Fluctuation theory for a layer of unstable phase in the planar Ising model

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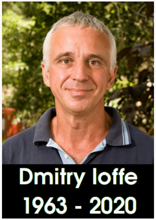
Joint work with Dmitry Ioffe, Sébastien Ott and Senya Shlosman



— Introduction —

Ising model

- **Box:** $\mathrm{B}_N = \{-N+1, \dots, N\}^2$

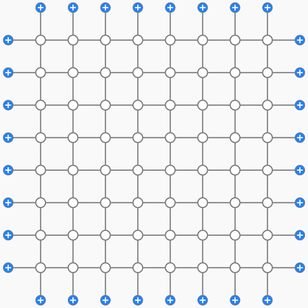
- ⊕ **boundary condition:**

$$\Omega_N^{\oplus} = \{\sigma = (\sigma_i)_{i \in \mathbb{Z}^2} \in \{\pm 1\}^{\mathbb{Z}^2} \;:\; \forall i \notin \mathrm{B}_N, \sigma_i = 1\}$$

- **Hamiltonian:** $\mathscr{H}_N(\sigma) = -\beta \sum_{\substack{\{i,j\} \cap \mathrm{B}_N \neq \varnothing \\ i \sim j}} \sigma_i \sigma_j$

- **Gibbs measure:** Probability measure on $\Omega_N^{\oplus}$ s.t.

$$\mu_{N;\beta}^{\oplus}(\sigma) = \frac{1}{\mathscr{Z}_{N;\beta}^{\oplus}} \mathrm{e}^{-\mathscr{H}_N(\sigma)}$$



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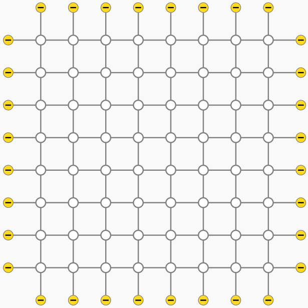


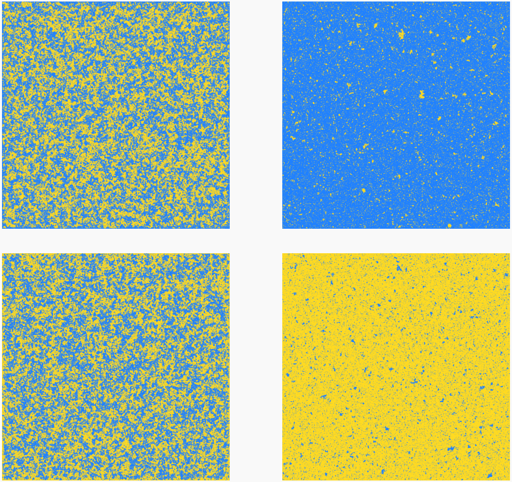
Extends trivially to other boundary conditions.

For instance, the ⊖ **boundary condition**: $\mu_{N;\beta}^{\ominus}, \dots$

Phase transition

Let $\beta_c = \frac{1}{2}\log(1+\sqrt{2})$. **Typical configurations** at $\beta \in [0,\infty)$ for $N > N_0(\beta)$:

$\beta < \beta_c$	$\beta > \beta_c$	



under $\mu^{\oplus}_{N;\beta}$

under $\mu^{\ominus}_{N;\beta}$

— Phase coexistence —

Phase coexistence: from a constraint on the magnetization

- Let $\beta > \beta_c$. $m^*_\beta = \lim_{N\to\infty} \mu^{+}_{N;\beta}(\sigma_0) > 0$ is the **spontaneous magnetization**.

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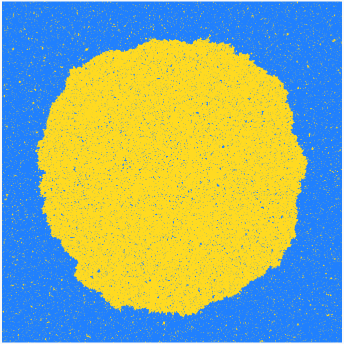
- Typical configurations contain a **unique macroscopic droplet** of $-$ phase, whose shape becomes deterministic in the continuum limit.



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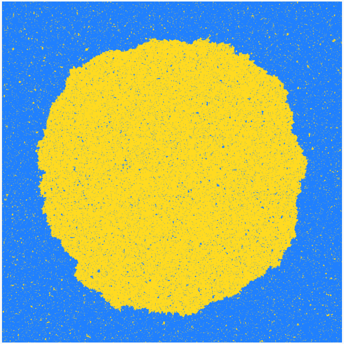
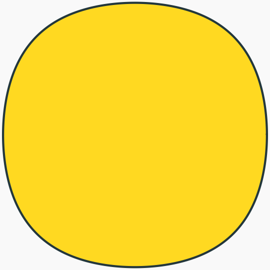
- Typical configurations contain a **unique macroscopic droplet** of $-$ phase, whose shape becomes deterministic in the continuum limit. Limiting shape is the **Wulff shape**.

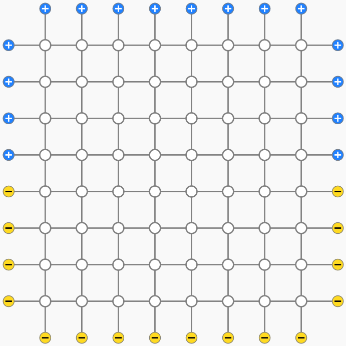


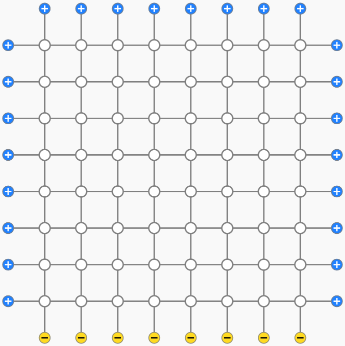


Well understood for planar Ising model since 1990s ([Dobrushin, Kotecký, Shlosman '92], [Pfister '91], [Ioffe '94, '95], [Pfister, V. '97], [Dobrushin, Hryniv '97], [Ioffe, Schonmann '98], ...)

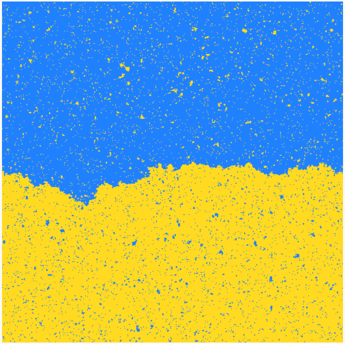
Phase coexistence: from boundary conditions

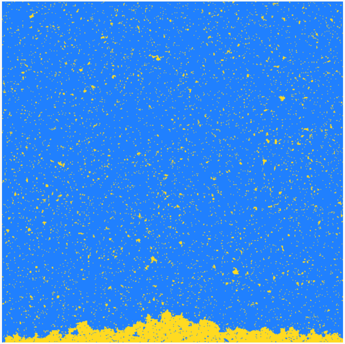
An alternative way of enforcing spatial coexistence is to consider various types of **Dobrushin boundary condition**:





Typical configurations induced by these boundary conditions when $\beta > \beta_c$





Phase coexistence: scaling limit of the interface

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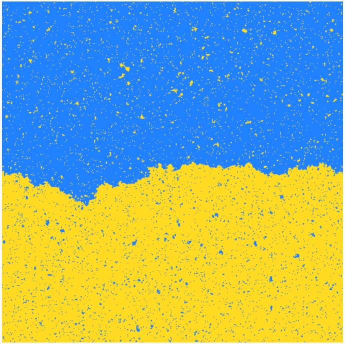
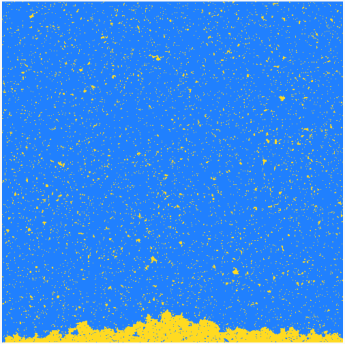




Corresponding **(diffusive) scaling limits** of the interface

Brownian bridge

[Greenberg, Ioffe 2005]

Brownian excursion

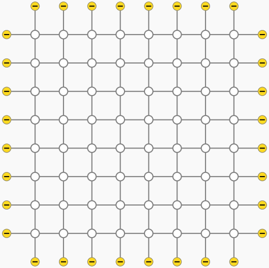
[Ioffe, Ott, V., Wachtel 2020]

Diffusive constant = curvature χ_β of the Wulff shape at its apex: 

— Metastability —

Effect of a magnetic field: metastability

- Let us consider again the ⊖ boundary condition



but let us add to the Hamiltonian a **magnetic field term**

$$-h \sum_{i \in B_N} \sigma_i$$

with $\boldsymbol{h > 0}$.

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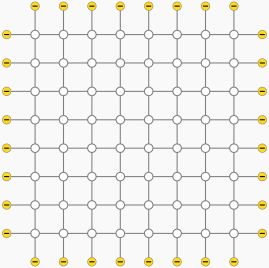

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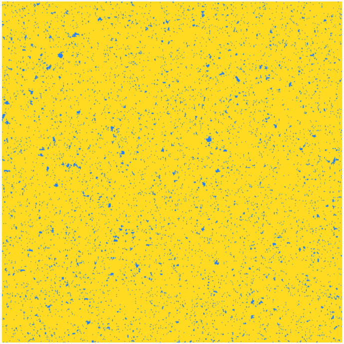
- This induces a **competition between the boundary condition and the magnetic field**:

effect of the boundary condition $\sim N$ effect of the field $\sim hN^2$

competition if $h \sim 1/N$

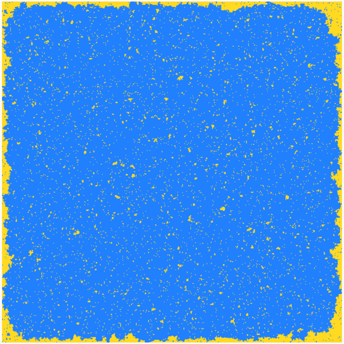
Effect of a magnetic field: metastability

- Let $h = \lambda/N$. [Schonmann and Shlosman 1996] proved: $\exists \lambda_c \in (0, \infty)$ such that



$\lambda < \lambda_c$

— phase is **metastable**



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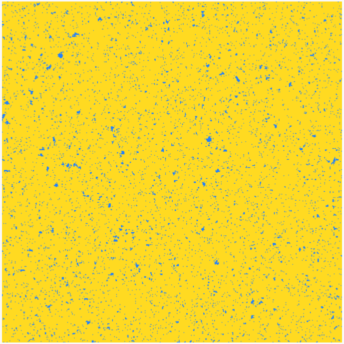

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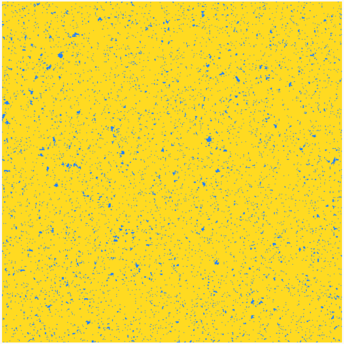

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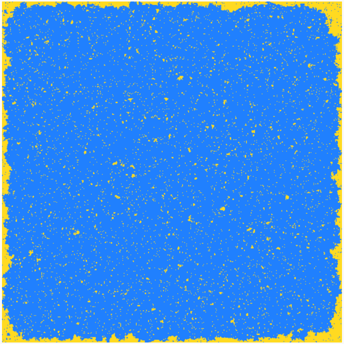

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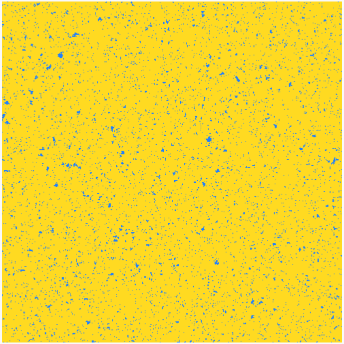

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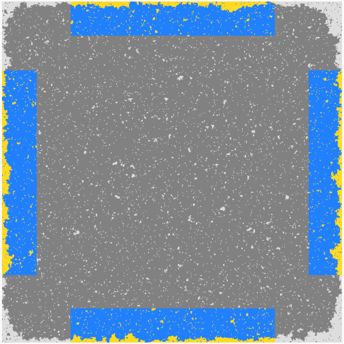


$\lambda > \lambda_c$

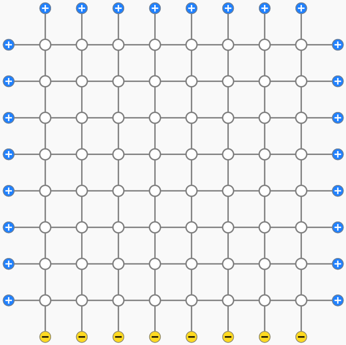
— phase is **unstable**

- **Question:** Behavior of the layer of unstable — phase **along the walls**?

— Behavior of an unstable layer —

Simpler geometry

We consider again the boundary condition

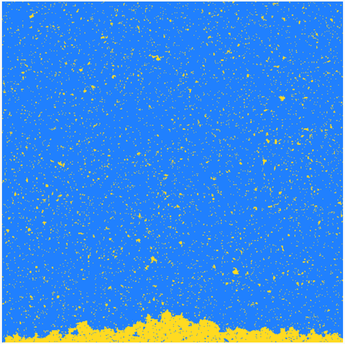


but add to the Hamiltonian a **magnetic field term**

$$-h \sum_{i \in \mathrm{B}_N} \sigma_i$$

with $\boldsymbol{h} > \mathbf{0}$.

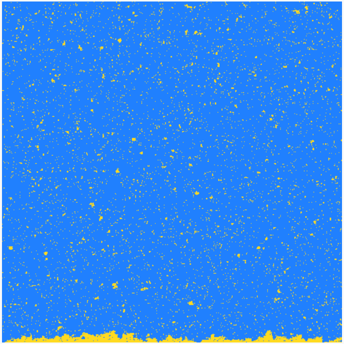
Let $\beta > \beta_c$. Since $h > 0$, the layer of $-$ phase becomes **unstable**:



$h = 0$

mesoscopic layer

average width $= O(N^{1/2})$

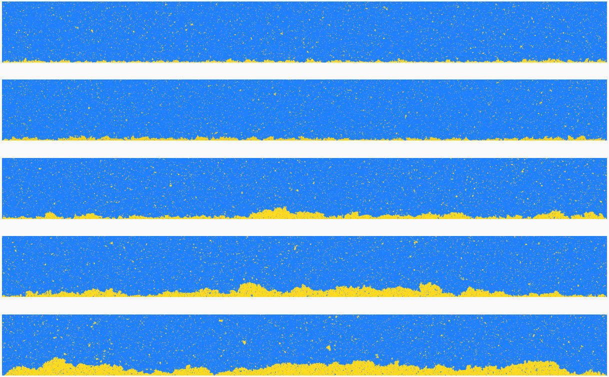


$h > 0$

microscopic layer

average width $= O(1)$

- The width of the layer increases as h decreases:



Critical prewetting

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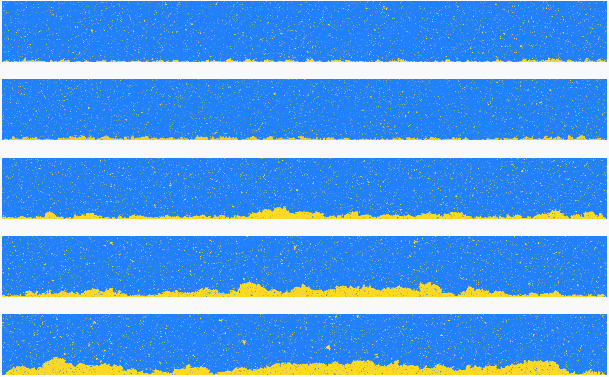


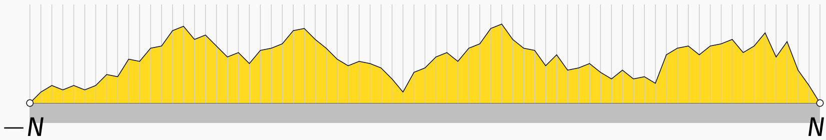
- To get a meaningful scaling limit and mimic the Schonmann–Shlosman setting, we choose $h = h(N)$ to be of the form

$$h = \frac{\lambda}{N}$$

for some $\lambda > 0$.

Earlier rigorous results

- This type of problem was first studied for **effective models** in
  - [Abraham, Smith 1986] specific integrable model: width $\sim N^{1/3}$, corr. length $\sim N^{2/3}$
  - [Hryniv, V. 2004] general class: width $\sim N^{1/3}$, correlation length $\sim N^{2/3}$
  - [Ioffe, Shlosman, V. 2015] general class: weak convergence to Ferrari–Spohn diffusion



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- Results for the **2d Ising model** were obtained in
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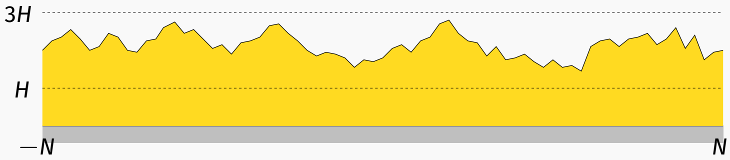
Goal of this work: complete the analysis by proving weak convergence to a Ferrari–Spohn diffusion for the (scaled) 2d Ising interface

Parenthesis: why $N^{1/3}$?

In effective models, it is easy to understand heuristically the origin of the $N^{1/3}$ scaling.

- Consider a path staying in the tube $[-N, N] \times [H, 3H]$ for some fixed $H > 0$.



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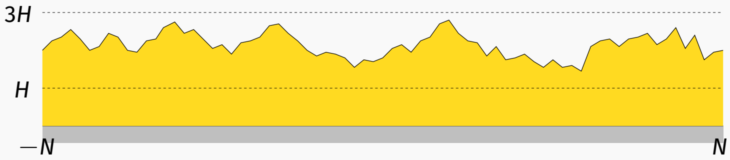

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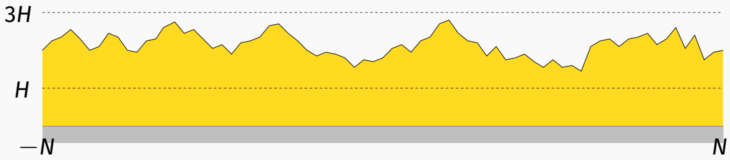

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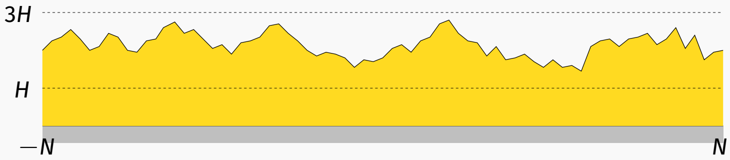

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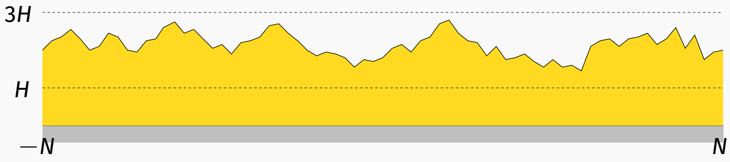

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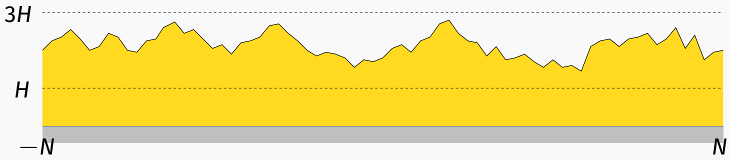

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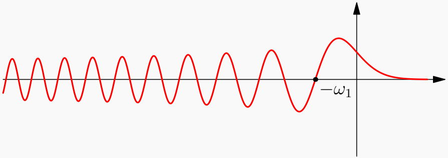
- This argument can be turned into a rigorous proof (for effective models).

The Ferrari–Spohn diffusion

- Remember that
  - m^*_β is the **spontaneous magnetization**
  - χ_β is the **curvature of the Wulff shape** at its apex.
- Consider the **Airy function** Ai and its first zero $-\omega_1$



- Set $\varphi_0(r) = \mathrm{Ai}\big((4\lambda m^*_\beta \sqrt{\chi_\beta})^{1/3}\, r - \omega_1\big)$.
- The relevant **Ferrari–Spohn diffusion** in the present context is the diffusion on $(0,\infty)$ with generator

$$L_\beta = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}r^2} + \frac{\varphi_0'}{\varphi_0}\frac{\mathrm{d}}{\mathrm{d}r}$$

and Dirichlet boundary condition at 0.

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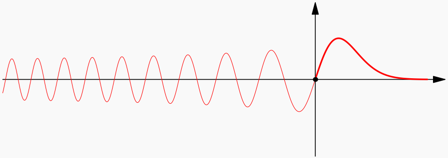

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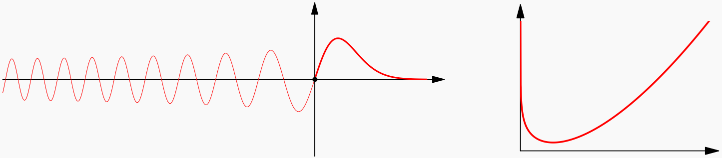
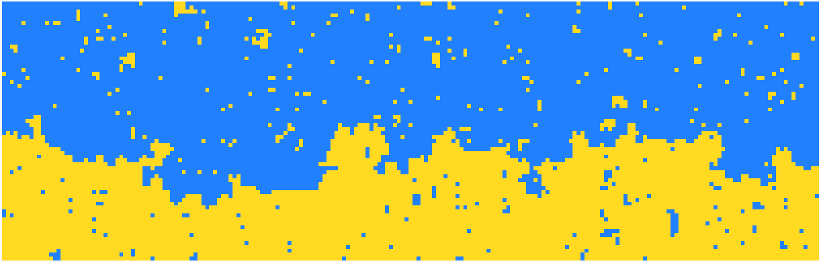


- Set $\varphi_0(r) = \mathrm{Ai}\big((4\lambda m^*_\beta \sqrt{\chi_\beta})^{1/3}\, r - \omega_1\big)$.
- The relevant **Ferrari–Spohn diffusion** in the present context is the diffusion on $(0, \infty)$ with generator

$$L_\beta = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}r^2} + \frac{\varphi_0'}{\varphi_0}\frac{\mathrm{d}}{\mathrm{d}r}$$

and Dirichlet boundary condition at 0.

- We want to prove weak convergence of the interface towards the FS diffusion, but the interface is not the graph of a function:

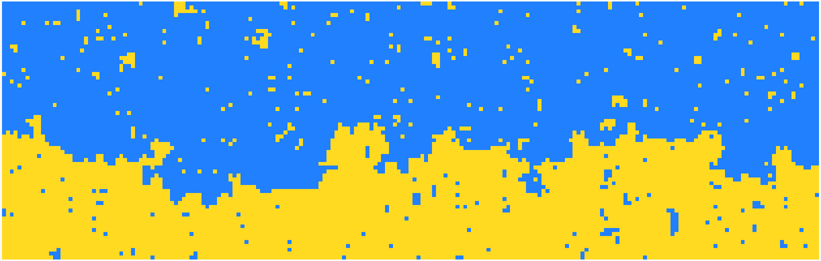


[zoom on a piece of interface]

- We thus need to explain what we mean by the above-mentioned convergence.

Structure of the interface

- We consider the upper and lower envelopes, whose linear interpolations are graphs of functions from $\mathbb{R}$ to $\mathbb{R}$.



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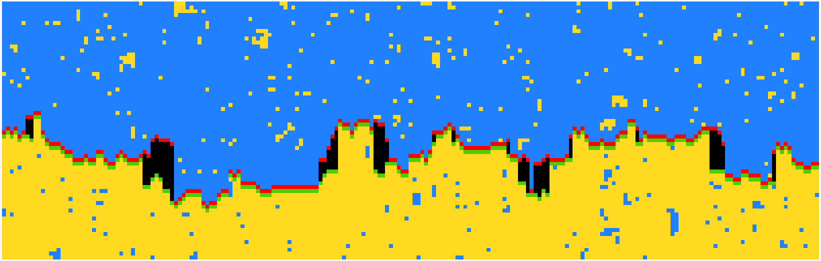

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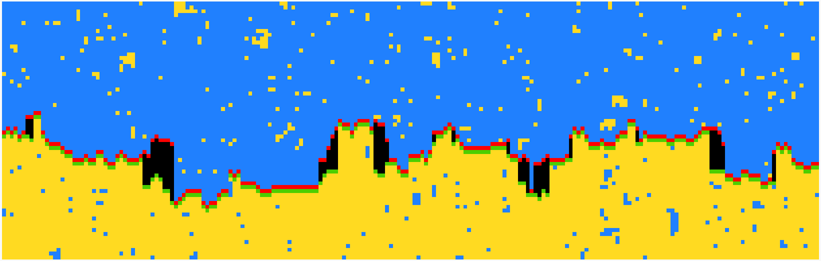


▶ It can be shown that there exists $K = K(\beta)$ such that the probability that these two envelopes differ by less than $K \log N$ everywhere tends to 1 as $N \to \infty$.

▶ Since the relevant vertical scale for our scaling will be $N^{1/3}$, one can use any of these envelopes for the weak convergence.

Main result

Theorem (Informal statement [Ioffe, Ott, Shlosman, V. 2020]**)**

Fix $\beta > \beta_c$ and $\lambda > 0$. Let $\hat{\gamma}^+ : \mathbb{R} \to \mathbb{R}$ be the function obtained from the (linearly interpolated) upper envelope by

- *scaling it horizontally by $N^{-2/3}$*
- *scaling it vertically by $\chi_\beta^{-1/2} N^{-1/3}$*

Then, as $N \to \infty$, the distribution of $\hat{\gamma}^+$ converges weakly to that of the trajectories the stationary Ferrari–Spohn diffusion introduced in a previous slide.

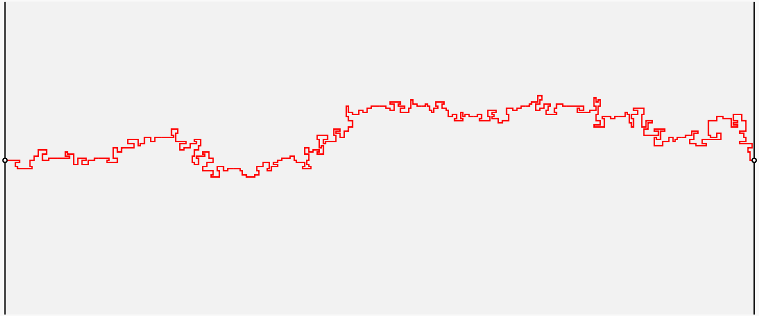
— Sketch of proof —

Sketch of proof — General strategy and difficulties

- The proofs of all the previously mentioned convergence theorems follow the same pattern: the **reduction to an effective model**.
- More precisely, one constructs a **coupling between the interface and the trajectories of a directed random walk** on $\mathbb{Z}^2$ (subject to suitable constraints or external potentials).
- This coupling is strong enough that **the desired convergence follows from the corresponding statement for the random walk**. This is useful, since establishing the latter is both easier and more classical.
- The construction of the coupling is **based on the Ornstein–Zernike theory** as developed, in particular, in [Campanino, Ioffe, V. 2003] and [Ott, V. 2018].

Sketch of proof — General strategy and difficulties

- Let us first consider, as a warm-up, the case of the standard Dobrushin b.c. in the absence of an external magnetic field.
- One can decompose the interface into pieces



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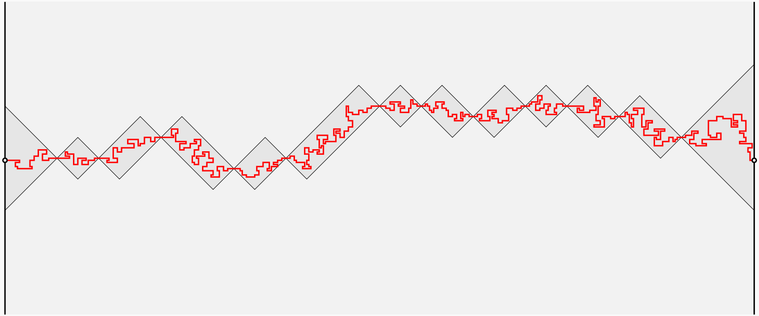

  - pieces are small (diameters have exponential moments)
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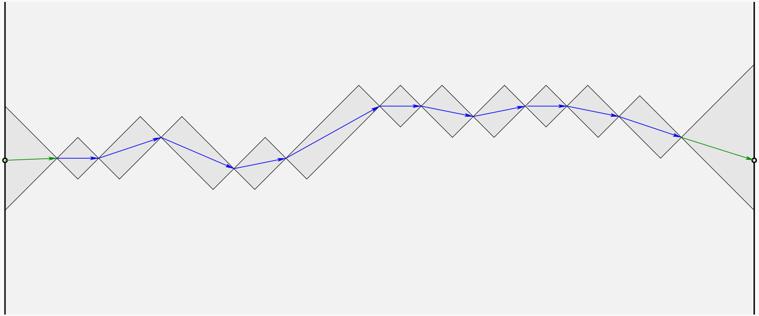


  - pieces are small (diameters have exponential moments)
  - pieces are i.i.d. (except the two extremal ones, which have a different law)
  - interface is contained inside the rectangles
- This leads to a directed RW on $\mathbb{Z}^2$ with increments having exponential moments.

Sketch of proof — General strategy and difficulties

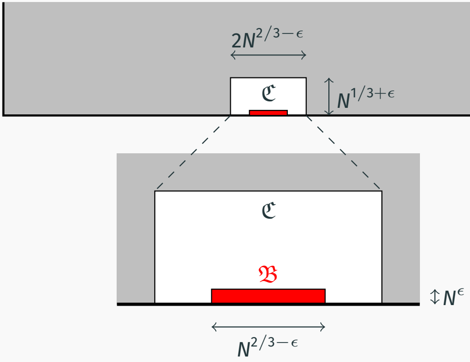
- Main difficulties in extending this to our case of interest:
  - The interface lies along the bottom wall. This leads to a **spatially inhomogeneous RW**, with transition probabilities depending (in a complicated way) on the distance to the wall.
  - The presence of a magnetic field **prevents a direct use of the Ornstein–Zernike theory**.

Sketch of proof — General strategy and difficulties

- Main difficulties in extending this to our case of interest:
  - The interface lies along the bottom wall. This leads to a **spatially inhomogeneous RW**, with transition probabilities depending (in a complicated way) on the distance to the wall.
  - The presence of a magnetic field **prevents a direct use of the Ornstein–Zernike theory**.
- In the next slides, I sketch the solutions to these two problems, namely
  - derivation of an a priori (rough) entropic repulsion bound that guarantees that **the interface stays far away from the bottom wall**, which restores (asymptotic) spatial homogeneity of the effective RW;
  - proof that the magnetic-field results in a **simple effective weight**, which allows reduction to the case $h = 0$.

Sketch of proof — Step 1: Entropic repulsion

First, given the following setting: for any fixed (small) $\epsilon > 0$,

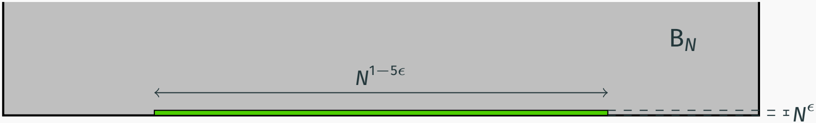


we show that, with high probability, γ does not intersect $\mathfrak{B}$, using the following facts:

- ▷ we can restrict to the same event in the box $\mathfrak{C}$ (by FKG)
- ▷ in the box $\mathfrak{C}$, the magnetic field is irrelevant ($\frac{\lambda}{N}|\mathfrak{C}| = 2\lambda$ is of order 1)
- ▷ this allows us to use weak convergence of the interface to Brownian excursion proved in [Ioffe, Ott, V., Wachtel 2020]

Sketch of proof — Step 1: Entropic repulsion

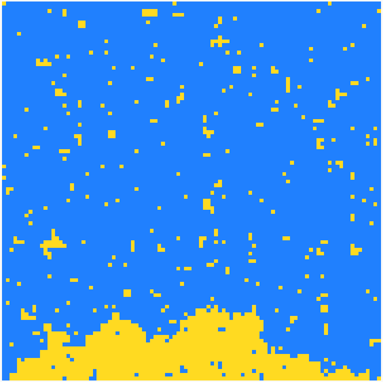
- A union bound then allows one to conclude that, with probability tending to 1, γ stays above the following green rectangle:



- This will turn out to be very useful when deriving the effective model later... Let us first analyze the effect of the magnetic field on the distribution of the interface γ.

Sketch of proof — Step 2: Effective weight induced by the magnetic field

- Any realization of the interface γ splits the box B_N into two sets:
  - ▷ $B_N^+[\gamma]$ **above** γ
  - ▷ $B_N^-[\gamma]$ **below** γ



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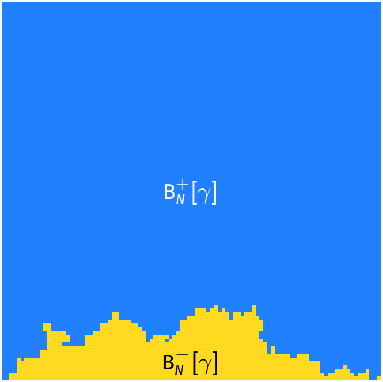

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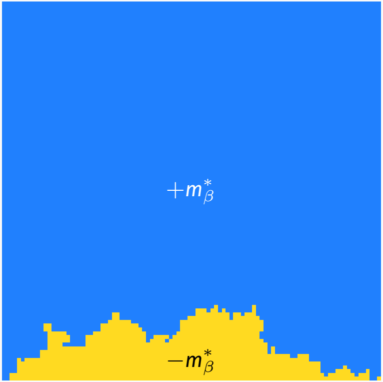

- Conditionally on a typical realization of γ, we expect the **empirical magnetization density** in $B_N^\pm[\gamma]$ to be very close to $\pm m_\beta^*$. We first make this precise.

Sketch of proof — Step 2: Effective weight induced by the magnetic field

- **Claim:** there exists $\kappa = \kappa(\beta)$ such that, apart from γ,

all contours have diameter at most κ log *N*

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▷ Not so clear inside $B_N^-[\gamma]$: the $-$ phase is not stable $\rightsquigarrow$ may be favorable to create giant droplets of $+$ phase!

▷ However, the critical droplet of $+$ phase is a “square” of sidelength D such that $2\beta \cdot 4D \lesssim 2\frac{\lambda}{N} \cdot D^2$, that is, $D \gtrsim \frac{4\beta}{\lambda} N$.
$\rightsquigarrow$ for a typical realization of γ, there is not enough room in $B_N^-[\gamma]$ to accommodate a critical droplet and **the layer of $-$ phase is metastable!**

Sketch of proof — Step 2: Effective weight induced by the magnetic field

- Since all contours are small, we can prove that, **conditionally on the realization of γ**, the magnetization concentrates (using results from [Ioffe, Schonmann 1998]):

$$\sum_{i \in \mathrm{B}_N} \sigma_i \approx m^*_\beta |\mathrm{B}_N^+[\gamma]| - m^*_\beta |\mathrm{B}_N^-[\gamma]| = m^*_\beta |\mathrm{B}_N| - 2m^*_\beta |\mathrm{B}_N^-[\gamma]|$$

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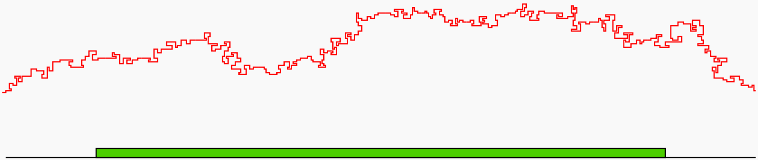
- From this, we deduce an **effective probability** for the contour γ in terms of the probability when $h = 0$: up to negligible corrections,

$$\mathrm{Prob}_{\beta, h=\lambda/N}(\gamma) \propto \exp\left[-\frac{\lambda}{N} \cdot 2m^*_\beta |\mathrm{B}_N^-[\gamma]|\right] \mathrm{Prob}_{\beta, h=0}(\gamma)$$

- Since $h = 0$, we can use the Ornstein–Zernike approach to couple the interface γ with a directed random walk on $\mathbb{Z}^2$.



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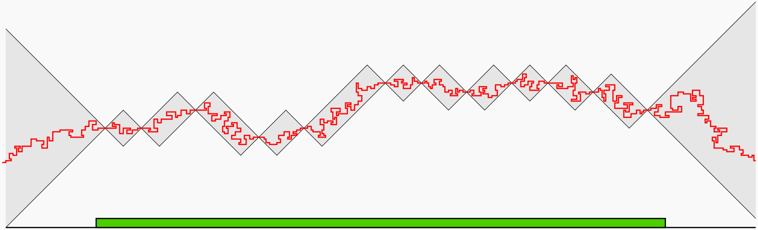

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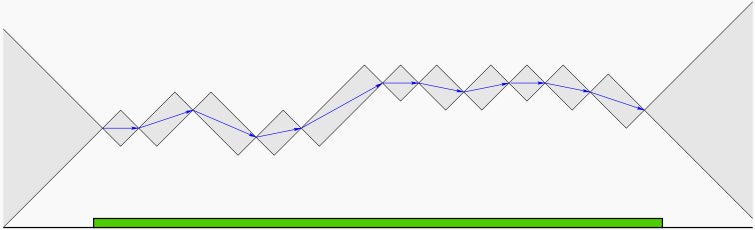

- Entropic repulsion bound: above the green rectangle, the distance between γ and the bottom wall is at least N^{ϵ}.

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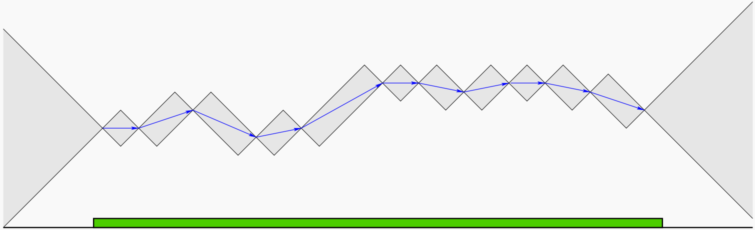

▶ Entropic repulsion bound: above the green rectangle, the distance between γ and the bottom wall is at least N^ϵ.

▶ It follows that the **finite-volume weights are well approximated by infinite-volume weights**. Therefore, the resulting effective random walk can be taken spatially homogeneous.

Sketch of proof — Step 3: Full effective model

- Remember that

$$\mathrm{Prob}_{\beta,h=\lambda/N}(\gamma) \propto \exp\left[-\frac{2\lambda m_\beta^*}{N}|\mathsf{B}_N^-[\gamma]|\right] \mathrm{Prob}_{\beta,h=0}(\gamma)$$

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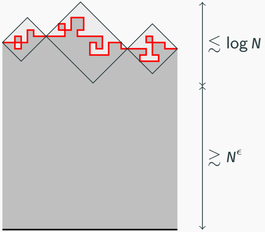
- This leads, *in the presence of the magnetic field λ/N*, to a coupling between γ and an **effective RW model subject to an area-tilt**: roughly speaking,

$$\mathrm{Prob}_{\mathrm{RW};h=\lambda/N}(X) \propto \exp\left[-\frac{2\lambda m_\beta^*}{N}\,\mathrm{Area}(X)\right] \mathrm{Prob}_{\mathrm{RW}}(X)$$



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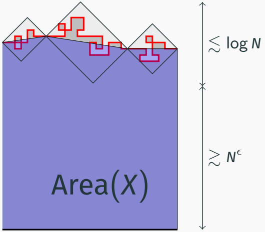

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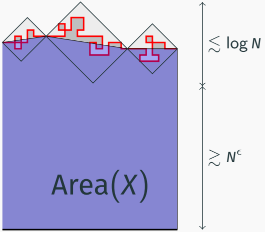

- This reduces our task to proving the desired weak convergence for this effective model.

Sketch of proof — Step 4: convergence for the effective model

- This part is done in a way very similar to the analysis in [Ioffe, Shlosman, V. 2015] and [Ioffe, V., Wachtel 2018]:
  - Express the relevant partition functions in terms of powers of a suitable **transfer operator**.
  - Compute the scaling limit of these quantities in terms of the **scaling limit of the generator of the induced semigroup**, which can be computed explicitly.
  - Deduce **convergence of finite-dimensional distributions**.
  - Complete the analysis with a proof of **tightness** (rough probabilistic estimates).

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  - Deduce **convergence of finite-dimensional distributions**.
  - Complete the analysis with a proof of **tightness** (rough probabilistic estimates).
- The main difference is that in our earlier work, the path was the space-time trajectory of a 1d random walk rather than the spatial trajectory of a directed 2d random walk. Mainly, this results in a **random number of steps** in the present situation, which adds technicalities but does not affect the general scheme.

Some open problems

- Consider $h = N^{-\alpha}$ for other values of α, in particular $\alpha = 0$ (i.e., first the limit $N \to \infty$, then the limit $h \downarrow 0$).

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- Consider $h = N^{-\alpha}$ for other values of α, in particular $\alpha = 0$ (i.e., first the limit $N \to \infty$, then the limit $h \downarrow 0$).
- Extend the analysis to the case of ⊖ boundary condition when $\lambda > \lambda_c$ (Schonmann–Shlosman geometry).
- In the case of ⊖ boundary condition when $\lambda > \lambda_c$, determine the limiting process at the junction between one arc of the droplet of ⊕ phase and the layer along the boundary. Fluctuations of all orders from $N^{1/2}$ to $N^{1/3}$ are expected to occur.

Thank you for your attention!

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