1. Using Proposition 3.3, we get

$$P(\ell_1 \leqslant n-m) = \frac1n \sum_{j=m}^{n-1} \theta_{n-j} \frac{h_j}{h_n} \leqslant B \sum_{j=m}^{n-1} \frac1j \frac{\theta_{n-j}\theta_j}{\theta_n} \leqslant B \sum_{j=m}^{n/2} \frac{c_j}{j} + B \sum_{j=n/2}^{n-1} \frac{c_{n-j}}{j}. \tag{3.10}$$

The last term goes to zero as $n \to \infty$. The first term goes to zero as $n \to \infty$ and $m \to \infty$. □

Proof of Theorem 3.2. From Eq. (2.3),

$$P(\ell_1 = n-m) = \frac1n \theta_{n-m} \frac{h_m}{h_n} = \frac{\theta_{n-m}}{\theta_n} \frac{\theta_n}{n h_n} h_m. \tag{3.11}$$

Further, (2.4) can be written as

$$\frac{n h_n}{\theta_n} = \sum_{j=1}^{n/2} \Big(\frac{\theta_{n-j} h_j}{\theta_n} + \frac{\theta_j h_{n-j}}{\theta_n} \Big). \tag{3.12}$$

This is actually correct for odd n only; there is an unimportant correction for even n coming from $j = n/2$. Since $h_j \leqslant B \frac{\theta_j}{j}$ (Proposition 3.3), the summand is less than $B c_j (\frac1j + \frac1{n-j}) \leqslant 2B \frac{c_j}{j}$. For each j and as $n \to \infty$, we have $\frac{\theta_{n-j}}{\theta_n} \to 1$ and $\frac{\theta_j h_{n-j}}{\theta_n} \leqslant B \frac{c_j}{n-j} \to 0$. The right side of (3.12) then converges to $\sum_j h_j$ by dominated convergence. We can now take the limit $n \to \infty$ in (3.11) and we obtain the claim indeed. □

4. Quickly diverging parameters

Here we treat parameters $\theta_j = \mathrm{e}^{-\alpha_j}$ with α_j diverging quickly, or equivalently θ_j decaying quickly. More precisely, we shall make the following two assumptions: for some $M > 0$, all $k \geqslant 1$, and two coprime numbers $j_1, j_2 \geqslant 4$,

$$\begin{aligned} &0 \leqslant \theta_k \leqslant \frac{\mathrm{e}^{Mk}}{k!}, \\ &\theta_{j_1} > 0,\, \theta_{j_2} > 0. \end{aligned} \tag{4.1}$$

It is necessary to impose some kind of aperiodicity condition on the set of indices corresponding to nonvanishing coefficients θ_j. This prevents us from prescribing e.g. permutations with only even lengths of cycles. In this case, we have $h_n = 0$ for all odd n, as can be easily seen from the recursion (2.4); Proposition 4.5 below would fail.

Our assumptions allow to get the asymptotics of h_n using the saddle point method. We write down the steps explicitly in order to keep the article self-contained. A slightly shorter path would be to prove that our assumptions imply that e^f, with $f(z) = \sum_{j=0}^\infty \theta_j z^j$, is "Hayman admissible", and to use standard results [11]. That Hayman admissibility holds is implicitly used in our proof.

We describe general results in Subsection 4.1, relegating proofs to Subsection 4.2. The general results turn out to be somewhat abstract, so we use them to study the particularly interesting class $\alpha_j = j^\gamma$, $\gamma > 1$, in Subsection 4.3.

4.1. **Main properties.** We now describe three general theorems about cycle lengths. In all theorems the conditions (4.1) are silently assumed. The first statement concerns the absence of macroscopic cycles.

Theorem 4.1. *For arbitrarily small $\delta > 0$ and arbitrarily large $k > 0$, there exists $n_0 = n_0(\delta, k)$ such that*

$$P\Big(\max_{1 \leqslant i \leqslant n} \ell_i \;\geqslant\; \delta n\Big) \;\leqslant\; n^{-k},$$

for all $n \geqslant n_0$.

More precise information about typical cycle lengths can be extracted from the following result. Let $r_n > 0$ be defined by the equation

$$\sum_{j \geqslant 1} \theta_j r_n^j = n. \tag{4.2}$$

That such r_n exists uniquely is immediate.

Theorem 4.2. *Let $a(n), b(n)$ be such that*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{a(n)} \theta_j r_n^{j+1/2} = 0, \qquad \lim_{n\to\infty} \frac{1}{n} \sum_{j=b(n)}^{n} \theta_j r_n^{j+1/2} = 0.$$

Then

$$\lim_{n\to\infty} P(\ell_1 \in [a(n), b(n)]) = 1.$$

When the information about the coefficients θ_j is sufficiently detailed, some control on r_n is possible and Theorem 4.2 can be used to obtain sharp results. This is exemplified in Subsection 4.3 for the special case $\alpha_j = \alpha(j) = j^\gamma$ with $\gamma > 1$. In such cases, the sum $\sum_{j=1}^\infty \theta_j r_n^{j+1/2}$ (whose value is $r_n^{1/2} n$) is dominated by the terms corresponding to indices j close to the solution $j_{\max}$ of the equation $\alpha'(j) = \log r_n$.

Finally, it is also possible to extract from Theorem 4.2 a general result proving absence of small cycles.

Theorem 4.3.

$$\lim_{n\to\infty} P\Big(\ell_1 \;\leqslant\; \frac{\log n}{\log r_n} - \tfrac{3}{4}\Big) = 0.$$

We shall see below that the proof of Theorem 4.3 is straightforward; nonetheless, the result is quite strong. In the case where only finitely many θ_j are nonzero, we find $r_n \sim n^{1/j_0}$, where j_0 is the last index with nonzero θ_j. Thus $\log n / \log r_n \approx j_0$, and we obtain that the probability that $\ell_1 \;\leqslant\; j_0 - 1$ is zero. It follows that almost all cycles have length j_0, a fact already observed in [17, 2]. On the other hand, if infinitely many θ_j are nonzero, it is easy to see that $\log n / \log r_n$ diverges. Thus ℓ_1 goes to infinity in probability. To summarize, the only way to force a positive fraction of indices to lie in finite cycles is to forbid infinite cycles altogether, in which case typical cycles have the maximal length that is allowed.

4.2. **Proofs of the main properties.** We now prove Theorems 4.1 – 4.3. We use the following elementary result, which is a consequence of the first assumption in (4.1).

Lemma 4.4. *Let $f(x) = \sum_{k=0}^\infty c_k x^k$ with Taylor coefficients that satisfy $0 \;\leqslant\; c_k \;\leqslant\; \mathrm{e}^{Mk}\, k^{-k}$ for some $M > 0$ and all $k \;\geqslant\; 1$. Then for all $\delta > 0$ and all $x \;\geqslant\; 0$, we have*

$$f'(x) \;\leqslant\; (1+\delta)\,\mathrm{e}^M\, f(x) + \mathrm{e}^M / \delta.$$

Proof. Let $k_0 = k_0(x) = \lfloor (1+\delta)\,\mathrm{e}^M\, x \rfloor$. We decompose

$$f'(x) = \sum_{k=1}^{\infty} c_k k x^{k-1} = \sum_{k=1}^{k_0} c_k k x^{k-1} + R(x).$$

By our assumptions,

$$\begin{aligned} R(x) = \sum_{k=k_0+1}^{\infty} c_k k x^{k-1} \ &\leqslant\ \mathrm{e}^M \sum_{k=k_0+1}^{\infty} \left(\frac{x\,\mathrm{e}^M}{k} \right)^{k-1} \\ &\leqslant\ \mathrm{e}^M \sum_{k=k_0+1}^{\infty} \left(\frac{1}{1+\delta} \right)^{k} \ \leqslant\ \mathrm{e}^M /\delta. \end{aligned}$$

On the other hand, for the terms up to k_0 we have $k \leqslant k_0 \leqslant (1+\delta)x\,\mathrm{e}^M$, and thus

$$\sum_{k=1}^{k_0} c_k k x^{k-1} \ \leqslant\ (1+\delta)\,\mathrm{e}^M \sum_{k=0}^{k_0} c_k x^k \ \leqslant\ (1+\delta)\,\mathrm{e}^M f(x).$$

This completes the proof. □

Let us define the functions

$$I_\beta(z) = \sum_{j=1}^{\infty} j^\beta \theta_j z^j$$

for $\beta \in \mathbb{R}$. $\phi(z) := I_{-1}(z)$ plays a special role, since the generating function of (h_n) is given by $G_h(z) = \exp(\phi(z))$. All I_β are analytic by the first assumption in (4.1), monotone increasing and positive on $\{z > 0\}$ together with all their derivatives, and $I_{\beta+1}(z) = z I'_\beta(z)$. Lemma 4.2 implies that for each $\beta > 0$, there exists C such that for all $z \geqslant 0$, we have

$$I'_\beta(z) \ \leqslant\ C I_\beta(z). \tag{4.3}$$

Recall that $r_n = I_0^{-1}(n)$, where I_0^{-1} denote the inverse function.

Proposition 4.5. *We have*

$$h_n = \frac{r_n^{-n}}{\sqrt{2\pi I_1(r_n)}}\,\mathrm{e}^{\phi(r_n)}\,(1+o(1)).$$

Proof. The condition (4.1) on Taylor coefficients implies that $I_0(z) < \tilde{D}\exp(Cz)$. Then

$$r_n \ \geqslant\ c \log n \tag{4.4}$$

for some $c > 0$. On the other hand, r_n diverges more slowly than $n^{1/4}$ since $I_0(x)$ diverges faster than x^4 by (4.1).

For the saddle point method, we use Cauchy's formula and we obtain

$$\begin{aligned} h_n &= \frac{1}{2\pi r^n} \int_{-\pi}^{\pi} \mathrm{e}^{\phi(r\,\mathrm{e}^{\mathrm{i}\gamma}) - n\mathrm{i}\gamma}\,\mathrm{d}\gamma \\ &= \frac{\mathrm{e}^{\phi(r)}}{2\pi r^n} \Big[\int_{-\gamma_0}^{\gamma_0} \mathrm{e}^{\phi(r\,\mathrm{e}^{\mathrm{i}\gamma}) - \phi(r) - n\mathrm{i}\gamma}\,\mathrm{d}\gamma + 2\int_{\gamma_0}^{\pi} \mathrm{e}^{\phi(r\,\mathrm{e}^{\mathrm{i}\gamma}) - \phi(r) - n\mathrm{i}\gamma}\,\mathrm{d}\gamma \Big], \end{aligned} \tag{4.5}$$

for any $r > 0$ and any $0 < \gamma_0 < \pi$. We choose the $r = r_n$ defined by Eq. (4.2) since it is the minimum point of $r^{-n}\,\mathrm{e}^{\phi(r)}$, and $\gamma_0 = \gamma_0(n) = r_n^{-(1+\delta)}$ for some $0 < \delta < 1/2$. The leading order of the first term above can be found by expanding $\phi(z) - n\log z$ around $\gamma = 0$. We have

$$\phi(r_n\,\mathrm{e}^{\mathrm{i}\gamma}) - \phi(r_n) - n\mathrm{i}\gamma = \sum_{j \geqslant 1} \frac{\theta_j}{j} r_n^j \big(\mathrm{e}^{\mathrm{i}j\gamma} - 1 - \mathrm{i}j\gamma \big). \tag{4.6}$$

Expanding $\mathrm{e}^{\mathrm{i}j\gamma} - 1 - \mathrm{i}j\gamma = -\frac{1}{2}j^2\gamma^2 + R(j\gamma)$ with $|R(j\gamma)| \leqslant \frac{1}{3!}(j\gamma)^3$, we get

$$\phi(r_n \mathrm{e}^{\mathrm{i}\gamma}) - \phi(r_n) - n\mathrm{i}\gamma = -\tfrac{1}{2}\gamma^2 \sum_{j \geqslant 1} j\theta_j r_n^j + A(\gamma) = -\tfrac{1}{2}\gamma^2 I_1(r_n) + A(\gamma), \tag{4.7}$$

with

$$|A(\gamma)| \leqslant \frac{\gamma_0^3}{3!} \sum_{j \geqslant 1} j^2 \theta_j r_n^j = \frac{\gamma^2}{r_n^{1+\delta} 3!} I_2(r_n) \tag{4.8}$$

for all $\gamma \leqslant \gamma_0$. Now by (4.3), we have $I_2(r_n) \leqslant C r_n I_1(r_n)$. Thus, as $n \to \infty$, the term $A(\gamma)$ is negligible compared to $\gamma^2 I_1(r_n)$ in the first integral, which is therefore given by

$$\int_{-\gamma_0}^{\gamma_0} \mathrm{e}^{-\frac{1}{2}\gamma^2 I_1(r_n)(1+o(1))} \,\mathrm{d}\gamma = \frac{1}{\sqrt{I_1(r_n)}} \int_{-\gamma_0\sqrt{I_1(r_n)}}^{\gamma_0\sqrt{I_1(r_n)}} \mathrm{e}^{-\frac{1}{2}\xi^2(1+o(1))} \,\mathrm{d}\xi = \sqrt{\frac{2\pi}{I_1(r_n)}}(1+o(1)). \tag{4.9}$$

The last equality is justified by the fact that $\gamma_0(n) I_1(r_n) \geqslant r_n^{-1-\delta} I_0(r_n) \geqslant r_n^{-2} n$, which diverges as $n \to \infty$.

We now turn to the second term in (4.5). We want to show that it is negligible, and we estimate it by replacing the integral by π times the maximum of the integrand. In view of (4.9) it is enough to show that

$$\lim_{n\to\infty} \tfrac{1}{2} \log I_1(r_n) - \mathrm{Re}(\phi(r_n) - \phi(r_n \mathrm{e}^{\mathrm{i}\gamma})) = -\infty \tag{4.10}$$

for all $\gamma \in [\gamma_0, \pi]$. For the first term, we have $\log I_1(r_n) \leqslant \log(C r_n I_0(r_n)) \leqslant \tilde{C} \log n$. For the second term, we have

$$\begin{aligned} \mathrm{Re}(\phi(r_n) - \phi(r_n \mathrm{e}^{\mathrm{i}\gamma})) &= \sum_{j \geqslant 1} \frac{1}{j} \theta_j r_n^j (1 - \cos(\gamma j)) \\ &\geqslant \frac{\theta_{j_1}}{j_1} r_n^{j_1} (1 - \cos(\gamma j_1)) + \frac{\theta_{j_2}}{j_2} r_n^{j_2} (1 - \cos(\gamma j_2)), \end{aligned} \tag{4.11}$$

where j_1 and j_2 are picked according to (4.1). The right side is zero at $\gamma = 0$, and it is strictly positive when $\gamma \in (0, \pi]$ (j_1 and j_2 are coprime); so its minimum is attained at γ_0 when n is sufficiently large (recall that $\gamma_0 \to 0$ when $n \to \infty$). Expanding the cosine, we get

$$\mathrm{Re}(\phi(r_n) - \phi(r_n \mathrm{e}^{\mathrm{i}\gamma})) \geqslant c' r_n^4 \gamma_0^2 = c' r_n^{2-2\delta} \geqslant cc' (\log n)^{2-2\delta}. \tag{4.12}$$

This dominates the first term of (4.10) since $\delta < 1/2$, and this completes the proof. □

Proof of Theorem 4.1. Clearly,

$$P(\max_i \ell_i > \delta n) \leqslant n P(\ell_1 > \delta n). \tag{4.13}$$

We have $I_1(r_n) \leqslant C^2 r_n^2 \phi(r_n)$ by (4.3), and thus Proposition 4.5 gives $h_n \geqslant C' r_n^{-n-1}$ for n large enough. Since all the h_{n-j}'s are clearly bounded by some $D > 0$, we have by (2.3)

$$n P(\ell_1 > \delta n) \leqslant D r_n^{n+1} \sum_{j=\delta n}^{n} \Big(\frac{\mathrm{e}^M}{j} \Big)^j \leqslant D r_n^{n+1} n \Big(\frac{\mathrm{e}^M}{\delta n} \Big)^{\delta n} \leqslant D n \Big(\frac{\mathrm{e}^M r_n^{2/\delta}}{\delta n} \Big)^{\delta n}. \tag{4.14}$$

The statement is trivial (and seen directly from (2.3)) if only finitely many θ_j are nonzero; thus we may assume there are infinitely many nonzero θ_j. Then $I_0(z)$ grows faster at infinity than any power of z, and r_n diverges more slowly than any power of n. The last bracket is less than 1 for n large enough, so that the right side vanishes in the limit $n \to \infty$. □

In order to make more precise statements about the length of typical cycles, we need a better control over the terms appearing in (2.3). By the previous result it suffices to consider the case where j is not too close to n.

Proposition 4.6. *For each $\delta > 0$ there exists C_δ such that, for all $n \in \mathbb{N}$ and all $j < (1-\delta)n$, we have*

$$\frac{h_{n-j}}{h_n} \leqslant C_\delta r_n^{j+1/2}.$$

Proof. By Proposition 4.5 we have

$$\begin{aligned}\frac{h_{n-j}}{h_n} &\approx r_n^j \Big(\tfrac{r_n}{r_{n-j}}\Big)^{n-j} \Big(\tfrac{I_1(r_n)}{I_1(r_{n-j})}\Big)^{1/2} \, \mathrm{e}^{\phi(r_{n-j})-\phi(r_n)} \\ &= r_n^j \exp\Big(- \big(\phi(r_n) - \phi(r_{n-j}) - (n-j)(\ln(r_n) - \ln(r_{n-j}))\big)\Big) \Big(\tfrac{I_1(r_n)}{I_1(r_{n-j})}\Big)^{1/2} = \qquad (4.15) \\ &= r_n^j \exp\Big(- \big(\phi(r_n) - \phi(r_{n-j}) - \phi'(r_{n-j}) r_{n-j} \ln(\tfrac{r_n}{r_{n-j}})\big)\Big) \Big(\tfrac{I_1(r_n)}{I_1(r_{n-j})}\Big)^{1/2},\end{aligned}$$

when both n and $n-j$ are large. Put $r_{n-j} = x$ and $r_n = x+u$. Since $n \mapsto r_n$ is increasing, we have $u > 0$. The exponent above then has the form

$$\phi(x+u) - \phi(x) - x\phi'(x)\ln(\tfrac{x+u}{x}) = (\phi(x+u) - \phi(x) - \phi'(x)u) + \phi'(x)(u - x\ln(\tfrac{x+u}{x})). \quad (4.16)$$

The first bracket in the right side is greater than $\frac12 u^2 \phi''(x)$, since all derivatives of ϕ are positive on $\mathbb{R}^+$. The second bracket is always positive. Thus for all $n \in \mathbb{N}$ and all $j \leqslant (1-\delta)n$, there exists $C'_\delta > 0$ such that

$$\frac{h_{n-j}}{h_n} \leqslant C'_\delta r_n^j \, \mathrm{e}^{-\frac12 (r_n - r_{n-j})^2 \phi''(r_{n-j})} \Big(\tfrac{I_1(r_n)}{I_1(r_{n-j})}\Big)^{1/2} \quad (4.17)$$

By (4.3), $I_1(x) = xI_0'(x) \leqslant CxI_0(x)$. We also have $I_0(x) \leqslant I_1(x)$. Since $I_0(r_n) = n$, we get

$$\frac{I_1(r_n)}{I_1(r_{n-j})} \leqslant C r_n \frac{n}{n-j} \leqslant \frac{C}{\delta} r_n. \quad (4.18)$$

This proves the claim. ☐

Proof of Theorem 4.2. The claims follows immediately from (2.3) and Proposition 4.6. ☐

Proof of Theorem 4.3. Let $m = \log n / \log r_n - \frac34$. We use Eq. (2.3), bounding θ_j by a constant and using Proposition 4.6 for the ratio of normalization factors. Since r_n diverges, we have

$$P(\ell_1 \leqslant m) \leqslant \tfrac{C}{n} \sum_{j=1}^m r_n^{j+1/2} = \tfrac{C}{n} r_n^{3/2} \frac{r_n^m - 1}{r_n - 1} \leqslant \tfrac{C'}{n} r_n^{m+1/2}, \quad (4.19)$$

if n is large enough. The right side is equal to $C' r_n^{-1/4}$ and it vanishes in the limit $n \to \infty$. ☐

4.3. **An explicit example.** In this subsection, we treat explicitly the case $\alpha_j = \alpha(j) = j^\gamma$ with $\gamma > 1$, as an example of application of the previous general results. We first observe that the assumptions (4.1) are trivially satisfied, so that the general results in this section apply.

The main result of this subsection is that typical cycles are of size $(\frac{1}{\gamma-1}\log n)^{1/\gamma}$, to leading order.

Theorem 4.7. *Let $\alpha_j = j^\gamma$, with $\gamma > 1$. Then*

$$\frac{\ell_1}{(\frac{1}{\gamma-1}\log_n)^{1/\gamma}} \to 1 \quad (4.20)$$

in probability.

Let us define

$$\Delta(j) = \alpha(j) - \alpha(j_{\max}) - (j - j_{\max}) \log r_n. \quad (4.21)$$

The proof of Theorem 4.7 follows from two simple technical estimates.

Lemma 4.8. *Let $j_{\max} \in \mathbb{R}$ be such that $\alpha'(j_{\max}) = \log r_n$.*

(a) *Assume that* $\gamma \geqslant 2$. *Then, for all* $j \geqslant 1$, *there exists* $c = c(\gamma) > 0$ *such that*

$$\Delta(j) \geqslant c\alpha''(j_{\max})(j - j_{\max})^2. \tag{4.22}$$

(When $j \geqslant j_{\max}$, *one can choose* $c = \frac{1}{2}$.*)*

(b) *Assume that* $\gamma \in (1,2)$. *Then, for all* $1 \leqslant j \leqslant 2j_{\max}$, *there exists* $c = c(\gamma) > 0$ *such that*

$$\Delta(j) \geqslant c\alpha''(j_{\max})(j - j_{\max})^2. \tag{4.23}$$

(When $j \leqslant j_{\max}$, *one can choose* $c = \frac{1}{2}$.*) Moreover, for all* $j > 2j_{\max}$, *there exists* $c = c(\gamma) > 0$ *such that*

$$\Delta(j) \geqslant cj^\gamma. \tag{4.24}$$

Proof. We start with the case $\gamma \geqslant 2$. First of all, since $j_{\max} = (\alpha')^{-1}(\log r_n)$, we have, for any $j > j_{\max}$,

$$\begin{aligned}\Delta(j) &= \alpha(j) - \alpha(j_{\max}) - (j - j_{\max})\log r_n = \alpha(j) - \alpha(j_{\max}) - (j - j_{\max})\alpha'(j_{\max}) \\ &= \int_{j_{\max}}^{j} \mathrm{d}s \int_{j_{\max}}^{s} \alpha''(t)\mathrm{d}t \geqslant \tfrac{1}{2}\alpha''(j_{\max})(j - j_{\max})^2,\end{aligned} \tag{4.25}$$

since α'' is an increasing function. Similarly, we have, for any $\frac{1}{2}j_{\max} \leqslant j < j_{\max}$,

$$\Delta(j) = \int_{j}^{j_{\max}} \mathrm{d}s \int_{s}^{j_{\max}} \alpha''(t)\mathrm{d}t \geqslant \tfrac{1}{2}\alpha''(\tfrac{1}{2}j_{\max})(j - j_{\max})^2 = 2^{1-\gamma}\alpha''(j_{\max})(j - j_{\max})^2. \tag{4.26}$$

Finally, for $0 \leqslant j < \frac{1}{2}j_{\max}$, we use

$$\begin{aligned}\Delta(j) = \int_{j}^{j_{\max}} \mathrm{d}s \int_{s}^{j_{\max}} \alpha''(t)\mathrm{d}t &\geqslant \int_{j_{\max}/2}^{j_{\max}} \mathrm{d}s \int_{s}^{j_{\max}} \alpha''(t)\mathrm{d}t \\ &\geqslant \tfrac{1}{2}\alpha''(\tfrac{1}{2}j_{\max})\tfrac{1}{4}j_{\max}^2 \geqslant 2^{-\gamma-1}\alpha''(j_{\max})(j - j_{\max})^2.\end{aligned} \tag{4.27}$$

Let us now turn to the case $\gamma \in (1,2)$. The proof is completely similar. When $j \leqslant j_{\max}$, we use (observe that α'' is a decreasing function now)

$$\Delta(j) = \int_{j}^{j_{\max}} \mathrm{d}s \int_{s}^{j_{\max}} \alpha''(t)\mathrm{d}t \geqslant \tfrac{1}{2}\alpha''(j_{\max})(j - j_{\max})^2. \tag{4.28}$$

When $j_{\max} < j \leqslant 2j_{\max}$, we use

$$\Delta(j) = \int_{j_{\max}}^{j} \mathrm{d}s \int_{j_{\max}}^{s} \alpha''(t)\mathrm{d}t \geqslant \tfrac{1}{2}\alpha''(2j_{\max})(j - j_{\max})^2 = 2^{\gamma-3}\alpha''(j_{\max})(j - j_{\max})^2. \tag{4.29}$$

Finally, when $j > 2j_{\max}$, we have

$$\Delta(j) = \int_{j_{\max}}^{j} \mathrm{d}s \int_{j_{\max}}^{s} \alpha''(t)\mathrm{d}t \geqslant \tfrac{1}{2}\alpha''(j)(j - j_{\max})^2 \geqslant \tfrac{1}{8}\alpha''(j)j^2 = \tfrac{1}{8}\gamma(\gamma-1)j^\gamma. \tag{4.30}$$

□

Corollary 4.9. *For any* $\gamma > 1$, *we have, as* $n \to \infty$,

$$j_{\max} = (\frac{1}{\gamma-1}\log n)^{1/\gamma}\,(1 + o(1)), \tag{4.31}$$

$$\log r_n = \alpha'(j_{\max}) = \gamma(\frac{1}{\gamma-1}\log n)^{(\gamma-1)/\gamma}\,(1 + o(1)), \tag{4.32}$$

$$\mathrm{e}^{-\alpha(j_{\max})}\,r_n^{j_{\max}} = n^{1+o(1)}. \tag{4.33}$$

Proof. We start with the case $\gamma \geqslant 2$. Using the previous lemma, it immediately follows that

$$I_0(r_n) = \sum_{j \geqslant 1} e^{-\alpha(j)} r_n^j \leqslant e^{-\alpha(j_{\max})} r_n^{j_{\max}} \sum_{j \geqslant 1} e^{-c\alpha''(j_{\max})(j-j_{\max})^2} \leqslant C_1\, e^{-\alpha(j_{\max})} r_n^{j_{\max}} \tag{4.34}$$

Since, for $j < j_{\max}$, $\Delta(j) \leqslant \frac{1}{2}\alpha''(j_{\max})(j-j_{\max})^2$, we also have

$$I_0(r_n) \geqslant e^{-\alpha(\lfloor j_{\max}\rfloor)} r_n^{\lfloor j_{\max}\rfloor} \geqslant e^{-\frac{1}{2}\alpha''(j_{\max})}\, e^{-\alpha(j_{\max})} r_n^{j_{\max}}. \tag{4.35}$$

Using the relation $I_0(r_n) = n$, (4.34) and (4.35) immediately imply the claimed asymptotics.

Let us now turn to the case $\gamma \in (1,2)$. The lemma implies that

$$I_0(r_n) = e^{-\alpha(j_{\max})} r_n^{j_{\max}} \sum_{j \geqslant 1} e^{-\Delta(j)} \leqslant C_2\, e^{-\alpha(j_{\max})} r_n^{j_{\max}} \big\{ \alpha''(j_{\max})^{-1/2} + \sum_{j > 2j_{\max}} e^{-cj^\gamma} \big\}. \tag{4.36}$$

Since $j_{\max} \nearrow \infty$ as $n \to \infty$, we see that $\sum_{j>2j_{\max}} e^{-cj^\gamma} \ll \alpha''(j_{\max})^{-1/2}$ and thus that, for large n,

$$I_0(r_n) \leqslant C_3\, \alpha''(j_{\max})^{-1/2}\, e^{-\alpha(j_{\max})} r_n^{j_{\max}}. \tag{4.37}$$

As above, we also have

$$I_0(r_n) \geqslant e^{-\alpha(\lceil j_{\max}\rceil)} r_n^{\lceil j_{\max}\rceil} \geqslant e^{-\frac{1}{2}\alpha''(j_{\max})}\, e^{-\alpha(j_{\max})} r_n^{j_{\max}} \geqslant C_4\, e^{-\alpha(j_{\max})} r_n^{j_{\max}}. \tag{4.38}$$

The claimed asymptotics follow as before. □

Proof of Theorem 4.7. Let $\epsilon > 0$. It is sufficient to check that Theorem 4.2 applies with $a(n) = (1-\epsilon) j_{\max}$ and $b(n) = (1+\epsilon) j_{\max}$. It follows from Lemma 4.8 and Corollary 4.9 that

$$\frac{1}{n} \sum_{j=b(n)}^{\infty} e^{-\alpha(j)} r_n^{j+1/2} \leqslant n^{o(1)} \sum_{j=b(n)}^{\infty} e^{-c\alpha''(j_{\max})(j-j_{\max})^2}, \tag{4.39}$$

which goes to 0 as $n \to \infty$, since

$$e^{-c\alpha''(j_{\max})(b(n)-j_{\max})^2} = n^{-c\epsilon^2\gamma(1+o(1))}. \tag{4.40}$$

Similarly,

$$\frac{1}{n} \sum_{j=1}^{a(n)} e^{-\alpha(j)} r_n^{j+1/2} \leqslant n^{o(1)} \sum_{j=1}^{a(n)} e^{-c\alpha''(j_{\max})(j-j_{\max})^2} \leqslant n^{o(1)} e^{-c\alpha''(j_{\max}) j_{\max}^2 \epsilon^2}, \tag{4.41}$$

which again goes to 0 as $n \to \infty$. □

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