Percolation with a line of defects

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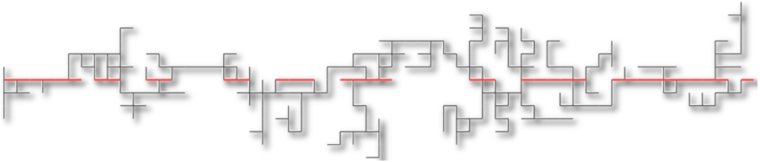
Université de Genève



joint work with
Sacha Friedli and Dmitry Ioffe

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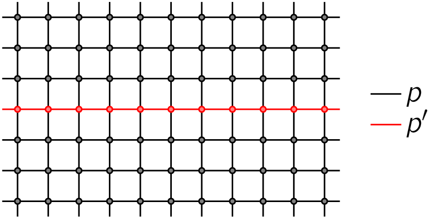
The model

Inhomogeneous independent bond percolation model

$\mathcal{L} = \{n\,\vec{e}_1 \ : \ n \in \mathbb{Z}\}$

$(\omega_e)_{e \in \mathbb{Z}^d}$, $\omega_e \in \{0,1\}$, indep.

$$\mathbb{P}_{p,p'}(\omega_e = 1) = \begin{cases} p & \text{if } e \not\subset \mathcal{L}, \\ p' & \text{if } e \subset \mathcal{L}. \end{cases}$$



When $p' = p$, we simply write $\mathbb{P}_p \equiv \mathbb{P}_{p,p}$.

Main question

Let $p_c = p_c(d)$ be the critical value of the homogeneous model ($p' = p$).

Earlier works on this model dealt with the case $p = p_c(d)$ and proved that there is no percolation for any $p' < 1$ when

- $d = 2$ [Zhang, AoP '94],
- d large [Newman& Wu, AoP '97].

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We are interested in the case $p < p_c(d)$, $d \geq 2$.

- Of course, there is no percolation in any dimension for any $p' < 1$ in that case.
- Instead, what concerns us here is the rate of exponential decay of connectivities along $\mathcal{L}$:

$$\xi_{p,p'} = -\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}_{p,p'}(0 \longleftrightarrow n\vec{e}_1).$$

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What is the behavior of $\xi_{p,p'}$ as a function of p' for fixed $p < p_c(d)$?

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Basic properties

- $\xi_{p,p'}$ exists by sub-additivity.
- $\xi_p \equiv \xi_{p,p} > 0$ for all $p < p_c$[Menshikov '86, Aizenman&Barsky '87].
- $\xi_{p,p'}$ is non-increasing in p'. In particular,

$$p' \leq p \implies \xi_{p,p'} \geq \xi_p,$$
$$p' \geq p \implies \xi_{p,p'} \leq \xi_p.$$

Existence of a transition

Fact #1

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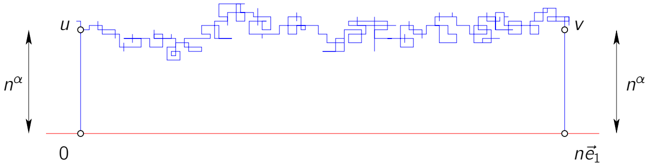

Let $\alpha \in (\frac{1}{2}, 1)$, $u = n^\alpha \mathbf{e}_2$, $v = n\vec{e}_1 + n^\alpha \mathbf{e}_2$. By FKG,

$$\mathbb{P}_{p,p'}(0 \longleftrightarrow n\vec{e}_1) \geq p^{2n^\alpha} \mathbb{P}_{p,p'}(u \longleftrightarrow v).$$

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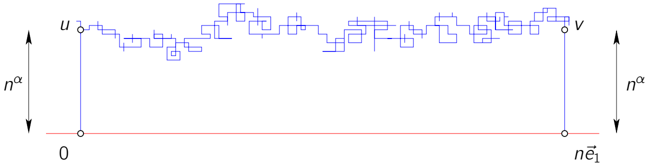

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But

$$\begin{aligned} \mathbb{P}_{p,p'}(u \longleftrightarrow v) &\geq \mathbb{P}_{p,p'}(u \longleftrightarrow v, u \not\longleftrightarrow \mathcal{L}) \\ &= \mathbb{P}_p(u \longleftrightarrow v, u \not\longleftrightarrow \mathcal{L}) \\ &= (1 - o(1))\, \mathbb{P}_p(u \longleftrightarrow v) \\ &= e^{-\xi_p n(1+o(1))}. \end{aligned}$$

This implies that $\xi_{p,p'} \leq \xi_p$, since $\mathbb{P}_{p,p'}(0 \longleftrightarrow n\vec{e}_1) \leq e^{-\xi_{p,p'} n}$.

Continuity

Fact #2

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- Let $p/2 \leq p'_1 < p'_2 \leq 1$. From Russo's formula,

  $$\frac{\mathbb{P}_{p,p'_2}(0 \longleftrightarrow n\vec{e}_1)}{\mathbb{P}_{p,p'_1}(0 \longleftrightarrow n\vec{e}_1)} = \exp\left\{\int_{p'_1}^{p'_2} \frac{1}{s}\mathbb{E}_{p,s}\left[\#\mathrm{Piv}_{\mathcal{L}}(0 \longleftrightarrow n\vec{e}_1) \,\middle|\, 0 \longleftrightarrow n\vec{e}_1\right] \mathrm{d}s\right\},$$

  where $\mathrm{Piv}_{\mathcal{L}}(0 \longleftrightarrow n\vec{e}_1)$ is the set of pivotal edges, for the event $0 \longleftrightarrow n\vec{e}_1$, contained in $\mathcal{L}$.

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where $\mathrm{Piv}_{\mathcal{L}}(0 \longleftrightarrow n\vec{e}_1)$ is the set of pivotal edges, for the event $0 \longleftrightarrow n\vec{e}_1$, contained in $\mathcal{L}$.
- It is easy to show that $|C(0,n\vec{e}_1) \cap \mathcal{L}| \le 2n$, with high probability. This implies that
$$\frac{\mathbb{P}_{p,p_2'}(0 \longleftrightarrow n\vec{e}_1)}{\mathbb{P}_{p,p_1'}(0 \longleftrightarrow n\vec{e}_1)} \le \exp\Big\{\frac{8}{p}(p_2' - p_1')n\Big].$$

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- Therefore

  $$0 \le \xi_{p,p'_1} - \xi_{p,p'_2} \le \frac{8}{p}(p'_2 - p'_1).$$

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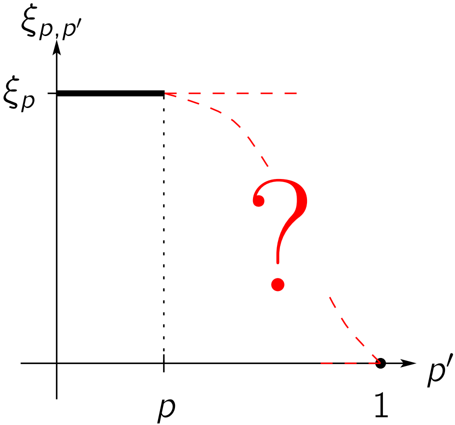
Summary



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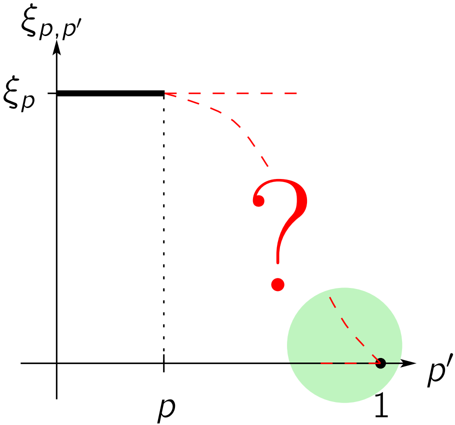

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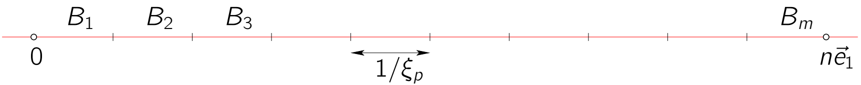

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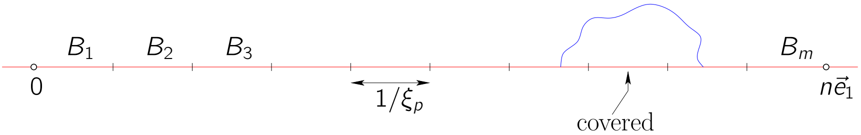

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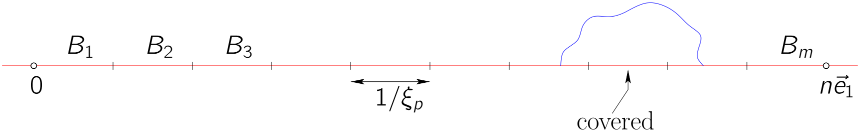

Up to a probability at most e^{-cn},

- Positive fraction of uncovered blocks.

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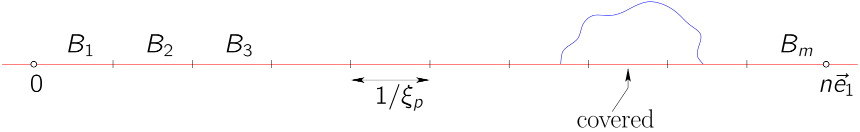

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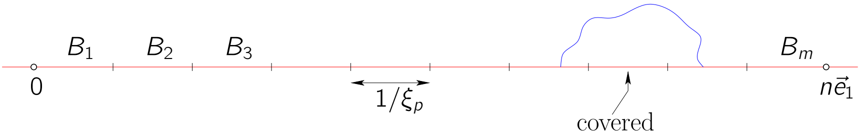

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The event $\{0 \longleftrightarrow n\vec{e}_1\}$ occurs only if there are no uncovered blocks with all their edges closed, which is exponentially unlikely.

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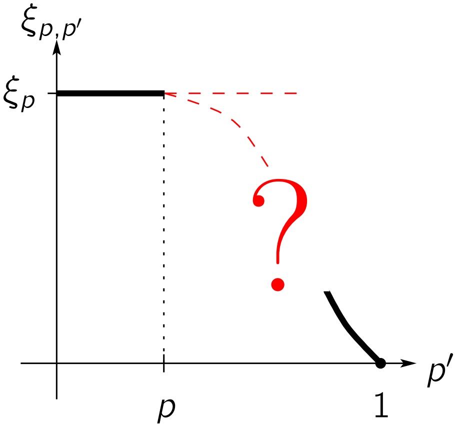

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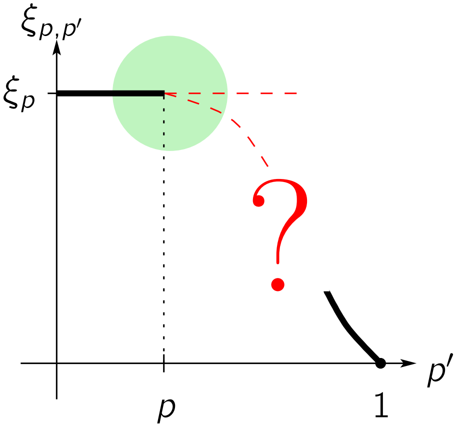

Critical point

Let $p'_c = p'_c(d) = \sup\{p' \,:\, \xi_{p,p'} = \xi_p\}$.

Fact #4

$$p'_c(2) = p'_c(3) = p, \qquad \forall d \geq 4: \; p'_c(d) \in (p, 1).$$

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Let $p_c' = p_c'(d) = \sup\{p' : \xi_{p,p'} = \xi_p\}$.

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This amounts to determining whether

$$\frac{\mathbb{P}_{p,p'}(0 \longleftrightarrow n\vec{e}_1)}{\mathbb{P}_p(0 \longleftrightarrow n\vec{e}_1)}$$

grows exponentially fast with n when p' is slightly larger than p.

Critical point

- Observe that

$$\frac{\mathbb{P}_{p,p'}(0 \longleftrightarrow n\vec{e}_1)}{\mathbb{P}_p(0 \longleftrightarrow n\vec{e}_1)} = \mathbb{E}_p\big[e^{\tilde{L}(C_{0,n\vec{e}_1})} \mid 0 \longleftrightarrow n\vec{e}_1\big],$$

where

$$\tilde{L}(C) = \log(p'/p)\,|C \cap \mathcal{L}| + \log((1-p')/(1-p))\,|\partial C \cap \mathcal{L}|,$$

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- Superficially similar to the pinning problem for a $(d-1)$-dimensional RW $(X_n)_{n\geq 0}$: determine the growth rate of

$$E_{RW}[e^{\epsilon L_N} | X_N = 0]\,,$$

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- **Major difference:** above, $\log(p'/p)$ and $\log((1-p')/(1-p))$ have opposite signs, which results in *both attractive and repulsive components*.

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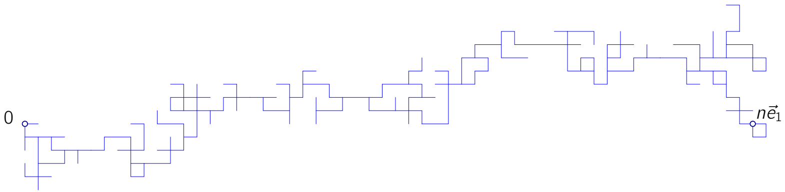
Essential tool: random walk representation of subcritical percolation clusters [Campanino, Ioffe&V., AoP '08].

Let $p < p_c$ and $n \in \mathbb{N}$. Then, up to an event of exponentially small $\mathbb{P}_p(\cdot \,|\, 0 \longleftrightarrow n\vec{e}_1)$-probability, $C_{0,n\vec{e}_1}$ admits the following decomposition:



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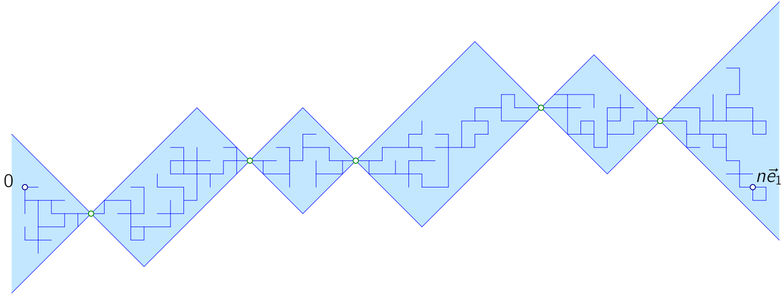

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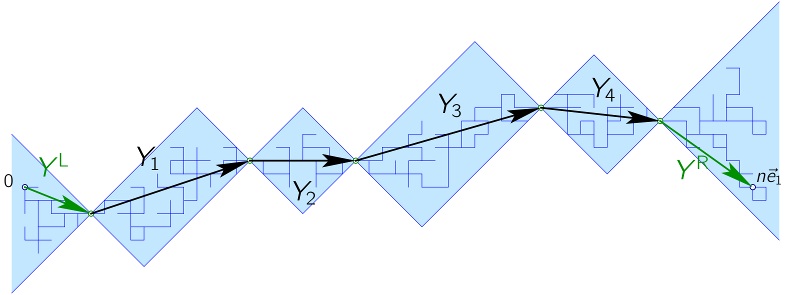

$$\{0 \longleftrightarrow n\vec{e}_1\} = \{Y^L + Y_1 + \cdots + Y_N + Y^R = n\vec{e}_1\},$$

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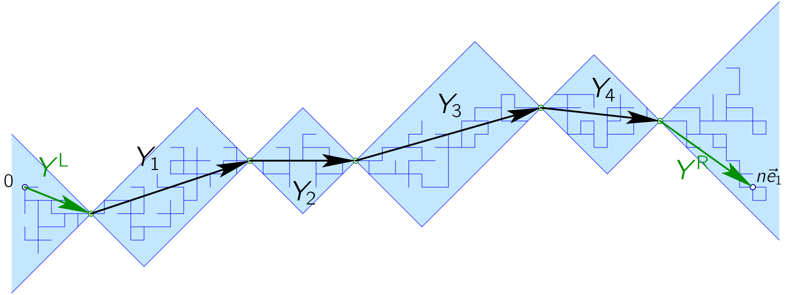

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In the sequel, I'll always ignore the boundary terms Y^L and Y^R.

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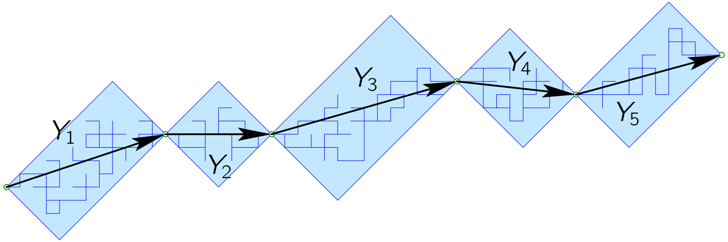
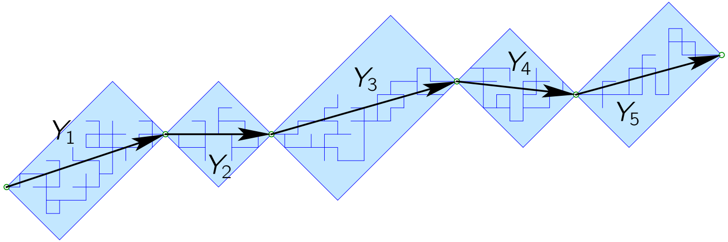


We write $Y_k = (Y_k^{\parallel}, Y_k^{\perp}) \in \mathbb{Z} \times \mathbb{Z}^{d-1}$.



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Properties of the effective random walk Y:

- $\mathrm{P}(Y_1^{\parallel} \geq 1) = 1$;
- $\mathrm{P}(|Y_1| > t) \leq e^{-\nu t}$ for some $\nu = \nu(p, d) > 0$;
- for any $z^{\perp} \in \mathbb{Z}^{d-1}$, $\mathrm{P}\big(Y_1^{\perp} = z^{\perp}\big) = \mathrm{P}\big(Y_1^{\perp} = -z^{\perp}\big)$.

Critical point

Assume that $d \geq 4$. We already know that $p'_c < 1$, by continuity.

To prove that $p'_c > p$, we return to the observation that

$$\frac{\mathbb{P}_{p,p'}(0 \longleftrightarrow n\vec{e}_1)}{\mathbb{P}_p(0 \longleftrightarrow n\vec{e}_1)} = \mathbb{E}_p\left[e^{\tilde{L}(C_{0,n\vec{e}_1})} \,\middle|\, 0 \longleftrightarrow n\vec{e}_1\right]$$

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with

$$\hat{L}(C) = \underbrace{\log(p'/p)}_{\equiv \epsilon > 0} |C \cap \mathcal{L}|.$$

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Rewriting the previous expression in terms of the effective RW yields:

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$$\leq \mathrm{E}\Big[e^{\epsilon\sum_{i=1}^{T_n}|D(Y_i,Y_{i-1})\cap\mathcal{L}|} \;\Big|\; \exists N \geq 1 : Y_N = n\vec{e}_1\Big],$$

where

- $D(Y_{i-1}, Y_i)$ denotes the "diamond" containing the piece of cluster between Y_{i-1} and Y_i;
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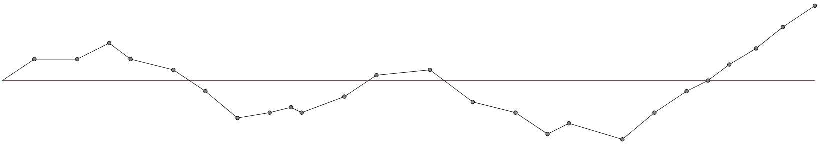

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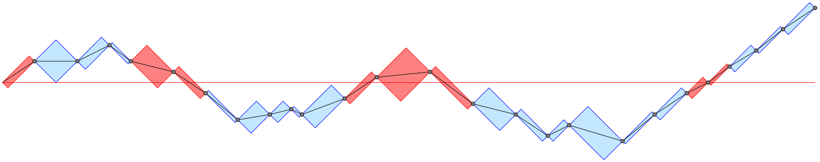

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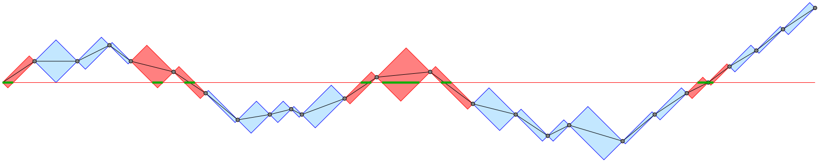

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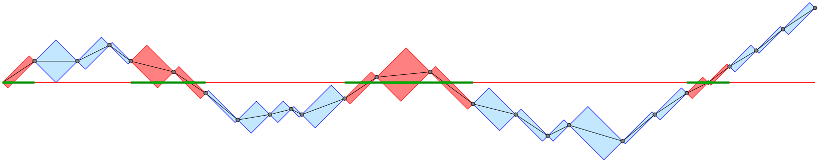
$$\frac{\mathbb{P}_{p,p'}(0 \longleftrightarrow n\vec{e}_1)}{\mathbb{P}_p(0 \longleftrightarrow n\vec{e}_1)} \leq \mathbb{E}_p\left[e^{\epsilon|C\cap\mathcal{L}|} \mid 0 \longleftrightarrow n\vec{e}_1\right]$$

$$\leq \mathrm{E}\left[e^{\epsilon\sum_{i=1}^{T_n}|D(Y_i,Y_{i-1})\cap\mathcal{L}|} \mid \exists N \geq 1 : Y_N = n\vec{e}_1\right],$$

where

- $D(Y_{i-1}, Y_i)$ denotes the "diamond" containing the piece of cluster between Y_{i-1} and Y_i;
- $T_n = \min\{k \geq 1\ :\ Y_k = n\vec{e}_1\} \leq n$.



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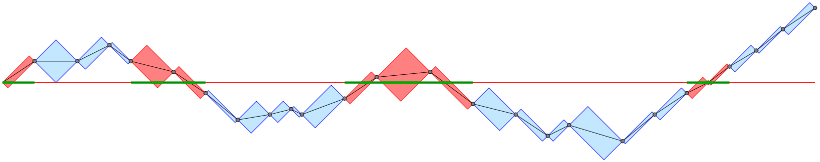

We're essentially back to the pinning problem for a RW in dimension $3+1$ or more, for which the claim is easy.

Critical point

Let us turn now to the proof that $p_c' = p$ when $d = 2, 3$.

We introduce a suitable event $\mathcal{M}_\delta \subset \{0 \longleftrightarrow n\vec{e}_1\}$ and write

$$\frac{\mathbb{P}_{p,p'}(0 \longleftrightarrow n\vec{e}_1)}{\mathbb{P}_p(0 \longleftrightarrow n\vec{e}_1)} \geq \frac{\mathbb{P}_{p,p'}(\mathcal{M}_\delta)}{\mathbb{P}_p(0 \longleftrightarrow n\vec{e}_1)}$$

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We'll choose $\mathcal{M}_\delta$ (δ small) in such a way that

$$\frac{\mathbb{P}_{p,p'}(\mathcal{M}_\delta)}{\mathbb{P}_p(\mathcal{M}_\delta)} \geq e^{c\delta(p'-p)n}.$$

and

$$\mathbb{P}_p(\mathcal{M}_\delta | 0 \longleftrightarrow n\vec{e}_1) \geq \begin{cases} e^{-c\,\delta^2 n} & \text{if } d = 2, \\ e^{-c\,(\delta/|\log\delta|)n} & \text{if } d = 3. \end{cases}$$

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The conclusion follows for small enough δ, since $\delta \gg \delta^2, \delta/|\log\delta|$.

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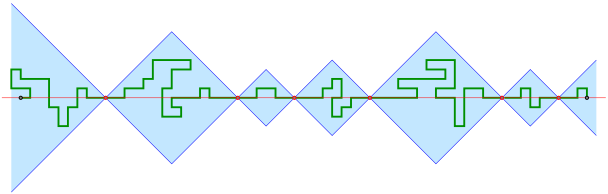
We choose for $\mathcal{M}_\delta \subset \{0 \longleftrightarrow n\vec{e}_1\}$ the event

There exists a self-avoiding path $\gamma \subset C_{0,n\vec{e}_1}$ possessing at least δn cone-points on $\mathcal{L}$



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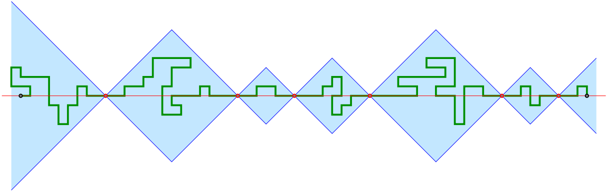

Entropy estimate:

- What is the probability that the effective random walk Y visits $\mathcal{L}$ at least δn times before reaching $n\vec{e}_1$?
- Not difficult to obtain estimates of the correct order.

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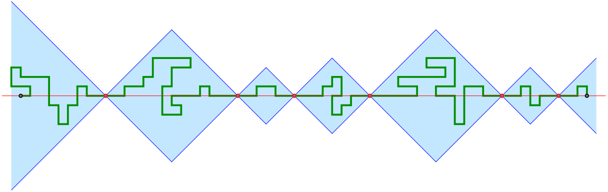

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Let's see how the energy bound is established...

Critical point

Since $\mathcal{M}_\delta$ is an increasing event, we can again use Russo's formula

$$\frac{\mathbb{P}_{p,p'}(\mathcal{M}_\delta)}{\mathbb{P}_p(\mathcal{M}_\delta)} = \exp \int_p^{p'} \frac{1}{s} \mathbb{E}_{p,s}\left[\#\mathrm{Piv}_{\mathcal{L}}(\mathcal{M}_\delta) \mid \mathcal{M}_\delta\right] \mathrm{d}s$$

$$\geq \exp \int_p^{p'} \frac{1}{s} \mathbb{E}_{p,s}\left[\#\mathrm{Piv}_{\mathcal{L}}(0 \longleftrightarrow n\vec{e}_1) \mid \mathcal{M}_\delta\right] \mathrm{d}s \,.$$

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The problem is thus reduced to proving that there are in average $O(\delta n)$ pivotal edges on $\mathcal{L}_n$ for the event $\{0 \longleftrightarrow n\vec{e}_1\}$, when $\mathcal{M}_\delta$ occurs.

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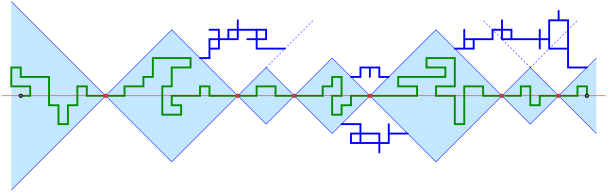

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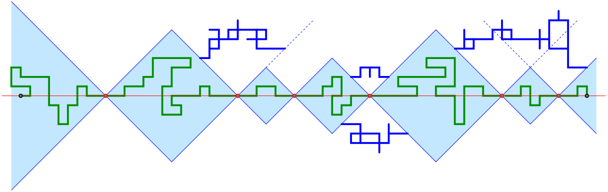
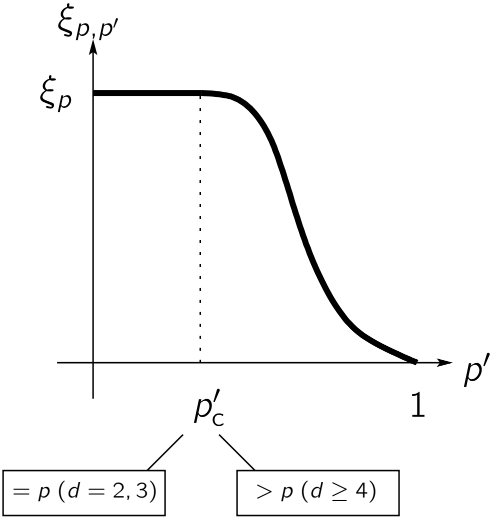


Claim: since $p < p_c$, a positive fraction of the cone-points of the path on $\mathcal{L}$ are not covered, with high probability. Since the edges incident on such cone-points are necessarily pivotal for the connection, we obtain:

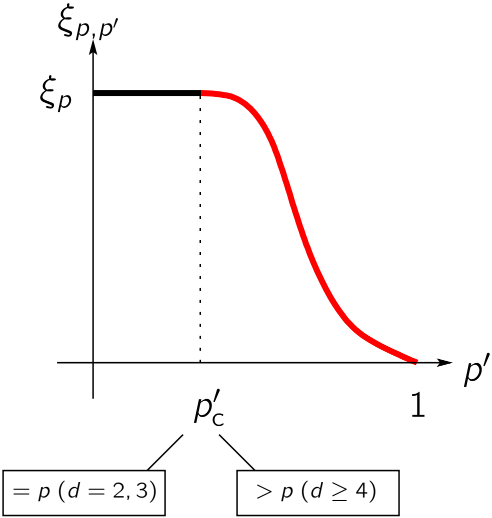
$$\mathbb{E}_{p,s}\left(\#\mathrm{Piv}_{\mathcal{L}}(0 \longleftrightarrow n\vec{e}_1) \;\middle|\; 0 \longleftrightarrow n\vec{e}_1, \mathcal{M}_\delta\right) \geq c\delta n,$$

for some $c = c(p) > 0$. The conclusion follows.

Summary



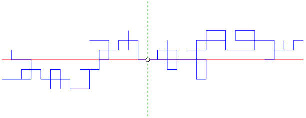
Summary



Additional informations about the red part?

Behavior on $(p'_c, 1)$

Let $\#\mathrm{CutPts}_{\mathcal{L}}(C_{0,n\vec{e}_1})$ denote the number of cut-points of $C_{0,n\vec{e}_1}$ on the line $\mathcal{L}$



Fact #5

Assume that $p' > p'_c$. Then, there exist $\rho, c > 0$ such that

$$\mathbb{P}_{p,p'}(\#\mathrm{CutPts}_{\mathcal{L}}(C_{0,n\vec{e}_1}) < \rho n \,|\, 0 \longleftrightarrow n\vec{e}_1) \leq e^{-cn}.$$

Three steps:

- A self-avoiding path $\pi : 0 \to n\vec{e}_1$ makes typically only small excursions away from $\mathcal{L}$.
- Conditionally on such a path π, most of the cluster remains close to π.
- Surgery to ensure the presence of many cut-points on $\mathcal{L}$.

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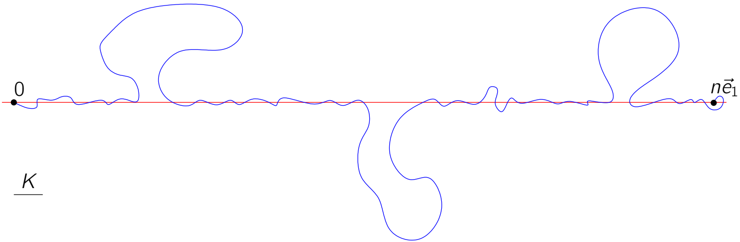
Step 1. Let us consider a self-avoiding path $\pi : 0 \to n\vec{e}_1$. We want to show that π typically leaves $\mathcal{L}$ only for small excursions.

Fix $K > 0$ large (depending on p, p'). We coarse-grain π as follows:



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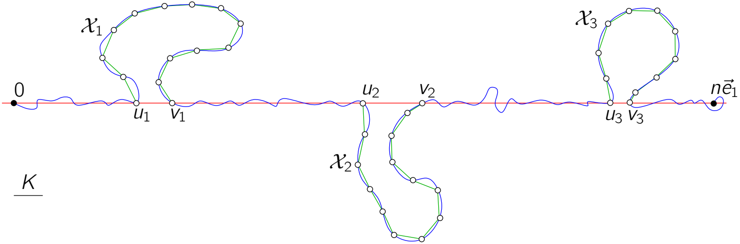
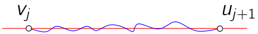


The resulting broken line is the **skeleton** associated to π.

Behavior on $(p_c', 1)$

Probabilistic cost of the pieces (remember that $\xi_p > \xi_{p,p'}$):

- Of a stretch along the line:



$$\leq e^{-\xi_{p,p'}|u_{j+1}-v_j|}$$

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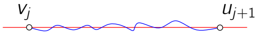

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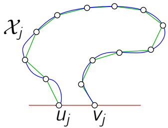
- Of an excursion $\mathcal{X}_j$ with $|\mathcal{X}_j|$ K-steps away from the line:



$$\leq e^{-\xi_{p,p'}|v_j-u_j|-c|\mathcal{X}_j|K}$$

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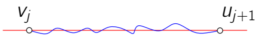

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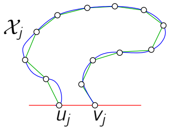

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Therefore the $\mathbb{P}_{p,p'}$-probability of a skeleton is bounded above by

$$e^{-\xi_{p,p'} n - cK \sum_{j=1}^{M} |\mathcal{X}_j|},$$

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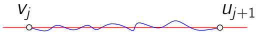

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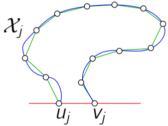

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and thus its $\mathbb{P}_{p,p'}(\,\cdot\,|\,0 \longleftrightarrow n\vec{e}_1)$-probability is bounded above by

$$\frac{e^{-\xi_{p,p'}n-cK\sum_{j=1}^{M}|\mathcal{X}_j|}}{e^{-\xi_{p,p'}n(1-o(1))}} \leq e^{-cK\sum_{j=1}^{M}|\mathcal{X}_j|+o(n)}.$$

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Behavior on $(p_c', 1)$

On can deduce from the above (and a control over the entropy of such skeletons) that

$$\mathbb{P}_{p,p'}\Big(\exists\pi : 0 \to n\vec{e}_1 \text{ s.t. } \sum_{j=1}^{M} K|\mathcal{X}_j| \geq \epsilon n \;\Big|\; 0 \longleftrightarrow n\vec{e}_1\Big) \leq e^{-c(\epsilon)n}.$$

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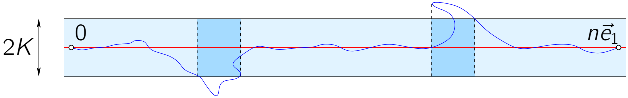

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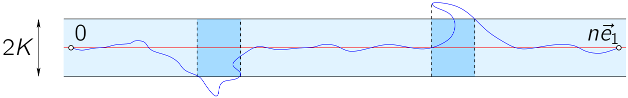
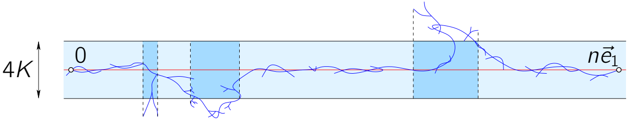
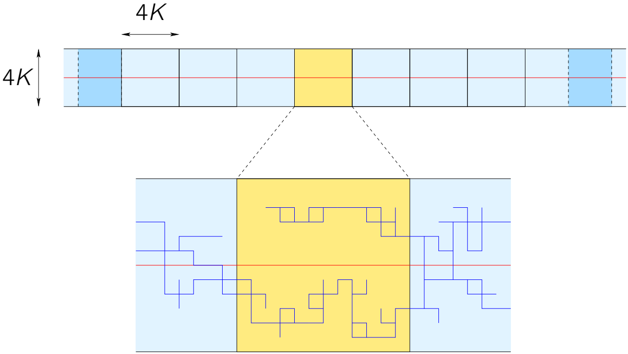


A similar coarse-graining argument shows that the same is true for $C_{0,n\vec{e}_1}$:



Behavior on $(p_c', 1)$

The conclusion follows from a surgery argument:



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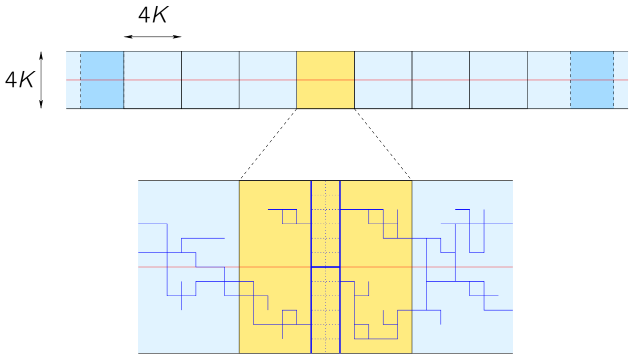

This has a positive probability of occuring in any box, uniformly in what happens elsewhere. Therefore a positive fraction of the boxes must contain a cut-point on $\mathcal{L}$.

Behavior on $(p'_c, 1)$

Fact #6

$\xi_{p,p'}$ is strictly decreasing and real analytic on $(p'_c, 1)$.

We use, once more, Russo's formula:

$$\frac{\partial}{\partial p'} \log \mathbb{P}_{p,p'}(0 \longleftrightarrow n\vec{e}_1) = \mathbb{E}_{p,p'}\left[\#\mathrm{Piv}_{\mathcal{L}}(0 \longleftrightarrow n\vec{e}_1) \;\middle|\; 0 \longleftrightarrow n\vec{e}_1\right].$$

Behavior on $(p'_c, 1)$

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Since $p' > p'_c$,

$$\mathbb{E}_{p,p'}\left[\#\mathrm{Piv}_{\mathcal{L}}(0 \longleftrightarrow n\vec{e}_1) \,\middle|\, 0 \longleftrightarrow n\vec{e}_1\right] \geq \tfrac{\rho}{2} n.$$

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Consequently,

$$\begin{aligned}\xi_{p,p'_2} - \xi_{p,p'_1} &= -\lim_{n\to\infty} \frac{1}{n}\left(\log \mathbb{P}_{p,p'_2}(0 \longleftrightarrow n\vec{e}_1) - \log \mathbb{P}_{p,p'_1}(0 \longleftrightarrow n\vec{e}_1)\right) \\ &\leq -\frac{\rho}{2}(p'_2 - p'_1)\,.\end{aligned}$$

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$$\frac{\partial}{\partial p'} \log \mathbb{P}_{p,p'}(0 \longleftrightarrow n\vec{e}_1) = \mathbb{E}_{p,p'}\left[\#\mathrm{Piv}_{\mathcal{L}}(0 \longleftrightarrow n\vec{e}_1) \;\middle|\; 0 \longleftrightarrow n\vec{e}_1\right].$$

Since $p' > p'_c$,

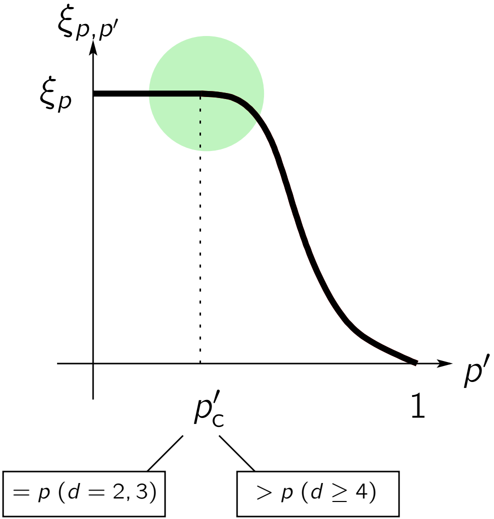
$$\mathbb{E}_{p,p'}\left[\#\mathrm{Piv}_{\mathcal{L}}(0 \longleftrightarrow n\vec{e}_1) \;\middle|\; 0 \longleftrightarrow n\vec{e}_1\right] \geq \tfrac{\rho}{2} n.$$

Consequently,

$$\begin{aligned}\xi_{p,p'_2} - \xi_{p,p'_1} &= -\lim_{n\to\infty} \frac{1}{n}\left(\log \mathbb{P}_{p,p'_2}(0 \longleftrightarrow n\vec{e}_1) - \log \mathbb{P}_{p,p'_1}(0 \longleftrightarrow n\vec{e}_1)\right)\\ &\leq -\frac{\rho}{2}(p'_2 - p'_1)\,.\end{aligned}$$

Analyticity follows from the renewal structure of $C_{0,n\vec{e}_1}$.

Summary



What about the critical behavior as $p' \downarrow p'_c$?

Critical behavior

Only in dimensions 2 and 3, unfortunately.

Fact #7

There exist constants $c_2^\pm, c_3^\pm > 0$ such that, as $p' \downarrow p'_c$,

$$c_2^-(p' - p'_c)^2 \leq \xi_p - \xi_{p,p'} \leq c_2^+(p' - p'_c)^2 \qquad (d = 2), \qquad (1)$$

$$e^{-c_3^-/(p'-p'_c)} \leq \xi_p - \xi_{p,p'} \leq e^{-c_3^+/(p'-p'_c)} \qquad (d = 3). \qquad (2)$$

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This actually follows from the estimates on p'_{c} done before, by taking care of the dependence on p' of the various constants...

Sharp asymptotics

Fact #8

For all $d \geq 2$ and for all $p' > p'_c$, there exists $r_d = r_d(p, p') > 0$ such that

$$\mathbb{P}_{p,p'}(0 \longleftrightarrow n\vec{e}_1) = r_d \, e^{-\xi_{p,p'} n} \, (1 + o(1)) \, .$$

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This should be contrasted with the behavior in the homogeneous case [Campanino&Chayes², PTRF '91]: for all $d \geq 1$,

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The proof of these purely exponential asymptotics relies on the renewal structure of $C_{0,n\vec{e}_1}$.

Open problems

- Properties of $\xi_{p,p'}$:
  - Analyze the behavior of $\xi_{p,p'}$ as $p' \downarrow p'_c$, in dimensions $d \geq 4$.
  - Analyze the behavior of $\xi_{p,p'}$ as a function of both p and p'. In particular, for (p, p') close to the critical line $p \mapsto p'_c(p)$.
- More general defects:
  - Defect line not coinciding with a coordinate axis; higher-dimensional defects (e.g., hyperplanes of given codimension).
  - Half-space percolation, with the defect line (or hyperplane) at the boundary of the system.
- Sharp asymptotics of the connectivity function $\mathbb{P}_{p,p'}(0 \longleftrightarrow n\vec{e}_1)$ for $p' \leq p'_c$, and the corresponding scaling limit of the cluster $C_{0,n\vec{e}_1}$.
- Extension to other models: e.g., a version for FK-percolation would provide an extension to Ising/Potts models.

Thank you!