Submitted to the Annals of Probability
arXiv: math.PR/1409.5199

A QUANTITATIVE BURTON-KEANE ESTIMATE UNDER STRONG FKG CONDITION

By Hugo Duminil-Copin Dmitry Ioffe and Yvan Velenik

University of Geneva and Techion

We consider translationally-invariant percolation models on $\mathbb{Z}^d$ satisfying the finite energy and the FKG properties. We provide explicit upper bounds on the probability of having two distinct clusters going from the endpoints of an edge to distance n (this corresponds to a finite size version of the celebrated Burton-Keane [8] argument proving uniqueness of the infinite-cluster). The proof is based on the generalization of a reverse Poincaré inequality proved in [10]. As a consequence, we obtain upper bounds on the probability of the so-called four-arm event for planar random-cluster models with cluster-weight $q \geq 1$.

1. Introduction and main result. This article is devoted to deriving a weak reverse Poincaré-type inequality for percolation models satisfying strong association and finite-energy properties, and examining some of its consequences. Let Λ be a finite set and consider a percolation model on Λ, *i.e.*, a random binary field $\omega \in \{0,1\}^\Lambda$. The value of the field at $i \in \Lambda$ is denoted by ω_i, and the field on the complementary set $\Lambda \setminus i$ is denoted by ω^i. The law of ω on $\{0,1\}^\Lambda$ is denoted by $\mathbb{P}$. There is a standard partial order $\prec$ on $\{0,1\}^\Lambda$, and a function f on $\{0,1\}^\Lambda$ is said to be non-decreasing if $f(\omega) \leq f(\psi)$ whenever $\omega \prec \psi$. An event $A \subset \{0,1\}^\Lambda$ is said to be non-decreasing if its indicator function $\mathbf{1}_A$ is.

We will be interested in percolation models satisfying the following two conditions:

(FE) *Finite Energy:* There exists $c_{\mathsf{FE}} > 0$ such that, for any $i \in \Lambda$ and $\omega \in \{0,1\}^\Lambda$,

$$\mathbb{P}\left(\omega_i = 1 \,|\, \omega_j, j \neq i\right) \in \left(c_{\mathsf{FE}}, 1 - c_{\mathsf{FE}}\right). \tag{1.1}$$

(FKG) *Strong Positive Association:* For any $i \in \Lambda$ and $\xi \prec \psi$ in $\{0,1\}^{\Lambda\setminus\{i\}}$,

$$\mathbb{P}\left(\omega_i = 1 \,|\, \omega_j = \xi_j, j \neq i\right) \leq \mathbb{P}\left(\omega_i = 1 \,|\, \omega_j = \psi_j, j \neq i\right). \tag{1.2}$$

Recall that (1.2) is equivalent to the so-called FKG lattice condition; see [18, Theorem (2.24)]. For a discussion of the relation between this condition and the weaker condition of positive association (characterized by the FKG inequality), we refer the reader to [2].

Before stating the theorem, let us define two more objects. For a configuration ω and $i \in \Lambda$, define the configurations $\omega^i \times 1$ and $\omega^i \times 0$ obtained from ω by changing the state i to 1 and 0 respectively. Define

$$\nabla_i f(\omega) \stackrel{\text{def}}{=} f(\omega^i \times 1) - f(\omega^i \times 0).$$

MSC 2010 subject classifications: Primary 60K35; secondary 82B20, 82B43

Keywords and phrases: reverse Poincaré inequality, dependent percolation, FK percolation, random cluster model, four-arms event, Burton-Keane theorem, negative association

Also, for a non-decreasing event A, define the set $\mathrm{Piv}_i(A)$ of configurations ω such that $\omega^i \times 1 \in A$ and $\omega^i \times 0 \notin A$. Equivalently, $\mathrm{Piv}_i(A) \stackrel{\text{def}}{=} \{\omega : \nabla_i \mathbf{1}_A(\omega) = 1\}$.

THEOREM 1.1. *Consider a percolation model on a finite set* Λ *satisfying* (FE) *and* (FKG). *Then, there exists* $c_{\mathsf{P}} = c_{\mathsf{P}}(c_{\mathsf{FE}}) > 0$ *such that, for any non-decreasing function* $f : \{0,1\}^\Lambda \longrightarrow \mathbb{R}$,

$$\mathbb{V}\mathrm{ar}\big(f(\omega)\big) \geq c_{\mathsf{P}} \sum_{i \in \Lambda} \big(\mathbb{E}[\nabla_i f]\big)^2. \tag{1.3}$$

In particular, for any non-decreasing event A

$$\mathbb{P}(A)\big(1 - \mathbb{P}(A)\big) \geq c_{\mathsf{P}} \sum_{i \in \Lambda} \mathbb{P}\big(\mathrm{Piv}_i(A)\big)^2. \tag{1.4}$$

We emphasize that the constant c_{P} is not depending on the size of Λ. One may think of this theorem as a weak reverse Poincaré inequality. Indeed, when $\{\omega_i : i \in \Lambda\}$ are independent, the standard discrete Poincaré inequality, see for instance [16], states that

$$\mathbb{P}(A)\big(1 - \mathbb{P}(A)\big) \leq \frac{1}{4} \sum_{i \in \Lambda} \mathbb{P}\big(\mathrm{Piv}_i(A)\big). \tag{1.5}$$

In the independent case, $\mathbb{P}\big(\mathrm{Piv}_i(A)\big) = \mathbb{P}(A|\omega_i = 1) - \mathbb{P}(A|\omega_i = 0) \stackrel{\text{def}}{=} I_A(i)$ is the so-called *influence* of i on A. Let us also mention that some inequalities for influences in models with dependency have been obtained by encoding strongly positively-associated measures in terms of the Lebesgue measure on the hypercube $[0,1]^\Lambda$. Nevertheless, these inequalities bound influences from below; see [18, Theorem (2.28)] and [17]. They are therefore not directly relevant here.

Inequality (1.3) was derived in the independent Bernoulli case in [10]. The latter work was one of the motivations for our study of dependent models here.

Our proof of (1.3) hinges on the following simple but apparently new observation, which may be of independent interest. Fix $0 < p < 1$; given a realization $\omega \in \{0,1\}^\Lambda$ of the percolation model, we construct a field $\sigma \in \{0,1\}^\Lambda$ of (conditionally on ω) independent random variables, σ_i taking value 1 with probability

$$\widehat{\mathbb{P}}\left(\sigma_i = 1|\omega\right) = \frac{p \cdot \omega_i}{\mathbb{P}(\omega_i = 1|\omega^i)},$$

for each $i \in \Lambda$. Then, the distribution of the field σ, once integrated over ω, enjoys a form of *negative* dependence. Namely, for any $i \in \Lambda$ and for any non-decreasing functions $f : \{0,1\}^{\Lambda \setminus \{i\}} \to \mathbb{R}$ and $g : \{0,1\} \to \mathbb{R}$,

$$\widehat{\mathbb{E}}\big[f(\sigma^i) g(\sigma_i)\big] \leq \widehat{\mathbb{E}}\big[f(\sigma^i)\big] \widehat{\mathbb{E}}\big[g(\sigma_i)\big].$$

This is proved in Theorem 3.1 below, together with additional relevant properties.

2. Applications.

2.1. *Some examples of percolation models.* Our applications to percolation models will be mostly dealing with connectivity properties of the graph induced by $\{i \in \Lambda : \omega_i = 1\}$. For simplicity, we will focus on *bond percolation* models – similar results would also hold for so-called *site percolation* models. The set Λ is now the edge-set E_G of a finite graph $G = (V_G, E_G)$. The edge i is said to be *open* (resp. *closed*) if $\omega_i = 1$ (resp. $\omega_i = 0$). The configuration ω can therefore be seen as a subgraph of G with vertex set V_G and edge set composed of open edges. Two vertices x and y are said to be *connected* if they belong to the same connected component of ω (we denote the event that x and y are connected by $x \longleftrightarrow y$). Connected components of ω are called *clusters*.

The most classical example of a percolation model is provided by Bernoulli percolation. This model was introduced by Broadbent and Hammersley in the fifties [7]. In this model, each edge i is open with probability p, and closed with probability $1-p$ independently of the states of the other edges. For general background on Bernoulli percolation, we refer the reader to the books [19, 20].

More generally, the states of edges may not be independent. In such case, we speak of a *dependent* percolation model. Among classical examples, we mention the *random-cluster model* (or *Fortuin-Kasteleyn percolation*) introduced by Fortuin and Kasteleyn in [14]. Let $o(\omega)$ be the number of open edges in ω, $c(\omega)$ be the number of closed edges and $k(\omega)$ be the number of clusters. The probability measure $\phi_{p,q,G}$ of the random-cluster model on a finite graph G with parameters $p \in [0,1]$ and $q > 0$ is defined by

$$\phi_{p,q,G}(\{\omega\}) \overset{\text{def}}{=} \frac{p^{o(\omega)}(1-p)^{c(\omega)}q^{k(\omega)}}{Z_{p,q,G}}$$

for every configuration ω on G, where $Z_{p,q,G}$ is a normalizing constant referred to as the *partition function*.

The random-cluster models satisfy (FE) for $q > 0$ and (FKG) for any $q \geq 1$. For this reason, random-cluster models are good examples of models satisfying our two assumptions, but they are not the only ones. The uniform spanning tree ($\mathbb{P}$ is simply the uniform measure on trees containing every vertices of G) is a typical example of a model not satisfying (FE).

Our applications provide upper bounds on the probability of having two distinct clusters from the inner to the outer boundaries of annuli. In two dimensions because of dual connections, the usual name would be four-arm type events, namely probabilities of having two long disjoint clusters attached to two vertices of a given edge.

In order to deduce such estimates for individual bonds from (1.3) or (1.4), we need to assume some form of translation invariance.

2.2. *First application.* To give a simple illustration of how Theorem 1.1 might be put to work, let us mention the following result. Consider the d-dimensional torus $\mathbb{T}_n^{(d)}$ of size $2n+1$ and denote by $\tilde{A}_2^e(n)$ the event that the edge e is pivotal for the existence of an open circuit of non-trivial homotopy in $\mathbb{T}_n^{(d)}$.

PROPOSITION 2.1. *Let $d \geq 2$, there exists $c_{\tilde{\mathsf{A}}_2} = c_{\tilde{\mathsf{A}}_2}(c_{\mathsf{FE}}, d) > 0$ such that, for every $n \geq 1$ and any edge e of $\mathbb{T}_n^{(d)}$,*

$$\mathbb{P}[\tilde{A}_2^e(n)] \leq \frac{c_{\tilde{\mathsf{A}}_2}}{n^{d/2}},$$

where $\mathbb{P}$ is the law of an arbitrary translation invariant percolation model on $\mathbb{T}_n^{(d)}$ satisfying (FE) *and* (FKG)*.*

Note that $\tilde{A}_2^e(n)$ is basically the event that there are two disjoint clusters emanating from the end-points of e and going to distance n, with some additional topological requirement on the macroscopic structure of these clusters (among these requirements, they should join into a cluster of $\mathbb{T}_n^{(d)}$). This additional condition is not so nice, and it does not directly apply to models on $\mathbb{Z}^d$. We would like to replace this by the event that there are two disjoint clusters going from the end-points of *some fixed bond* e to distance n. Let $A_2^e(n)$ be the event that there are two disjoint clusters going from the endpoints of the edge e to distance n.

In the next two applications we explain two ways of deriving upper bounds on $\mathbb{P}\left(A_2^e(n)\right)$.

2.3. *A quantitative Burton-Keane argument.* Our second application is an extension of the results of [10] to arbitrary bond percolation models $\mathbb{P}$ on $\mathbb{Z}^d$ which satisfy (FE), (FKG) and are invariant under translations:

(TI) The measure $\mathbb{P}$ is invariant under shift $\tau_x : \{0,1\}^{\mathbb{Z}^d} \to \{0,1\}^{\mathbb{Z}^d}$ defined by

$$\tau_x(\omega)_{(u,v)} \stackrel{\text{def}}{=} \omega_{(u+x,v+x)} \quad \forall u, v \in \mathbb{Z}^d.$$

THEOREM 2.1. *Consider a percolation model on $\mathbb{Z}^d$ satisfying* (FE), (FKG) *and* (TI)*. Then there exists $c_{\mathsf{BK}} > 0$ such that, for any edge e,*

$$\mathbb{P}[A_2^e(n)] \leq \frac{c_{\mathsf{BK}}}{(\log n)^{d/2}}. \tag{2.1}$$

As we have already mentioned a quantitative Burton-Keane argument leading to (2.1) for Bernoulli percolation-type models was developed in [10]. In the case of Bernoulli site percolation, polynomial order upper bounds on $\mathbb{P}[A_2^e(n)]$ were derived in the recent paper [9] via a clever refinement of techniques introduced by [4] and [15].

2.4. *Continuity of percolation probabilities away from critical points.* Consider a one-parametric family $\{\mathbb{P}_\alpha\}_{\alpha\in(a,b)}$ of bond or site strong-FKG percolation models on $\mathbb{Z}^d$. Define percolation probabilities

$$\theta(\alpha) \stackrel{\text{def}}{=} \mathbb{P}_\alpha\left(0 \leftrightarrow \infty\right). \tag{2.2}$$

Assume that the measures $\mathbb{P}_\alpha$ satisfy the finite energy condition (FE) uniformly over compact intervals of (a,b), and assume that $\theta > 0$ on (a,b). At last, assume that $\alpha \mapsto \mathbb{P}_\alpha$ is increasing (in the FKG-sense), that is, assume that $\mathbb{P}_\alpha$ is stochastically dominated by $\mathbb{P}_\beta$ whenever $a < \alpha \leq \beta < b$. We shall say that $\alpha \mapsto \mathbb{P}_\alpha$ is continuous at $\alpha_0 \in (a,b)$ if the map $\alpha \mapsto \mathbb{P}_\alpha(f)$ is continuous at α_0 for any local function f.

THEOREM 2.2. *Under the above conditions: $\alpha \mapsto \theta(\alpha)$ cannot have jumps at continuity points of $\alpha \mapsto \mathbb{P}_\alpha$.*

In the case of Bernoulli percolation, continuity comes for free and Theorem 2.2 implies continuity of percolation probabilities away from critical points, as it was originally proved in [4]. In the case of FK-percolation for the Ising model on $\mathbb{Z}^d$, proving continuity of measures seems to be on the same level of difficulty as proving continuity of percolation probabilities [6]. On the other hand, in view of [6] and [3], Theorem 2.2 does imply continuity of the site +-spin percolation away from critical inverse temperature for the latter.

2.5. *Spanning clusters and polynomial decay.* Proposition 2.1 and Theorem 2.1 are based on (1.4). Yet, (1.3) provides us with additional degrees of freedom: one can try various model-dependent monotone functions f.

Let $\mathbb{P}$ be a bond percolation measure on $\mathbb{Z}^d$. Consider the boxes $\Lambda_k \stackrel{\text{def}}{=} [-k,\ldots,k]^d$ and the annuli $A_{m,n} \stackrel{\text{def}}{=} \Lambda_n \setminus \Lambda_m$ for $0 < m < n$. Let $N = N_{m,n}$ be the number of distinct clusters of $\partial\Lambda_n$ *crossing* $A_{m,n}$ in the *restriction* of the percolation configuration to the bonds of Λ_n. In the sequel we shall use η for a percolation configuration on $\mathbb{Z}^d \setminus \Lambda_m$, ω for a percolation configuration on Λ_m, and $\eta \times \omega$ for the configuration obtained by merging the two previous configurations. For a given η the function $\omega \mapsto N(\eta \times \omega) \stackrel{\text{def}}{=} N^\eta(\omega)$ is decreasing. Hence, (1.3) implies that

$$\mathbb{V}\mathrm{ar}\left(N^\eta(\omega)\middle|\eta\right) \geq c_{\mathsf{P}} \sum_{e\in\mathcal{E}_{\Lambda_m}} \left(\mathbb{E}\left(\nabla_e N^\eta(\omega)\middle|\eta\right)\right)^2. \tag{2.3}$$

Above $\mathcal{E}_{\Lambda_m}$ is the set of nearest neighbor bonds of Λ_m. Note that, for any $e \in \mathcal{E}_{\Lambda_m}$,

$$-\nabla_e N^\eta_{m,n}(\omega) \geq \mathbf{1}_{A_2^e(2n)}(\eta\times\omega).$$

Therefore, we infer from (2.3) the following corollary.

THEOREM 2.3. *Consider a percolation model on $\mathbb{Z}^d$ satisfying* (FE)*,* (FKG) *and* (TI)*. Then, for any edge e,*

$$\mathbb{P}\left(A_2^e(2n)\right) \leq \sqrt{\frac{1}{c_{\mathsf{P}}(2m)^d}\mathbb{E}\left(\mathbb{V}\mathrm{ar}\left(N^\eta_{m,n}(\omega)\middle|\eta\right)\right)} \leq \sqrt{\frac{1}{c_{\mathsf{P}}(2m)^d}\mathbb{V}\mathrm{ar}\left(N_{m,n}\right)}, \tag{2.4}$$

for any $0 < m < n$.

Of course, (2.4) is useful only when one is able to control the number of crossing clusters of $A_{m.n}$, specifically $\mathbb{E}\left(\mathbb{V}\mathrm{ar}\left(N^\eta_{m,n}(\omega)\middle|\eta\right)\right)$. This requires work: a trivial upper bound of order $m^{2(d-1)}$ gives nothing even in two dimensions. Settling this in any dimension would be a feat even in the case of Bernoulli percolation, see [1]. For the moment it is not clear to us that a nice closed form bound can be obtained in the full generality suggested by (FE), (FKG) and (TI), even if one requires ergodicity instead of just translation invariance.

In the case of Bernoulli site percolation, the following bound was derived using very different methods based on independence (see [9]):

$$\mathbb{P}\left(A_2^e(2n)\right) \leq \frac{c\log n}{n^{d/2}}\,\mathbb{E}\left(\sqrt{N_{n,2n}}\right). \tag{2.5}$$

Unlike (2.4), (2.5) always gives a non-trivial polynomial decay, even if the roughest possible bound $N_{n,2n} \le cN^{d-1}$ is used.

2.6. *Four-arm event for critical planar random-cluster models with $q \ge 1$.* Using very recent results of [12, 13] for the random-cluster model on $\mathbb{Z}^2$, the distribution of the number of crossing clusters can be controlled, and the upper bound (2.4) implies the following refinement of Theorem 2.1, which is of the same order as the bound of Proposition 2.1:

THEOREM 2.4. *Let $d = 2$, $q \in [1, 4]$, there exists $c_{\mathsf{A}_2} = c_{\mathsf{A}_2}(p, q) > 0$ such that, for any edge e and every $n \ge 1$,*

$$\phi_{p,q,\mathbb{Z}^2}[A_2^e(n)] \le \frac{c_{\mathsf{A}_2}}{n},$$

where $\phi_{p,q,\mathbb{Z}^2}$ is the unique infinite volume random-cluster measure with edge-weight p and cluster-weight q (see Section 4.4 for a precise definition).

The proof is easy whenever $p \ne p_c$. In the critical case $p = p_c$, the proof is based on Russo-Seymour-Welsh (RSW) bounds obtained in [12, 13]. We give two arguments: one is based on the implied mixing properties of $\phi_{p,q,\mathbb{Z}^2}$ and on a subsequent reduction to Proposition 2.1. The second directly relies on RSW bounds to check that one can fix $\epsilon > 0$ such that $\left\{\phi_{p,q,\mathbb{Z}^2}\left(N^2_{\epsilon n,n}\right)\right\}$ is a bounded sequence. Then (2.3) applies.

Note that the phase transition is expected to be discontinuous for $q > 4$ (see the discussion in Section 4.4) and the probability of $A_2(n)$ should decay exponentially fast at every p.

3. Proof of Theorem 1.1.

We shall prove Theorem 1.1 with $c_{\mathsf{P}} = \frac{c_{\mathsf{FE}}{}^3}{(2-c_{\mathsf{FE}})^2}$.

From now on in this section, we fix a finite set Λ. Consider a percolation model on $\Omega \stackrel{\text{def}}{=} \{0,1\}^\Lambda$ satisfying (FE) and (FKG) and let $\mathbb{P}$ be the law of the random configuration ω. Furthermore, for $I \subset \Lambda$, we define $\omega_I \stackrel{\text{def}}{=} \{\omega_i : i \in I\}$ and $\omega^I \stackrel{\text{def}}{=} \{\omega_i : i \notin I\}$. To keep notation compatible, we set ω_i and ω^i when $I = \{i\}$.

Recall that $\omega^i \times 1$ and $\omega^i \times 0$ denote the configurations obtained from ω by setting the value of ω_i to 1 and 0 respectively.

To lighten the notation, we write $p = c_{\mathsf{FE}}/2$ for the rest of this section.

3.1. *A representation of fields satisfying* (FE) *and* (FKG). In order to prove Theorem 1.1 we introduce an auxiliary Bernoulli field $\sigma \in \{0,1\}^\Lambda$ and utilize the projection method of [10] with respect to σ-algebras generated by this auxiliary field. The efficiency of such approach hinges on the fact that σ_i-s happen to be negatively correlated in the sense specified in $\mathsf{P3}$ of Theorem 3.1 below.

DEFINITION 1. *Consider a probability space $(\widehat{\Omega}, \widehat{\mathbb{P}})$ containing $(\Omega, \mathbb{P})$ and an additional field $\sigma \in \{0,1\}^\Lambda$ which, conditionally on $\omega \in \Omega$, has independent entries satisfying, for every $i \in \Lambda$,*

$$\widehat{\mathbb{P}}(\sigma_i = 1|\omega) = \frac{p \cdot \omega_i}{\mathbb{P}(\omega_i = 1|\omega^i)}. \tag{3.1}$$

Note that by our choice of p, which is adjusted to the finite energy property (1.1), the right hand side of (3.1) always belongs to $[0,1]$.

We claim that σ enjoys the following set of properties:

THEOREM 3.1. *Let ω be a field satisfying* (FE) *and* (FKG)*. Then,*

- P1 *For each $i \in \Lambda$, if $\sigma_i = 1$, then $\omega_i = 1$.*
- P2 *For any $i \in \Lambda$ and any non-decreasing function $f : \Omega \to \mathbb{R}$, the conditional expectation $\widehat{\mathbb{E}}\left(f(\omega) \,|\, \sigma_i\right)$ is also non-decreasing.*
- P3 *For any $i \in \Lambda$ and for any non-decreasing functions $f : \{0,1\}^{\Lambda\setminus\{i\}} \to \mathbb{R}$ and $g : \{0,1\} \to \mathbb{R}$,*

$$\widehat{\mathbb{E}}\big[f(\sigma^i)g(\sigma_i)\big] \le \widehat{\mathbb{E}}\big[f(\sigma^i)\big]\widehat{\mathbb{E}}\big[g(\sigma_i)\big]. \tag{3.2}$$

- P4 *The family $\{\sigma_i - p : i \in \Lambda\}$ is free in $\mathbb{L}^2(\widehat{\Omega}, \widehat{\mathbb{P}})$.*

Property P3 provides a form of negative association. It is weaker than the usual form of negative association (which corresponds to the analogue of (3.2) with i and $\Lambda\setminus\{i\}$ replaced by arbitrary disjoint subsets $A, B \subset \Lambda$), but stronger than other related notions, such as totally negative dependence (see [11] for this and other forms of negative dependence).

PROOF. *Property* P1. The first property follows directly from the definition of σ.

Property P2. Let us first prove that, for each $i \in \Lambda$, σ_i is a Bernoulli random variable of parameter p, independent of ω^i. This follows from (3.1) and the following computation

$$\widehat{\mathbb{P}}\left(\sigma_i = 1; \omega^i\right) = \widehat{\mathbb{P}}\left(\sigma_i = 1; \omega_i = 1; \omega^i\right) = \frac{p}{\mathbb{P}\left(\omega_i = 1 \,|\, \omega^i\right)}\, \mathbb{P}\left(\omega_i = 1; \omega^i\right) = p\, \mathbb{P}\left(\omega^i\right).$$

Hence σ_i is indeed a Bernoulli random variable of parameter p (also σ_i and ω^i are independent). Now, let us simplify the notation by setting $f_i(\sigma_i) \stackrel{\text{def}}{=} \widehat{\mathbb{E}}\left(f(\omega) \,|\, \sigma_i\right)$. Then, using $\widehat{\mathbb{P}}(\sigma_i = 1) = p$ in the second equality below,

$$\begin{aligned}
(1-p)(f_i(1) - f_i(0)) &\stackrel{\text{def}}{=} (1-p)\left[\frac{\widehat{\mathbb{E}}\left(f(\omega)\mathbf{1}_{\sigma_i=1}\right)}{\widehat{\mathbb{P}}(\sigma_i = 1)} - \frac{\widehat{\mathbb{E}}\left(f(\omega)\mathbf{1}_{\sigma_i=0}\right)}{\widehat{\mathbb{P}}(\sigma_i = 0)}\right] \\
&= (\tfrac{1}{p} - 1)\widehat{\mathbb{E}}\left(f(\omega)\mathbf{1}_{\sigma_i=1}\right) - \widehat{\mathbb{E}}\left(f(\omega)\mathbf{1}_{\sigma_i=0}\right) \\
&\stackrel{(\mathsf{P1})}{=} \tfrac{1}{p}\widehat{\mathbb{E}}\left(f(\omega)\mathbf{1}_{\omega_i=1}\mathbf{1}_{\sigma_i=1}\right) - \mathbb{E}\left(f(\omega)\right) \\
&\stackrel{(3.1)}{=} \tfrac{1}{p}\mathbb{E}\Big(f(\omega)\mathbf{1}_{\omega_i=1}\frac{p}{\mathbb{P}\left(\omega_i = 1 \,|\, \omega^i\right)}\Big) - \mathbb{E}\left(f(\omega)\right) \\
&= \mathbb{E}\left(f(\omega^i \times 1)\right) - \mathbb{E}\left(f(\omega)\right) \stackrel{(\mathsf{FKG})}{\ge} 0.
\end{aligned} \tag{3.3}$$

Property P3. We wish to prove the negative association formula (3.2). Since g is a non-decreasing function of only one site, we only need to treat the case $g = \mathrm{id}$ (any non-decreasing function of one site is of the form $\alpha\,\mathrm{id} + \beta$ with $\alpha \ge 0$). For $\omega_I \in \{0,1\}^I$ and $\omega^I \in \{0,1\}^{\Lambda\setminus I}$, let $\omega_I \times \omega^I$ be the configuration in Ω coinciding with ω_I on I and ω^I on $\Lambda \setminus I$.

Claim. *For any subset $I \subset \Lambda$, the following happens: If $\omega_I \succ \widetilde{\omega}_I$, then, for any ω^I and for any non-decreasing function $f : \{0,1\}^{\Lambda\setminus I} \to \mathbb{R}$,*

$$\widehat{\mathbb{E}}\left(f(\sigma^I)\,|\,\omega_I \times \omega^I\right) \le \widehat{\mathbb{E}}\left(f(\sigma^I)\,|\,\widetilde{\omega}_I \times \omega^I\right). \tag{3.4}$$

Proof of the Claim. Under the conditional measure $\widehat{\mathbb{P}}(\cdot\,|\,\omega)$, the sequence σ is simply a collection of independent Bernoulli random variables with probabilities of success specified by (3.1). By (1.2),

$$\frac{p}{\mathbb{P}\big(\omega_i = 1\big|(\omega_I \times \omega^I)^i\big)} \le \frac{p}{\mathbb{P}\big(\omega_i = 1\big|(\widetilde{\omega}_I \times \omega^I)^i\big)},$$

for any $i \notin I$ and $\omega_I \succ \widetilde{\omega}_I$, a fact which implies that the random variable σ_I conditioned on $\omega_I \times \omega^I$ is stochastically dominated by the random variable σ_I conditioned on $\widetilde{\omega}_I \times \omega^I$. The claim follows by definition of stochastic domination. □

In particular, the claim yields that if f is a non-decreasing function of σ^i, then, for any i and any ω^i,

$$\widehat{\mathbb{E}}\left(f(\sigma^i)\,|\,\omega^i \times 1\right) \le \widehat{\mathbb{E}}\left(f(\sigma^i)\,|\,\omega^i\right). \tag{3.5}$$

As a result, we infer:

$$\begin{aligned}
\widehat{\mathbb{E}}(f(\sigma^i)\sigma_i) &= \widehat{\mathbb{E}}(f(\sigma^i)\mathbf{1}_{\sigma_i=1}\mathbf{1}_{\omega_i=1})\\
&\overset{(3.1)}{=} \sum_{\omega^i} p\,\widehat{\mathbb{E}}\left(f(\sigma^i)\,|\,\omega^i \times 1\right)\mathbb{P}\left(\omega^i\right)\\
&\overset{(3.5)}{\le} p\sum_{\omega^i}\widehat{\mathbb{E}}\left(f(\sigma^i)\,|\,\omega^i\right)\mathbb{P}\left(\omega^i\right)\\
&= p\,\widehat{\mathbb{E}}[f(\sigma^i)] \overset{(\mathsf{P1})}{=} \widehat{\mathbb{E}}[\sigma_i]\,\widehat{\mathbb{E}}[f(\sigma^i)].
\end{aligned}$$

Property P4. Assume that $X \stackrel{\text{def}}{=} \sum_{i\in\Lambda}\lambda_i(\sigma_i - p) = 0$. We deduce that

$$0 = \widehat{\mathbb{E}}[X^2] = \widehat{\mathbb{E}}[(X - \widehat{\mathbb{E}}[X|\omega])^2] + \widehat{\mathbb{E}}[\widehat{\mathbb{E}}[X|\omega]^2] \ge \widehat{\mathbb{E}}[(X - \widehat{\mathbb{E}}[X|\omega])^2].$$

Now,

$$X - \widehat{\mathbb{E}}[X|\omega] = \sum_{i\in\Lambda}\lambda_i(\sigma_i - \widehat{\mathbb{E}}[\sigma_i|\omega])$$

and, conditionally on ω, the random variables $\sigma_i - \widehat{\mathbb{E}}[\sigma_i|\omega]$ are independent and have mean 0. We deduce that

$$\begin{aligned}
\widehat{\mathbb{E}}[(X - \widehat{\mathbb{E}}[X|\omega])^2] &= \widehat{\mathbb{E}}\big[\widehat{\mathbb{E}}[(X - \widehat{\mathbb{E}}[X|\omega])^2|\omega]\big]\\
&= \widehat{\mathbb{E}}\Big[\widehat{\mathbb{E}}\big[\big(\sum_{i\in\Lambda}\lambda_i(\sigma_i - \widehat{\mathbb{E}}[\sigma_i|\omega])\big)^2 \;\big|\; \omega\big]\Big]\\
&= \widehat{\mathbb{E}}\Big[\sum_{i\in\Lambda}\lambda_i^2\widehat{\mathbb{E}}\big[(\sigma_i - \widehat{\mathbb{E}}[\sigma_i|\omega])^2\big|\omega\big]\Big]\\
&= \sum_{i\in\Lambda}\lambda_i^2\widehat{\mathbb{E}}[(\sigma_i - \widehat{\mathbb{E}}[\sigma_i|\omega])^2].
\end{aligned}$$

Now, $\widehat{\mathbb{E}}[\sigma_i|\omega] \le 1/2$ by (3.1) and (1.1), and therefore

$$\widehat{\mathbb{E}}[(\sigma_i - \widehat{\mathbb{E}}[\sigma_i|\omega])^2] = \mathbb{E}\big[\widehat{\mathbb{E}}[\sigma_i|\omega](1 - \widehat{\mathbb{E}}[\sigma_i|\omega])\big] \ge \tfrac{1}{2}\mathbb{E}[\widehat{\mathbb{E}}[\sigma_i|\omega]] = \tfrac{p}{2} > 0, \tag{3.6}$$

which implies that $\lambda_i = 0$ for every $i \in \Lambda$. □

3.2. *Proof of Theorem 1.1.* Recall our notation $f_i(\sigma_i) \overset{\text{def}}{=} \widehat{\mathbb{E}}\left(f(\omega)\,|\,\sigma_i\right)$. We may represent $f(\omega)$ as

$$f = \sum_i \gamma_i f_i(\sigma_i) + f^{\perp},$$

with $f^{\perp}$ orthogonal to the subspace of $\mathbb{L}^2(\widehat{\Omega}, \widehat{\mathbb{P}})$ spanned by $(f_i(\sigma_i), i \in \Lambda)$. Since $\widehat{\mathbb{E}}[f_i(\sigma_i)] = 0$ and $\widehat{\mathbb{E}}[\sigma_i] = p$ by $\mathsf{P1}$ of Theorem 3.1, we deduce that

$$f_i(\sigma_i) = (f_i(1) - f_i(0))(\sigma_i - p). \tag{3.7}$$

Should the random variables σ_i be independent, we would immediately infer that $\gamma_i \equiv 1$ and

$$\mathbb{V}\mathrm{ar}(f(\omega)) \ge p(1-p)\sum_i (f_i(1) - f_i(0))^2,$$

which, by (3.3), would be the end of the proof. This is precisely the computation done in the Bernoulli case in [10]. In our case, however, the random variables σ_i are dependent, and we need additional information and more care in order to control both the coefficients γ_i and the cross-terms. It is precisely at this stage that negative dependence, as stated in $\mathsf{P3}$ of Theorem 3.1, becomes crucial.

Before proceeding with the proof of the theorem, let us formulate and prove the following elementary lemma.

LEMMA 3.1. *Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a finite dimensional Hilbert space, and let $\{\mathsf{f}_1, \dots, \mathsf{f}_n\}$ be a normalized basis of $\mathcal{H}$, such that $\langle \mathsf{f}_i, \mathsf{f}_j\rangle \le 0$ for every pair $i \neq j$. Then, for any $\mathsf{f} = \sum_{i=1}^n \lambda_i \mathsf{f}_i$ such that $\langle \mathsf{f}, \mathsf{f}_i\rangle \ge 0$ for every $1 \le i \le n$, we have that $\lambda_i \ge \langle \mathsf{f}, \mathsf{f}_i\rangle$ for every $1 \le i \le n$.*

Note that when the basis $\{\mathsf{f}_1, \dots, \mathsf{f}_n\}$ is orthogonal, we find that $\lambda_i = \langle \mathsf{f}, \mathsf{f}_i\rangle$ for every $1 \le i \le n$.

PROOF. Lemma 3.1 relies on the following transparent geometric fact:

Obtuse cone property. Consider a positive cone $\mathcal{C} = \mathcal{C}(\mathsf{f}_1, \dots, \mathsf{f}_n)$ spanned by vectors f_i. The cone is called obtuse if $\langle \mathsf{f}_i, \mathsf{f}_j\rangle \le 0$ for any $i \neq j$. Then, $\langle \mathsf{f}, \mathsf{f}_i\rangle \ge 0$ for every i implies that $\mathsf{f} \in \mathcal{C}$, that is, all the coefficients λ_j in the decomposition $\mathsf{f} = \sum_j \lambda_j \mathsf{f}_j$ are non-negative. □

Taking scalar product with normalized vectors f_i yields

$$\langle \mathsf{f}, \mathsf{f}_i\rangle = \lambda_i + \sum_{j\neq i} \lambda_j \langle \mathsf{f}_j, \mathsf{f}_i\rangle \le \lambda_i$$

and one then gets the conclusion of Lemma 3.1 .

Perhaps the simplest way to prove the above obtuse cone property is by induction on the cardinality of the basis. The two dimensional case is straightforward. Let us now consider the basis $\{\mathsf{f}_1, \dots, \mathsf{f}_{n+1}\}$. For $i \geq 2$, let

$$\mathsf{F}_i \stackrel{\text{def}}{=} \frac{\mathsf{f}_i - \langle \mathsf{f}_i, \mathsf{f}_1 \rangle \mathsf{f}_1}{1 - \langle \mathsf{f}_1, \mathsf{f}_i \rangle^2} \tag{3.8}$$

be the normalized projection of f_i on $\mathrm{vec}(\mathsf{f}_1)^\perp$. Also, write $\mathsf{f} = \langle \mathsf{f}, \mathsf{f}_1 \rangle \mathsf{f}_1 + \mathsf{F}$ where $\mathsf{F} \in \mathrm{vec}(\mathsf{F}_2, \dots, \mathsf{F}_{n+1})$. We wish to apply the induction hypothesis with F and $\mathsf{F}_2, \dots, \mathsf{F}_{n+1}$. For that, simply observe that, for $i \neq j$,

$$\langle \mathsf{F}_i, \mathsf{F}_j \rangle = \frac{\langle \mathsf{f}_i, \mathsf{f}_j \rangle - \langle \mathsf{f}_i, \mathsf{f}_1 \rangle \langle \mathsf{f}_j, \mathsf{f}_1 \rangle}{(1 - \langle \mathsf{f}_1, \mathsf{f}_i \rangle^2)(1 - \langle \mathsf{f}_1, \mathsf{f}_j \rangle^2)} \leq 0$$

(we used that $\langle \mathsf{f}_i, \mathsf{f}_j \rangle \leq 0$ for $i \neq j$) and

$$\langle \mathsf{F}, \mathsf{F}_i \rangle = \langle \mathsf{f}, \mathsf{F}_i \rangle = \frac{\langle \mathsf{f}, \mathsf{f}_i \rangle - \langle \mathsf{f}_i, \mathsf{f}_1 \rangle \langle \mathsf{f}, \mathsf{f}_1 \rangle}{1 - \langle \mathsf{f}_i, \mathsf{f}_1 \rangle^2} \geq \langle \mathsf{f}, \mathsf{f}_i \rangle \geq 0, \tag{3.9}$$

where the first equality is due to the orthogonality of f_1 and F_i, and the first inequality to the fact that $\langle \mathsf{f}_i, \mathsf{f}_1 \rangle \leq 0$ and $\langle \mathsf{f}, \mathsf{f}_1 \rangle \geq 0$. Therefore, by the induction assumption, F lies in the cone $\mathcal{C}(\mathsf{F}_2, \dots, \mathsf{F}_{n+1})$. By (3.8), all F_i-s lie in $\mathcal{C}(\mathsf{f}_1, \dots, \mathsf{f}_{n+1})$ and hence $\mathsf{F} \in \mathcal{C}(\mathsf{f}_1, \dots, \mathsf{f}_{n+1})$ as well. Since the coefficient $\langle \mathsf{f}, \mathsf{f}_1 \rangle$ in $\mathsf{f} = \langle \mathsf{f}, \mathsf{f}_1 \rangle \mathsf{f}_1 + \mathsf{F}$ is non-negative, we are done. □

Proof of Theorem 1.1. Consider a field $\omega \in \{0,1\}^\Lambda$ satisfying (FE) and (FKG). We keep the notations from the previous section. In particular, on the probability space $(\widehat{\Omega}, \widehat{\mathbb{P}})$, we associate the field σ to ω.

Let f be a square-integrable non-decreasing function of ω. As before, we set $f_i(\sigma_i) \stackrel{\text{def}}{=} \widehat{\mathbb{E}}[f(\omega)|\sigma_i]$. Without loss of generality, we assume that $\mathbb{E}[f(\omega)] = 0$ and, consequently, $\widehat{\mathbb{E}}[f_i(\sigma_i)] = 0$ for every $i \in \Lambda$.

Let I be the set of indices i satisfying $\mathbb{E}[f_i(\sigma_i)^2] > 0$. Consider $V = \mathrm{vec}(f_i(\sigma_i) : i \in I)$ and write

$$f = \sum_{i \in I} \gamma_i f_i(\sigma_i) + f^\perp,$$

where $f^\perp \in V^\perp$. Properties P2 and P3 of Theorem 3.1 show that $\langle f_i(\sigma_i), f_j(\sigma_j) \rangle \leq 0$ for every $i \neq j$ in I. Furthermore,

$$\langle f(\omega), f_i(\sigma_i) \rangle = \widehat{\mathbb{E}}[f(\omega) f_i(\sigma_i)] = \widehat{\mathbb{E}}[f_i(\sigma_i)^2] \geq 0.$$

(The last equality is due to the definition of the conditional expectation.) Last but not least, the family $\{f_i(\sigma_i) : i \in I\}$ forms a basis of V. Indeed, this directly follows from (3.7).

Now, $f_i(1) - f_i(0) > 0$ for $i \in I$, and the family $\{\sigma_i - p : i \in I\}$ is free thanks to P4. We are therefore in position to apply the previous lemma with $\mathsf{f}_i(\sigma_i) \stackrel{\text{def}}{=} f_i(\sigma_i)/\|f_i(\sigma_i)\|$ in order to obtain that

$$\gamma_i \geq \frac{\langle f(\omega), f_i(\sigma_i) \rangle}{\|f_i(\sigma_i)\|^2} = \frac{\widehat{\mathbb{E}}[f_i(\sigma_i)^2]}{\|f_i(\sigma_i)\|^2} = 1. \tag{3.10}$$

We deduce that

$$\begin{aligned}
\widehat{\mathbb{E}}[f(\omega)^2] &\geq \widehat{\mathbb{E}}\Big[\Big(\sum_{i\in I}\gamma_i f_i(\sigma_i)\Big)^2\Big] \\
&= \widehat{\mathbb{E}}\Big[\Big(\sum_{i\in I}\gamma_i(f_i(\sigma_i)-\widehat{\mathbb{E}}[f_i(\sigma_i)|\omega])\Big)^2\Big] + \widehat{\mathbb{E}}\Big[\Big(\sum_{i\in I}\gamma_i\widehat{\mathbb{E}}[f_i(\sigma_i)|\omega]\Big)^2\Big] \\
&\geq \widehat{\mathbb{E}}\Big[\Big(\sum_{i\in I}\gamma_i(f_i(\sigma_i)-\widehat{\mathbb{E}}[f_i(\sigma_i)|\omega])\Big)^2\Big] \\
&= \sum_{i\in I}\gamma_i^2\widehat{\mathbb{E}}[(f_i(\sigma_i)-\widehat{\mathbb{E}}[f_i(\sigma_i)|\omega])^2] \\
&\overset{(3.10)}{\geq} \sum_{i\in I}\widehat{\mathbb{E}}[(f_i(\sigma_i)-\widehat{\mathbb{E}}[f_i(\sigma_i)|\omega])^2] \\
&= \sum_{i\in\Lambda}\widehat{\mathbb{E}}[(f_i(\sigma_i)-\widehat{\mathbb{E}}[f_i(\sigma_i)|\omega])^2],
\end{aligned}$$

where the second equality is due to the fact that, conditionally on ω, the random variables $\{f_i(\sigma_i)-\widehat{\mathbb{E}}[f_i(\sigma_i)|\omega] : i\in I\}$ are orthogonal (since the $\sigma_i-\widehat{\mathbb{E}}[\sigma_i|\omega]$ are), and the last equality to the observation that, for $i\notin I$,

$$0\leq \widehat{\mathbb{E}}[(f_i(\sigma_i)-\widehat{\mathbb{E}}[f_i(\sigma_i)|\omega])^2]\leq \widehat{\mathbb{E}}[f_i(\sigma_i)^2]=0.$$

We are now ready to conclude. Similarly to (3.7), we find that (remember that we chose $p=c_{\mathsf{FE}}/2$)

$$\begin{aligned}
\widehat{\mathbb{E}}\big[(f_i(\sigma_i)-\widehat{\mathbb{E}}[f_i(\sigma_i)|\omega])^2\big] &= (f_i(1)-f_i(0))^2\widehat{\mathbb{E}}\big[(\sigma_i-\widehat{\mathbb{E}}[\sigma_i|\omega])^2\big] \\
&\overset{(3.6)}{\geq} \tfrac{p}{2}(f_i(1)-f_i(0))^2 \\
&\overset{(3.3)}{=} \frac{p}{2(1-p)^2}\big(\mathbb{E}[f(\omega^i\times 1)]-\mathbb{E}[f(\omega)]\big)^2 \\
&= \frac{p}{2(1-p)^2}\big(\mathbb{E}[(f(\omega^i\times 1)-f(\omega^i\times 0))\mathbf{1}_{\omega_i=0}]\big)^2 \\
&\overset{(1.1)}{\geq} \frac{2p^3}{(1-p)^2}\big(\mathbb{E}[f(\omega^i\times 1)-f(\omega^i\times 0)]\big)^2. \qquad (3.11)
\end{aligned}$$

Overall, we find that

$$\mathbb{V}\mathrm{ar}(f(\omega))=\widehat{\mathbb{E}}[f(\omega)^2]\geq \frac{2p^3}{(1-p)^2}\sum_{i\in\Lambda}\big(\mathbb{E}[f(\omega^i\times 1)-f(\omega^i\times 0)]\big)^2.$$

□

REMARK 1. *Observe that, up to* (3.11)*, the proof only made use of the lower bound* $\mathbb{P}(\omega_i=1\,|\,\omega^i)\geq c_{\mathsf{FE}}=2p$ *in* (1.1).

4. Applications.

4.1. *Proof of Proposition 2.1.*

PROOF OF PROPOSITION 2.1. Let E_n be the event that there exists an open circuit with non trivial homotopy in $\mathbb{T}_n$. Theorem 1.1 implies that

$$\sum_{i\in E_{\mathbb{T}_n}} \mathbb{P}[\mathrm{Piv}_i(E_n)]^2 \le \frac{1}{c_{\mathsf{P}}}\mathbb{P}(E_n)(1-\mathbb{P}(E_n)) \le \frac{1}{c_{\mathsf{P}}}.$$

Note that edges are of two types: either "vertical" or "horizontal". By shift invariance of $\mathbb{P}$ and of the event E_n we therefore obtain, for an horizontal edge e (the same reasoning can be applied to vertical edges),

$$|\mathbb{T}_n|\cdot\mathbb{P}[\tilde{A}_2^e(n)]^2 \le \sum_{i\in E_{\mathbb{T}_n}} \mathbb{P}[\mathrm{Piv}_i(E_n)]^2 \le \frac{1}{c_{\mathsf{P}}}.$$

□

4.2. *Quantitative Burton-Keane argument.* Recall that we are working with (nearest neighbor) bond percolation models. For $x\in\Lambda_n$, the set $\mathcal{C}_n(x)$ is the connected component of x in the restriction of the percolation configuration to the edges with at least one end-point in Λ_n.

For $x\in\Lambda_n$ let $\mathrm{Trif}_n(x)$ be the event that:

a) There are *exactly* three open bonds incident to x.
b) $\mathcal{C}_n(x)\setminus x$ is a disjoint union of *exactly* three connected clusters, and each of these three clusters is connected to $\partial\Lambda_{n+1}\stackrel{\mathrm{def}}{=}\Lambda_{n+1}\setminus\Lambda_n$.

Recall the following classical fact [8]:

LEMMA 4.1. *Consider a percolation model on Λ_n. Then, for any ω,*

$$\sum_{x\in\Lambda_n}\mathbf{1}_{\mathrm{Trif}_n(x)}(\omega)\le|\partial\Lambda_{n+1}|. \tag{4.1}$$

Note that this lemma has the following useful consequence: if $\mathbb{P}$ is in fact a translation invariant measure on the whole plane or on a torus, it implies that

$$\mathbb{P}[\mathrm{Trif}_{2n}(0)]\le\frac{|\partial\Lambda_{n+1}|}{|\Lambda_n|}. \tag{4.2}$$

Let $k\le n$. Define the event $\mathrm{CoarseTrif}_{k,n}$ that there are at least three distinct clusters in the annulus $A_{k,n}\stackrel{\mathrm{def}}{=}\Lambda_n\setminus\Lambda_k$ connecting the inner to the outer boundaries of $A_{k,n}$. In other words, there are at least three distinct crossing clusters of $A_{k,n}$.

COROLLARY 4.1. *Consider a percolation model on $\mathbb{Z}^d$ satisfying* (FE), (FKG) *and* (TI). *There exists $c_1=c_1(c_{\mathsf{FE}})>0$ such that, for any $0\le k\le n$,*

$$\mathbb{P}[\mathrm{CoarseTrif}_{k,n}]\le\frac{\exp(c_1k)}{n}.$$

Proof. By conditioning on the clusters in $A_{k,n}$, one may easily check that

$$\mathbb{P}[\mathrm{Trif}_n(0)|\mathrm{CoarseTrif}_{k,n}] \geq c_{\mathsf{FE}}{}^{6dk}.$$

(There may be some problems if the three clusters reach $\partial\Lambda_k$ near the corner, yet such cases can be treated separately.) The result follows readily from Lemma 4.1. □

Proof of Theorem 2.1. Set $\varepsilon \stackrel{\text{def}}{=} 1/(2c_1)$ and let $k = \lfloor \varepsilon \log n \rfloor$. Let E_n be the event that there are exactly two clusters in $A_{k,n}$ from the inner to the outer boundaries. On E_n, let $\mathcal{C}$ be the set of vertices of $A_{k,n}$ connected to the boundary of Λ_n by an open path. Since there are only two distinct clusters connecting the inside and outside boundaries of $A_{k,n}$, the vertices of $\mathcal{C} \cap \Lambda_k$ can be divided into two subsets $E_1 \stackrel{\text{def}}{=} E_1(\mathcal{C})$ and $E_2 \stackrel{\text{def}}{=} E_2(\mathcal{C})$ depending on which clusters they belong to.

For every possible realization C of $\mathcal{C}$ so that $\{\mathcal{C} = C\} \subset E_n$, define $\mathrm{Cross}(C)$ to be the event that E_1 and E_2 are connected by an open path inside Λ_k. Theorem 1.1 applied to $\mathbb{P}(\cdot|\mathcal{C} = C)$ and $\mathrm{Cross}(C)$ gives

$$\sum_{e \in E_{\Lambda_k}} \mathbb{P}[\mathrm{Piv}_e(\mathrm{Cross}(C))|\mathcal{C} = C]^2 \leq \tfrac{1}{4},$$

which, by Cauchy-Schwarz, implies that

$$\sum_{e \in E_{\Lambda_k}} \mathbb{P}[\mathrm{Piv}_e(\mathrm{Cross}(C))|\mathcal{C} = C] \leq \tfrac{1}{2}\sqrt{|E_{\Lambda_k}|}.$$

For an edge e', let $A_2(n, e')$ be the event that the two end-points of e' are connected to the boundary of Λ_n by two disjoint clusters. By definition of C, we have that $\mathrm{Piv}_{e'}(\mathrm{Cross}(C)) \cap \{\mathcal{C} = C\} = A_2(n, e') \cap \{\mathcal{C} = C\}$ which, by summing over all possible C, implies that

$$\sum_{e' \in E_{\Lambda_k}} \mathbb{P}[A_2(n, e'), E_n] \leq \tfrac{1}{2}\sqrt{|E_{\Lambda_k}|}\,\mathbb{P}(E_n) \leq \tfrac{1}{2}\sqrt{|E_{\Lambda_k}|} \leq \sqrt{d|\Lambda_k|/2}.$$

It only remains to see that Corollary 4.1 implies

$$\mathbb{P}[A_2(n, e'), \mathrm{CoarseTrif}_{k,n}] \leq \mathbb{P}[\mathrm{CoarseTrif}_{k,n}] \leq \frac{\exp(c_1 \varepsilon \log n)}{n} = \frac{1}{\sqrt{n}}.$$

At the end, we find that

$$\frac{1}{|\Lambda_k|} \sum_{e' \in E_{\Lambda_k}} \mathbb{P}[A_2(n, e')] \leq \frac{\sqrt{d/2}}{\sqrt{|\Lambda_k|}} + \frac{1}{\sqrt{n}}.$$

Consider for a moment that the edge e involved in $A_2^e(2n)$ is horizontal. Since $A_2^e(2n)$ is included in a translate of $A_2(n, e')$ for any horizontal edge $e' \in E_{\Lambda_k}$, we conclude that

$$\mathbb{P}[A_2^e(2n)] \leq \frac{1}{|\Lambda_k|} \sum_{e' \in E_{\Lambda_k}} \mathbb{P}[A_2(n, e')] \leq \frac{\sqrt{d/2}}{\sqrt{|\Lambda_k|}} + \frac{1}{\sqrt{n}},$$

which proves the theorem with $c_{\mathsf{BK}} = c_{\mathsf{BK}}(c_{\mathsf{FE}}) > 0$ small enough thanks to our choice for k. □

4.3. *Continuity of percolation probabilities away from critical points.* Theorem 2.2 is an easy consequence of the quantitative Burton-Keane bound (2.1). Let $\alpha_0 \in (a,b)$. Pick $\epsilon > 0$ such that $[\alpha_0 - \epsilon, \alpha_0 + \epsilon]$ is still in (a,b). By our assumptions on the family $\{\mathbb{P}_\alpha\}$,

$$\limsup_{n\to\infty} \sup_{\alpha\in[\alpha_0-\epsilon,\alpha_0+\epsilon]} \mathbb{P}_\alpha\left(\partial\Lambda_n \not\leftrightarrow \infty\right) = \limsup_{n\to\infty} \mathbb{P}_{\alpha_0-\epsilon}\left(\partial\Lambda_n \not\leftrightarrow \infty\right) = 0. \tag{4.3}$$

On the other hand, for any $N > n$ and any $\alpha \in [\alpha_0 - \epsilon, \alpha_0 + \epsilon]$,

$$\begin{aligned} \theta(\alpha) &\stackrel{\text{def}}{=} \mathbb{P}_\alpha\left(0 \leftrightarrow \infty\right) = \mathbb{P}_\alpha\left(0 \leftrightarrow \partial\Lambda_N\right) \\ &\quad - \mathbb{P}_\alpha\left(0 \leftrightarrow \partial\Lambda_N; 0 \not\leftrightarrow \infty; \partial\Lambda_n \leftrightarrow \infty\right) - \mathbb{P}_\alpha\left(0 \leftrightarrow \partial\Lambda_N; \partial\Lambda_n \not\leftrightarrow \infty\right). \end{aligned} \tag{4.4}$$

The event $\{0 \leftrightarrow \partial\Lambda_N; 0 \not\leftrightarrow \infty; \partial\Lambda_n \leftrightarrow \infty\}$ implies the existence of at least two disjoint crossings of the annulus $A_{n,N}$. If we choose $N = \mathrm{e}^{Cn}$ for some sufficiently large constant C, then the second term in (4.4) tends to zero as $n \to \infty$, uniformly in $\alpha \in [\alpha_0 - \epsilon, \alpha_0 + \epsilon]$; this follows from (2.1), which in view of the assumed uniformity of (FE) on compact subintervals, yields uniform upper bounds for $\alpha \in [\alpha_0 - \epsilon, \alpha_0 + \epsilon]$. The third term in (4.4) is controlled by (4.3). Hence, since the events $\{0 \leftrightarrow \partial\Lambda_N\}$ are local, continuity of $\mathbb{P}_\alpha$ at α_0 implies that θ is continuous at α_0 as well. □

4.4. *Proof of Theorem 2.4.* Let us recall some additional facts on the random-cluster model. First, let us introduce random-cluster measures with boundary conditions. Fix a finite graph G. *Boundary conditions* ξ are given by a partition $P_1 \sqcup \cdots \sqcup P_k$ of ∂G. Two vertices are *wired in* ξ if they belong to the same P_i. The graph obtained from the configuration ω by identifying the wired vertices together is denoted by ω^ξ. Let $k(\omega^\xi)$ be the number of connected components of the graph ω^ξ. The probability measure $\mathbb{P}^\xi_{p,q,G}$ of the random-cluster model on G with *edge-weight* $p \in [0,1]$, *cluster-weight* $q > 0$ and boundary conditions ξ is defined by

$$\mathbb{P}^\xi_{p,q,G}(\{\omega\}) \stackrel{\text{def}}{=} \frac{p^{o(\omega)}(1-p)^{c(\omega)}q^{k(\omega^\xi)}}{Z^\xi_{p,q,G}} \tag{4.5}$$

for every configuration ω on G. The constant $Z^\xi_{p,q,G}$ is a normalizing constant, referred to as the *partition function*, defined in such a way that the sum over all configurations equals 1. For $q \geq 1$, infinite-volume random-cluster measures can be defined as weak limits of random-cluster measures on larger and larger boxes.

Recall that the planar random-cluster model possesses a dual model on the dual graph $(\mathbb{Z}^2)^\star$. The configuration $\omega^\star \in \{0,1\}^{E_{(\mathbb{Z}^2)^\star}}$ is defined as follows: each dual-edge $e^\star \in (\mathbb{Z}^2)^\star$ is dual-open in $\omega^\star$ if and only if the edge of $\mathbb{Z}^2$ passing through its middle (there is a unique such edge) is closed in ω. If the law of ω is $\mathbb{P}^\xi_{p,q,G}$, then the law of the dual model is $\mathbb{P}^{\xi^\star}_{p^\star,q,G^\star}$ for some dual boundary conditions $\xi^\star$. We will only use that free and wired boundary conditions are dual to each other.

On $\mathbb{Z}^2$, the random-cluster model undergoes a phase transition at some parameter $p_c(q)$ satisfying, for every infinite-volume random-cluster model $\mathbb{P}_{p,q,\mathbb{Z}^2}$ with parameters p and q,

$$\mathbb{P}_{p,q,\mathbb{Z}^2}[0 \longleftrightarrow \infty] = \begin{cases} \theta(p,q) > 0 & \text{if } p > p_c(q), \\ 0 & \text{if } p < p_c(q). \end{cases}$$

The critical point of the planar random-cluster model on $\mathbb{Z}^2$ is known to correspond to the self-dual point of the model, *i.e.*, $p_c(q) \stackrel{\text{def}}{=} \sqrt{q}/(1+\sqrt{q})$ [5]. Also, for $q \in [1,4]$, the behavior at criticality is known to be the following (see [13]): there is a unique infinite-volume measure and, for any numbers $1 < a < b \leq \infty$, there exists $c_{\mathsf{RSW}} = c_{\mathsf{RSW}}(a,b) > 0$ such that, for all $n \geq 1$ and any boundary conditions ξ,

$$c_{\mathsf{RSW}} \leq \mathbb{P}^{\xi}_{p_c,q,\widehat{R}_n}\big[\mathrm{Cross}(R_n)\big] \leq 1 - c_{\mathsf{RSW}}, \tag{4.6}$$

where $R_n = R_n[a] \stackrel{\text{def}}{=} [-an, an] \times [-n, n]$, $\widehat{R}_n = \widehat{R}_n[b] \stackrel{\text{def}}{=} [-bn, bn] \times [-2n, 2n]$ and $\mathrm{Cross}(R_n)$ is the event that the left and right sides of R_n are connected by an open path in R_n.

PROOF OF THEOREM 2.4. Fix $q \in [1,4]$. Note that, for $p < p_c(q)$, there exists $c = c(p,q) > 0$ such that, for any edge e and every $n \geq 1$,

$$\mathbb{P}_{p,q,\mathbb{Z}^2}[A_2^e(n)] \leq \mathbb{P}_{p,q,\mathbb{Z}^2}[0 \longleftrightarrow \partial\Lambda_n] \leq e^{-c(p,q)n},$$

thanks to exponential decay of correlations, see [5] one more time. Similarly, when $p > p_c(q)$, for any edge e and every $n \geq 1$,

$$\mathbb{P}_{p,q,\mathbb{Z}^2}[A_2^e(n)] = \mathbb{P}_{p^\star,q,(\mathbb{Z}^2)^\star}[A_2(n)] \leq \mathbb{P}_{p^\star,q,(\mathbb{Z}^2)^\star}[u \stackrel{\star}{\longleftrightarrow} \partial\Lambda_n^\star] \leq e^{-c(p^\star,q)n}.$$

The only interesting case is therefore the critical point $p = p_c$.

Proof using mixing properties and Proposition 2.1. Recall that, by Proposition 2.1, we already know that, for any edge e' of $\mathbb{T}_n^{(2)}$ and every $n \geq 1$,

$$\mathbb{P}_{p_c,q,\mathbb{T}_n^{(2)}}[\tilde{A}_2^{e'}(n)] \leq \frac{c_{\tilde{\mathsf{A}}_2}}{n},$$

where $\mathbb{P}_{p_c,q,\mathbb{T}_n^{(2)}}$ is the random-cluster measure on $\mathbb{T}_n^{(2)}$. We therefore only need to show that there exists $C > 0$ such that

$$\mathbb{P}_{p_c,q,\mathbb{Z}^2}[A_2^e(n)] \leq C\,\mathbb{P}_{p_c,q,\mathbb{T}_n^{(2)}}[\tilde{A}_2^{e'}(n)].$$

Embed $\mathbb{T}_n^{(2)}$ into $\mathbb{Z}^2$ in such a way that the vertex set is $\Lambda_n \stackrel{\text{def}}{=} [-n,n]^2$ and e is an edge having 0 as an endpoint. First, we wish to highlight that (4.6) (more precisely the mixing result [12, Theorem 5.45]) classically implies the existence of $c_1 > 0$ such that, for every boundary conditions ξ and $n \geq 1$,

$$\mathbb{P}_{p_c,q,\mathbb{Z}^2}[A_2^e(n/2)] \leq c_1 \mathbb{P}^{\xi}_{p_c,q,\Lambda_n}[A_2^e(n/2)].$$

In particular, this is also true for so-called *periodic boundary conditions*, so that

$$\mathbb{P}_{p_c,q,\mathbb{Z}^2}[A_2^e(n/2)] \leq c_1 \mathbb{P}_{p_c,q,\mathbb{T}_n^{(2)}}[A_4^e(n/2)]. \tag{4.7}$$

Now, introduce the event $A_2^{\mathrm{sep}}(n/2)$ that there exist two open paths $\gamma, \tilde{\gamma}$ and two dual-open dual-paths $\gamma^\star$ and $\tilde{\gamma}^\star$, originating from the endpoints of e and e^* respectively, satisfying:

- the endpoints (on the boundary of $\Lambda_{n/2}$ and $\Lambda^\star_{n/2}$ respectively) x, $\tilde{x}$, $x^\star$ and $\tilde{x}^\star$ of the paths are at distance larger or equal to $\frac{n}{10}$ from each others.
- x and $\tilde{x}$ are connected to $\partial\Lambda_{\frac{3n}{5}}$ in $x+\Lambda_{\frac{n}{10}}$ and $\tilde{x}+\Lambda_{\frac{n}{10}}$,
- $x^\star$ and $\tilde{x}^\star$ are connected to $\partial\Lambda^\star_{\frac{3n}{5}}$ in $x^\star+\Lambda_{\frac{n}{10}}$ and $\tilde{x}^\star+\Lambda_{\frac{n}{10}}$.

Classically, (4.6) implies that there exists $c_2>0$ such that, for any $n\geq 1$,

$$\mathbb{P}_{p_c,q,\mathbb{T}_n^{(2)}}[A_2^e(n/2)] \leq c_2 \mathbb{P}_{p_c,q,\mathbb{T}_n^{(2)}}[A_2^{\rm sep}(n/2)]. \tag{4.8}$$

See [21] for a treatment in the case of Bernoulli percolation and [12, Theorems 10.22 and 10.23] for the FK-Ising model (the proofs of the theorems apply *mutatis mutandis* to any random-cluster model with $1\leq q\leq 4$).

It remains to see that there exists $c_3>0$ such that, for any $n\geq 1$,

$$\mathbb{P}_{p_c,q,\mathbb{T}_n^{(2)}}[A_2^{\rm sep}(n/2)] \leq c_3 \mathbb{P}_{p_c,q,\mathbb{T}_n^{(2)}}[\tilde{A}_2^e(n)].$$

In order to do so, mimic the classical argument to prove quasi-multiplicativity of arm-probabilities for Bernoulli percolation (see [21] again and Figure 1).

We only sketch the proof. Condition on $A_2^{\rm sep}(n/2)$. Consider a thin area S of width $\frac{n}{10}$ going from $x+\Lambda_{\frac{n}{10}}$ to $\tilde{x}+\Lambda_{\frac{n}{10}}$ outside $\Lambda_{\frac{n}{2}}$, an a thin dual area $S^\star$ of width $\frac{n}{10}$ going from $x^\star+\Lambda_{\frac{n}{10}}$ to $\tilde{x}^\star+\Lambda_{\frac{n}{10}}$ outside $\Lambda_{\frac{n}{2}}$ so that these two areas do not intersect. Now, (4.6) implies that there exist, with probability $c_4>0$, a primal path in S connecting the two paths γ and $\tilde{\gamma}$, and a dual path in $S^\star$ connecting $\gamma^\star$ and $\tilde{\gamma}^\star$. But whenever this occurs, the event $\tilde{A}_2^e(n)$ is satisfied, so that

$$\mathbb{P}_{p_c,q,\mathbb{T}_n^{(2)}}[\tilde{A}_4^e(n)] \geq c_4 \mathbb{P}_{p_c,q,\mathbb{T}_n^{(2)}}[A_2^{\rm sep}(n/2)].$$

It only remains to invoke (4.7) and (4.8) to conclude. □

Proof using bounds on $\mathbb{E}_{p_c,q,\mathbb{Z}^2}\left(N^2_{m,n}\right)$ and (2.4). We shall check that there exists $c_{\sf N}=c_{\sf N}(c_{\sf RSW})<\infty$, such that, uniformly in m,

$$\mathbb{E}_{p_c,q,\mathbb{Z}^2}\left(N^2_{m,5m}\right) \leq c_{\sf N}. \tag{4.9}$$

A substitution into (2.4) yields the claim.

Consider the annulus $A_{m,5m}$ and the four rectangles

$$\begin{aligned} S_{m,{\sf U}} &= [-5m,5m]\times[m,5m], \quad S_{m,{\sf R}} = [-5m,-m]\times[-5m,5m],\\ S_{m,{\sf L}} &= [m,5m]\times[-5m,5m], \quad S_{m,{\sf D}} = [-5m,5m]\times[-5m,-m]. \end{aligned} \tag{4.10}$$

For $*\in\{{\sf U},{\sf R},{\sf L},{\sf D}\}$, let $N_{m,*}$ be the number of distinct short-side crossing clusters of S_{m*}. For instance $N_{m,{\sf U}}$ is the number of distinct clusters which connect $[-5m,5m]\times\{m\}$ to $[-5m,5m]\times\{5m\}$ in the restriction of the percolation configuration to the rectangle $S_{m,{\sf U}}$. Clearly,

$$N_{m,5m} \leq \sum_{*\in\{{\sf U},{\sf R},{\sf L},{\sf D}\}} N_{m,*},$$

and, by symmetry, it remains to give an upper bound on $\mathbb{E}_{p_c,q,\mathbb{Z}^2}\left(N^2_{m,{\sf U}}\right)$.

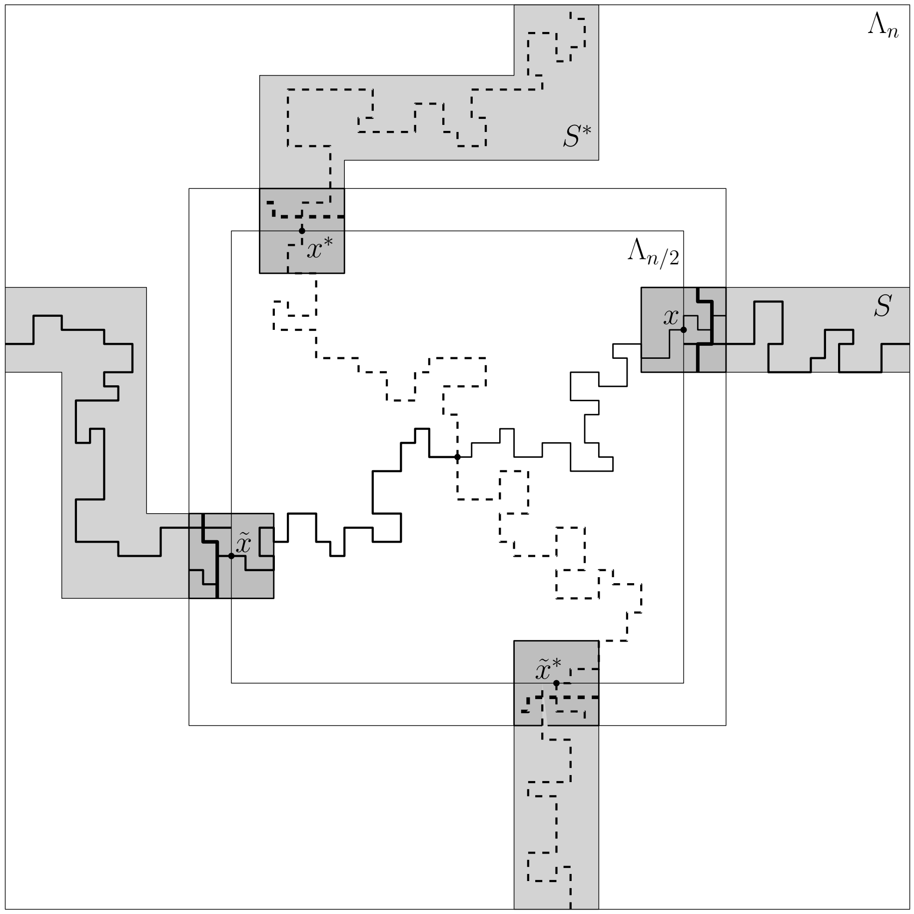

Fig 1. *The event $A_2^{\mathrm{sep}}(n/2)$ together with the extension of the four paths using the sets S and S^*. Estimates on crossing probabilities available from [13] show that these extensions cost a mutiplicative constant (not depending on n).*

Lemma 4.2. *The RSW bound (4.6) implies:*

$$\mathbb{P}_{p_c,q,\mathbb{Z}^2}\left(N_{m,\mathsf{U}} \geq k\right) \leq \left(1 - c_{\mathsf{RSW}}(5,\infty)\right)^{k-1}, \tag{4.11}$$

uniformly in $k > 1$ and m.

Proof. Let us introduce the events $\mathcal{R}_k \stackrel{\mathrm{def}}{=} \{N_{m,\mathsf{U}} \geq k\}$. We claim that

$$\mathbb{P}_{p_c,q,\mathbb{Z}^2}\left(\mathcal{R}_k \middle| \mathcal{R}_{k-1}\right) \leq 1 - c_{\mathsf{RSW}}(5,\infty), \tag{4.12}$$

uniformly in m and $k > 1$. Indeed, distinct crossing clusters which show up in any percolation configuration from $\mathcal{R}_{k-1}$ are naturally ordered from left to right. There are at least $(k-1)$ such clusters. The following somewhat standard construction, which we sketch below, is depicted on Figure 2:

Consider the disjoint decomposition $\mathcal{R}_{k-1} = \cup \mathcal{R}_{k-1}^{u,v}$, where u (respectively v) is the rightmost vertex of the $(k-1)$-th crossing cluster on the bottom (respectively top) side of $S_{m,\mathsf{U}}$. The event $\mathcal{R}_{k-1}^{u,v}$ implies that there is the left-most dual crossing $\gamma_*^{u,v}$ from u^* to v^*,

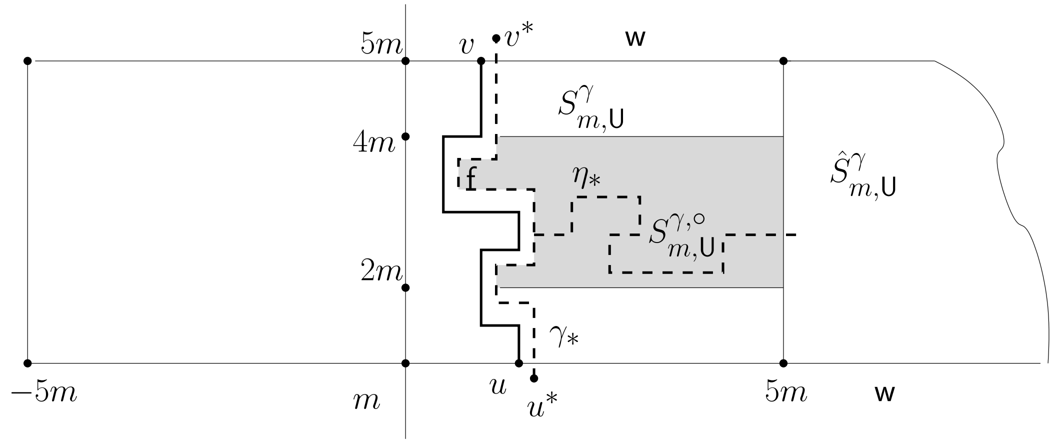

FIG 2. *The dual path $\gamma_* = \gamma_*^{u,v}$. Event $\mathcal{D}^\circ$: a dual path η_* crosses from left to right the middle section $S^{\gamma,\circ}_{m,\mathsf{U}}$ of $S^{\gamma}_{m,\mathsf{U}}$ and, as such, rules out the occurrence of the event $\mathcal{R}_k$.*
The boundary conditions (for the direct model) on the semi-infinite strip $\hat{S}^{\gamma}_{m,\mathsf{U}}$ are w *on the upper and lower parts, and* f *on γ_*.*

where $u^* \stackrel{\text{def}}{=} u + \frac{1}{2}(1,-1)$ and $v^* \stackrel{\text{def}}{=} v + \frac{1}{2}(1,1)$. Consider the remaining part, denoted by $S^{\gamma}_{m,\mathsf{U}}$, of the rectangle $S_{m,\mathsf{U}}$ to the right of $\gamma_*^{u,v}$. Let $S^{\gamma,\circ}_{m,\mathsf{U}}$ be the middle section of $S^{\gamma}_{m,\mathsf{U}}$, *i.e.*, $S^{\gamma,\circ}_{m,\mathsf{U}} \stackrel{\text{def}}{=} S^{\gamma}_{m,\mathsf{U}} \cap (\mathbb{Z} \times [2m, 4m])$. Finally consider the infinite strip extension $\hat{S}^{\gamma}_{m,\mathsf{U}}$ to the right of $S^{\gamma}_{m,\mathsf{U}}$.

Let $\mathcal{D}^\circ$ be the event that there is a left to right dual crossing of $S^{\gamma,\circ}_{m,\mathsf{U}}$. By the FKG property of the random-cluster model,

$$\mathbb{P}_{p_c,q,\mathbb{Z}^2}\left(\mathcal{R}_k \middle| \mathcal{R}_{k-1}\right) \le 1 - \min_{\gamma} \mathbb{P}^{\mathsf{w},\mathsf{f}}_{p_c,q,\hat{S}^{\gamma}_{m,\mathsf{U}}}(\mathcal{D}^\circ),$$

where the boundary conditions are *direct* boundary conditions on the semi-infinite strip $\hat{S}^{\gamma}_{m,\mathsf{U}}$: wired on upper and lower parts and free on γ. Note that the model is self-dual at criticality. Hence, for any possible realization of γ,

$$\mathbb{P}^{\mathsf{w},\mathsf{f}}_{p_c,q,\hat{S}^{\gamma}_{m,\mathsf{U}}}(\mathcal{D}^\circ) \ge \mathbb{P}^{\mathsf{f}}_{p_c,q,\hat{R}_m[\infty]}\left(\mathrm{Cross}(R_m[5])\right),$$

and (4.6) applies. □

REMARK 2. *Let us highlight the fact that SLE predictions, see [12, Section 13.3.2], suggest that $\mathbb{P}_{p,q,\mathbb{Z}^2}[A_4^e(n)] = n^{-\xi_{1010}+o(1)}$, where*

$$\xi_{1010} \stackrel{\text{def}}{=} \tfrac{3\sigma^2+10\sigma+3}{4(1+\sigma)} \quad \text{with} \quad \sigma \stackrel{\text{def}}{=} \tfrac{2}{\pi}\arcsin(\sqrt{q}/2).$$

This implies that $\mathbb{P}_{p,q,\mathbb{Z}^2}[A_4^e(n)] \gg \frac{1}{n}$ for $q < 2\arcsin[\pi\frac{2-\sqrt{3}}{\sqrt{3}}] \approx 0.459$. This illustrates the fact that the claim of Theorem 1.1 can fail to hold when the condition (FKG) *is dropped.*

Acknowledgments. The research of H. D.-C. and Y. V. was supported by the NCCR SwissMap founded by the swiss NSF. The research of D.I. was supported by Technion research grant 2020225. The authors also thank Michael Aizenman for suggesting the use of the number of distinct clusters in Corollary 2.3.

References.

[1] Aizenman, M. (1997). On the number of incipient spanning clusters. *Nuclear Phys. B* **485** 551–582. MR1431856 (98j:82030)

[2] Aizenman, M., Chayes, J. T., Chayes, L. and Newman, C. M. (1988). Discontinuity of the magnetization in one-dimensional $1/|x-y|^2$ Ising and Potts models. *J. Statist. Phys.* **50** 1–40. MR939480 (89f:82072)

[3] Aizenman, M., Duminil-Copin, H. and Sidoravicius, V. (2014). Random currents and continuity of Ising model's spontaneous magnetization. *Communications in Mathematical Physics* 1–24.

[4] Aizenman, M., Kesten, H. and Newman, C. M. (1987). Uniqueness of the infinite cluster and continuity of connectivity functions for short and long range percolation. *Comm. Math. Phys.* **111** 505–531. MR901151 (89b:82060)

[5] Beffara, V. and Duminil-Copin, H. (2012). The self-dual point of the two-dimensional random-cluster model is critical for $q \geq 1$. *Probab. Theory Related Fields* **153** 511–542. MR2948685

[6] Bodineau, T. (2006). Translation invariant Gibbs states for the Ising model. *Probability theory and related fields* **135** 153–168.

[7] Broadbent, S. R. and Hammersley, J. M. (1957). Percolation processes. I. Crystals and mazes. *Proc. Cambridge Philos. Soc.* **53** 629–641. MR0091567 (19,989e)

[8] Burton, R. M. and Keane, M. (1989). Density and uniqueness in percolation. *Comm. Math. Phys.* **121** 501–505. MR990777 (90g:60090)

[9] Cerf, R. (2013). A lower bound on the two-arms exponent for critical percolation on the lattice. Preprint arXiv:1306.3105.

[10] Chatterjee, S. and Sen, S. (2013). Minimal spanning trees and Stein's method. Preprint, arXiv:1307.1661.

[11] Daly, F. (2015). Negative dependence and stochastic orderings. arXiv:1504.06493.

[12] Duminil-Copin, H. (2013). *Parafermionic observables and their applications to planar statistical physics models. Ensaios Matematicos* **25**. Brazilian Mathematical Society.

[13] Duminil-Copin, H., Sidoravicius, V. and Tassion, V. (2013). Continuity of the phase transition for planar Potts models with $1 \leq q \leq 4$. *Preprint.*

[14] Fortuin, C. M. and Kasteleyn, P. W. (1972). On the random-cluster model. I. Introduction and relation to other models. *Physica* **57** 536–564. MR0359655 (50 ##12107)

[15] Gandolfi, A., Grimmett, G. and Russo, L. (1988). On the uniqueness of the infinite cluster in the percolation model. *Comm. Math. Phys.* **114** 549–552. MR929129 (89e:60199)

[16] Garban, C. and Steif, J. E. (2012). Noise sensitivity and percolation. In *Probability and statistical physics in two and more dimensions. Clay Math. Proc.* **15** 49–154. Amer. Math. Soc., Providence, RI. MR3025390

[17] Graham, B. and Grimmett, G. (2011). Sharp thresholds for the random-cluster and Ising models. *Ann. Appl. Probab.* **21** 240–265. MR2759201 (2012e:60255)

[18] Grimmett, G. (2006). *The random-cluster model. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]* **333**. Springer-Verlag, Berlin. MR2243761 (2007m:60295)

[19] Grimmett, G. R. (1999). Inequalities and entanglements for percolation and random-cluster models. In *Perplexing problems in probability. Progr. Probab.* **44** 91–105. Birkhäuser Boston, Boston, MA. MR1703126 (2000i:60115)

[20] Kesten, H. (1982). *Percolation theory for mathematicians. Progress in Probability and Statistics* **2**. Birkhäuser Boston, Mass. MR692943 (84i:60145)

[21] Nolin, P. (2008). Near-critical percolation in two dimensions. *Electron. J. Probab.* **13** no. 55, 1562–1623. MR2438816

Section de Mathématiques
Université de Genève
1211 Genève 4, Switzerland
E-mail: hugo.duminil@unige.ch; yvan.velenik@unige.ch

Faculty of IE& M
Technion
Haifa 32000, Israel
E-mail: ieioffe@ie.technion.ac.il