2D models of statistical physics with continuous symmetry: the case of singular interactions.

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Abstract

We show the absence of continuous symmetry breaking in 2D lattice systems without any smoothness assumptions on the interaction. We treat certain cases of interactions with integrable singularities. We also present cases of singular interactions with continuous symmetry, when the symmetry is broken in the thermodynamic limit.

Keywords and phrases: continuous symmetry, percolation, compact Lie group, recurrent random walks.

Running title: 2D models with continuous symmetry and singular interactions.

1 Introduction and results

1.1 The invariance problem: an overview

In this paper we are studying the two-dimensional lattice models of statistical mechanics, which are defined by a G-invariant interaction, where G is some compact connected Lie group. We shall investigate both the cases of finite and infinite range interactions. The general class of finite-range models to be considered is given by the following Hamiltonians:

$$\mathcal{H}(\phi) = \sum_{x\in\mathbb{Z}^2} U_\Lambda\left(\phi_{\cdot+x}|_\Lambda\right). \tag{1}$$

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Here $\phi = \left\{ \phi_y, y \in \mathbb{Z}^2 \right\}$ is the field, taking values in some compact topological space S, Λ is a fixed finite subset of $\mathbb{Z}^2$ and the translation-invariant interaction $\mathcal{U} = \left\{ U_{\Lambda + x} \left(\cdot \right), x \in \mathbb{Z}^2 \right\}$ is specified by a real function U_Λ on S^Λ. We suppose that a continuous action of a compact connected Lie group G on S is given, $a : G \times S \to S$, and for $\phi \in S$, $g \in G$ we introduce the notation $g\phi = a\left(g, \phi\right)$. This action defines the action of G on S^k for every k by $g\left(\phi_1, ..., \phi_k\right) = \left(g\phi_1, ..., g\phi_k\right)$, and the main assumption is that the function U_Λ is invariant under this action on S^Λ: for every $g \in G$

$$U_\Lambda \left(g \left(\phi_1, ..., \phi_{|\Lambda|} \right) \right) = U_\Lambda \left(\phi_1, ..., \phi_{|\Lambda|} \right). \tag{2}$$

Of course, we suppose that the free measure $d\phi$ on S is G-invariant as well.

The best known examples of such models are XY model (or plane rotator model) and XYZ model (or classical Heisenberg model). For the XY model $S = \mathbb{S}^1 \subset \mathbb{R}^2$ is the unit circle, and for the XYZ model $S = \mathbb{S}^2 \subset \mathbb{R}^3$ is the unit sphere. The Hamiltonians are

$$\mathcal{H}\left(\phi\right) = - \sum_{\substack{x,y \in \mathbb{Z}^2 \\ |x-y|=1}} J\left(\phi_x \circ \phi_y\right), \tag{3}$$

where J is a positive constant, $(.. \circ ..)$ stands for the scalar product, and the free measures are just the Lebesgue measures on the spheres.

The first rigorous result in this field is the well-known Mermin-Wagner theorem, which for the models (3) states the absence of spontaneous magnetization. Then in [DS1] a stronger result was proven, stating that under some **smoothness conditions** (see (4) below) on the function U_Λ every Gibbs state of the model defined by the Hamiltonian (1) is G-invariant under the natural action of G on $S^{\mathbb{Z}^2}$. Later it was proven in [MS] that for the model (3) the correlations decay at least as a power law. In [S78] the same power law was obtained for the general model (1), again under the smoothness condition (4). This result was reproved later in [N] for the case $G = SO(n)$, by means of the complex translations method of McBryan and Spencer. Another proof of G-invariance of the Gibbs states of models of type (1) was found in [P, FP]; with this technique it was possible to prove the result for long range interactions decaying as slowly as r^{-4} (in fact better, see the remark after Theorem 2). This result is optimal since it is known [KP] that in the low temperature XY model with interactions decaying as $r^{-4+\alpha}$, $\alpha > 0$, there is spontaneous symmetry breaking. On the other hand, this technique seems unable to yield the algebraic decay of correlations. In the case of $SO(N)$-symmetric models, the technique of [MS] can be extended to cover such long-range interactions, see [MMR].

An alternative approach to these problems is via Bogoliubov inequalities, see [M, KLS, BPK, I]; it also permits to prove absence of continuous symmetry breaking for long-range interactions. As the technique of [P] however, they seem unable to yield algebraic decay of correlations.

The smoothness condition on the interaction, which was crucial for all the results mentioned above, is the following. Let $M \subset \Lambda$ be a subset, and $\phi_\Lambda =$

$\left(\phi_M \cup \phi_{\Lambda\setminus M}\right)$ be an arbitrary configuration. Then one requires that for every choice of the subset M and the configuration ϕ_Λ the functions[1]

$$V_{\phi_\Lambda,M}\left(g\right) = U_\Lambda\left(\phi_M \cup g\phi_{\Lambda\setminus M}\right) \text{ are } \mathcal{C}^2 \text{ functions on } G. \tag{4}$$

Moreover, the second derivatives of the functions $V_{\phi_\Lambda,M}\left(\cdot\right)$, taken along any tangent direction in G, have to be bounded from above, uniformly in ϕ_Λ and M. In the next section we are explaining how that condition can be used in the proof of the G-invariance.

1.2 No breaking of continuous symmetry for singular interactions

The Main Result of the present paper is that the smoothness property (4) is in fact not necessary for the G-invariance, and the latter is implied by the mere continuity of the functions $V_{\phi_\Lambda,M}\left(g\right)$ and the invariance (2). Moreover, even the continuity is not necessary, and a certain integrability condition on $V_{\phi_\Lambda,M}\left(g\right)$ is enough (see relation (26) below). For example, for G being a circle, $\mathbb{S}^1$, with U nearest neighbour interaction, $U\left(\phi_1,\phi_2\right) = U\left(\phi_1-\phi_2\right)$, the singularity $U\left(\phi\right) \sim \ln\left|\phi\right|$ at $\phi = 0$ does not destroy $\mathbb{S}^1$-invariance of the corresponding Gibbs measures. Jumps are allowed as well. However, if the interaction U is "even more singular", then the G-invariance can be destroyed, as the Theorem 4 below shows.

To formulate our main result, we will introduce the notation μ^{U_Λ} for a Gibbs measure on $S^{\mathbb{Z}^2}$ corresponding to the formal Hamiltonian (1) and the free measure $d\phi$ on S, which is supposed to be G-invariant. We will denote the integration operation with respect to μ^{U_Λ} by $\left\langle\cdot\right\rangle^{U_\Lambda}$. By $\mu^{U_\Lambda}_{x,y}\left(d\phi_x, d\phi_y\right)$ we denote the restriction of the measure μ^{U_Λ} to the product $S\times S$ of the state spaces of the field variables ϕ_x and ϕ_y.

We will prove the following

Theorem 1 *Suppose that:*

- *the finite range interaction function U_Λ is continuous, bounded on $S^{|\Lambda|}$, and satisfies the G-invariance property (2),*
- *the free measure $d\phi$ on S is G-invariant.*

  Then the measure μ^{U_Λ} is G-invariant: for every $g\in G$, every finite $V\subset\mathbb{Z}^2$ and every μ^{U_Λ}-integrable function f on S^V

$$\left\langle f\left(g\cdot\right)\right\rangle^{U_\Lambda} = \left\langle f\left(\cdot\right)\right\rangle^{U_\Lambda}. \tag{5}$$

  Moreover, it has the following correlation decay: for every $A,B\subset S$ the conditional distributions of the measure $\mu^{U_\Lambda}_{x,y}$ satisfy for every $g\in G$ the

[1] In fact, as noted in [P], the proofs really only use the fact that these functions are $\mathcal{C}^1$, with a first derivative satisfying a Lipschitz condition.

estimate

$$\left|\frac{\mu_{x,y}^{U_\Lambda}\left(\phi_x \in gA \;\middle|\; \phi_y \in B\right)}{\mu_{x,y}^{U_\Lambda}\left(\phi_x \in A \;\middle|\; \phi_y \in B\right)} - 1\right| \leq C\left(U_\Lambda\right)|x-y|^{-c(U_\Lambda)}, \tag{6}$$

with $C(U_\Lambda) < \infty$, $c(U_\Lambda) > 0$. *In case when the space* S *is a homogeneous space of the group* G *(e.g.* $S = G$*), the measure* $\mu_{x,y}^{U_\Lambda}(d\phi_x, d\phi_y)$ *can be written as a convex sum of two probability measures:*

$$\mu_{x,y}^{U_\Lambda}(d\phi_x, d\phi_y) = c_{xy}\hat{\mu}_{x,y}^{U_\Lambda}(d\phi_x, d\phi_y) + (1 - c_{xy})\,\tilde{\mu}_{x,y}^{U_\Lambda}(d\phi_x, d\phi_y).$$

The measure $\hat{\mu}_{x,y}^{U_\Lambda}(d\phi_x, d\phi_y)$ *can be singular, but the number* c_{xy} *is very small:* $0 \leq c_{xy} \leq \exp\left\{-\sqrt{|x-y|}\right\}$, *while the measure* $\tilde{\mu}_{x,y}^{U_\Lambda}(d\phi_x, d\phi_y)$ *has a density* $p_{x,y}^{U_\Lambda}(\phi_x, \phi_y)$ *with respect to the measure* $d\phi_x d\phi_y$, *which for every conditioning* $\phi_y = \psi$ *satisfies the estimate*

$$\left|p_{x,y}^{U_\Lambda}\left(\phi_x \;\middle|\; \phi_y = \psi\right) - 1\right| \leq C(U_\Lambda)|x-y|^{-c(U_\Lambda)},$$

with $c(U_\Lambda) > 0$. *In particular, for the case* $G = SO(n)$, $S = \mathbb{S}^{n-1} \subset \mathbb{R}^n$ *with* $n \geq 2$ *we have*

$$0 \leq \langle(\phi_x \circ \phi_y)\rangle^{U_\Lambda} \leq C(U_\Lambda)|x-y|^{-c(U_\Lambda)}. \tag{7}$$

We remind the reader that the homogeneous space is a manifold of the classes of conjugacy of a compact subgroup $H \subset G$.

The G-invariance (5) does not imply the uniqueness of the Gibbs state with the interaction U_Λ. The reason is that the interaction U_Λ may possess an additional discrete symmetry, which may be broken. An example is constructed in [S80].

The estimate (7) cannot be improved in general. Indeed, Fröhlich and Spencer have obtained the power law decay of the pair correlations in XY model (3) for large values of the coupling constant J, see [FS]. On the other hand, for XYZ model it is expected that the pair correlations decay exponentially for all values of J.

1.3 Infinite range case

The preceding theorem is restricted to finite-range interactions. Let us now turn to the long-range case. The formal Hamiltonian is supposed to be of the form

$$\mathcal{H}(\phi) = \sum_{x,y} J_{x-y}\, U(\phi_x, \phi_y). \tag{8}$$

More general Hamiltonians (e.g., without separating the spatial and spin part of the interaction, or with more than 2-body interactions) could also be treated along the lines of the approach we develop here, but for the sake of simplicity we shall restrict ourselves to the case of (8). Since the coupling constants $\{J_{\cdot}\}$ have to satisfy the summability condition, we can make an additional normalization assumption

$$\sum_{x\neq 0} |J_x| = 1. \tag{9}$$

Let $X_{\cdot}$ be the random walk on $\mathbb{Z}^2$ with transition probabilities from x to y given by $|J_{x-y}|$.

We then have the following

Theorem 2 *Suppose that*

- *The random walk $X_{\cdot}$ is recurrent.*
- *The 2-body interaction function U is continuous on $S \times S$, and satisfies the invariance property (2).*
- *The free measure $d\phi$ on S is G-invariant.*

Then all Gibbs states, corresponding to the Hamiltonian (8), are G-invariant.

The recurrency condition is known to be optimal even in the case of smooth U, in the sense that there are examples of systems for which the continuous symmetry is broken as soon as the underlying random-walk is transient, see [BPK] or Theorem (20.15) in [G].

Recurrence of the underlying random-walk is not a very explicit condition. Explicit examples have been given in [P]. Namely, it follows from the latter that Theorem 2 applies if there exists $p < \infty$ such that the coupling constants decays for large $\|x\|_\infty$ at least like

$$\|x\|_\infty^{-4} \log_2 \|x\|_\infty \dots \log_p \|x\|_\infty \,,$$

where $\log_k x = \log \log_{k-1} x$, and $\log_2 x = \log \log x$. On the other hand, it follows from [FILS] that the continuous symmetry is broken for the low temperature XY model with coupling constants behaving, for large $\|x\|_\infty$, like

$$\|x\|_\infty^{-4} \log_2 \|x\|_\infty \dots \left(\log_p \|x\|_\infty\right)^{1+\varepsilon} \,,$$

for any $p < \infty$ and $\varepsilon > 0$.

1.4 Non-compact symmetry group: non-existence of 2D Gibbs states

Finally we mention the case of connected non-compact Lie group G. The case of the smooth interaction was treated in [DS2], and the corresponding long-range result was obtained in [FP]. Technically the compact and the non-compact cases are very similar, but the results are quite different. The reason is that while in the compact case the Haar measure on G can be normalized to a probability measure, in the non-compact case it is not possible. Therefore, there are no G-invariant probability measures on G for G non-compact. This is the main reason behind the result of [DS2] and [FP]: the corresponding 1D and 2D Gibbs measures do not exist.

Below we are formulating the simplest such result for the non-compact case and singular interaction that our technique can produce. The field ϕ will be real-valued, $G = \mathbb{R}^1$, and

$$\mathcal{H}(\phi) = \sum_{x,y \in \mathbb{Z}^2} J_{x-y} \bar{U}(\phi_x - \phi_y), \tag{10}$$

with the function $\bar{U}$ satisfying

- $\bar{U}(\phi) = \bar{U}(-\phi)$,
- $\bar{U}(\phi) = U(\phi) - \upsilon(\phi)$, where U is a $\mathcal{C}^2$ function with uniformly bounded second derivative, and $0 \leq \upsilon \leq \varepsilon_0$, where ε_0 is some technical constant, which is small,

and the coupling constants $\{J_{\cdot}\}$ satisfy the same hypothesis as in Theorem 2 .

Theorem 3 *There are no two-dimensional Gibbs fields, corresponding to the Hamiltonian (10), with interaction* $\bar{U}$ *and coupling constants* $J_{\cdot}$ *as above.*

In particular, the last theorem covers the case of the (non-convex) interactions

$$\bar{U}(\phi) = |\phi|^{\alpha}, \ 0 < \alpha \leq 1,$$

and so answers a question which was left open in the paper [BLL]. In fact, all the results of [BLL] concerning the non-existence of the 2D Gibbs fields for interactions growing at most quadratically in ϕ follow from the above theorem. Notice that our techniques also allow to obtain lower bounds with the correct behavior for the variance of the field in a finite box.

The general formulation of the above theorem and its proof will be published in a separate paper.

1.5 Continuous symmetry breaking in 2D

Our results on continuous symmetry breaking are taking place for the Patrascioiu-Seiler model [PS]. Namely, it was argued there, and was rigorously proven later by M. Aizenman [A], that the following holds. Consider the case when $S = G = \mathbb{S}^1$, with the interaction $U(\phi_1, \phi_2) = U(\phi_1 - \phi_2)$ given by

$$U(\phi) = \begin{cases} -\cos\phi & \text{if } |\phi| \leq \theta, \\ +\infty & \text{if } |\phi| > \theta. \end{cases} \tag{11}$$

Then in the 2D case, the statement is that the two-point pair correlations in the state with free or periodic b.c. decay at most as a power law, at all temperatures including infinite temperature, provided $|\theta| < \frac{\pi}{4}$.

It would be interesting to know whether the Gibbs states $\mu^{\mathbf{0}}$ of this model with zero b.c., i.e. $\phi \equiv 0$, are $\mathbb{S}^1$-invariant. To the best of our knowledge this question is open. However, one can prove the following simple:

Theorem 4 *Suppose that $|\theta| < \frac{\pi}{4}$. Then at any temperature there exist Gibbs states, corresponding to the interaction (11), which are not $\mathbb{S}^1$-invariant.*

2 Proofs

2.1 Theorem 1: Smooth case.

We begin by reminding the reader the main ideas of the proof for the case of smooth interaction. The proof for the general case would be built upon it. We follow [DS1], with simplifications made in [Si].

For simplicity we will consider the case when both the space S and the group G will be a circle, $\mathbb{S}^1$. The general case follows easily from this special one, see [DS1], since for every element $g \in G$ there is a compact commutative subgroup (torus) $T \subset G$, such that $g \in T$. We also suppose that the interaction $\mathcal{U}$ is a nearest neighbour translation invariant interaction, given by a symmetric function U of two variables: $U(\phi_1, \phi_2) = U(\phi_2, \phi_1)$. The generalization to a finite range interaction is straightforward. The $\mathbb{S}^1$-invariance of $\mathcal{U}$ means that $U(\phi_1, \phi_2) = U(\phi_1 + \psi, \phi_2 + \psi)$ for every $\psi \in \mathbb{S}^1$, so in fact we can say that U is a function of one variable, $U(\phi_1, \phi_2) = U(\phi_1 - \phi_2)$, with $U(\phi) = U(-\phi)$. The smoothness we need is the following: we suppose that U has the second derivative, which is bounded from above:

$$U''(\phi) \leq \bar{C}. \tag{12}$$

Let Λ_n be the box $\left\{x \in \mathbb{Z}^2 : ||x||_\infty \leq n\right\}$, and $\bar{\phi}$ be an arbitrary boundary condition outside Λ_n. Let $\langle \cdot \rangle_{n,\bar{\phi}}$ be the Gibbs state in Λ_n corresponding to the interaction U and the boundary condition $\bar{\phi}$. Let V be an arbitrary finite subset of $\mathbb{Z}^2$, containing the origin. Our theorem will be proven for the interaction U once we obtain the following estimate:

Lemma 5 *For every function $f(\phi) = f(\phi_V)$, which depends only on the configuration ϕ inside V, we have for every $\psi \in \mathbb{S}^1$*

$$\left|\langle f(\phi + \psi)\rangle_{n,\bar{\phi}} - \langle f(\phi)\rangle_{n,\bar{\phi}}\right| \leq C\left(\bar{C}, V\right) ||f||_\infty n^{-N(U)} \tag{13}$$

for some $C\left(\bar{C}, V\right) > 0$, while the functional $N(\cdot)$ is positive for every U smooth.

Proof. Our system in the box Λ_n has $(2n+1)^2$ degrees of freedom, which is hard to study. We are going to fix $(2n+1)^2 - (n+1)$ of them, leaving only $n+1$ degrees of freedom, and we will show that for every choice Φ of the degrees frozen we have

$$\left|\langle f(\phi + \psi) | \Phi\rangle_{n,\bar{\phi}} - \langle f(\phi) | \Phi\rangle_{n,\bar{\phi}}\right| \leq C\left(\bar{C}, V\right) ||f||_\infty n^{-N(U)} \tag{14}$$

uniformly in Φ. From that (13) evidently follows by integration. These degrees of freedom are introduced in the following way.

For every $k = 0, 1, 2, ...$ we define the layer $L_k \subset \mathbb{Z}^2$ as the subset $L_k = \left\{ x \in \mathbb{Z}^2 : ||x||_\infty = k \right\}$. For a configuration ϕ in Λ_n we denote by $\Phi_k, k = 0, 1, 2, ..., n$ its restrictions to the layers L_k :

$$\Phi_k = \phi|_{L_k}.$$

We define now the action $(\psi_0, \psi_1, ..., \psi_n)\,\phi$ of the group $\left(\mathbb{S}^1\right)^{n+1}$ on configurations ϕ in Λ_n by

$$((\psi_0, \psi_1, ..., \psi_n)\,\phi)\,(x) = \phi\,(x) + \psi_{k(x)}$$

where $k\,(x) = ||x||_\infty$ is the number of the layer to which the site x belongs. We define the torus $\Phi\,(\phi)$ to be the orbit of the configuration ϕ under this action. In other words, $\Phi\,(\phi)$ is the set of configurations $\Phi_0 + \psi_0, \Phi_1 + \psi_1, ..., \Phi_n + \psi_n$, for all possible values of the angles ψ_i, where the configuration $\Phi_k + \psi_k$ on the layer L_k is defined by $(\Phi_k + \psi_k)\,(x) = \phi\,(x) + \psi_k$.

Let us fix for every orbit Φ one representative, ϕ, so $\Phi = \Phi\,(\phi)$, and let $\Phi_0, \Phi_1, ..., \Phi_n$ be the restrictions, $\Phi_k = \phi|_{L_k}$.

We will study the conditional Gibbs distribution $\langle\cdot|\Phi\,(\phi) = \Phi\rangle_{n,\bar{\phi}}$. This distribution is again a Gibbs measure on $\left(\mathbb{S}^1\right)^{n+1} = \{(\psi_0, \psi_1, ..., \psi_n)\}$, corresponding to the nearest neighbour interaction $\mathcal{W}_{\Phi,\bar{\phi}} = \{W_k, k = 1, 2, ..., n\}$. It is defined for $k < n$ by

$$W_k\,(\psi_k, \psi_{k+1}) = \sum_{\substack{x\in L_k, y\in L_{k+1}: \\ |x-y|=1}} U\,[(\Phi_k + \psi_k)\,(x)\,, (\Phi_{k+1} + \psi_{k+1})\,(y)]\,, \tag{15}$$

while

$$W_n\,(\psi_n) = \sum_{\substack{x\in L_n, y\in L_{n+1}: \\ |x-y|=1}} U\left[(\Phi_n + \psi_n)\,(x)\,, \bar{\phi}\,(y)\right]. \tag{16}$$

(Note for the future, that the interactions along the bonds which are contained within one layer do not contribute to W-s.) We are going to show that for every k the distribution of the random variable ψ_k under $\langle\cdot|\Phi\,(\phi) = \Phi\rangle_{n,\bar{\phi}}$ has a density $p_k\,(t)$ with respect to the Lebesgue measure on $\mathbb{S}^1$, which satisfies

$$\sup_{t\in\mathbb{S}^1} |p_k\,(t) - 1| \leq C\sqrt{k}\left(\frac{n}{k}\right)^{-N(U)}, \tag{17}$$

uniformly in $\Phi, \bar{\phi}$, with $C = C\left(\bar{C}\right)$. That implies (14).

To show (17) we note that due to $\mathbb{S}^1$-invariance of U we have

$$W_k\,(\psi_k, \psi_{k+1}) = W_k\,(\psi_k + \alpha, \psi_{k+1} + \alpha)$$

for every $\alpha \in \mathbb{S}^1$. Hence $W_k\,(\psi_k, \psi_{k+1}) = W_k\,(\psi_k - \psi_{k+1}, 0)$, and therefore the random variables

$$\chi_k = \begin{cases} \psi_k - \psi_{k+1} & \text{for } k < n \\ \psi_n & \text{for } k = n \end{cases}$$

are independent. Since evidently

$$\psi_k = \chi_k + \chi_{k+1} + ... + \chi_n, \tag{18}$$

we are left with the question about the distribution of the sum of independent random elements of $\mathbb{S}^1$. Were the independent random elements χ_i identically distributed, with the distribution having density, the statement (17) would be immediate. However, they are not identically distributed, so we need to work further.

Introducing $W_k(\chi_k) = W_k(\chi_k, 0)$ for $k < n$, we have that for all $k \le n$ the distribution of the random element χ_k is given by the density

$$q_k(t) = \frac{\exp\{-W_k(t)\}}{\int \exp\{-W_k(t)\}\, dt}.$$

Let $t_{\min}$ be (any) global minimum of the function $W_k(\cdot)$. Then for every t the Taylor expansion implies the estimate

$$W_k(t_{\min}) \le W_k(t) \le W_k(t_{\min}) + 8\bar{C}(k+1)|t - t_{\min}|^2, \tag{19}$$

due to (12), (15), (16). (This is the point where both smoothness and two-dimensionality are crucial.) Hence

$$\max q_k(t) \le C_1\sqrt{k+1} \tag{20}$$

for some $C_1 = C_1(\bar{C})$.

Because of (18), $p_k(t) = (q_k * ... * q_n)(t)$, where $*$ stays for convolution. Therefore it is natural to study the Fourier coefficients

$$a_s(q_l) = \frac{1}{2\pi}\int_0^{2\pi} q_l(t)\, e^{ist}\, dt,$$

$s = 0, \pm 1, \pm 2, ...$, since

$$a_s(p_k) = \prod_{l=k}^{n} a_s(q_l). \tag{21}$$

We want to show that for every $s \neq 0$ the last product goes to 0 as $n \to \infty$, uniformly in s. To estimate the coefficients $|a_s(q_l)|$ we use the following straightforward

Lemma 6 *Let P_C be the set of all probability densities $q(\cdot)$ on a circle, satisfying*

$$\sup_{t\in\mathbb{S}^1} q(t) \le C,$$

and s be an integer. Then the functional on P_C, given by the integral

$$\frac{1}{2\pi}\int_0^{2\pi} q(t)\cos(st)\, dt,$$

attains its maximal value at the density

$$q_C(t) = \begin{cases} C & \text{if } \left|t - \frac{2\pi k}{s}\right| \le \frac{1}{2Cs} \text{ for some } k = 0, ..., s-1, \\ 0 & \text{otherwice.} \end{cases}$$

Using this lemma and the estimate (20), we obtain that

$$\sup\left\{|a_s(q_l)| : s \neq 0\right\} \tag{22}$$
$$\leq 2C_1\sqrt{l+1}\int_0^{\frac{1}{2}\left(C_1\sqrt{l+1}\right)^{-1}}\left(1-\frac{t^2}{3}\right)dt = 1-\frac{1}{36\left(C_1\right)^2(l+1)}.$$

Since

$$\sup_{t\in\mathbb{S}^1}|p_k(t)-1| \leq \sum_{s\neq 0}|a_s(p_k)|,$$

we are almost done. Namely, note that due to Parseval identity and (20) we have for every l

$$1+\sum_{s\neq 0}|a_s(q_l)|^2 = \int (q_l(t))^2\,dt \leq C_1\sqrt{l+1}.$$

Let us introduce now the densities $p_{k,r}(t) = (q_k * ... * q_r)(t)$, $k \leq r \leq n$. Due to Cauchy inequality,

$$1+\sum_{s\neq 0}|a_s(p_{k,k+1})| \leq C_1\sqrt[4]{(k+1)(k+2)}.$$

Therefore by (22) and (21)

$$\sup_{t\in\mathbb{S}^1}|p_{k,r}(t)-1| \leq C_1\sqrt[4]{(k+1)(k+2)}\prod_{l=k+2}^{r}\left(1-\frac{1}{36\left(C_1\right)^2(l+1)}\right), \tag{23}$$

which ends the proof of (17), with $C = 2C_1\left(\bar{C}\right)$ and $N(U) = \frac{1}{36\left(C_1\left(\bar{C}\right)\right)^2}$. ∎

2.2 Theorem 1: Singular case.

The key step in the above proof was the use of the Taylor expansion, to bound the densities q_r. There the existence of the second derivative of U and its boundedness was used in a crucial way. Yet, one can use essentially the same arguments to treat the general case, without smoothness assumption. The main idea is to represent the singular interaction as a small perturbation of a smooth one, smallness being understood in the L_1 sense. Another version of this idea was used earlier in [BI, BCPK, DV, IV].

Namely, we will consider the nearest neighbour interaction

$$\bar{U}(\phi) = U(\phi) - \upsilon(\phi), \tag{24}$$

where U is a smooth function with a bounded second derivative, as above, while $\upsilon \geq 0$ is a "small" singular component. The precise meaning of smallness will be made explicit a bit later, see (26). However, already now we can say that every continuous function $\bar{U}$ can be written in the form (24), with U twice differentiable and with υ satisfying

$$0 \leq \upsilon(\cdot) \leq \varepsilon, \tag{25}$$

with $\varepsilon > 0$ arbitrarily small. That follows immediately for example from the Weierstrass theorem, stating that the trigonometric polynomials are everywhere dense in the space of continuous functions on the circle. Clearly, the estimate (25) implies L_1-smallness of υ, whatever the latter may mean.

We will denote by $\bar{\mathcal{H}}$ the Hamiltonian corresponding to the singular interaction $\bar{U}$, while $\mathcal{H}$ will be the Hamiltonian defined by the smooth part of interaction, U. To proceed with the expansion, we introduce the set $\mathcal{E}_n$ to be the collection of all bonds of $\mathbb{Z}^2$ with at least one end in the box Λ_n, and rewrite the partition function $Z_n^{\bar{U},\bar{\phi}}$ in Λ_n, corresponding to the interaction $\bar{U}$ and the boundary conditions $\bar{\phi}$, as follows:

$$\begin{aligned} Z_n^{\bar{U},\bar{\phi}} &= \int_{\Omega_n} \exp\left\{-\bar{\mathcal{H}}\left(\phi|\bar{\phi}\right)\right\} d\phi \\ &= \int_{\Omega_n} \exp\left\{-\mathcal{H}\left(\phi|\bar{\phi}\right)\right\} \prod_{\langle x,y\rangle\in\mathcal{E}_n} \left[1 + \left(e^{\upsilon(\phi(x)-\phi(y))} - 1\right)\right] d\phi \\ &= \sum_{A\subset\mathcal{E}_n} \int_{\Omega_n} \exp\left\{-\mathcal{H}\left(\phi|\bar{\phi}\right)\right\} \prod_{\langle x,y\rangle\in A} \left(e^{\upsilon(\phi(x)-\phi(y))} - 1\right) d\phi \\ &\equiv \sum_{A\subset\mathcal{E}_n} Z_n^{U,\bar{\phi},A}. \end{aligned}$$

For every subset $A \subset \mathcal{E}_n$ we now introduce the probability distribution

$$\mu_n^{U,\bar{\phi},A}(d\phi) = \frac{1}{Z_n^{U,\bar{\phi},A}} \exp\left\{-\mathcal{H}\left(\phi|\bar{\phi}\right)\right\} \prod_{\langle x,y\rangle\in A} \left(e^{\upsilon(\phi(x)-\phi(y))} - 1\right) d\phi.$$

Then we have for the original Gibbs state $\mu_n^{\bar{U},\bar{\phi}}$ the following decomposition:

$$\mu_n^{\bar{U},\bar{\phi}} = \sum_{A\subset\mathcal{E}_n} \pi_n(A)\, \mu_n^{U,\bar{\phi},A},$$

with the probabilities $\pi_n(\cdot)$ given by

$$\pi_n(A) = \frac{Z_n^{U,\bar{\phi},A}}{Z_n^{\bar{U},\bar{\phi}}}.$$

Note that the states $\mu_n^{U,\bar{\phi},A}$ are themselves Gibbs states in Λ_n, corresponding to the boundary condition $\bar{\phi}$ and the (non-translation invariant) nearest neighbour interaction $\mathcal{U}^A$, which for bonds outside A is given by our smooth function $U(\phi_s - \phi_t)$, while on bonds from A it equals to $U(\phi_s - \phi_t) - \ln\left(e^{\upsilon[\phi_s-\phi_t]} - 1\right)$.

(Here the positivity of the function υ is used.) Let us now introduce the bond percolation process $\mathcal{A}$ on $\mathcal{E}_n$, defining its probability distribution $\mathbb{P}_n$ by

$$\mathbb{P}_n(\mathcal{A} = A) = \pi_n(A).$$

This process is of course a dependent percolation process. Happily, it turns out that it is dominated by independent bond percolation, with probability of a bond to be open very small! Our claim would follow once we check that the conditional probabilities

$$\mathbb{P}_n\left(b \in \mathcal{A} \middle| (\mathcal{E}_n \backslash b) \cap \mathcal{A} = \mathcal{D}\right)$$

are small uniformly in $\mathcal{D}$. We will show this under the following condition on the smallness of the singular part υ of the interaction $\bar{U}$. We suppose that

- $\bar{U}(\phi) = U(\phi) - \upsilon(\phi)$, with U having bounded second derivative,
- $\upsilon \geq 0$,
- for every choice of the four values $\phi_1, \phi_2, \phi_3, \phi_4$

$$\frac{\int \exp\left\{-\sum_{i=1}^4 U(\phi - \phi_i) + \sum_{i=1}^4 \upsilon(\phi - \phi_i)\right\} d\phi}{\int \exp\left\{-\sum_{i=1}^4 U(\phi - \phi_i)\right\} d\phi} \leq 1 + \varepsilon, \tag{26}$$

  with ε small enough.

In words, the last condition says that the expectation of the observable $\exp\left\{\sum_{i=1}^4 \upsilon(\phi - \phi_i)\right\}$ with respect to a single site conditional Gibbs distribution corresponding to the (smooth) interaction U and any boundary condition $\phi_1, \phi_2, \phi_3, \phi_4$ around that site, is smaller than $1 + \varepsilon$. A straightforward calculation implies that under (26)

$$\mathbb{P}_n\left(b \in \mathcal{A} \middle| (\mathcal{E}_n \backslash b) \cap \mathcal{A} = \mathcal{D}\right) \leq \varepsilon, \tag{27}$$

uniformly in $\mathcal{D}$. We denote by $\mathbb{Q}_\varepsilon$ the distribution of the corresponding independent bond percolation process, η..

The strategy of the remainder of this subsection is the following:

- we will show that if the set A is sparse enough, then for the measure $\mu_n^{U,\bar{\phi},A}$ the analog of the estimate (13) holds.
- such sparse sets A constitute the dominant contribution to the distribution $\mathbb{P}_n$.

Let us formulate now the sparseness condition on A we need.

In what follows, by a path we will mean a sequence of pairwise distinct bonds of our lattice, such that any two consecutive bonds share a site. A path with coinciding beginning and end is called a loop. If a loop surrounds the origin, we will call it a circuit. Any two objects of the above will be called disjoint, if they share neither a bond nor a site. The same objects, associated with the dual lattice will be called d-sites, d-bonds, d-paths, d-loops and d-circuits.

Suppose the set A is given, and $\lambda_1, \lambda_2, ..., \lambda_\nu$ be a collection of disjoint d-circuits, avoiding A. The latter means that no d-bond of any λ_k crosses any of the bonds from A. We suppose that these d-circuits are ordered by "inclusion". Then we introduce layers L_k by

$$L_k = \left\{ x \in \mathbb{Z}^2 : x \in \operatorname{Int}(\lambda_k) \setminus \operatorname{Int}(\lambda_{k-1}) \right\}, \; k = 1, 2, ..., \nu + 1,$$

with the convention that $\operatorname{Int}(\lambda_0) = \emptyset$ and $\operatorname{Int}(\lambda_{\nu+1}) = \mathbb{Z}^2$. (Note that these layers are connected sets of sites, and they surround the origin in the same way as the "old" layers did.) For every configuration ϕ in Λ_n we introduce, as in the previous section, the layer configurations $\Phi_k, k = 1, 2, ..., \nu + 1$ as its restrictions to the layers L_k, the layer angles $\psi_1, ..., \psi_\nu$, the ν-dimensional torus $\Phi(\phi)$, and we note that the distribution of ψ-s under the condition that the orbit $\Phi(\phi)$ is fixed, is a (one-dimensional) Gibbs distribution. Moreover, it is defined by the nearest neighbour interaction $\mathcal{W}_{\Phi,\bar{\phi}} = \{W_k, k = 1, 2, ..., \nu\}$, given by almost the same formula as (15): for $k < \nu$

$$W_k(\psi_k, \psi_{k+1}) = \sum_{\substack{x \in L_k, y \in L_{k+1}: \\ |x-y|=1}} U\left[(\Phi_k + \psi_k)(x), (\Phi_{k+1} + \psi_{k+1})(y)\right], \qquad (28)$$

while for $k = \nu$

$$W_\nu(\psi_\nu) = \sum_{\substack{x \in L_\nu, y \in L_{\nu+1}: \\ |x-y|=1}} U\left[(\Phi_n + \psi_n)(x), \left(\phi \vee \bar{\phi}\right)(y)\right]. \qquad (29)$$

(Here the configuration $\phi \vee \bar{\phi}$ equals to ϕ inside Λ_n and to $\bar{\phi}$ outside Λ_n.) Note that the singular part of the interaction $\mathcal{U}^A$ does not enter in these formulas, precisely because the d-circuits λ_k avoid the set A! Hence we can conclude that for every k the distribution of the random variable ψ_k under the measure $\langle \cdot | \Phi(\phi) = \Phi \rangle_{n,\bar{\phi}}$ has a density $p_k(t)$ on $\mathbb{S}^1$, which satisfies the following analog of (23):

$$\sup_{t \in \mathbb{S}^1} |p_k(t) - 1| \leq C_1 \sqrt[4]{|\lambda_k| \, |\lambda_{k+1}|} \exp\left\{ -\frac{1}{36 (C_1)^2} \sum_{l=k+2}^{\nu} \frac{1}{|\lambda_l|} \right\}, \qquad (30)$$

uniformly in $\Phi, \bar{\phi}$. The last relation suggests the following

Definition 7 ***of sparseness:*** *The set A of bonds in $\mathcal{E}_n$ is τ-sparse, if there exists a family of $\nu(A)$ disjoint d-circuits λ_l in Λ_n, avoiding A, and such that*

$$\sum_{l=1}^{\nu(A)} \frac{1}{|\lambda_l|} \geq \tau \ln n.$$

Therefore we will be done, once we show the following:

Proposition 8 *For any κ, $1 > \kappa > 0$, there exists a value $\tau = \tau(\kappa) > 0$, such that*

$$\mathbb{P}_n(\mathcal{A} \text{ is not } \tau\text{-sparse}) \leq e^{-n^{\kappa}}. \tag{31}$$

The proof of this proposition is the content of the following subsections.

2.2.1 τ-sparseness is typical.

For every $l = 2, 3, ...$ let us define the northern rectangle

$$R_N^l = [-2^l, \ldots, 2^l] \times [2^{l-1} + 1, \ldots, 2^l],$$

and let the eastern, southern and western rectangles R_E^l, R_S^l and R_W^l be the clock-wise rotations of R_N^l by, respectively, $\pi/2$, π and $3\pi/2$ with respect to the origin. Define the l-th shell T^l by

$$T^l \;=\; R_N^l \cup R_E^l \cup R_S^l \cup R_W^l.$$

Clearly, $T^l \subset \Lambda_n$ once $n \geq 2^l$, while different T^l-s are disjoint.

Let a configuration A of bonds be given. By a good crossing of a rectangle $R_\cdot^\cdot$ we will mean a d-path, joining the two short sides of $R_\cdot^\cdot$ and avoiding A. We denote the set of such crossings by $\mathcal{R}^{\longleftrightarrow}$. Let $\lambda_N^l, \lambda_E^l, \lambda_S^l, \lambda_W^l$ be four good crossings of the rectangles $R_N^l, R_E^l, R_S^l, R_W^l$ respectively. Then the collection of those d-bonds of the union $\lambda_N^l \cup \lambda_E^l \cup \lambda_S^l \cup \lambda_W^l$, which are seen from the origin, form a d-circuit avoiding A. Therefore we want to get a

2.2.2 Lower bound on the number of disjoint good crossings of a rectangle

We claim that for all ε sufficiently small there exist $\alpha = \alpha(\varepsilon) > 0$ and $c_1 = c_1(\varepsilon) > 0$ such that at each scale k the $\mathbb{Q}_\varepsilon$-probability that there are less than $\alpha 2^k$ disjoint good crossings of R_N^k is smaller than $e^{-c_1 2^k}$, where $\mathbb{Q}_\varepsilon$ is the measure of the independent bond percolation process $\eta_\cdot$, defined after (27).

Indeed, by the Ford-Fulkerson min-cut/max-flow Theorem (see e.g. [R]), the number of disjoint good crossings of R_N^k (which by definition are left-to-right crossings by d-paths) is bounded from below by

$$\frac{1}{2} \min_{\widetilde{\lambda} \in \mathcal{R}^{\updownarrow}} \left\{ \left|\widetilde{\lambda}\right| - \left|\widetilde{\lambda} \cap A\right| \right\},$$

where the minimum is taken over the set $\mathcal{R}^{\updownarrow}$ of all "cuts", which are just paths in R_N^k, joining the bottom and top sides of R_N^k. The min-cut quantity $\min_{\widetilde{\lambda}} \left\{ \left|\widetilde{\lambda}\right| - \left|\widetilde{\lambda} \cap A\right| \right\}$ equals to the maximal left-to-right flow by d-paths, avoiding A, and the factor 1/2 accounts for the fact that the corresponding d-paths might share the same d-sites, so in order to estimate the number of disjoint paths we have to take a half of the total flow.

Evidently,

$$\mathbb{Q}_\varepsilon\left(\exists\widetilde{\lambda}\in\mathcal{R}^{\updownarrow}\text{ with }\left|\widetilde{\lambda}\right|-\left|\widetilde{\lambda}\cap A\right|\leq\alpha 2^k\right)\leq\sum_{\widetilde{\lambda}\in\mathcal{R}^{\updownarrow}}\mathbb{Q}_\varepsilon\left(\left|\widetilde{\lambda}\right|-\left|\widetilde{\lambda}\cap A\right|\leq\alpha 2^k\right), \tag{32}$$

while for every $\widetilde{\lambda}$

$$\mathbb{Q}_\varepsilon\left(\left|\widetilde{\lambda}\right|-\left|\widetilde{\lambda}\cap A\right|\leq\alpha 2^k\right)\leq 2^{|\widetilde{\lambda}|}\varepsilon^{|\widetilde{\lambda}|-\alpha 2^k}\ \leq\ e^{-c_2|\widetilde{\lambda}|},$$

since any top-to-bottom crossing contains at least 2^{k-1} bonds. Here $c_2 = c_2(\alpha,\varepsilon)>0$ satisfies

$$\lim_{\varepsilon\to 0}c_2(\alpha,\varepsilon)\ =\ \infty,$$

once $\alpha<1/2$. Thus, choosing $\alpha<1/2$ and ε sufficiently small, we infer that there exists $c_1>0$, such that the right hand side of (32) is bounded above by

$$2^k\sum_{l=2^{k-1}}^{\infty}3^l e^{-c_2 l}\ \leq\ e^{-c_1 2^k}.$$

Thus, the min-cut/max-flow theorem insures that up to the $\mathbb{Q}_\varepsilon$-probability $1-e^{-c_1 2^k}$, there are at least $\alpha 2^{k-1}$ disjoint good crossings λ_i of R_N^k. Moreover, observe that at least $\alpha 2^{k-2}$ of these d-paths have the length bounded above by $\alpha^{-1}2^{k+3}$. Indeed, should this not be the case,

$$\sum_i|\lambda_i|>\alpha 2^{k-2}\,\frac{1}{\alpha}2^{k+3}=2|R_N^k|$$

which in view of the disjointedness of λ_i-s is impossible.

Let us say that a left-to-right crossing d-path λ of the k-th scale is α-short, if $|\lambda|<\alpha^{-1}2^{k+3}$, and define the event

$$\mathcal{T}_N^{k,\alpha}\ =\ \left\{A:\text{there are at least }\alpha 2^{k-2}\text{ disjoint good }\alpha\text{-short crossings of }R_N^k\right\}.$$

What we have proved up to now can be summarized as follows:

There exists $c_1>0$, such that uniformly in k,

$$\mathbb{Q}_\varepsilon\left(\mathcal{T}_N^{k,\alpha}\right)\ \geq 1-\ e^{-c_1 2^k}, \tag{33}$$

as soon as α and ε are sufficiently small.

2.2.3 Proof of Proposition 8

Consider now the event

$$\mathcal{T}^{k,\alpha}=\mathcal{T}_N^{k,\alpha}\cap\mathcal{T}_E^{k,\alpha}\cap\mathcal{T}_S^{k,\alpha}\cap\mathcal{T}_W^{k,\alpha}.$$

From the previous argument one knows that for ε close enough to 0 the $\mathbb{Q}_\varepsilon$-probability of the event $\mathcal{T}^{k,\alpha}$ is at least $1-4\,e^{-c_1 2^k}$. Note that under $\mathcal{T}^{k,\alpha}$

there are at least $\alpha 2^{k-2}$ disjoint d-circuits in T^k, avoiding A, all of which have length at most $2^{k+5}/\alpha$. Also, the events $\mathcal{T}^{k,\alpha}$ are non-decreasing, therefore their $\mathbb{P}_n$-probability is at least $1 - 4\,e^{-c_1 2^k}$ as well.

The claim of Proposition 8 is now an immediate consequence: Let $1 > \rho > 0$. Then, for every $n = 2, 3, \ldots$ the event

$$\mathcal{T}^{n,\rho,\alpha} = \bigcap_{k=[\rho \log_2 n]}^{[\log_2 n]} \mathcal{T}^{k,\alpha}$$

has, by (33), $\mathbb{P}_n$-probability at least $1 - c_3 e^{-c_4 n^\rho}$. However, by the very construction, the occurrence of the event $\mathcal{T}^{n,\rho,\alpha}$ ensures that in each shell T_k, $k \in \{[\rho \log_2 n], \ldots, [\log_2 n]\}$, it is possible to find a family of disjoint d-circuits avoiding A and such that the sum of the inverse of their lengths is at least $\alpha^2/128$. Their total is at least

$$\frac{\alpha^2}{128} \frac{1-\rho}{2} \log_2 n.$$

The conclusion (31) follows.

2.2.4 General finite-range interactions

We briefly describe the main modifications to the proof given above, which are needed in order to treat the case of finite-range, non nearest-neighbour interactions $\bar{U}_\Lambda$, $\Lambda \Subset \mathbb{Z}^2$. As in (24), we decompose $\bar{U}_\Lambda = U_\Lambda - \upsilon_\Lambda$ to a smooth part U_Λ and a small singular part $0 \le \upsilon_\Lambda \le \varepsilon$. Notice that the choice of $\varepsilon = \varepsilon(r_\Lambda)$ will in general depend on the diameter $r_\Lambda = \mathrm{diam}(\Lambda)$ of the interaction set Λ.

The singular part of the interaction will be controlled by a dependent *site* percolation process, which we construct in two steps as follows. Define $\bar{\Lambda}_n = \{x : x + \Lambda \cap \Lambda_n \ne \emptyset\}$.

Step 1. As in the nearest-neighbour case, write

$$\begin{aligned} Z_n^{\bar{U},\bar{\phi}} &= \sum_{A \subset \bar{\Lambda}_n} \int_{\Omega_n} \exp\left\{-\mathcal{H}\left(\phi|\bar{\phi}\right)\right\} \prod_{x\in A} \left(e^{\upsilon_\Lambda(\phi_{\cdot+x})} - 1\right) d\phi \\ &\triangleq \sum_{A \subset \bar{\Lambda}_n} Z_n^{U,\bar{\phi},A}. \end{aligned}$$

Then, exactly as before, it is easy to show that the probability distribution

$$\mathbb{P}_n\left(\mathcal{A} = A\right) \triangleq \frac{Z_n^{U,\bar{\phi},A}}{Z_n^{\bar{U},\bar{\phi}}} \tag{34}$$

on $\{0,1\}^{\bar{\Lambda}_n}$ is stochastically dominated by the Bernoulli site percolation process $\mathbb{Q}_\varepsilon$ with density ε.

Step 2. Let us split $\mathbb{Z}^2$ into the disjoint union of the shifts of squares $B_\Lambda \stackrel{\triangle}{=} \{-2r_\Lambda, \ldots, 2r_\Lambda\}^2$,

$$\mathbb{Z}^2 = \bigvee_x (4r_\Lambda\, x + B_\Lambda) \ .$$

Given a realization A of the random set $\mathcal{A}$ (distributed according to (34)) let us say that $x \in \mathbb{Z}^2$ is good if $4r_\Lambda\, x + B_\Lambda \cap A = \emptyset$. Thus, for every n, $\mathcal{A}$ induces a probability distribution on $\{0,1\}^{\mathbb{Z}^2}$, which stochastically dominates Bernoulli site percolation with density $1 - (1-\varepsilon)^{16r_\Lambda^2}$.

This dictates the choice of ε in terms of the diameter of the interaction r_Λ: For example, $\varepsilon = 1/(Cr_\Lambda^2)$ for C large enough qualifies.

The end of the proof is a straightforward modification of the one in the nearest-neighbour case.

2.3 Long-range case: Proof of Theorem 2

In this section we study the long-range case, by adapting the technique of [P, FP] to the setting of singular interaction. As in the previous section, we restrict our attention to the case of $\mathbb{S}^1$-valued spins (the extension to the general case is done in the same way as before). We give here a proof only for the case when all the interactions J_x in (8) are nonnegative. The proof in the general case is then straightforward.

Let again Λ_n be the box $\left\{x \in \mathbb{Z}^2 : ||x||_\infty \leq n\right\}$, $\mathcal{E}_n^J = \{\{x,y\} : J_{x-y} \neq 0, \{x,y\} \cap \Lambda_n \neq \emptyset\}$, and let $\bar\phi$ be an arbitrary boundary condition outside Λ_n. The relative Hamiltonian takes the form

$$\bar{\mathcal{H}}\left(\phi_{\Lambda_n}|\bar\phi\right) = \sum_{\substack{\{x,y\}\in\mathcal{E}_n^J \\ \{x,y\}\subset\Lambda_n}} J_{x-y}\, \bar U(\phi_x - \phi_y) + \sum_{\substack{\{x,y\}\in\mathcal{E}_n^J \\ \{x,y\}\not\subset\Lambda_n}} J_{x-y}\, \bar U(\phi_x - \bar\phi_y)\,,$$

where as in (24) the interaction $\bar U$ consists of smooth part U and small part v.

Recall that due to the normalization assumption (9), we can interpret the numbers $j(x) \stackrel{\triangle}{=} J_x$ as the transition probabilities of a symmetric random-walk $X_\cdot$ on $\mathbb{Z}^2$. We denote by $\mathbb{E}_X$ expectation w.r.t. this random-walk conditioned to start at the origin at time 0. Our assumption on the coupling constants $J_\cdot$ is that $X_\cdot$ is recurrent.

Let $\langle\cdot\rangle_{n,\bar\phi}$ be the Gibbs state in Λ_n corresponding to the interaction $\bar U$ and the boundary condition $\bar\phi$. To prove the theorem, it is enough to show that, for any $\delta > 0$, any bounded local function $f(\phi)$ and any $\psi \in \mathbb{S}^1$,

$$\lim_{n\to\infty} \left|\langle f(\phi+\psi)\rangle_{n,\bar\phi} - \langle f(\phi)\rangle_{n,\bar\phi}\right| \leq \delta\,. \qquad (35)$$

2.3.1 Expansion of the measure

As in Subsection 2.2, we expand the Gibbs measure as

$$\mu_n^{\bar U,\bar\phi} = \sum_{A\subset\mathcal{E}_n^J} \pi_n(A)\, \mu_n^{U,\bar\phi,A},$$

with the probabilities $\pi_n(\cdot)$ given by

$$\pi_n(A) = \frac{Z_n^{U,\bar{\phi},A}}{Z_n^{\bar{U},\bar{\phi}}},$$

and consider the bond percolation process $\mathcal{A}$ on $\mathcal{E}_n^J$ with probability distribution

$$\mathbb{P}_n(\mathcal{A} = A) = \pi_n(A).$$

Exactly as before, we can show that this process is stochastically dominated by independent bond percolation process $\mathbb{Q}_{J,\varepsilon}$ on $\mathcal{E}_n^J$ with probabilities

$$\mathbb{Q}_{J,\varepsilon}(\{x,y\} \in \mathcal{A}) = \varepsilon J_{x-y}.$$

From now on, we always assume that ε is chosen strictly smaller than 1.
We will use the following notation for the connectivities of the process $\mathbb{Q}_{J,\varepsilon}$:

$$p_{x,\varepsilon} = \mathbb{Q}_{J,\varepsilon}\left(0 \overset{\mathcal{A}}{\leftrightarrow} x\right).$$

Notice that

$$p_{x,\varepsilon} \le \sum_{n=1}^{\infty} \varepsilon^n j^{(n)}(x) \triangleq d_\varepsilon(x),$$

where $j^{(n)}$ are the n-steps transition probabilities of the random-walk $X_\cdot$. Therefore

$$c(\varepsilon) \triangleq \sum_x p_{x,\varepsilon} \le \sum_x d_\varepsilon(x) = \sum_{n=1}^{\infty} \varepsilon^n = \frac{\varepsilon}{1-\varepsilon}, \tag{36}$$

and the numbers $c(\varepsilon)^{-1} p_{x,\varepsilon}$ can be considered as the transition probabilities of a new random-walk on $\mathbb{Z}^2$, which we denote by $Y_\cdot$; expectation w.r.t. $Y_\cdot$ conditioned to start at 0 at time 0 is denoted by $\mathbb{E}_Y$. The following lemma plays an essential role in the sequel:

Lemma 9 *$X_\cdot$ recurrent $\Longrightarrow$ $Y_\cdot$ recurrent.*

Proof. The recurrence of $X_\cdot$ is equivalent (see Th. 8.2 in Chapter II of [Sp]) to

$$\int_{\mathbb{T}^2} \frac{d\theta}{1-\phi(\theta)} = \infty, \tag{37}$$

where

$$\phi(\theta) \triangleq \mathbb{E}_X e^{i(\theta,X_1)} = \sum_x e^{i(\theta,x)} j(x) = \sum_x \cos((\theta,x))\, j(x),$$

One has to show that

$$\int_{\mathbb{T}^2} \frac{d\theta}{1-\mathbb{E}_Y e^{i(\theta,Y_1)}} = \infty. \tag{38}$$

Now, $Y.$ is symmetric. Thus

$$
\begin{aligned}
1 - \mathbb{E}_Y e^{i(\theta, Y_1)} &= \mathbb{E}_Y \left(1 - \cos\left((\theta, Y_1)\right)\right) \\
&= \tfrac{1}{c(\varepsilon)} \sum_x \left(1 - \cos\left((\theta, x)\right)\right) p_{x,\varepsilon} \\
&\le \tfrac{1}{c(\varepsilon)} \sum_x \sum_{n=1}^{\infty} \left(1 - \cos\left((\theta, x)\right)\right) \varepsilon^n j^{(n)}(x) \\
&= \tfrac{1}{c(\varepsilon)} \sum_{n=1}^{\infty} \left(1 - \phi^n(\theta)\right) \varepsilon^n \\
&= \frac{1-\phi(\theta)}{c(\varepsilon)} \sum_{n=1}^{\infty} \varepsilon^n \left(1 + \phi(\theta) + \cdots + \phi^{n-1}(\theta)\right) \\
&= \frac{1-\phi(\theta)}{c(\varepsilon)} \sum_{n=0}^{\infty} \phi^n(\theta) \sum_{k>n} \varepsilon^k \\
&= \frac{(1-\phi(\theta))\varepsilon}{c(\varepsilon)(1-\varepsilon)(1-\varepsilon\phi(\theta))} ,
\end{aligned}
$$

which implies that (38) follows from (37). ■

2.3.2 The spin-wave

Let us denote by V the support of f .

Given a subset $A \subseteq \mathcal{E}_n^J$, we define the equivalence relation $\overset{A}{\leftrightarrow}$ between sites of $\mathbb{Z}^2$ by saying that $x \overset{A}{\leftrightarrow} y$ iff there is a path made from the bonds of A, which connects the sites x and y. By definition, $x \overset{A}{\leftrightarrow} x$ for any A. For every $x \in \Lambda_n$ we define

$$
r_A(x) = \sup\{\|y\|_\infty \,:\, y \in \mathbb{Z}^2 \text{ and } y \overset{A}{\leftrightarrow} x\}\,.
$$

Clearly, $\|x\|_\infty \le r_A(x) \le \infty$. We define

$$
\rho_V = \max\{\|x\|_\infty;\, x \in V\} \vee 1 \text{ and } r_A(V) = \max\{r_A(x);\, x \in V\} \vee 1\,.
$$

Let $R(\delta)$ be the smallest number such that

$$
\mathbb{Q}_{J,\varepsilon}\left(r_{\mathcal{A}}(V) > R(\delta)\right) \le \frac{\delta}{2\|f\|_\infty}\,.
$$

Notice that $R(\delta) < \infty$ since

$$
\mathbb{Q}_{J,\varepsilon}\left(r_{\mathcal{A}}(V) > R(\delta)\right) \le |V| \sum_{y:\, \|y\|_\infty > R(\delta) - \rho_V} p_{y,\varepsilon}\,,
$$

and $\sum_x p_{x,\varepsilon} = c(\varepsilon) < \infty$, see (36).

By recurrence of the random-walk $Y.$, which was established above, one can find, for any $\delta > 0$ and $0 < \psi < \infty$, a sequence of non-negative functions $\Psi_{n,\delta,\psi}$

on $\mathbb{Z}^2$ – *the spin-waves* – such that $\Psi_{n,\delta,\psi}(x) = 0$ if $x \notin \Lambda_n$, $\Psi_{n,\delta,\psi}(x) = \psi$ if $\|x\|_\infty < R(\delta)$, and

$$\lim_{n\to\infty} \sum_{x\in\Lambda_n} \sum_{y\in\mathbb{Z}^2} p_{x-y,\varepsilon} \left(\Psi_{n,\delta,\psi}(x) - \Psi_{n,\delta,\psi}(y)\right)^2 = 0\,. \tag{39}$$

The most natural candidate for such a spin-wave is given by

$$\Psi_{n,\delta,\psi}(x) = \psi\, \mathbb{P}_Y^x \left(\tau_{\Lambda_R} < \tau_{\Lambda_n^c}\right), \tag{40}$$

where $\mathbb{P}_Y^x$ denotes the law of Y-random walk starting at x, whereas τ_{Λ_R} and $\tau_{\Lambda_n^c}$ are the first hitting times of $\Lambda_{R(\delta)}$ and of the exterior $\Lambda_n^c = Z^2 \setminus \Lambda_n$ respectively. Then (39) is related to the vanishing, as $n \to \infty$, of the escape probability from Λ_n.

The function $\Psi_{n,\delta,\psi}(\cdot)$ in (40) also represents the voltage distribution (c.f. [DoS] on the interpretation of recurrence in terms of electric networks) in the network on the graph $\left(\mathbb{Z}, \mathcal{E}^J\right)$ with bond conductances $p_{x-y,\varepsilon}$, once all the sites in $\Lambda_{R(\delta)}$ are kept at the constant voltage ψ, whereas all the sites in Λ_n^c are grounded. In this language the vanishing of the limit in (39) means zero conductance from $\Lambda_{R(\delta)}$ to infinity, which is a characteristic property of electric networks corresponding to recurrent random walks.

Let us fix a spin-wave sequence $\{\Psi_{n,\delta,\psi}(x)\}$ so that (39) holds.

For any n and any $A \subset \mathbb{Z}^2$ such that $r_A(V) \leq R(\delta)$, we define the corresponding A-deformed spin-wave by

$$\widetilde{\Psi}_{n,\delta,\psi,A}(x) \stackrel{\triangle}{=} \min_{y : x \stackrel{A}{\leftrightarrow} y} \Psi_{n,\delta,\psi}(y)\,. \tag{41}$$

When A is such that $r_A(V) > R(\delta)$, we simply set $\widetilde{\Psi}_{n,\delta,\psi,A} \equiv 0$.

For any $x \in \Lambda_n$ we denote by $t_A(x) \in \mathbb{Z}^2$ one of the sites $y : x \stackrel{A}{\leftrightarrow} y$, at which the minimum in (41) is attained. (This is a slight abuse of notation, since in fact the site $t_A(x)$ depends also on the function $\Psi_{n,\delta,\psi}(\cdot)$.)

The deformed spin-wave is less regular than $\Psi_{n,\delta,\psi}$, but has the property, crucial for us, that $\widetilde{\Psi}_{n,\delta,\psi,A}(x) = \widetilde{\Psi}_{n,\delta,\psi,A}(y)$ whenever $x \stackrel{A}{\leftrightarrow} y$. In particular, $\widetilde{\Psi}_{n,\delta,\psi,A}(x) = 0$ whenever x is A-connected to the outside of Λ_n.

We introduce the tilted measure

$$\mu_n^{U,\bar\phi,A,\widetilde{\Psi}}(\,\cdot\,) = \mu_n^{U,\bar\phi,A}(\,\cdot\, + \widetilde{\Psi}_{n,\delta,\psi,A})\,.$$

Notice that $\mu_n^{U,\bar\phi,A,\widetilde{\Psi}} = \mu_n^{U,\bar\phi,A}$ whenever A is such that $r_A(V) > R(\delta)$. On the other hand, if $r_A(V) \leq R(\delta)$, then

$$\langle f(\phi + \psi)\rangle_n^{U,\bar\phi,A} = \langle f(\phi)\rangle_n^{U,\bar\phi,A,\widetilde{\Psi}}\ .$$

Consequently the following estimate holds:

$$\left|\langle f\left(\phi+\psi\right)\rangle_{n,\bar\phi} - \langle f\left(\phi\right)\rangle_{n,\bar\phi}\right| \leq \mathbb{E}_n \left|\langle f(\phi)\rangle_n^{U,\bar\phi,A} - \langle f(\phi)\rangle_n^{U,\bar\phi,A,\widetilde{\Psi}}\right| + 2\|f\|_\infty\, \mathbb{P}_n\left(r_A(V) > R(\delta)\right).$$

Our target assertion (35) is a consequence of the following two results:

$$\lim_{n\to\infty} \mathbb{E}_n \left| \langle f(\phi)\rangle_n^{U,\bar\phi,A} - \langle f(\phi)\rangle_n^{U,\bar\phi,A,\widetilde\Psi} \right| = 0\,, \tag{42}$$

and

$$2\|f\|_\infty\, \mathbb{P}_n\left(r_A(V) > R(\delta)\right) \le \delta\,. \tag{43}$$

The second bound readily follows from the stochastic domination by the Bernoulli percolation process $\mathbb{Q}_J$ and the definition of $R(\delta)$. The next subsection is devoted to the proof of (42). Our approach is essentially that of [P, FP], but with some simplifications. The main difference between the latter works and ours is that, using a suitable relative entropy inequality, we obtain estimates on difference of expectations in finite volume; in this way, (42) follows immediately by taking the thermodynamic limit, instead of using the general theory of infinite-volume Gibbs states.

2.3.3 Relative entropy estimate

By the well known inequality (see e.g. [F], f-la (3.4) on p.133),

$$\left| \langle f(\phi)\rangle_n^{U,\bar\phi,A} - \langle f(\phi)\rangle_n^{U,\bar\phi,A,\widetilde\Psi} \right| \le \|f\|_\infty \sqrt{2\mathrm{H}(\mu_n^{U,\bar\phi,A,\widetilde\Psi}\,|\,\mu_n^{U,\bar\phi,A})},$$

where $\mathrm{H}(\mu_n^{U,\bar\phi,A,\widetilde\Psi}\,|\,\mu_n^{U,\bar\phi,A})$ is the relative entropy of $\mu_n^{U,\bar\phi,A,\widetilde\Psi}$ with respect to $\mu_n^{U,\bar\phi,A}$. By Jensen's inequality it suffices to show that

$$\lim_{n\to\infty} \mathbb{E}_n \mathrm{H}\left(\mu_n^{U,\bar\phi,\mathcal{A},\widetilde\Psi}\,|\,\mu_n^{U,\bar\phi,\mathcal{A}}\right) = 0\,. \tag{44}$$

From now on we assume that we are working on the event $r_A(V) \le R(\delta)$ (otherwise the relative entropy is 0). We follow [P], and we write:

$$\begin{aligned}\mathrm{H}(\mu_n^{U,\bar\phi,A,\widetilde\Psi}\,|\,\mu_n^{U,\bar\phi,A}) &\le \mathrm{H}(\mu_n^{U,\bar\phi,A,\widetilde\Psi}\,|\,\mu_n^{U,\bar\phi,A}) + \mathrm{H}(\mu_n^{U,\bar\phi,A,-\widetilde\Psi}\,|\,\mu_n^{U,\bar\phi,A}) \\ &= \left\langle \left(\mathcal{H}(\phi + \widetilde\Psi_{n,\delta,\psi,A}\,|\,\bar\phi) + \mathcal{H}(\phi - \widetilde\Psi_{n,\delta,\psi,A}\,|\,\bar\phi) - 2\mathcal{H}(\phi\,|\,\bar\phi)\right)\right\rangle_{n,\bar\phi,A}\,,\end{aligned}$$

where, as before, $\mathcal{H}(\phi|\bar\phi)$ is the Hamiltonian defined by the smooth part of the interaction. Taylor expansion yields

$$\begin{aligned}&\mathcal{H}(\phi + \widetilde\Psi_{n,\delta,\psi,A}\,|\,\bar\phi) + \mathcal{H}(\phi - \widetilde\Psi_{n,\delta,\psi,A}\,|\,\bar\phi) - 2\mathcal{H}(\phi\,|\,\bar\phi) \\ &\qquad\le c_4 \sum_{\substack{x\in\Lambda_n\\ y\in\mathbb{Z}^2}} J_{x-y}\left(\Psi_{n,\delta,\psi}(t_A(x)) - \Psi_{n,\delta,\psi}(t_A(y))\right)^2\end{aligned}$$

with $c_4 = \max|U''|$. By Jensen's inequality,

$$\begin{aligned}&\left(\Psi_{n,\delta,\psi}(t_A(x)) - \Psi_{n,\delta,\psi}(t_A(y))\right)^2 \\ &\quad\le 3\Big\{\left(\Psi_{n,\delta,\psi}(t_A(x)) - \Psi_{n,\delta,\psi}(x)\right)^2 + \left(\Psi_{n,\delta,\psi}(t_A(y)) - \Psi_{n,\delta,\psi}(y)\right)^2 \\ &\qquad\qquad + \left(\Psi_{n,\delta,\psi}(x) - \Psi_{n,\delta,\psi}(y)\right)^2\Big\}\,.\end{aligned} \tag{45}$$

The sum of the third terms of (45) is bounded by

$$\sum_{\substack{x\in\Lambda_n\\ y\in\mathbb{Z}^2}} J_{x-y}\ \left(\Psi_{n,\delta,\psi}(x)-\Psi_{n,\delta,\psi}(y)\right)^2 \le \frac{1}{\varepsilon}\sum_{\substack{x\in\Lambda_n\\ y\in\mathbb{Z}^2}} p_{x-y,\varepsilon}\ \left(\Psi_{n,\delta,\psi}(x)-\Psi_{n,\delta,\psi}(y)\right)^2\ ,$$

and therefore, by the very definition of $\Psi_{n,\delta,\psi}$, goes to zero as $n\to\infty$. The contribution of the remaining two terms of (45) to $\mathbb{E}_n\mathrm{H}$ is bounded by

$$\begin{aligned}
2\,\mathbb{E}_n \sum_{\substack{x\in\Lambda_n\\ y\in\mathbb{Z}^2}} J_{x-y}\ &\left(\Psi_{n,\delta,\psi}(t_A(x))-\Psi_{n,\delta,\psi}(x)\right)^2\\
&\le C\sum_{x\in\Lambda_n}\mathbb{E}_n\ \left(\Psi_{n,\delta,\psi}(t_A(x))-\Psi_{n,\delta,\psi}(x)\right)^2\\
&\le C\sum_{\substack{x\in\Lambda_n\\ y\in\mathbb{Z}^2}}\mathbb{Q}_{J,\varepsilon}\left(x\overset{\mathcal{A}}{\leftrightarrow}y\right)\left(\Psi_{n,\delta,\psi}(y)-\Psi_{n,\delta,\psi}(x)\right)^2\\
&= C\sum_{\substack{x\in\Lambda_n\\ y\in\mathbb{Z}^2}} p_{x-y,\varepsilon}\ \left(\Psi_{n,\delta,\psi}(y)-\Psi_{n,\delta,\psi}(x)\right)^2\ ,
\end{aligned}$$

and the result follows again from the definition (39) of $\Psi_{n,\delta,\psi}$.

2.4 Continuous symmetry breaking: proof of Theorem 4

We construct the whole family of spontaneously magnetized states μ_ν by prescribing the corresponding boundary conditions. Let Λ_n be the box $\left\{\mathbf{x}\in\mathbb{Z}^2 : ||\mathbf{x}||_\infty \le n\right\}$. We define first the boundary condition $\tilde{\phi}_\tau$ by

$$\tilde{\phi}_\tau\,(x_1,x_2)=x_2\tau\theta,\ \tau\in[0,1]\,,$$

see (11). It is easy to see that the unique configuration in Λ_n with finite energy with respect to the b.c. $\tilde{\phi}_{\tau=1}$ outside Λ_n is the one which coincides with $\tilde{\phi}_{\tau=1}$ inside Λ_n. In principle that means that the atomic measure $\mu_{\nu=1}$, concentrated on the configuration $\tilde{\phi}_{\tau=1}$, is itself a Gibbs state for interaction (11), for any temperature, so we are already done. However, this measure has its finite-dimensional distributions singular with respect to the Lebesgue measure. To present a more aesthetically appealing example we proceed as follows.

Consider the measure $\mu^{\mathbf{0}}$ corresponding to zero boundary conditions $\tilde{\phi}_0=0$. In case it is not S^1-invariant at some finite temperature β^{-1} and has nonzero spontaneous magnetization, we are done again. In the opposite case (which seems to us to be the true one) we have that for every arc γ on a circle $\mathbb{S}^1$

$$\mathbb{P}_{0,\Lambda_n}\left\{\phi\,(0,0)\in\gamma\right\}\to\frac{|\gamma|}{2\pi}\text{ as }n\to\infty,$$

where we denote by P_{τ,Λ_n} the conditional Gibbs distribution in Λ_n subject to boundary conditions $\tilde{\phi}_\tau$ outside Λ_n, corresponding to inverse temperature β. Let us fix γ to be the arc $\left[-\frac{\pi}{6}, \frac{\pi}{6}\right] \subset \mathbb{S}^1$, say, so $|\gamma| = \frac{\pi}{3}$. Then $P_{0,\Lambda_n}\{\phi(0,0) \in \gamma\} \to \frac{1}{6}$ as $n \to \infty$. Note now, that for every n fixed, the function $P_{\tau,\Lambda_n}\{\phi(0,0) \in \gamma\}$ is continuous in τ, with $P_{\tau,\Lambda_n}\{\phi(0,0) \in \gamma\} \to 1$ as $\tau \to 1$. Therefore for every n big enough we can define the value $\tau(n,\nu)$ to be the solution of the equation

$$\mathbb{P}_{\tau(n,\nu),\Lambda_n}\left\{\phi(0,0) \in \left[-\frac{\pi}{6}, \frac{\pi}{6}\right]\right\} = \nu,$$

where $\frac{1}{6} < \nu < 1$. Denote by μ_ν any weak limit of the sequence of the finite-dimensional Gibbs distributions $P_{\tau(n,\nu),\Lambda_n}$. Then μ_ν is of course a Gibbs state. Since evidently $\mu_\nu\left\{\phi(0,0) \in \left[-\frac{\pi}{6}, \frac{\pi}{6}\right]\right\} = \nu$, this state is not $\mathbb{S}^1$-invariant once $\nu > \frac{1}{6}$.

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