Potts model with a defect line

Yvan Velenik

joint work with Sébastien Ott

Université de Genève

The Potts model & basic properties

The model

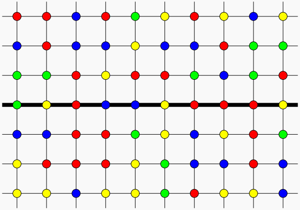
q-state Potts model on $\mathbb{Z}^d$

- The line: $\mathcal{L} = \{k\mathbf{e}_1 \in \mathbb{Z}^d \,:\, k \in \mathbb{Z}\}$
- Configurations:

$$\Omega = \{\sigma = (\sigma_i)_{i\in\mathbb{Z}^d} \,:\, \sigma_i \in \{1,\dots,q\}\}$$

- Coupling constants:

$$J_{ij} = \begin{cases} 1 & \text{if } i \sim j, \{i,j\} \not\subset \mathcal{L} \\ J & \text{if } i \sim j, \{i,j\} \subset \mathcal{L} \\ 0 & \text{otherwise} \end{cases}$$



- Energy of $\sigma \in \Omega$ in the box $\Lambda \Subset \mathbb{Z}^d$:

$$\mathcal{H}_{\Lambda;J}(\sigma) = \sum_{\{i,j\}\cap\Lambda\neq\emptyset} J_{ij}\, \mathbb{1}_{\{\sigma_i\neq\sigma_j\}}$$

The model

- Gibbs measure in $\Lambda \Subset \mathbb{Z}^d$ with boundary condition $\eta \in \Omega$:

$$\mu^{\eta}_{\Lambda;\beta,J}(\sigma) = \begin{cases} \frac{1}{\mathcal{Z}^{\eta}_{\Lambda;\beta,J}} e^{-\beta \mathcal{H}_{\Lambda;J}(\sigma)} & \text{if } \sigma_i = \eta_i \ \forall i \notin \Lambda \\ 0 & \text{otherwise} \end{cases}$$

- Infinite-volume Gibbs measure: any probability measure μ on Ω s.t.

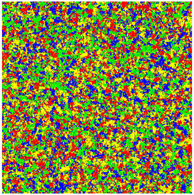
$$\mu(\,\cdot\,|\,\sigma_i = \eta_i \ \forall i \notin \Lambda) = \mu^{\eta}_{\Lambda;\beta,J}(\,\cdot\,)$$

for all $\Lambda \Subset \mathbb{Z}^d$ and μ-a.e. $\eta \in \Omega$.

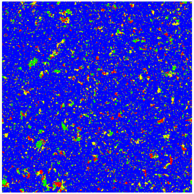
Phase transition

For all $d \geq 2$, there exists $\beta_c = \beta_c(d) \in (0, \infty)$ s.t.

- for all $\beta < \beta_c$, there is a unique infinite-volume Gibbs measure



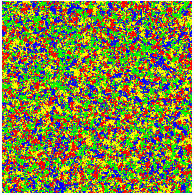
- for all $\beta > \beta_c$, there exist distinct infinite-volume Gibbs measures $\mu_\beta^1, \ldots, \mu_\beta^q$ displaying long-range order: $\inf_{i \in \mathbb{Z}^d} \mu_\beta^k(\sigma_0 = \sigma_i) > \frac{1}{q}$



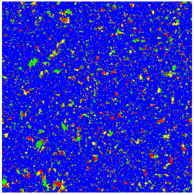
Phase transition

For all $d \geq 2$, there exists $\beta_c = \beta_c(d) \in (0, \infty)$ s.t.

- for all $\beta < \beta_c$, there is a unique infinite-volume Gibbs measure



- for all $\beta > \beta_c$, there exist distinct infinite-volume Gibbs measures $\mu_\beta^1, \ldots, \mu_\beta^q$ displaying long-range order: $\inf_{i \in \mathbb{Z}^d} \mu_\beta^k(\sigma_0 = \sigma_i) > \frac{1}{q}$



From now on, we fix some $\boxed{\beta < \beta_c}$ and denote by μ the unique infinite-volume Gibbs measure

Some facts about the homogeneous $[J = 1]$ model

Exponential decay of correlations

$$\left|\mu(\sigma_0 = \sigma_{n\mathbf{e}_1}) - \frac{1}{q}\right|$$

Some facts about the homogeneous $[J = 1]$ model

Exponential decay of correlations

$$\xi = -\lim_{n\to\infty} \frac{1}{n} \log \left| \mu(\sigma_0 = \sigma_{n\mathbf{e}_1}) - \frac{1}{q} \right| > 0$$

[Duminil-Copin, Raoufi, Tassion '17]

ξ is the inverse correlation length (in the $\mathbf{e}_1$ direction)

Some facts about the homogeneous $[J = 1]$ model

Exponential decay of correlations

$$\xi = -\lim_{n\to\infty} \frac{1}{n} \log \left| \mu(\sigma_0 = \sigma_{n\mathbf{e}_1}) - \frac{1}{q} \right| > 0$$

[Duminil-Copin, Raoufi, Tassion '17]

ξ is the inverse correlation length (in the $\mathbf{e}_1$ direction)

Ornstein–Zernike asymptotics

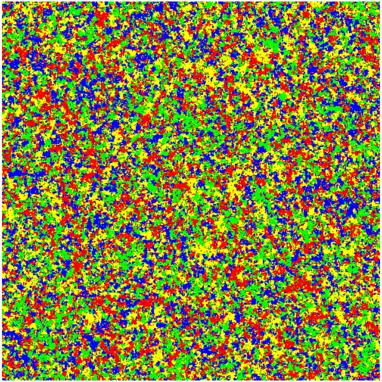
There exists $C = C(\beta)$ such that, as $n \to \infty$,

$$\mu(\sigma_0 = \sigma_{n\mathbf{e}_1}) = \frac{1}{q} + \frac{C}{n^{(d-1)/2}} e^{-\xi n} (1 + o(1))$$

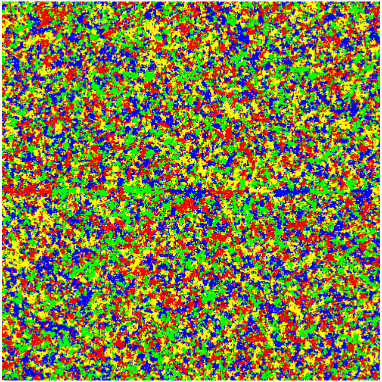
[Campanino, Ioffe, V. '08]

Effect of the defect line on the correlation length

Impact of the defect line: basic question



$J = 1$



$J = 3$

Impact of the defect line: basic question

Fix $\beta < \beta_c$; let $J \geq 0$ and denote by μ_J the unique infinite-volume Gibbs measure.

Longitudinal inverse correlation length:

$$\xi(J) = -\lim_{n\to\infty} \frac{1}{n} \log\left|\mu_J(\sigma_0 = \sigma_{n\mathbf{e}_1}) - \frac{1}{q}\right|$$

(exists by FKG + subadditivity)

Impact of the defect line: basic question

Fix $\beta < \beta_c$; let $J \geq 0$ and denote by μ_J the unique infinite-volume Gibbs measure.

Longitudinal inverse correlation length:

$$\xi(J) = -\lim_{n\to\infty} \frac{1}{n} \log\left|\mu_J(\sigma_0 = \sigma_{n\mathbf{e}_1}) - \frac{1}{q}\right|$$

(exists by FKG + subadditivity)

Main question

How does $\xi(J)$ vary as J grows from 0 to $+\infty$?

Impact of the defect line: basic question

Fix $\beta < \beta_c$; let $J \geq 0$ and denote by μ_J the unique infinite-volume Gibbs measure.

Longitudinal inverse correlation length:

$$\xi(J) = -\lim_{n\to\infty} \frac{1}{n} \log\left|\mu_J(\sigma_0 = \sigma_{n\mathbf{e}_1}) - \frac{1}{q}\right|$$

(exists by FKG + subadditivity)

Main question

How does $\xi(J)$ vary as J grows from 0 to $+\infty$?

This question was first addressed, using **exact computations** in the **two-dimensional Ising model** in [McCoy, Perk '80]. There were many follow-ups (same model, various settings, exact computations).

We consider the problem for **general Potts models**, in **any dimension** $d \geq 2$.

Basic properties

Theorem (Ott, V. '17)

- $J \mapsto \xi(J)$ *is positive, Lipschitz-continuous and nonincreasing*
- $\xi(J) = \xi(1) \equiv \xi$ *for all* $J \leq 1$
- $\xi(J) \sim e^{-2\beta J}$ *as* $J \to \infty$

Impact of the defect line: main results

Basic properties

Theorem (Ott, V. '17)

- *$J \mapsto \xi(J)$ is positive, Lipschitz-continuous and nonincreasing*
- $\xi(J) = \xi(1) \equiv \xi$ *for all* $J \leq 1$
- $\xi(J) \sim e^{-2\beta J}$ *as* $J \to \infty$

Basic properties

Theorem (Ott, V. '17)

- $J \mapsto \xi(J)$ *is positive, Lipschitz-continuous and nonincreasing*
- $\xi(J) = \xi(1) \equiv \xi$ *for all* $J \leq 1$
- $\xi(J) \sim e^{-2\beta J}$ *as* $J \to \infty$

Basic properties

Theorem (Ott, V. '17)

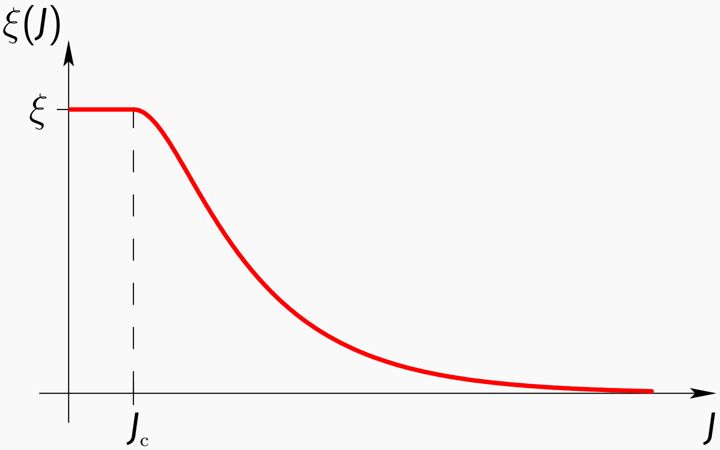
- *$J \mapsto \xi(J)$ is positive, Lipschitz-continuous and nonincreasing*
- $\xi(J) = \xi(1) \equiv \xi$ *for all* $J \leq 1$
- $\xi(J) \sim e^{-2\beta J}$ *as* $J \to \infty$

In particular, there exists $J_c \in [1, \infty)$ such that

$\xi(J) = \xi$ for all $J \leq J_c$ and $\xi(J) < \xi$ for all $J > J_c$

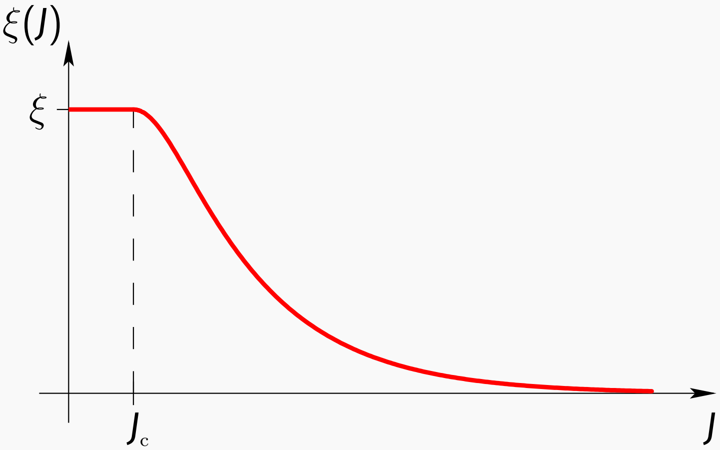
Impact of the defect line: main results

Typical graph



Impact of the defect line: main results

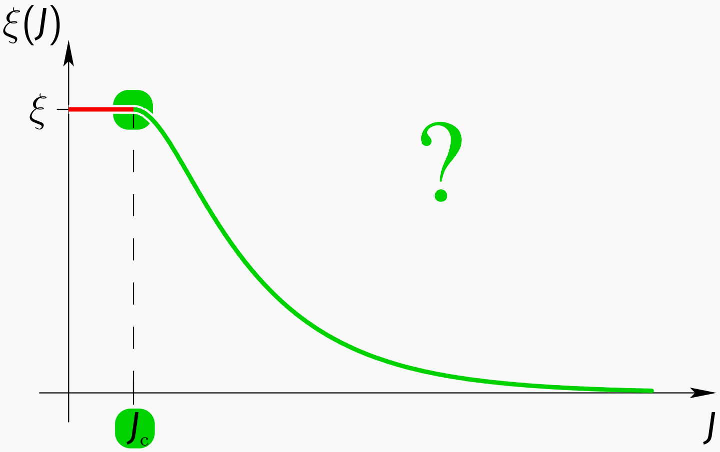
Typical graph



Actually: corresponds to the exact result for $2d$ Ising from [McCoy, Perk '80]

Impact of the defect line: main results

Typical graph



Actually: corresponds to the exact result for $2d$ Ising from [McCoy, Perk '80]

Theorem (Ott, V. '17)

1. $J_c = 1$ *when* $d = 2$ *or* 3, *but* $J_c > 1$ *when* $d \geq 4$
2. $J \mapsto \xi(J)$ *is* ***strictly decreasing*** *and* ***real-analytic*** *when* $J > J_c$
3. *There exist constants* $c_2^\pm, c_3^\pm > 0$ *such that, as* $J \downarrow J_c$,

$$c_2^-(J - J_c)^2 \leq \xi(J_c) - \xi(J) \leq c_2^+(J - J_c)^2 \qquad (d = 2)$$

$$e^{-c_3^-/(J-J_c)} \leq \xi(J_c) - \xi(J) \leq e^{-c_3^+/(J-J_c)} \qquad (d = 3)$$

4. *For any* $J > J_c$, *there exists* $C = C(J, \beta)$ *such that*

$$\mu(\sigma_0 = \sigma_{n\mathbf{e}_1}) = \frac{1}{q} + C\, e^{-\xi(J)n}\,(1 + o(1))$$

Impact of the defect line: main results

Theorem (Ott, V. '17)

1. *$J_c = 1$ when $d = 2$ or 3, but $J_c > 1$ when $d \geq 4$*
2. *$J \mapsto \xi(J)$ is strictly decreasing and real-analytic when $J > J_c$*
3. *There exist constants $c_2^{\pm}, c_3^{\pm} > 0$ such that, as $J \downarrow J_c$,*

$$c_2^-(J - J_c)^2 \leq \xi(J_c) - \xi(J) \leq c_2^+(J - J_c)^2 \qquad (d = 2)$$

$$e^{-c_3^-/(J-J_c)} \leq \xi(J_c) - \xi(J) \leq e^{-c_3^+/(J-J_c)} \qquad (d = 3)$$

4. *For any $J > J_c$, there exists $C = C(J, \beta)$ such that*

$$\mu(\sigma_0 = \sigma_{n\mathbf{e}_1}) = \frac{1}{q} + C\, e^{-\xi(J)n}\,(1 + o(1))$$

Impact of the defect line: main results

Theorem (Ott, V. '17)

1. $J_c = 1$ *when* $d = 2$ *or* 3, *but* $J_c > 1$ *when* $d \geq 4$
2. $J \mapsto \xi(J)$ *is* ***strictly decreasing*** *and* ***real-analytic*** *when* $J > J_c$
3. *There exist constants* $c_2^{\pm}, c_3^{\pm} > 0$ *such that, as* $J \downarrow J_c$,

$$c_2^-(J - J_c)^2 \leq \xi(J_c) - \xi(J) \leq c_2^+(J - J_c)^2 \qquad (d = 2)$$
$$e^{-c_3^-/(J-J_c)} \leq \xi(J_c) - \xi(J) \leq e^{-c_3^+/(J-J_c)} \qquad (d = 3)$$

4. *For any* $J > J_c$, *there exists* $C = C(J, \beta)$ *such that*

$$\mu(\sigma_0 = \sigma_{n\mathbf{e}_1}) = \frac{1}{q} + C\, e^{-\xi(J)n}\,(1 + o(1))$$

Impact of the defect line: main results

Theorem (Ott, V. '17)

1. $J_c = 1$ *when* $d = 2$ *or* 3, *but* $J_c > 1$ *when* $d \geq 4$
2. $J \mapsto \xi(J)$ *is* ***strictly decreasing*** *and* ***real-analytic*** *when* $J > J_c$
3. *There exist constants* $c_2^{\pm}, c_3^{\pm} > 0$ *such that, as* $J \downarrow J_c$,

$$c_2^-(J - J_c)^2 \leq \xi(J_c) - \xi(J) \leq c_2^+(J - J_c)^2 \qquad (d = 2)$$

$$e^{-c_3^-/(J-J_c)} \leq \xi(J_c) - \xi(J) \leq e^{-c_3^+/(J-J_c)} \qquad (d = 3)$$

4. *For any* $J > J_c$, *there exists* $C = C(J, \beta)$ *such that*

$$\mu(\sigma_0 = \sigma_{n\mathbf{e}_1}) = \frac{1}{q} + C\, e^{-\xi(J)n}\,(1 + o(1))$$

Reformulation as FK percolation

Reformulation as FK percolation

The proof relies on a reformulation in terms of **FK percolation**.

- Let $p, p' \in (0,1)$, $q \in [1,\infty)$. Set $x = p/(1-p)$ and $x' = p'/(1-p')$.

Reformulation as FK percolation

The proof relies on a reformulation in terms of **FK percolation**.

- Let $p, p' \in (0,1)$, $q \in [1,\infty)$. Set $x = p/(1-p)$ and $x' = p'/(1-p')$.
- Let ω be a collection of n.n. edges of $\{-M,\ldots,M\}^d$.

Reformulation as FK percolation

The proof relies on a reformulation in terms of **FK percolation**.

- Let $p, p' \in (0,1)$, $q \in [1,\infty)$. Set $x = p/(1-p)$ and $x' = p'/(1-p')$.
- Let ω be a collection of n.n. edges of $\{-M,\ldots,M\}^d$.
- Write $|\omega| = \#$ of edges in ω and $\mathrm{O}_{\mathcal{L}}(\omega) = \#$ of those edges lying on $\mathcal{L}$.

Reformulation as FK percolation

The proof relies on a reformulation in terms of **FK percolation**.

- Let $p, p' \in (0,1)$, $q \in [1, \infty)$. Set $x = p/(1-p)$ and $x' = p'/(1-p')$.
- Let ω be a collection of n.n. edges of $\{-M, \dots, M\}^d$.
- Write $|\omega| = \#$ of edges in ω and $\mathrm{O}_{\mathcal{L}}(\omega) = \#$ of those edges lying on $\mathcal{L}$.
- We associate to ω the probability

$$\nu^M_{x,x',q}(\omega) \propto x^{|\omega|} \left(\frac{x'}{x}\right)^{\mathrm{O}_{\mathcal{L}}(\omega)} q^{\mathcal{N}(\omega)}$$

where $\mathcal{N}(\omega)$ denotes the number of connected components (including isolated vertices).

Reformulation as FK percolation

The proof relies on a reformulation in terms of **FK percolation**.

- Let $p, p' \in (0,1)$, $q \in [1,\infty)$. Set $x = p/(1-p)$ and $x' = p'/(1-p')$.
- Let ω be a collection of n.n. edges of $\{-M,\ldots,M\}^d$.
- Write $|\omega| = \#$ of edges in ω and $\mathrm{O}_{\mathcal{L}}(\omega) = \#$ of those edges lying on $\mathcal{L}$.
- We associate to ω the probability

$$\nu^{M}_{x,x',q}(\omega) \propto x^{|\omega|} \left(\frac{x'}{x}\right)^{\mathrm{O}_{\mathcal{L}}(\omega)} q^{\mathcal{N}(\omega)}$$

where $\mathcal{N}(\omega)$ denotes the number of connected components (including isolated vertices).

- The weak limit as $M \to \infty$ exists and is denoted simply $\nu_{x'}$ (p and q being kept fixed).

Reformulation as FK percolation

Set $p = 1 - e^{-2\beta}$ and $p' = 1 - e^{-2\beta J}$.

The standard relation between FK percolation and the q-state Potts model implies that

$$\mu_J(\sigma_0 = \sigma_{n\mathbf{e}_1}) = \frac{1}{q} + \frac{q-1}{q}\,\nu_{x'}(0 \leftrightarrow n\mathbf{e}_1)$$

and thus, writing $\xi(x') \equiv \xi(J(x'))$,

$$\xi(x') = -\lim_{n\to\infty} \frac{1}{n} \log \nu_{x'}(0 \leftrightarrow n\mathbf{e}_1)$$

Reformulation as FK percolation

Set $p = 1 - e^{-2\beta}$ and $p' = 1 - e^{-2\beta J}$.

The standard relation between FK percolation and the q-state Potts model implies that

$$\mu_J(\sigma_0 = \sigma_{n\mathbf{e}_1}) = \frac{1}{q} + \frac{q-1}{q}\,\nu_{x'}(0 \leftrightarrow n\mathbf{e}_1)$$

and thus, writing $\xi(x') \equiv \xi(J(x'))$,

$$\boxed{\xi(x') = -\lim_{n\to\infty} \frac{1}{n} \log \nu_{x'}(0 \leftrightarrow n\mathbf{e}_1)}$$

The analysis of $\xi(x')$ in the case $q = 1$ was done in [Friedli, Ioffe, V. '13]

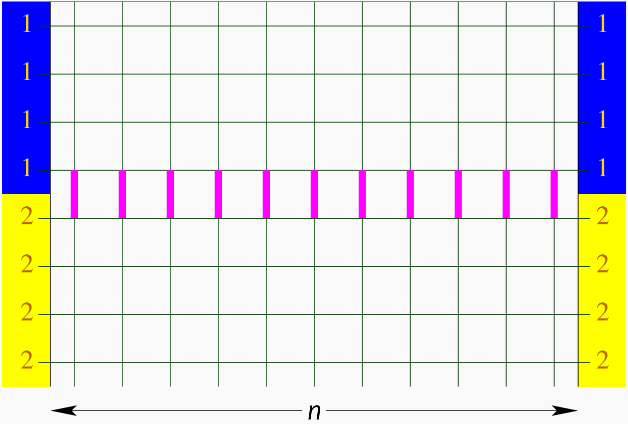
The extension to any $q \geq 1$ is made difficult by the lack of independence and of useful consequences, such as the BK inequality, which have to be substituted by suitable constructions exploiting **exponential mixing** properties.

Interface pinning in the $2d$ Potts model

Pinning in the subcritical Potts model on $\mathbb{Z}^2$

On $\mathbb{Z}^2$, the **self-duality** of FK percolation allows one to reformulate some of the results in terms of the properties of the **interface** in the $2d$ Potts model below its critical temperature.

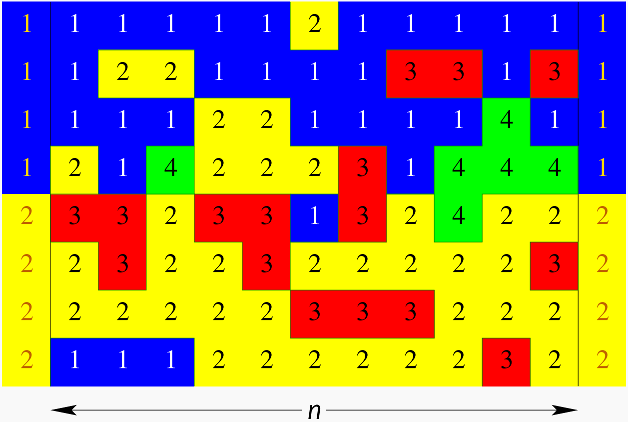
$2d$ Potts model with Dobrushin boundary condition



Only nearest-neighbor spins interact and the coupling constants are all equal to 1, except for those represented in purple, the value of which is $J \geq 0$. We assume that $\beta > \beta_c$.

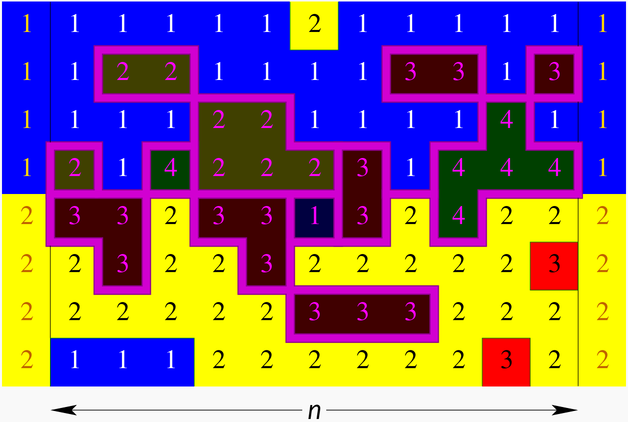
Pinning in the subcritical Potts model on $\mathbb{Z}^2$

$2d$ Potts model with Dobrushin boundary condition



Pinning in the subcritical Potts model on $\mathbb{Z}^2$

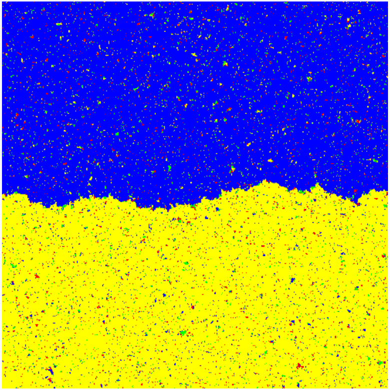
$2d$ Potts model with Dobrushin boundary condition



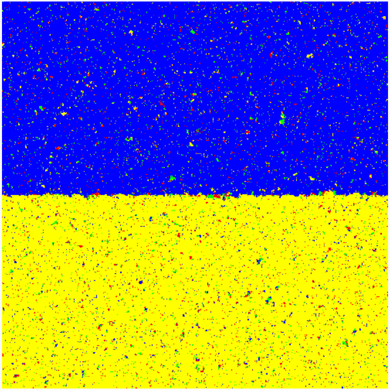
The interface is the connected set of “frustrated” bonds induced by the boundary condition. Weakly converges to Brownian bridge under diffusive scaling when $J = 1$. [Campanino, Ioffe, V. ’08]

Theorem (Ott, V. ’17)

The interface localizes (converges to a horizontal line under diffusive scaling) for all $J < 1$.



$J = 1$



$J = \frac{1}{2}$

Some historical remarks

In 1980–1981, Douglas Abraham published two papers

PHYSICAL REVIEW LETTERS

VOLUME 44 | 5 MAY 1980 | NUMBER 18

Solvable Model with a Roughening Transition for a Planar Ising Ferromagnet

D. B. Abraham[a]

J. Phys. A: Math. Gen. **14** (1981) L369–L372. Printed in Great Britain

LETTER TO THE EDITOR

Binding of a domain wall in the planar Ising ferromagnet

D B Abraham

Some historical remarks

In 1980–1981, Douglas Abraham published two papers

PHYSICAL REVIEW LETTERS

VOLUME 44 | 5 MAY 1980 | NUMBER 18

Solvable Model with a Roughening Transition for a Planar Ising Ferromagnet

D. B. Abraham[a]

J. Phys. A: Math. Gen. **14** (1981) L369–L372. Printed in Great Britain

LETTER TO THE EDITOR

Binding of a domain wall in the planar Ising ferromagnet

D B Abraham

that immediately triggered an intense activity (partial list from 1981!):

Some historical remarks

- Abraham's papers analyzed the interface pinning (and wetting) problem in the $2d$ Ising model, using exact computations.
- The subsequent papers dealt with effective (SOS/random walk) versions of the same problem, with two main goals: getting a better understanding of the mechanisms at play, and analyzing various extensions (higher-dimensional spaces, higher-dimensional interfaces, disordered pinning potential, etc.).
- This intense activity continues to date. See Giacomin's book *Random Polymer Models* for a recent account of the developments from the probabilistic point of view.

Our goal was to show that the current methods of rigorous statistical mechanics finally make it possible to **import back some of these results to actual lattice spin systems**, for which exact computations are not available (and to provide additional information even when they are available).

Link to RW pinning problem

Link to RW pinning problem — upper bound

- Let $\lambda = \log(x'/x)$. Clearly:

$$\xi - \xi_{x'} = \lim_{n\to\infty} \frac{1}{n} \log \frac{\nu_{x'}(0 \leftrightarrow n\mathbf{e}_1)}{\nu(0 \leftrightarrow n\mathbf{e}_1)} = \lim_{n\to\infty} \frac{1}{n} \log \frac{\nu\big[e^{\lambda \mathrm{o}_{\mathcal{L}}} \;\big|\; 0 \leftrightarrow n\mathbf{e}_1\big]}{\nu\big[e^{\lambda \mathrm{o}_{\mathcal{L}}}\big]}$$

Link to RW pinning problem — upper bound

- Let $\lambda = \log(x'/x)$. Clearly:

$$\xi - \xi_{x'} = \lim_{n\to\infty} \frac{1}{n} \log \frac{\nu_{x'}(0 \leftrightarrow n\mathbf{e}_1)}{\nu(0 \leftrightarrow n\mathbf{e}_1)} = \lim_{n\to\infty} \frac{1}{n} \log \frac{\nu\big[e^{\lambda o_{\mathcal{L}}} \ \big|\ 0 \leftrightarrow n\mathbf{e}_1\big]}{\nu\big[e^{\lambda o_{\mathcal{L}}}\big]}$$

- Using FKG, one can show that

$$\frac{\nu\big[e^{\lambda o_{\mathcal{L}}} \ \big|\ 0 \leftrightarrow n\mathbf{e}_1\big]}{\nu\big[e^{\lambda o_{\mathcal{L}}}\big]} \leq \nu\big[e^{\lambda|\mathbf{C}_0 \cap \mathcal{L}|} \ \big|\ 0 \leftrightarrow n\mathbf{e}_1\big]$$

Link to RW pinning problem — upper bound

- Let $\lambda = \log(x'/x)$. Clearly:

$$\xi - \xi_{x'} = \lim_{n\to\infty} \frac{1}{n} \log \frac{\nu_{x'}(0 \leftrightarrow n\mathbf{e}_1)}{\nu(0 \leftrightarrow n\mathbf{e}_1)} = \lim_{n\to\infty} \frac{1}{n} \log \frac{\nu\big[e^{\lambda o_{\mathcal{L}}} \;\big|\; 0 \leftrightarrow n\mathbf{e}_1\big]}{\nu\big[e^{\lambda o_{\mathcal{L}}}\big]}$$

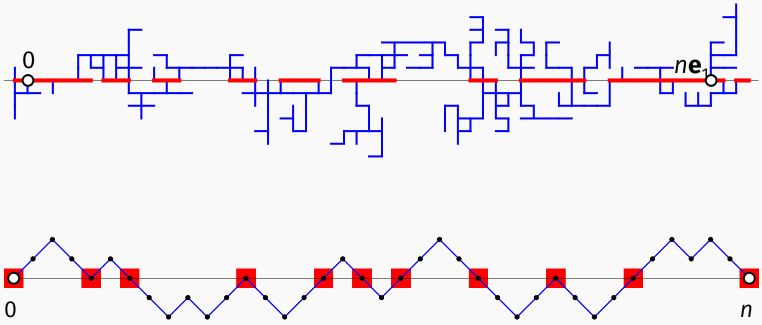
- Using FKG, one can show that

$$\frac{\nu\big[e^{\lambda o_{\mathcal{L}}} \;\big|\; 0 \leftrightarrow n\mathbf{e}_1\big]}{\nu\big[e^{\lambda o_{\mathcal{L}}}\big]} \leq \nu\big[e^{\lambda |\mathbf{C}_0 \cap \mathcal{L}|} \;\big|\; 0 \leftrightarrow n\mathbf{e}_1\big]$$

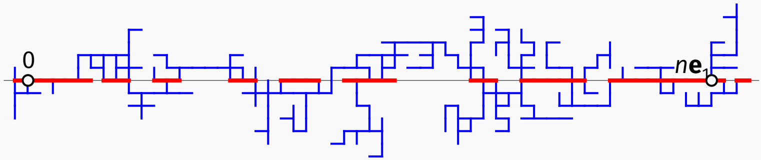
- This is formally analogous to free energy of **RW pinning problem**:

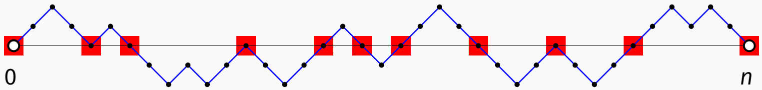
$$f_n(\epsilon) = \mathbb{E}\big[e^{\epsilon|\mathbb{X} \cap \mathcal{L}|} \,|\, X_n = 0\big]$$

where the expectation is w.r.t. ($\mathbb{Z}^{d-1}$-valued) random walk path $\mathbb{X} = (X_0 = 0, X_1 \ldots, X_n)$ and $\epsilon \in \mathbb{R}_+$ is the energetical reward (pinning parameter)


0
$n\mathbf{e}_1$
0
n

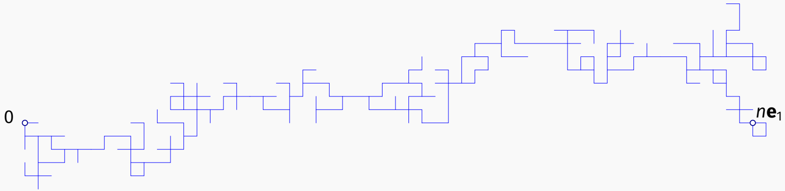
Link to RW pinning problem — upper bound



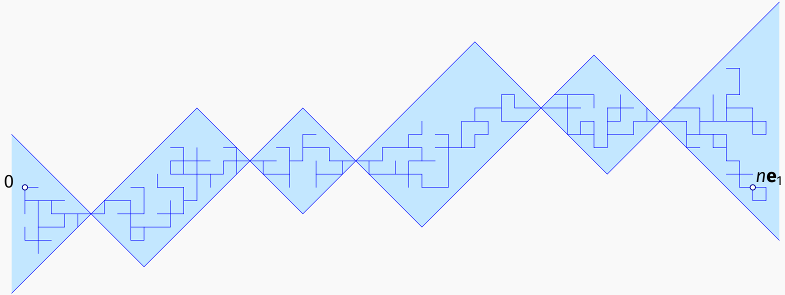


The analogy is made made more precise using an **effective random walk representation** for $\mathbf{C}_0$. This allows to use (extensions of) the arguments developed for RW pinning.

Let $p < p_c$ and $n \in \mathbb{N}$. Then, up to an event of exponentially small $\nu(\cdot \mid 0 \leftrightarrow n\mathbf{e}_1)$-probability, $\mathbf{C}_0$ admits the following decomposition:

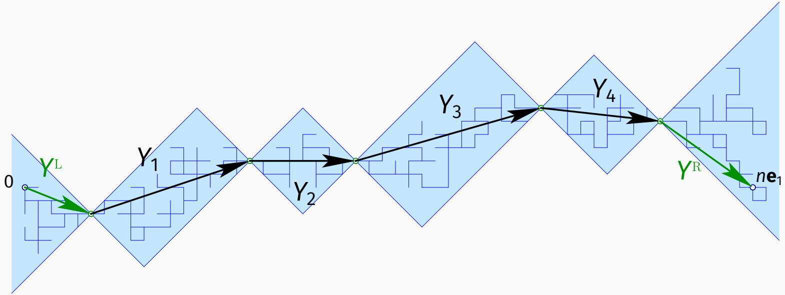


Let $p < p_{\mathrm{c}}$ and $n \in \mathbb{N}$. Then, up to an event of exponentially small $\nu(\cdot \mid 0 \leftrightarrow n\mathbf{e}_1)$-probability, $\mathbf{C}_0$ admits the following decomposition:



Link to RW pinning problem — Effective RW representation

Let $p < p_c$ and $n \in \mathbb{N}$. Then, up to an event of exponentially small $\nu(\cdot \mid 0 \leftrightarrow n\mathbf{e}_1)$-probability, $\mathbf{C}_0$ admits the following decomposition:

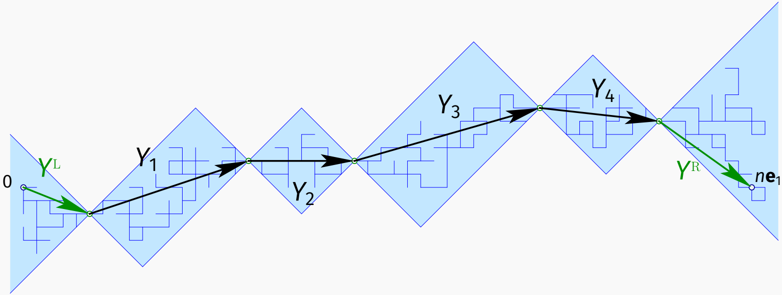


$$\{0 \leftrightarrow n\mathbf{e}_1\} = \{Y^{\mathrm{L}} + Y_1 + \cdots + Y_N + Y^{\mathrm{R}} = n\mathbf{e}_1\},$$

where $(Y_k)_{k\geq 1}$ is a random walk on $\mathbb{Z}^d$ with law P, and Y^{L}, Y^{R} are independent random variables with exponential tails.

Link to RW pinning problem — Effective RW representation

Let $p < p_\mathrm{c}$ and $n \in \mathbb{N}$. Then, up to an event of exponentially small $\nu(\cdot \mid 0 \leftrightarrow n\mathbf{e}_1)$-probability, $\mathbf{C}_0$ admits the following decomposition:



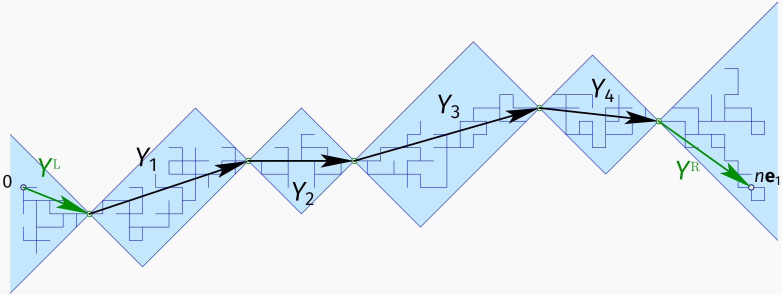
$$\{0 \leftrightarrow n\mathbf{e}_1\} = \{Y^{\mathrm{L}} + Y_1 + \cdots + Y_N + Y^{\mathrm{R}} = n\mathbf{e}_1\},$$

where $(Y_k)_{k\geq 1}$ is a random walk on $\mathbb{Z}^d$ with law P, and Y^{L}, Y^{R} are independent random variables with exponential tails.

This relies on the "RW" representation in [Campanino, Ioffe, V. '08]; independence of the increments is achieved by randomly aggregating the increments of the latter process in a suitable way.

Link to RW pinning problem — Effective RW representation

Let $p < p_c$ and $n \in \mathbb{N}$. Then, up to an event of exponentially small $\nu(\cdot \mid 0 \leftrightarrow n\mathbf{e}_1)$-probability, $\mathbf{C}_0$ admits the following decomposition:



Write $Y_k = (Y_k^{\parallel}, Y_k^{\perp}) \in \mathbb{Z} \times \mathbb{Z}^{d-1}$.

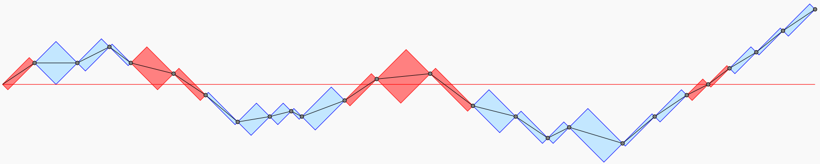
Properties of the effective random walk Y:

- $\mathrm{P}(Y_1^{\parallel} \geq 1) = 1$;
- $\mathrm{P}(\|Y_1\| > t) \leq e^{-\nu t}$ for some $\nu = \nu(p) > 0$;
- for any $z^{\perp} \in \mathbb{Z}^{d-1}$, $\mathrm{P}\big(Y_1^{\perp} = z^{\perp}\big) = \mathrm{P}\big(Y_1^{\perp} = -z^{\perp}\big)$.

Link to RW pinning problem — Effective RW representation

Remember that

$$\xi - \xi_{x'} \leq \lim_{n\to\infty} \frac{1}{n} \log \nu\left[e^{\lambda|\mathbf{c}_0 \cap \mathcal{L}|} \;\middle|\; 0 \leftrightarrow n\mathbf{e}_1\right]$$



We have basically **reduced the problem** (well, the upper bound) **to RW pinning**!

Link to RW pinning problem — Lower bound

Direct analogy with RW pinning **only holds for the upper bound**.

Problems with lower bound:

- interaction of $\mathbf{C}_0$ with the other clusters
- interaction between the other clusters and the line $\mathcal{L}$
  ($\rightsquigarrow$ effective interaction between $\mathbf{C}_0$ and $\mathcal{L}$ is not purely attractive)

$\rightsquigarrow$ requires a different approach...

Link to RW pinning problem — Lower bound

We will introduce a suitable event $\mathcal{M}_\delta \subset \{0 \leftrightarrow n\mathbf{e}_1\}$ and write

$$\exp\{(\xi - \xi_{x'})n\} \quad \asymp \quad \frac{\nu_{x'}(0 \leftrightarrow n\mathbf{e}_1)}{\nu(0 \leftrightarrow n\mathbf{e}_1)}$$

Link to RW pinning problem — Lower bound

We will introduce a suitable event $\mathcal{M}_\delta \subset \{0 \leftrightarrow n\mathbf{e}_1\}$ and write

$$\begin{aligned}
\exp\{(\xi - \xi_{x'})n\} &\asymp \frac{\nu_{x'}(0 \leftrightarrow n\mathbf{e}_1)}{\nu(0 \leftrightarrow n\mathbf{e}_1)} \\
&\geq \frac{\nu_{x'}(\mathcal{M}_\delta)}{\nu(0 \leftrightarrow n\mathbf{e}_1)}
\end{aligned}$$

Link to RW pinning problem — Lower bound

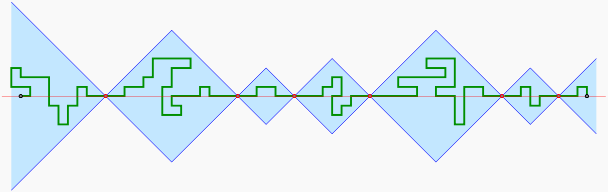
We will introduce a suitable event $\mathcal{M}_\delta \subset \{0 \leftrightarrow n\mathbf{e}_1\}$ and write

$$\begin{aligned}
\exp\{(\xi - \xi_{x'})n\} &\asymp \frac{\nu_{x'}(0 \leftrightarrow n\mathbf{e}_1)}{\nu(0 \leftrightarrow n\mathbf{e}_1)} \\
&\geq \frac{\nu_{x'}(\mathcal{M}_\delta)}{\nu(0 \leftrightarrow n\mathbf{e}_1)} \\
&= \underbrace{\frac{\nu_{x'}(\mathcal{M}_\delta)}{\nu(\mathcal{M}_\delta)}}_{\text{“Energy”}} \underbrace{\nu(\mathcal{M}_\delta \mid 0 \leftrightarrow n\mathbf{e}_1)}_{\text{“Entropy”}}
\end{aligned}$$

Link to RW pinning problem — Lower bound

We choose for $\mathcal{M}_\delta \subset \{0 \leftrightarrow n\mathbf{e}_1\}$ the event

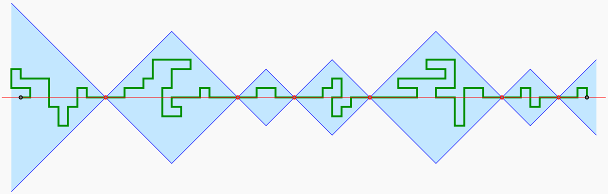
There exists a self-avoiding path $\gamma \subset C_{0,n\mathbf{e}_1}$
possessing at least δn cone-points on $\mathcal{L}$



Link to RW pinning problem — Lower bound

We choose for $\mathcal{M}_\delta \subset \{0 \leftrightarrow n\mathbf{e}_1\}$ the event

There exists a self-avoiding path $\gamma \subset C_{0,n\mathbf{e}_1}$
possessing at least δn cone-points on $\mathcal{L}$

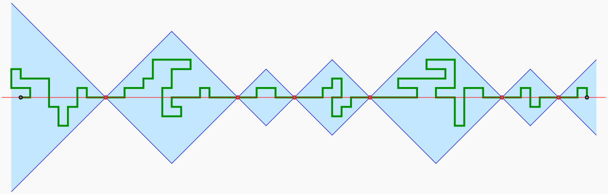


The **entropy term** $\nu(\mathcal{M}_\delta \mid 0 \leftrightarrow n\mathbf{e}_1)$ reduces to an estimate of the probability that the effective random walk Y visits $\mathcal{L}$ at least δn times before reaching $n\mathbf{e}_1$.

Link to RW pinning problem — Lower bound

We choose for $\mathcal{M}_\delta \subset \{0 \leftrightarrow n\mathbf{e}_1\}$ the event

There exists a self-avoiding path $\gamma \subset C_{0,n\mathbf{e}_1}$
possessing at least δn cone-points on $\mathcal{L}$



The **entropy term** $\nu(\mathcal{M}_\delta \mid 0 \leftrightarrow n\mathbf{e}_1)$ reduces to an estimate of the probability that the effective random walk Y visits $\mathcal{L}$ at least δn times before reaching $n\mathbf{e}_1$.

Let's see how the energy bound is established...

Link to RW pinning problem — Lower bound

We use the following Russo-formula-type inequality:

Lemma

Let A be an increasing event and take $x' > x$. Then:

$$\frac{\nu_{x'}(A)}{\nu(A)} \geq \exp\Big(\int_x^{x'} \frac{1}{s(1+s)} \sum_{e \in \mathcal{L}_{[0,n]}} \nu_s(e \in \mathrm{Piv}_A \,|\, A) \,\mathrm{d}s \Big).$$

Link to RW pinning problem — Lower bound

We use the following Russo-formula-type inequality:

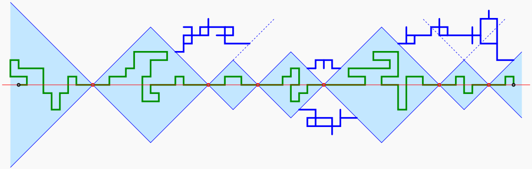
Lemma

Let A be an increasing event and take $x' > x$. Then:

$$\frac{\nu_{x'}(A)}{\nu(A)} \geq \exp\Big(\int_x^{x'} \frac{1}{s(1+s)} \sum_{e \in \mathcal{L}_{[0,n]}} \nu_s(e \in \mathrm{Piv}_A \,|\, A)\,\mathrm{d}s\Big).$$

Applying it with $A = \mathcal{M}_\delta$ reduces the problem to showing: $\exists \rho_1 > 0$ s.t.

$$\sum_{e \in \mathcal{L}_{[0,n]}} \nu_s(e \in \mathrm{Piv}_{\mathcal{M}_\delta} \,|\, \mathcal{M}_\delta) \geq \rho_1 \delta n$$



which follows from exponential decay under ν.

Open problems and extensions

Open problems

- Behavior of $\xi(J)$ in the neighborhood of J_c when $d \geq 4$.
- Sharp asymptotics of 2-point function when $J \leq J_c$ (and $J \neq 1$).
- Scaling limit of the interface in the 2d model when $J > J_c = 1$.

Open problems

- Behavior of $\xi(J)$ in the neighborhood of J_c when $d \geq 4$.
- Sharp asymptotics of 2-point function when $J \leq J_c$ (and $J \neq 1$).
- Scaling limit of the interface in the $2d$ model when $J > J_c = 1$.

Some extensions (work in progress)

- Defect located along the boundary of the system ($\rightsquigarrow$ wetting when $d = 2$).
- Defect of dimension $d' \in (1, d)$: long-range order along the defect is possible even when the bulk is disordered.
- Quenched random (ferromagnetic) coupling constants along the defect.

Thank you for your attention!