8 The Gaussian Free Field on $\mathbb{Z}^d$

The model studied in this chapter, the *Gaussian Free Field* (GFF), is the only one we will consider whose single-spin space, $\mathbb{R}$, is *non-compact*. Its sets of configurations in finite and infinite volume are therefore, respectively,

$$\Omega_\Lambda \stackrel{\text{def}}{=} \mathbb{R}^\Lambda \qquad \text{and} \qquad \Omega \stackrel{\text{def}}{=} \mathbb{R}^{\mathbb{Z}^d}.$$

Although most of the general structure of the DLR formalism developed in Chapter 6 applies, the *existence* of infinite-volume Gibbs measures is not guaranteed anymore under the most general hypotheses, and requires more care.

One possible physical interpretation of this model is as follows. In $d = 1$, the spin at vertex $i \in \Lambda$, $\omega_i \in \mathbb{R}$, can be interpreted as the *height of a random line* above the x-axis:

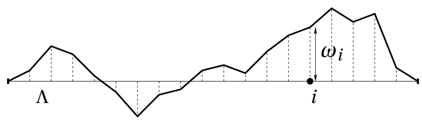

Figure 8.1: A configuration of the Gaussian Free Field in a one-dimensional box Λ, with boundary condition $\eta \equiv 0$.

The behavior of the model in large volumes is therefore intimately related to the *fluctuations* of the line away from the x-axis. Similarly, in $d = 2$, ω_i can be interpreted as the *height of a surface* above the (x, y)-plane:

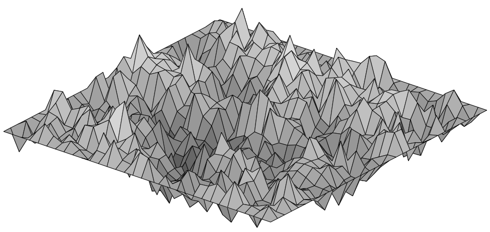

Figure 8.2: A configuration of the Gaussian Free Field in $d = 2$, in a 30×30 box with boundary condition $\eta \equiv 0$, which can be interpreted as a random surface.

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The techniques we will use to study the GFF will be very different from those used in the previous chapters. In particular, *Gaussian vectors* and *random walks* will play a central role in the analysis of the model. The basic results required about these two topics are collected in Appendices B.9 and B.13.

8.1 Definition of the model

We consider a configuration $\omega \in \Omega$ of the GFF, in which a variable $\omega_i \in \mathbb{R}$ is associated to each vertex $i \in \mathbb{Z}^d$; as usual, we will refer to ω_i as the **spin** at i. We define the interactions between the spins located inside a region $\Lambda \Subset \mathbb{Z}^d$, and between these spins and those located outside Λ. We motivate the definition of the Hamiltonian of the GFF by a few natural assumptions.

1. We first assume that only spins located at nearest-neighbors vertices of $\mathbb{Z}^d$ interact.
2. Our second requirement is that the interaction favors agreement of neighboring spins. This is achieved by assuming that the contribution to the energy due to two neighboring spins ω_i and ω_j is given by

$$\beta V(\omega_i - \omega_j), \tag{8.1}$$

   for some $V : \mathbb{R} \to \mathbb{R}_{\geq 0}$, which is assumed to be even, $V(-x) = V(x)$. Models with this type of interaction, depending only on the difference between neighboring spins, are often called **gradient models**. In the case of the GFF, the function V is chosen to be

$$V(x) \stackrel{\text{def}}{=} x^2 .$$

   An interaction of the type (8.1) has the following property: the interaction between two neighboring spins, ω_i and ω_j, does not change if the spins are shifted by the same value a: $\omega_i \mapsto \omega_i + a$, $\omega_j \mapsto \omega_j + a$. As will be explained later in Section 9.3, this invariance is at the origin of the mechanism that prevents the existence of infinite-volume Gibbs measures in low dimensions. The point is that local agreement between neighboring spins (that is, having $|\omega_j - \omega_i|$ small whenever $i \sim j$) does not prevent the spins from taking very large values. This is of course a consequence of the unboundedness of $\mathbb{R}$. One way to avoid this problem is to introduce some external parameter that penalizes large values of the spins.
3. To favor localization of the spin ω_i near zero, we introduce an additional term to the Hamiltonian, of the form

$$\lambda \omega_i^2 , \quad \lambda \geq 0 .$$

   This guarantees that when $\lambda > 0$, large values of $|\omega_i|$ represent large energies, and are therefore penalized.

We are thus led to consider a formal Hamiltonian of the following form:

$$\beta \sum_{\{i,j\} \in \mathscr{E}_{\mathbb{Z}^d}} (\omega_i - \omega_j)^2 + \lambda \sum_{i \in \mathbb{Z}^d} \omega_i^2 .$$

For convenience, we will replace β and λ by coefficients better suited to the manipulations that will come later.

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Definition 8.1. *The* ***Hamiltonian of the GFF in*** $\mathbf{\Lambda} \Subset \mathbb{Z}^{\boldsymbol{d}}$ *is defined by*

$$\mathscr{H}_{\Lambda;\beta,m}(\omega) \stackrel{\text{def}}{=} \frac{\beta}{4d} \sum_{\{i,j\}\in\mathscr{E}^{\text{b}}_{\Lambda}} (\omega_i - \omega_j)^2 + \frac{m^2}{2} \sum_{i\in\Lambda} \omega_i^2, \quad \omega \in \Omega, \tag{8.2}$$

where $\beta \geq 0$ *is the inverse temperature and* $m \geq 0$ *is the* ***mass***[1]. *The model is* ***massive*** *when* $m > 0$, ***massless*** *if* $m = 0$.

Once we have a Hamiltonian, finite-volume Gibbs measures are defined in the usual way. The measurable structures on Ω_Λ and Ω were defined in Section 6.10; we use the Borel sets $\mathscr{B}_\Lambda$ on Ω_Λ, and the σ-algebra $\mathscr{F}$ generated by cylinders on Ω. Since the spins are real-valued, a natural reference measure for the spin at site i is the Lebesgue measure, which we shall simply denote $\mathrm{d}\omega_i$. We remind the reader that $\omega_\Lambda \eta_{\Lambda^c} \in \Omega$ is the configuration that agrees with ω_Λ on Λ, and with η on Λ^c.

So, given $\Lambda \Subset \mathbb{Z}^d$ and $\eta \in \Omega$, the Gibbs distribution for the GFF in Λ with boundary condition η, at inverse temperature $\beta \geq 0$ and mass $m \geq 0$, is the probability measure $\mu^\eta_{\Lambda;\beta,m}$ on $(\Omega, \mathscr{F})$ defined by

$$\forall A \in \mathscr{F}, \quad \mu^\eta_{\Lambda;\beta,m}(A) = \int \frac{e^{-\mathscr{H}_{\Lambda;\beta,m}(\omega_\Lambda \eta_{\Lambda^c})}}{\mathbf{Z}^\eta_{\Lambda;\beta,m}} \mathbf{1}_A(\omega_\Lambda \eta_{\Lambda^c}) \prod_{i\in\Lambda} \mathrm{d}\omega_i. \tag{8.3}$$

The partition function is of course

$$\mathbf{Z}^\eta_{\Lambda;\beta,m} \stackrel{\text{def}}{=} \int e^{-\mathscr{H}_{\Lambda;\beta,m}(\omega_\Lambda \eta_{\Lambda^c})} \prod_{i\in\Lambda} \mathrm{d}\omega_i.$$

Exercise 8.1. *Show that* $\mathbf{Z}^\eta_{\Lambda;\beta,m}$ *is well-defined, for all* $\eta \in \Omega$, $\beta > 0$, $m \geq 0$.

Remark 8.2. In the previous chapters, we also considered other types of boundary conditions, namely free and periodic. As shown in the next exercise, this cannot be done for the massless GFF. Sometimes (in particular when using reflection positivity, see Chapter 10), it is nevertheless necessary to use periodic boundary conditions. In such situations, a common way of dealing with this problem is to take first the thermodynamic limit with a positive mass and then send the mass to zero: $\lim_{m\downarrow 0} \lim_{n\to\infty} \mu^{\text{per}}_{V_n;m}$, remember Definition 3.2. ⋄

Exercise 8.2. *Check that, for all nonempty* $\Lambda \Subset \mathbb{Z}^d$ *and all* $\beta > 0$,

$$\mathbf{Z}^\varnothing_{\Lambda;\beta,0} = \mathbf{Z}^{\text{per}}_{\Lambda;\beta,0} = \infty.$$

In particular, it is not possible to define the massless GFF with free or periodic boundary conditions.

Before pursuing, observe that the scaling properties of the Gibbs measure imply that one of the parameters, β or m, plays an irrelevant role when studying the GFF. Indeed, the change of variables $\omega'_i \stackrel{\text{def}}{=} \beta^{1/2}\omega_i$, $i \in \Lambda$, leads to

$$\mathbf{Z}^\eta_{\Lambda;\beta,m} = \beta^{-|\Lambda|/2} \mathbf{Z}^{\eta'}_{\Lambda;1,m'},$$

[1] The terminology "mass" is inherited from quantum field theory, where the corresponding quadratic terms in the Lagrangian indeed give rise to the mass of the associated particles.

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where $m' \stackrel{\text{def}}{=} \beta^{-1/2} m$ and $\eta' \stackrel{\text{def}}{=} \beta^{1/2}\eta$, and, similarly,

$$\mu^{\eta}_{\Lambda;\beta,m}(A) = \mu^{\eta'}_{\Lambda;1,m'}(\beta^{1/2}A), \quad \forall A \in \mathscr{F}.$$

This shows that there is no loss of generality in assuming that $\beta = 1$, which we will do from now on; of course, we will then also omit β from the notations.

The next step is to define infinite-volume Gibbs measures. We shall do so by using the approach described in detail in Chapter 6. Readers not comfortable with this material can skip to the next subsection. We emphasize that, although we will from time to time resort to this abstract setting in the sequel, most of our estimates actually pertain to finite-volume Gibbs measures, and therefore do not require this level of abstraction.

We proceed as in Section 6.10. First, the specification $\pi = \{\pi^m_\Lambda\}_{\Lambda \Subset \mathbb{Z}^d}$ of the GFF is defined by the kernels

$$\pi^m_\Lambda(\cdot \,|\, \eta) \stackrel{\text{def}}{=} \mu^{\eta}_{\Lambda;m}(\cdot).$$

Then, one defines the set of Gibbs measures compatible with π, by

$$\mathscr{G}(m) \stackrel{\text{def}}{=} \bigl\{\mu \in \mathscr{M}_1(\Omega) : \mu\pi^m_\Lambda = \mu \text{ for all } \Lambda \Subset \mathbb{Z}^d\bigr\}.$$

We remind the reader (see Remark 6.3.1) of the following equivalent characterization: $\mu \in \mathscr{G}(m)$ if and only if, for all $\Lambda \Subset \mathbb{Z}^d$ and all $A \in \mathscr{F}$,

$$\mu(A \,|\, \mathscr{F}_{\Lambda^c})(\omega) = \pi^m_\Lambda(A \,|\, \omega) \quad \text{for } \mu\text{-almost all } \omega. \tag{8.4}$$

Usually, a Gibbs measure in $\mathscr{G}(m)$ will be denoted μ_m, or μ^η_m when constructed via a limiting procedure using a boundary condition η. Expectation of a function f with respect to μ^η_m will be denoted $\mu^\eta_m(f)$ or $E^\eta_m[f]$.

8.1.1 Overview

The techniques used to study the GFF are very different from those used in previous chapters. Let us first introduce the random variables $\varphi_i : \Omega \to \mathbb{R}$, defined by

$$\varphi_i(\omega) \stackrel{\text{def}}{=} \omega_i, \quad i \in \mathbb{Z}^d.$$

Similarly to what was done in Chapter 3, we will consider first the distribution of $\varphi_\Lambda = (\varphi_i)_{i\in\Lambda}$ in a finite region $\Lambda \subset \mathsf{B}(n) \Subset \mathbb{Z}^d$, under $\mu^\eta_{\mathsf{B}(n);m}(\cdot)$. We will then determine under which conditions the random vector φ_Λ possesses a limiting distribution when $n \to \infty$. The first step will be to observe that, under $\mu^\eta_{\mathsf{B}(n);m}$, $\varphi_{\mathsf{B}(n)}$ is actually distributed as a *Gaussian vector*. This will give us access to various tools from the theory of Gaussian processes, in particular when studying the thermodynamic limit. Namely, as explained in Appendix B.9, the limit of a Gaussian vector, when it exists, is also Gaussian. This will lead to the construction, in the limit $n \to \infty$, of a *Gaussian field* $\varphi = (\varphi_i)_{i\in\mathbb{Z}^d}$. The distribution of this field, denoted μ^η_m, will be shown to be a Gibbs measure in $\mathscr{G}(m)$. But μ^η_m is entirely determined by its mean $E^\eta_m[\varphi_i]$ and by its covariance matrix, which measures the correlations between the variables φ_i:

$$\mathrm{Cov}^\eta_m(\varphi_i,\varphi_j) \stackrel{\text{def}}{=} E^\eta_m\bigl[(\varphi_i - E^\eta_m[\varphi_i])(\varphi_j - E^\eta_m[\varphi_j])\bigr].$$

It turns out that the mean and covariance matrix will take on a particularly nice form, with a probabilistic interpretation in terms of the *symmetric simple random*

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walk on $\mathbb{Z}^d$. This will make it possible to compute explicitly various quantities of interest. More precise statements will be given later, but the behavior established for the Gaussian Free Field will roughly be the following:

- *Massless case ($m = 0$), low dimensions:* In dimensions $d = 1$ and 2, the random variables φ_i, when considered in a large box $\mathsf{B}(n) = \{-n, \dots, n\}^d$ with an arbitrary fixed boundary condition, present large fluctuations, unbounded as $n \to \infty$. For example, the variance of the spin located at the center of the box is of order
$$\mathrm{Var}^{\eta}_{\mathsf{B}(n);0}(\varphi_0) \approx \begin{cases} n & \text{when } d = 1\,, \\ \log n & \text{when } d = 2\,. \end{cases}$$
In such a situation, the field is said to **delocalize**. As we will see, delocalization implies that *there are no infinite-volume Gibbs measures* in this case: $\mathscr{G}(0) = \varnothing$.

- *Massless case ($m = 0$), high dimensions:* In $d \geq 3$, the presence of a larger number of neighbors renders the field sufficiently more rigid to remain localized, in the sense that it has fluctuations of bounded variance. In particular, there exist (infinitely many extremal) infinite-volume Gibbs measures in this case. We will also show that the correlations under these measures are nonnegative and decay slowly with the distance:
$$\mathrm{Cov}^{\eta}_0(\varphi_i, \varphi_j) \approx \|j - i\|_2^{-(d-2)}\,.$$
In particular, the susceptibility is infinite:
$$\sum_{j \in \mathbb{Z}^d} \mathrm{Cov}^{\eta}_0(\varphi_i, \varphi_j) = +\infty\,.$$

- *Massive case ($m > 0$), all dimensions:* The presence of a mass term in the Hamiltonian prevents the delocalization observed in dimensions 1 and 2 in the massless case. However, we will show that, even in this case, there are infinitely many infinite-volume Gibbs measures. As we will see, the presence of a mass term also makes the correlations decay exponentially fast: there exist $c_+ = c_+(m) > 0$, $c_- = c_-(m) < \infty$, $C_+ = C_+(m) < \infty$ and $C_- = C_-(m) > 0$ such that
$$C_- e^{-c_- \|j-i\|_2} \leq \mathrm{Cov}^{\eta}_m(\varphi_i, \varphi_j) \leq C_+ e^{-c_+ \|j-i\|_2} \qquad \forall i, j \in \mathbb{Z}^d\,.$$
Moreover, $c_\pm(m) = O(m)$ as $m \downarrow 0$. This shows that the correlation length of the model is of the order of the inverse of the mass, m^{-1}, when the mass is small.

As seen from this short description, the GFF has no *uniqueness regime* (except in the trivial case $\beta = 0, m > 0$).

8.2 Parenthesis: Gaussian vectors and fields

Before pursuing, we recall a few generalities about Gaussian vectors, which in our case will be a family $(\varphi_i)_{i \in \Lambda}$ of random variables, indexed by the vertices of a finite region $\Lambda \Subset \mathbb{Z}^d$. A more detailed account of Gaussian vectors can be found in Appendix B.9.

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8.2.1 Gaussian vectors

Let $\varphi_\Lambda = (\varphi_i)_{i\in\Lambda} \in \Omega_\Lambda$ be a random vector, defined on some probability space. We do not yet assume that the distribution of this vector is Gibbsian. We consider the following scalar product on Ω_Λ: for $t_\Lambda = (t_i)_{i\in\Lambda}$, $\varphi_\Lambda = (\varphi_i)_{i\in\Lambda}$,

$$t_\Lambda \cdot \varphi_\Lambda \stackrel{\text{def}}{=} \sum_{i\in\Lambda} t_i \varphi_i \,.$$

Definition 8.3. *The random vector φ_Λ is **Gaussian** if, for all fixed t_Λ, $t_\Lambda \cdot \varphi_\Lambda$ is a Gaussian variable (possibly degenerate, that is, with zero variance).*

The distribution of a Gaussian variable X is determined entirely by its mean and variance, and its characteristic function is given by

$$E[e^{\mathrm{i}tX}] = \exp\bigl(\mathrm{i}\,tE[X] - \tfrac{1}{2}t^2\,\mathrm{Var}(X)\bigr)\,.$$

Let us assume that $\varphi_\Lambda = (\varphi_i)_{i\in\Lambda}$ is Gaussian, and let us denote its distribution by μ_Λ. Expectation (resp. variance, covariance) with respect to μ_Λ will be denoted E_Λ (resp. Var_Λ, Cov_Λ). The mean and variance of $t_\Lambda \cdot \varphi_\Lambda$ depend on t_Λ as follows:

$$E_\Lambda[t_\Lambda \cdot \varphi_\Lambda] = \sum_{i\in\Lambda} t_i E_\Lambda[\varphi_i] = t_\Lambda \cdot a_\Lambda\,, \tag{8.5}$$

where $a_\Lambda = (a_i)_{i\in\Lambda}$, $a_i \stackrel{\text{def}}{=} E_\Lambda[\varphi_i]$, is the **average** (or **mean**) of φ_Λ. Moreover,

$$\mathrm{Var}_\Lambda(t_\Lambda \cdot \varphi_\Lambda) = E_\Lambda\bigl[(t_\Lambda \cdot \varphi_\Lambda - E_\Lambda[t_\Lambda \cdot \varphi_\Lambda])^2\bigr] = \sum_{i,j\in\Lambda} \Sigma_\Lambda(i,j)\,t_i t_j = t_\Lambda \cdot \Sigma_\Lambda t_\Lambda\,, \tag{8.6}$$

where $\Sigma_\Lambda = (\Sigma_\Lambda(i,j))_{i,j\in\Lambda}$ is the **covariance matrix of** φ_Λ, defined by

$$\Sigma_\Lambda(i,j) \stackrel{\text{def}}{=} \mathrm{Cov}_\Lambda(\varphi_i,\varphi_j)\,. \tag{8.7}$$

Therefore, for each t_Λ, the characteristic function of $t_\Lambda \cdot \varphi_\Lambda$ is given by

$$E_\Lambda[e^{\mathrm{i}t_\Lambda\cdot\varphi_\Lambda}] = \exp\bigl(\mathrm{i}t_\Lambda \cdot a_\Lambda - \tfrac{1}{2}t_\Lambda \cdot \Sigma_\Lambda t_\Lambda\bigr) \tag{8.8}$$

and the moment generating function by

$$E_\Lambda[e^{t_\Lambda\cdot\varphi_\Lambda}] = \exp\bigl(t_\Lambda \cdot a_\Lambda + \tfrac{1}{2}t_\Lambda \cdot \Sigma_\Lambda t_\Lambda\bigr)\,. \tag{8.9}$$

The distribution of a Gaussian vector φ_Λ is thus entirely determined by the pair $(a_\Lambda, \Sigma_\Lambda)$; it is traditionally denoted by $\mathscr{N}(a_\Lambda, \Sigma_\Lambda)$. We say that φ_Λ is **centered** if $a_\Lambda \equiv 0$.

Clearly, Σ_Λ is symmetric: $\Sigma_\Lambda(i,j) = \Sigma_\Lambda(j,i)$. Moreover, since $\mathrm{Var}_\Lambda(t_\Lambda \cdot \varphi_\Lambda) \geq 0$, we see from (8.6) that Σ_Λ is nonnegative definite. In fact, to any $a_\Lambda \in \Omega_\Lambda$ and any symmetric nonnegative definite matrix Σ_Λ corresponds a (possibly degenerate) Gaussian vector φ_Λ having a_Λ as mean and Σ_Λ as covariance matrix. Moreover, when Σ_Λ is *positive* definite, the density with respect to the Lebesgue measure takes the following well-known form:

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Theorem 8.4. *Assume that $\varphi_\Lambda \sim \mathscr{N}(a_\Lambda, \Sigma_\Lambda)$, with a covariance matrix Σ_Λ which is positive definite (and, therefore, invertible). Then, the distribution of φ_Λ is absolutely continuous with respect to the Lebesgue measure $\mathrm{d}x_\Lambda$ (on Ω_Λ), with a density given by*

$$\frac{1}{(2\pi)^{|\Lambda|/2}\sqrt{|\det \Sigma_\Lambda|}} \exp\bigl(-\tfrac{1}{2}(x_\Lambda - a_\Lambda)\cdot \Sigma_\Lambda^{-1}(x_\Lambda - a_\Lambda)\bigr), \quad x_\Lambda \in \Omega_\Lambda . \tag{8.10}$$

Conversely, if φ_Λ is a random vector whose distribution is absolutely continuous with respect to the Lebesgue measure on Ω_Λ with a density of the form (8.10)*, then φ_Λ is Gaussian, with mean a_Λ and covariance matrix Σ_Λ.*

We emphasize that, once a vector is Gaussian, $\varphi_\Lambda \sim \mathscr{N}(a_\Lambda, \Sigma_\Lambda)$, various quantities of interest have immediate expressions in terms of the mean and covariance matrix. For example, to study the random variable φ_{i_0} (which is, of course, Gaussian) at some given vertex $i_0 \in \Lambda$, one can consider the vector $t_\Lambda = (\delta_{i_0 j})_{j\in\Lambda}$, write φ_{i_0} as $\varphi_{i_0} = t_\Lambda \cdot \varphi_\Lambda$ and conclude that the mean and variance of φ_{i_0} are given by

$$E_\Lambda[\varphi_{i_0}] = a_{i_0}, \qquad \mathrm{Var}_\Lambda(\varphi_{i_0}) = \Sigma_\Lambda(i_0, i_0) .$$

Although it will not be used in the sequel, the following exercise shows that correlation functions of Gaussian vectors enjoy a remarkable factorization property, known as *Wick's formula* or *Isserlis' theorem.*

Exercise 8.3. *Let φ_Λ be a centered Gaussian vector. Show that, for any $n \in \mathbb{N}$:*

1. *the **$2n+1$-point correlation functions** all vanish: for any collection of (not necessarily distinct) vertices $i_1, \dots, i_{2n+1} \in \Lambda$,*

$$E_\Lambda[\varphi_{i_1} \dots \varphi_{i_{2n+1}}] = 0; \tag{8.11}$$

2. *the **$2n$-point correlation function** can always be expressed in terms of the 2-point correlation functions: for any collection of (not necessarily distinct) vertices $i_1, \dots, i_{2n} \in \Lambda$,*

$$E_\Lambda[\varphi_{i_1} \dots \varphi_{i_{2n}}] = \sum_{\mathscr{P}} \prod_{\{\ell,\ell'\}\in\mathscr{P}} E_\Lambda[\varphi_{i_\ell}\varphi_{i_{\ell'}}], \tag{8.12}$$

*where the sum is over all **pairings** $\mathscr{P}$ of $\{1,\dots,2n\}$, that is, all families of n pairs $\{\ell,\ell'\} \subset \{1,\dots,2n\}$ whose union equals $\{1,\dots,2n\}$.* Hint: *Use* (8.9)*. Expanding the exponential in the right-hand side of that expression, determine the coefficient of $t^{i_1} \cdots t^{i_m}$.*

In fact, this factorization property characterizes Gaussian vectors.

Exercise 8.4. *Consider a random vector φ_Λ satisfying* (8.11) *and* (8.12)*. Show that φ_Λ is centered Gaussian.*

8.2.2 Gaussian fields and the thermodynamic limit.

Gaussian fields are infinite collections of random variables whose local behavior is Gaussian:

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Definition 8.5. *An infinite collection of random variables $\varphi = (\varphi_i)_{i\in\mathbb{Z}^d}$ is a* ***Gaussian field*** *if, for each $\Lambda \Subset \mathbb{Z}^d$, the restriction $\varphi_\Lambda = (\varphi_i)_{i\in\Lambda}$ is Gaussian. The distribution of a Gaussian field is called a* ***Gaussian measure****.*

Consider now the sequence of boxes $\mathsf{B}(n)$, $n \ge 0$, and assume that, for each n, a Gaussian vector $\varphi_{\mathsf{B}(n)}$ is given, $\varphi_{\mathsf{B}(n)} \sim \mathscr{N}(a_{\mathsf{B}(n)}, \Sigma_{\mathsf{B}(n)})$, whose distribution we denote by $\mu_{\mathsf{B}(n)}$. A meaning can be given to the thermodynamic limit as follows. We fix $\Lambda \Subset \mathbb{Z}^d$. If n is large, then $\mathsf{B}(n) \supset \Lambda$. Notice that the distribution of $\varphi_\Lambda = (\varphi_i)_{i\in\Lambda}$, seen as a collection of variables indexed by vertices of $\mathsf{B}(n)$, can be computed by taking a vector $t_{\mathsf{B}(n)} = (t_i)_{i\in\mathsf{B}(n)}$ for which $t_i = 0$ for all $i \in \mathsf{B}(n)\setminus\Lambda$. In this way,

$$E_{\mathsf{B}(n)}[e^{\mathrm{i}t_\Lambda\cdot\varphi_\Lambda}] = E_{\mathsf{B}(n)}[e^{\mathrm{i}t_{\mathsf{B}(n)}\cdot\varphi_{\mathsf{B}(n)}}] = e^{\mathrm{i}t_{\mathsf{B}(n)}\cdot a_{\mathsf{B}(n)} - \frac{1}{2}t_{\mathsf{B}(n)}\cdot\Sigma_{\mathsf{B}(n)}t_{\mathsf{B}(n)}}\,.$$

Remembering that only a fixed number of components of $t_{\mathsf{B}(n)}$ are non-zero, we see that the limiting distribution of φ_Λ can be controlled if $a_{\mathsf{B}(n)}$ and $\Sigma_{\mathsf{B}(n)}$ have limits as $n\to\infty$.

Theorem 8.6. *Let, for all n, $\varphi_{\mathsf{B}(n)} = (\varphi_i)_{i\in\mathsf{B}(n)}$ be a Gaussian vector, $\varphi_{\mathsf{B}(n)} \sim \mathscr{N}(a_{\mathsf{B}(n)}, \Sigma_{\mathsf{B}(n)})$. Assume that, for all $i, j \in \mathbb{Z}^d$, the limits*

$$a_i \stackrel{\text{def}}{=} \lim_{n\to\infty}(a_{\mathsf{B}(n)})_i \quad \textit{and} \quad \Sigma(i,j) \stackrel{\text{def}}{=} \lim_{n\to\infty}\Sigma_{\mathsf{B}(n)}(i,j)$$

exist and are finite. Then the following holds.

1. *For all $\Lambda \Subset \mathbb{Z}^d$, the distribution of $\varphi_\Lambda = (\varphi_i)_{i\in\Lambda}$ converges, when $n\to\infty$, to that of a Gaussian vector $\mathscr{N}(a_\Lambda, \Sigma_\Lambda)$, with mean and covariance given by the* ***restrictions***
$$a_\Lambda \stackrel{\text{def}}{=} (a_i)_{i\in\Lambda} \quad \textit{and} \quad \Sigma_\Lambda \stackrel{\text{def}}{=} (\Sigma(i,j))_{i,j\in\Lambda}\,.$$
2. *There exists a Gaussian field $\widetilde{\varphi}$ whose restriction $\widetilde{\varphi}_\Lambda$ to each $\Lambda \Subset \mathbb{Z}^d$ is a Gaussian vector with distribution $\mathscr{N}(a_\Lambda, \Sigma_\Lambda)$.*

Proof. The first claim is a consequence of Proposition B.56. For the second one, fix any $\Lambda \Subset \mathbb{Z}^d$, and let μ_Λ denote the limiting distribution of φ_Λ. By construction, the collection $\{\mu_\Lambda\}_{\Lambda\Subset\mathbb{Z}^d}$ is consistent in the sense of Kolmogorov's Extension Theorem (Theorem 6.6 and Remark 6.98). This guarantees the existence of a probability measure μ on $(\Omega, \mathscr{F})$ whose marginal on each $\Lambda \Subset \mathbb{Z}^d$ is exactly μ_Λ. Under μ, the random variables $\widetilde{\varphi}_i(\omega) \stackrel{\text{def}}{=} \omega_i$ then form a Gaussian field such that, for each Λ, $\widetilde{\varphi}_\Lambda = (\widetilde{\varphi}_i)_{i\in\Lambda}$ has distribution μ_Λ. □

Consider now the GFF in Λ, defined by the measure $\mu^\eta_{\Lambda;m}$ in (8.3). Although the latter is a probability measure on $(\Omega, \mathscr{F})$, it acts in a trivial way on the spins outside Λ (for each $j \notin \Lambda$, $\varphi_j = \eta_j$ almost surely). We will therefore, without loss of generality, consider it as a distribution on $(\Omega_\Lambda, \mathscr{F}_\Lambda)$.

By definition, $\mu^\eta_{\Lambda;m}$ is absolutely continuous with respect to the Lebesgue measure on Ω_Λ. We will show that it can be put in the form (8.10), which will prove that $(\varphi_i)_{i\in\Lambda}$ is a non-degenerate Gaussian vector. We thus need to reformulate the Hamiltonian $\mathscr{H}_{\Lambda;m}$ in such a way that it takes the form of the exponent that appears in the density (8.10). We do this following a step-by-step procedure that will take us on a detour.

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8.3 Harmonic functions and the Discrete Green Identities

Given a collection $f = (f_i)_{i\in\mathbb{Z}^d}$ of real numbers, we define, for each pair $\{i,j\}\in\mathscr{E}_{\mathbb{Z}^d}$, the **discrete gradient**

$$(\nabla f)_{ij} \stackrel{\text{def}}{=} f_j - f_i\,,$$

and, for all $i\in\mathbb{Z}^d$, the **discrete Laplacian**

$$(\Delta f)_i \stackrel{\text{def}}{=} \sum_{j:j\sim i} (\nabla f)_{ij}\,. \tag{8.13}$$

Lemma 8.7 (Discrete Green Identities)**.** *Let $\Lambda \Subset \mathbb{Z}^d$. Then, for all collections of real numbers $f = (f_i)_{i\in\mathbb{Z}^d}$, $g = (g_i)_{i\in\mathbb{Z}^d}$,*

$$\sum_{\{i,j\}\in\mathscr{E}^{\mathrm{b}}_\Lambda} (\nabla f)_{ij}(\nabla g)_{ij} = -\sum_{i\in\Lambda} g_i(\Delta f)_i + \sum_{\substack{i\in\Lambda\\ j\in\Lambda^{\mathrm{c}},j\sim i}} g_j(\nabla f)_{ij}\,, \tag{8.14}$$

$$\sum_{i\in\Lambda}\{f_i(\Delta g)_i - g_i(\Delta f)_i\} = \sum_{\substack{i\in\Lambda\\ j\in\Lambda^{\mathrm{c}},j\sim i}} \{f_j(\nabla g)_{ij} - g_j(\nabla f)_{ij}\}\,. \tag{8.15}$$

Remark 8.8. The continuous analogues of (8.14) and (8.15) are the classical *Green identities*, which, on a smooth domain $U\subset\mathbb{R}^n$, provide a higher-dimensional version of the classical integration by parts formula. That is, for all smooth functions f and g,

$$\int_U \nabla f\cdot\nabla g\,\mathrm{d}V = -\int_U g\Delta f\,\mathrm{d}V + \oint_{\partial U} g(\nabla f\cdot n)\,\mathrm{d}S,$$
$$\int_U \{f\Delta g - g\Delta f\}\,\mathrm{d}V = \oint_{\partial U}\{f\nabla g\cdot n - g\nabla f\cdot n\}\,\mathrm{d}S.$$

where n is the outward normal unit-vector and $\mathrm{d}V$ and $\mathrm{d}S$ denote respectively the volume and surface elements. ⋄

Proof of Lemma 8.7: Using the symmetry between i and j (in all the sums below, j is always assumed to be a nearest-neighbor of i):

$$\begin{aligned}\sum_{\{i,j\}\subset\Lambda}(\nabla f)_{ij}(\nabla g)_{ij} &= \sum_{\{i,j\}\subset\Lambda} g_j(f_j-f_i) - \sum_{\{i,j\}\subset\Lambda} g_i(f_j-f_i)\\ &= -\sum_{i\in\Lambda} g_i \sum_{j\in\Lambda}(f_j-f_i)\\ &= -\sum_{i\in\Lambda} g_i(\Delta f)_i + \sum_{i\in\Lambda} g_i\sum_{j\in\Lambda^{\mathrm{c}}}(f_j-f_i)\,.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{\{i,j\}\in\mathscr{E}^{\mathrm{b}}_\Lambda}(\nabla f)_{ij}(\nabla g)_{ij} &= \sum_{\{i,j\}\subset\Lambda}(\nabla f)_{ij}(\nabla g)_{ij} + \sum_{i\in\Lambda,j\in\Lambda^{\mathrm{c}}}(\nabla f)_{ij}(\nabla g)_{ij}\\ &= -\sum_{i\in\Lambda} g_i(\Delta f)_i + \sum_{i\in\Lambda,j\in\Lambda^{\mathrm{c}}} g_j(f_j-f_i)\,.\end{aligned}$$

The second identity (8.15) is obtained using the first one twice, interchanging the roles of f and g. □

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We can write the action of the Laplacian on $f = (f_i)_{i\in\mathbb{Z}^d}$ as:

$$(\Delta f)_i = \sum_{j\in\mathbb{Z}^d} \Delta_{ij} f_j\,, \quad i \in \mathbb{Z}^d\,,$$

where the matrix elements $(\Delta_{ij})_{i,j\in\mathbb{Z}^d}$ are defined by

$$\Delta_{ij} = \begin{cases} -2d & \text{if } i = j\,,\\ 1 & \text{if } i \sim j\,,\\ 0 & \text{otherwise.}\end{cases} \tag{8.16}$$

To obtain a representation of $\mathscr{H}_{\Lambda;m}$ in terms of the scalar product in Λ, we introduce the **restriction** of Δ to Λ, defined by $\Delta_\Lambda \stackrel{\text{def}}{=} (\Delta_{ij})_{i,j\in\Lambda}$.

Remark 8.9. Let $i \in \Lambda$. In what follows, it will be important to distinguish between $(\Delta f)_i$, defined in (8.13) and which may depend on some of the variables f_j located outside Λ, and $(\Delta_\Lambda f)_i$, which is a shorthand notation for $\sum_{j\in\Lambda}\Delta_{ij} f_j$ (and thus involves only variables f_j inside Λ). In particular, we will use the notation

$$f\cdot\Delta_\Lambda g \stackrel{\text{def}}{=} \sum_{i,j\in\Lambda} \Delta_{ij} f_i g_j\,,$$

which clearly satisfies

$$f\cdot\Delta_\Lambda g = (\Delta_\Lambda f)\cdot g\,. \tag{8.17}$$

◇

From now on, we assume that f coincides with η outside Λ and denote by B_Λ any **boundary term**, that is, any quantity (possibly changing from place to place) depending only on the values η_j, $j \in \Lambda^c$.

Let us see how the quadratic term in the Hamiltonian will be handled. Applying (8.14) with $f = g$ and rearranging terms, we get

$$\begin{aligned}\sum_{\{i,j\}\in\mathscr{E}^b_\Lambda} (f_j - f_i)^2 &= \sum_{\{i,j\}\in\mathscr{E}^b_\Lambda} (\nabla f)^2_{ij}\\ &= -f\cdot\Delta_\Lambda f - 2\sum_{i\in\Lambda}\sum_{j\in\Lambda^c, j\sim i} f_i f_j + B_\Lambda\,.\end{aligned} \tag{8.18}$$

One can then introduce $u = (u_i)_{i\in\mathbb{Z}^d}$, to be determined later, depending on η and Λ, and playing the role of the mean of f. Our aim is to rewrite (8.18) in the form $-(f-u)\cdot\Delta_\Lambda(f-u)$, up to boundary terms. We can, in particular, include in B_Λ any expression that depends only on the values of u. We have, using (8.17),

$$\begin{aligned}(f-u)\cdot\Delta_\Lambda(f-u) &= f\cdot\Delta_\Lambda f - 2f\cdot\Delta_\Lambda u + u\cdot\Delta_\Lambda u\\ &= f\cdot\Delta_\Lambda f - 2\sum_{i\in\Lambda} f_i(\Delta u)_i + 2\sum_{i\in\Lambda}\sum_{j\in\Lambda^c, j\sim i} f_i u_j + B_\Lambda\,.\end{aligned}$$

Comparing with (8.18), we deduce that

$$\begin{aligned}\sum_{\{i,j\}\in\mathscr{E}^b_\Lambda} (f_j - f_i)^2 &= -(f-u)\cdot\Delta_\Lambda(f-u)\\ &\quad - 2\sum_{i\in\Lambda} f_i \underbrace{(\Delta u)_i}_{\text{(i)}} + 2\sum_{i\in\Lambda}\sum_{j\in\Lambda^c, j\sim i} f_i \underbrace{(u_j - f_j)}_{\text{(ii)}} + B_\Lambda\,.\end{aligned} \tag{8.19}$$

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A look at the second line in this last display indicates exactly the restrictions one should impose on u in order for $-(f-u)\cdot\Delta_\Lambda(f-u)$ to be the one and only contribution to the Hamiltonian (up to boundary terms). To cancel the non-trivial terms that depend on the values of f inside Λ, we need to ensure that: (i) u is **harmonic in** $\boldsymbol{\Lambda}$:

$$(\Delta u)_i = 0 \quad \forall i \in \Lambda\,.$$

(ii) u coincides with f (hence with η) outside Λ. We have thus proved:

Lemma 8.10. *Let f coincide with η outside Λ. Assume that u* ***solves the Dirichlet problem in*** $\boldsymbol{\Lambda}$ ***with boundary condition*** $\boldsymbol{\eta}$*:*

$$\begin{cases} u \text{ harmonic in } \Lambda\,, \\ u_j = \eta_j \text{ for all } j \in \Lambda^{\mathrm{c}}\,. \end{cases} \tag{8.20}$$

Then,

$$\sum_{\{i,j\}\in\mathscr{E}^{\mathrm{b}}_\Lambda} (f_j - f_i)^2 = -(f-u)\cdot\Delta_\Lambda(f-u) + B_\Lambda\,. \tag{8.21}$$

Existence of a solution to the Dirichlet problem will be proved in Lemma 8.15. Uniqueness can be verified easily:

Lemma 8.11. (8.20) *has at most one solution.*

Proof. We first consider the boundary condition $\eta \equiv 0$, and show that $u \equiv 0$ is the unique solution. Namely, assume v is any solution, and let $i_* \in \Lambda$ be such that $|v_{i_*}| = \max_{j\in\Lambda}|v_j|$. With no loss of generality, assume $v_{i_*} \geq 0$. Since $(\Delta v)_{i_*} = 0$ implies $v_{i_*} = \frac{1}{2d}\sum_{j\sim i_*} v_j$, and $v_j \leq v_{i_*}$ for all $j \sim i_*$, we conclude that $v_j = v_{i_*}$ for all $j \sim i_*$. Repeating this procedure until the boundary of Λ is reached, we deduce that v must be constant, and this constant can only be 0. Let now u and v be two solutions of (8.20). Then, $h = u - v$ is a solution to the Dirichlet problem in Λ with boundary condition $\eta' \equiv 0$. By the previous argument, $h \equiv 0$ and thus $u = v$. □

Exercise 8.5. *Show that, when $d = 1$, the solution of the Dirichlet problem on an interval $\Lambda = \{a, \dots, b\}$ is of the form $u_i = ai + c$, for some $a, c \in \mathbb{R}$ determined by the boundary condition.*

8.4 The massless case

Let us consider the massless Hamiltonian $\mathscr{H}_{\Lambda;0}$, expressed in terms of the variables $\varphi = (\varphi_i)_{i\in\mathbb{Z}^d}$, which are assumed to satisfy $\varphi_i = \eta_i$ for all $i \notin \Lambda$. We apply Lemma 8.10 with $f = \varphi$, assuming for the moment that one can find a solution u to the Dirichlet problem (in Λ, with boundary condition η). Since it does not alter the Gibbs distribution, the constant B_Λ in (8.21) can always be subtracted from the Hamiltonian. We get

$$\mathscr{H}_{\Lambda;0} = \tfrac{1}{2}(\varphi - u)\cdot(-\tfrac{1}{2d}\Delta_\Lambda)(\varphi - u)\,. \tag{8.22}$$

Our next tasks are, first, to invert the matrix $-\frac{1}{2d}\Delta_\Lambda$, in order to obtain an explicit expression for the covariance matrix, and, second, to find an explicit expression for the solution u to the Dirichlet problem.

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8.4.1 The random walk representation

We need to determine whether there exists some positive-definite covariance matrix Σ_Λ such that $-\frac{1}{2d}\Delta_\Lambda = \Sigma_\Lambda^{-1}$. Observe first that

$$-\tfrac{1}{2d}\Delta_\Lambda = I_\Lambda - P_\Lambda\,,$$

where $I_\Lambda = (\delta_{ij})_{i,j\in\Lambda}$ is the identity matrix and $P_\Lambda = (P(i,j))_{i,j\in\Lambda}$ is the matrix with elements

$$P(i,j) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{2d} & \text{if } j \sim i\,,\\ 0 & \text{otherwise.}\end{cases}$$

The numbers $(P(i,j))_{i,j\in\mathbb{Z}^d}$ are the **transition probabilities of the symmetric simple random walk $X = (X_k)_{k\geq 0}$ on $\mathbb{Z}^d$**, which at each time step jumps to any one of its $2d$ nearest-neighbors with probability $\frac{1}{2d}$:

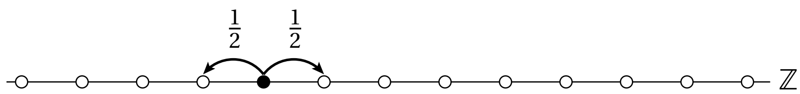

Figure 8.3: The one-dimensional symmetric simple random walk.

We denote by $\mathbb{P}_i$ the distribution of the walk starting at $i \in \mathbb{Z}^d$. That is, we have $\mathbb{P}_i(X_0 = i) = 1$ and, for $n \geq 0$,

$$\mathbb{P}_i(X_{n+1} = k \,|\, X_n = j) = P(j,k) \quad \forall j,k \in \mathbb{Z}^d.$$

(Information on the simple random walk on $\mathbb{Z}^d$ can be found in Appendix B.13.)

We will need to know that the walk almost surely exits a finite region in a finite time:

Lemma 8.12. *For $\Lambda \Subset \mathbb{Z}^d$, let $\tau_{\Lambda^c} \stackrel{\text{def}}{=} \inf\{k \geq 0 : X_k \in \Lambda^c\}$ be the **first exit time from Λ**. Then $\mathbb{P}_i(\tau_{\Lambda^c} < \infty) = 1$. More precisely, there exists $c = c(\Lambda) > 0$ such that, for all $i \in \Lambda$,*

$$\mathbb{P}_i(\tau_{\Lambda^c} > n) \leq e^{-cn}\,. \tag{8.23}$$

Proof. If we let $R = \sup_{l\in\Lambda}\inf_{k\in\Lambda^c}\|k-l\|_1$, then, starting from i, one can find a nearest-neighbor path of length at most R which exits Λ. This means that during any time-interval of length R, there is a probability at least $(2d)^{-R}$ that the random walk exits Λ (just force it to follow the path). In particular,

$$\mathbb{P}_i(\tau_{\Lambda^c} > n) \leq (1-(2d)^{-R})^{[n/R]}\,. \qquad \square$$

The next lemma shows that the matrix $I_\Lambda - P_\Lambda$ is invertible, and provides a probabilistic interpretation for its inverse:

Lemma 8.13. *The $|\Lambda|\times|\Lambda|$ matrix $I_\Lambda - P_\Lambda$ is invertible. Moreover, its inverse $G_\Lambda \stackrel{\text{def}}{=} (I_\Lambda - P_\Lambda)^{-1}$ is given by $G_\Lambda = (G_\Lambda(i,j))_{i,j\in\Lambda}$, the **Green function in Λ of the simple random walk on $\mathbb{Z}^d$**, defined by*

$$G_\Lambda(i,j) \stackrel{\text{def}}{=} \mathbb{E}_i\Big[\sum_{n=0}^{\tau_{\Lambda^c}-1} \mathbf{1}_{\{X_n=j\}}\Big]. \tag{8.24}$$

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The Green function $G_\Lambda(i,j)$ represents the average number of visits at j made by a walk started at i, before it leaves Λ.

Proof. To start, observe that (below, P_Λ^n denotes the nth power of P_Λ)

$$(I_\Lambda - P_\Lambda)(I_\Lambda + P_\Lambda + P_\Lambda^2 + \cdots + P_\Lambda^n) = I_\Lambda - P_\Lambda^{n+1}. \tag{8.25}$$

We claim that there exists $c = c(\Lambda)$ such that, for all $i, j \in \Lambda$ and all $n \geq 1$,

$$P_\Lambda^n(i,j) \leq e^{-cn}. \tag{8.26}$$

Indeed, for each $n \geq 1$,

$$P_\Lambda^n(i,j) = \sum_{i_1,\dots,i_{n-1}\in\Lambda} P(i,i_1)P(i_1,i_2)\cdots P(i_{n-1},j) = \mathbb{P}_i(X_n = j, \tau_{\Lambda^c} > n).$$

Since $\mathbb{P}_i(X_n = j, \tau_{\Lambda^c} > n) \leq \mathbb{P}_i(\tau_{\Lambda^c} > n)$, (8.26) follows from (8.23). This implies that the matrix $G_\Lambda = (G_\Lambda)_{i,j\in\Lambda}$, defined by

$$G_\Lambda(i,j) = (I_\Lambda + P_\Lambda + P_\Lambda^2 + \cdots)(i,j) = \sum_{n\geq 0} \mathbb{P}_i(X_n = j, \tau_{\Lambda^c} > n), \tag{8.27}$$

is well defined and, by (8.25), that it satisfies $(I_\Lambda - P_\Lambda)G_\Lambda = I_\Lambda$. Of course, by symmetry, we also have $G_\Lambda(I_\Lambda - P_\Lambda) = I_\Lambda$. The conclusion follows, since the right-hand side of (8.27) can be rewritten in the form given (8.24). □

Remark 8.14. The key ingredient in the above proof that $I_\Lambda - P_\Lambda$ is invertible is the fact that P_Λ is **substochastic**: $\sum_{j\in\Lambda} P(i,j) < 1$ for those vertices i which lie along the inner boundary of Λ. This property was crucial in establishing (8.26). ⋄

Let us now prove the existence of a solution to the Dirichlet problem (uniqueness was shown in Lemma 8.11), also expressed in terms of the simple random walk. Let $X_{\tau_{\Lambda^c}}$ denote the position of the walk at the time of first exit from Λ.

Lemma 8.15. *The solution to the Dirichlet problem* (8.20) *is given by the function* $u = (u_i)_{i\in\mathbb{Z}^d}$ *defined by*

$$u_i \stackrel{\text{def}}{=} \mathbb{E}_i[\eta_{X_{\tau_{\Lambda^c}}}] \quad \forall i \in \mathbb{Z}^d. \tag{8.28}$$

Proof. When $j \in \Lambda^c$, $\mathbb{P}_j(\tau_{\Lambda^c} = 0) = 1$ and, thus, $u_j = \mathbb{E}_j[\eta_{X_0}] = \eta_j$. When $i \in \Lambda$, by conditioning on the first step of the walk,

$$\begin{aligned} u_i = \mathbb{E}_i[\eta_{X_{\tau_{\Lambda^c}}}] &= \sum_{j\sim i} \mathbb{E}_i[\eta_{X_{\tau_{\Lambda^c}}}, X_1 = j] \\ &= \sum_{j\sim i} \mathbb{P}_i(X_1 = j)\mathbb{E}_i[\eta_{X_{\tau_{\Lambda^c}}} \mid X_1 = j] \\ &= \sum_{j\sim i} \tfrac{1}{2d}\mathbb{E}_j[\eta_{X_{\tau_{\Lambda^c}}}] = \tfrac{1}{2d}\sum_{j\sim i} u_j, \end{aligned}$$

which implies $(\Delta u)_i = 0$. □

Remark 8.16. Observe that if $\eta = (\eta_i)_{i\in\mathbb{Z}^d}$ is itself harmonic, then $u \equiv \eta$ *is* the solution to the Dirichlet problem in Λ with boundary condition η. ⋄

Theorem 8.17. *Under* $\mu^\eta_{\Lambda;0}$, $\varphi_\Lambda = (\varphi_i)_{i\in\Lambda}$ *is Gaussian, with mean* $u_\Lambda = (u_i)_{i\in\Lambda}$ *defined in* (8.28)*, and positive definite covariance matrix* $G_\Lambda = (G_\Lambda(i,j))_{i,j\in\Lambda}$ *given by the Green function* (8.24).

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Proof. The claim follows from the representation (8.22) of the Hamiltonian, the expression (8.28) for the solution to the Dirichlet problem, the expression (8.24) for the inverse of $I_\Lambda - P_\Lambda$ and Theorem 8.4. □

The reader should note the remarkable fact that the distribution of φ_Λ under $\mu^\eta_{\Lambda;0}$ depends on the boundary condition η only through its mean: the covariance matrix is only sensitive to the choice of Λ.

Example 8.18. Consider the GFF in dimension $d = 1$. Let η be any boundary condition, fixed outside an interval $\Lambda = \{a, \dots, b\} \Subset \mathbb{Z}$. As we saw in Exercise 8.5, the solution to the Dirichlet problem is the affine interpolation between (a, η_a) and (b, η_b). A typical configuration under $\mu^\eta_{\Lambda;0}$ should therefore be thought of as describing fluctuations around this line (however, these fluctuations can be large on the microscopic scale, as will be seen below):

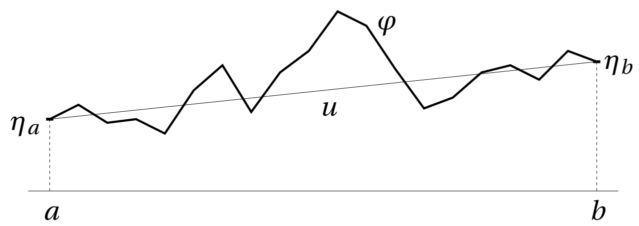

Figure 8.4: A configuration of the one-dimensional GFF under $\mu^\eta_{\Lambda;0}$, whose mean u_Λ is the harmonic function given by the linear interpolation between the values of η on the boundary.

◇

Before we start with our analysis of the thermodynamic limit, let us exploit the representation derived in Theorem 8.17 in order to study the fluctuations of $\varphi_{\mathsf{B}(n)}$ in a large box $\mathsf{B}(n)$.

For the sake of concreteness, consider the spin at the origin, φ_0. The latter is a Gaussian random variable with variance given by

$$\mathrm{Var}^\eta_{\mathsf{B}(n);0}(\varphi_0) = G_{\mathsf{B}(n)}(0,0)\,.$$

Notice first that the time $\tau_{\mathsf{B}(n)^c}$ it takes for the random walk, starting at 0, to leave the box $\mathsf{B}(n)$ is increasing in n and is always larger than n. Therefore, by monotone convergence,

$$\lim_{n\to\infty} G_{\mathsf{B}(n)}(0,0) = \mathbb{E}_0\Big[\sum_{k\geq 0} \mathbf{1}_{\{X_k=0\}}\Big] \tag{8.29}$$

is just the expected number of visits of the walk at the origin. In particular, the variance of φ_0 diverges in the limit $n \to \infty$, whenever the symmetric simple random walk is recurrent, that is, in dimensions 1 and 2 (see Appendix B.13.4). When this happens, the field is said to **delocalize**. A closer analysis of the Green function (see Theorem B.76) yields the following more precise information:

$$G_{\mathsf{B}(n)}(0,0) \approx \begin{cases} n & \text{if } d = 1, \\ \log n & \text{if } d = 2. \end{cases} \tag{8.30}$$

In contrast, in dimensions $d \geq 3$, the variance remains bounded, and therefore the field remains localized close to its mean value even in the limit $n \to \infty$.

In the next section, we will relate these properties with the problem of existence of infinite-volume Gibbs measures for the massless GFF.

Exercise 8.6. *Consider the one-dimensional GFF in* $\mathsf{B}(n)$ *with* 0 *boundary condition (see Figure 8.1). Interpreting the values of the field,* $\varphi_{-n},\dots,\varphi_n$*, as the successive positions of a random walk on* $\mathbb{R}$ *with Gaussian increments, starting at* $\varphi_{-n-1} = 0$ *and conditioned on* $\{\varphi_{n+1} = 0\}$*, prove directly (that is, without using the random walk representation of* $G(0,0)$*) that* $\varphi_0 \sim \mathcal{N}(0, n+1)$.

8.4.2 The thermodynamic limit

We explained, before Theorem 8.6, how the thermodynamic limit $n \to \infty$ can be expressed in terms of the limits of the means $u_{\mathsf{B}(n)}$ and of the covariance matrices $G_{\mathsf{B}(n)}$, when the latter exist:

$$\lim_{n\to\infty} \mathbb{E}_i[\eta_{X_{\tau_{\mathsf{B}(n)^c}}}], \quad \lim_{n\to\infty} G_{\mathsf{B}(n)}(i,j), \tag{8.31}$$

for all fixed pairs $i, j \in \mathbb{Z}^d$.

Low dimensions. We have seen that, when $d = 1$ or $d = 2$, $\lim_{n\to\infty} G_{\mathsf{B}(n)}(0,0) = \infty$. This has the following consequence:

Theorem 8.19. *When* $d = 1$ *or* $d = 2$*, the massless Gaussian Free Field has no infinite-volume Gibbs measures:* $\mathscr{G}(0) = \varnothing$.

Proof. Assume there exists a probability measure $\mu \in \mathscr{G}(0)$. Since $\mu = \mu\pi^0_{\mathsf{B}(n)}$ for all n, we have

$$\mu(\varphi_0 \in [a,b]) = \mu\pi^0_{\mathsf{B}(n)}(\varphi_0 \in [a,b]) = \int \mu^\eta_{\mathsf{B}(n);0}(\varphi_0 \in [a,b])\,\mu(\mathrm{d}\eta),$$

for any interval $[a,b] \subset \mathbb{R}$. But, uniformly in η,

$$\begin{aligned}\mu^\eta_{\mathsf{B}(n);0}(\varphi_0 \in [a,b]) &= \frac{1}{\sqrt{2\pi G_{\mathsf{B}(n)}(0,0)}} \int_a^b \exp\Big\{-\frac{\big(x - \mu^\eta_{\mathsf{B}(n);0}(\varphi_0)\big)^2}{2G_{\mathsf{B}(n)}(0,0)}\Big\}\,\mathrm{d}x \\ &\le \frac{b-a}{\sqrt{2\pi G_{\mathsf{B}(n)}(0,0)}}\,. \end{aligned} \tag{8.32}$$

In dimensions 1 and 2, the right-hand side tends to 0 as $n \to \infty$. We conclude that $\mu(\varphi_0 \in [a,b]) = 0$, for all $a < b$, and thus $\mu(\varphi_0 \in \mathbb{R}) = 0$, which contradicts the assumption that μ is a probability measure. □

Remark 8.20. The lack of infinite-volume Gibbs measures for the massless GFF $\varphi = (\varphi_i)_{i\in\mathbb{Z}^d}$ in dimensions 1 and 2, as seen above, is due to the fact that the fluctuations of each spin φ_i become unbounded when $\mathsf{B}(n) \uparrow \mathbb{Z}^d$. This is not incompatible, nevertheless, with the fact that some *random* translation of the field does have a well-defined thermodynamic limit. Namely, define the random variables $\tilde\varphi = (\tilde\varphi_i)_{i\in\mathbb{Z}^d}$ by

$$\tilde\varphi_i \stackrel{\text{def}}{=} \varphi_i - \varphi_0\,. \tag{8.33}$$

Then, as shown in the next exercise, these random variables have a well-defined thermodynamic limit, even when $d = 1, 2$. ⋄

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Exercise 8.7. *Consider the variables* $\widetilde{\varphi}_i$ *defined in* (8.33). *Show that under* $\mu^0_{\Lambda;0}$ *(zero boundary condition),* $(\widetilde{\varphi}_i)_{i\in\Lambda}$ *is Gaussian, centered, with covariance matrix given by*

$$\tilde{G}_\Lambda(i,j) \stackrel{\text{def}}{=} G_\Lambda(i,j) - G_\Lambda(i,0) - G_\Lambda(0,j) + G_\Lambda(0,0)\,. \tag{8.34}$$

It can be shown that the matrix elements $\tilde{G}_\Lambda(i,j)$ in (8.34) have a finite limit when $\Lambda \uparrow \mathbb{Z}^d$; see the comments in Section 8.7.2.

Higher dimensions. When $d \geq 3$, transience of the symmetric simple random walk implies that

$$G(i,j) \stackrel{\text{def}}{=} \lim_{n\to\infty} G_{\mathsf{B}(n)}(i,j) = \mathbb{E}_i\Bigl(\sum_{k\geq 0} \mathbf{1}_{\{X_k=j\}}\Bigr) \tag{8.35}$$

is finite. This will allow us to construct infinite-volume Gibbs measures. We say that $\eta = (\eta_i)_{i\in\mathbb{Z}^d}$ is **harmonic (in $\mathbb{Z}^d$)** if

$$(\Delta\eta)_i = 0 \quad \forall i \in \mathbb{Z}^d\,.$$

Theorem 8.21. *In dimensions* $d \geq 3$*, the massless Gaussian Free Field possesses infinitely many infinite-volume Gibbs measures:* $|\mathscr{G}(0)| = \infty$*. More precisely, given any harmonic function* η *on* $\mathbb{Z}^d$*, there exists a Gaussian Gibbs measure* μ^η_0 *with mean* η *and covariance matrix* $G = (G(i,j))_{i,j\in\mathbb{Z}^d}$ *given in* (8.35).

Remark 8.22. It can be shown that the Gibbs measures μ^η_0 of Theorem 8.21 are precisely the extremal elements of $\mathscr{G}(0)$: $\mathrm{ex}\mathscr{G}(0) = \{\mu^\eta_0 : \eta \text{ harmonic}\}$. ⋄

Clearly, there exist infinitely many harmonic functions. For example, any constant function is harmonic, or, more generally, any function of the form

$$\eta_i \stackrel{\text{def}}{=} \alpha_1 i_1 + \cdots + \alpha_d i_d + c\,, \qquad \forall i = (i_1,\dots,i_d) \in \mathbb{Z}^d\,, \tag{8.36}$$

with $\alpha = (\alpha_1,\dots,\alpha_d) \in \mathbb{R}^d$. But, in $d \geq 2$, the variety of harmonic functions is much larger:

Exercise 8.8. *Show that all harmonic functions* $u : \mathbb{Z}^d \to \mathbb{R}$ *can be obtained by fixing arbitrary values of* u_i *at all vertices* i *belonging to the strip*

$$\bigl\{i = (i_1, i_2, \dots, i_d) \in \mathbb{Z}^d : i_d \in \{0,1\}\bigr\}$$

and extending u *to the whole of* $\mathbb{Z}^d$ *using* $\Delta u = 0$.

An analysis of the Green function $G(i,j)$ as $\|j-i\|_1 \to \infty$ (see Theorem B.76) yields the following information on the asymptotic behavior of the covariance.

Proposition 8.23. *Assume that* $d \geq 3$ *and* $m = 0$*. Then, the infinite-volume Gibbs measures* μ^η_0 *of Theorem 8.21 satisfy, as* $\|i-j\|_2 \to \infty$,

$$\mathrm{Cov}^\eta_0(\varphi_i,\varphi_j) = \frac{r(d)}{\|j-i\|_2^{d-2}}(1+o(1)) \tag{8.37}$$

for some constant $r(d) > 0$.

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Proof of Theorem 8.21: Fix some harmonic function η. The restriction of η to any finite box $\mathsf{B}(n)$ obviously solves the Dirichlet problem on $\mathsf{B}(n)$ (with boundary condition η). Since the limits $\lim_{n\to\infty} G_{\mathsf{B}(n)}(i,j)$ exist when $d \geq 3$, we can use Theorem 8.6 to construct a Gaussian field $\varphi = (\varphi_i)_{i\in\mathbb{Z}^d}$ whose restriction to any finite region Λ is a Gaussian vector $(\varphi_i)_{i\in\Lambda}$ with mean $(\eta_i)_{i\in\Lambda}$ and covariance matrix $G_\Lambda = (G(i,j))_{i,j\in\Lambda}$. If we let μ_0^η denote the distribution of φ, then $\mu_0^\eta(\varphi_i) = \eta_i$ and

$$\mathrm{Cov}_0^\eta(\varphi_i,\varphi_j) = G(i,j)\,. \tag{8.38}$$

It remains to show that $\mu_0^\eta \in \mathscr{G}(0)$. This could be done following the same steps used to prove Theorem 6.26. For pedagogical reasons, we will give a different proof relying on the Gaussian properties of φ.

We use the criterion (8.4), and show that, for all $\Lambda \Subset \mathbb{Z}^d$ and all $A \in \mathscr{F}$,

$$\mu_0^\eta(A\,|\,\mathscr{F}_{\Lambda^c})(\omega) = \mu_{\Lambda;0}^\omega(A) \quad \text{for } \mu_0^\eta\text{-almost all } \omega.$$

For that, we will verify that the field $\varphi = (\varphi_i)_{i\in\mathbb{Z}^d}$, when conditioned on $\mathscr{F}_{\Lambda^c}$, remains Gaussian (Lemma 8.24 below), and that, for all t_Λ,

$$E_0^\eta\big[e^{\mathrm{i}t_\Lambda\cdot\varphi_\Lambda}\;\big|\;\mathscr{F}_{\Lambda^c}\big](\omega) = e^{\mathrm{i}t_\Lambda\cdot a_\Lambda(\omega) - \frac{1}{2}t_\Lambda\cdot G_\Lambda t_\Lambda}\,, \tag{8.39}$$

where $a_i(\omega) = \mathbb{E}_i[\omega_{X_{\tau_{\Lambda^c}}}]$ is the solution of the Dirichlet problem in Λ with boundary condition ω.

Lemma 8.24. *Let φ be the Gaussian field constructed above. Let, for all $i \in \Lambda$,*

$$a_i(\omega) \stackrel{\text{def}}{=} E_0^\eta[\varphi_i\,|\,\mathscr{F}_{\Lambda^c}](\omega)\,.$$

Then, μ-almost surely, $a_i(\omega) = \mathbb{E}_i[\omega_{X_{\tau_{\Lambda^c}}}]$. In particular, each $a_i(\omega)$ is a finite linear combination of the variables ω_j and $(a_i)_{i\in\mathbb{Z}^d}$ is a Gaussian field.

Proof. When $i \in \Lambda$, we use the characterization of the conditional expectation given in Lemma B.50: up to equivalence, $E_0^\eta[\varphi_i|\mathscr{F}_{\Lambda^c}]$ is the unique $\mathscr{F}_{\Lambda^c}$-measurable random variable ψ for which

$$E_0^\eta\big[(\varphi_i - \psi)\varphi_j\big] = 0 \quad \text{for all } j \in \Lambda^c. \tag{8.40}$$

We verify that this condition is indeed satisfied when $\psi(\omega) = \mathbb{E}_i[\omega_{X_{\tau_{\Lambda^c}}}]$. By (8.38),

$$\begin{aligned} E_0^\eta\big[(\varphi_i - \mathbb{E}_i[\varphi_{X_{\tau_{\Lambda^c}}}])\varphi_j\big] &= E_0^\eta[\varphi_i\varphi_j] - E_0^\eta\big[\mathbb{E}_i[\varphi_{X_{\tau_{\Lambda^c}}}]\varphi_j\big] \\ &= G(i,j) + \eta_i\eta_j - E_0^\eta\big[\mathbb{E}_i[\varphi_{X_{\tau_{\Lambda^c}}}]\varphi_j\big]\,. \end{aligned}$$

Using again (8.38),

$$\begin{aligned} E_0^\eta\big[\mathbb{E}_i[\varphi_{X_{\tau_{\Lambda^c}}}]\varphi_j\big] &= \sum_{k\in\partial^{\mathrm{ex}}\Lambda} E_0^\eta[\varphi_k\varphi_j]\,\mathbb{P}_i[X_{\tau_{\Lambda^c}} = k] \\ &= \mathbb{E}_i\big[E_0^\eta[\varphi_{X_{\tau_{\Lambda^c}}}\varphi_j]\big] \\ &= \mathbb{E}_i\big[G(X_{\tau_{\Lambda^c}},j)\big] + \mathbb{E}_i\big[E_0^\eta[\varphi_{X_{\tau_{\Lambda^c}}}]E_0^\eta[\varphi_j]\big]\,. \end{aligned} \tag{8.41}$$

On the one hand, since $i \in \Lambda$ and $j \in \Lambda^c$, any trajectory of the random walk that contributes to $G(i,j)$ must intersect $\partial^{\mathrm{ex}}\Lambda$ at least once, so the strong Markov Property gives

$$G(i,j) = \sum_{k\in\partial^{\mathrm{ex}}\Lambda} \mathbb{P}_i(X_{\tau_{\Lambda^c}} = k)G(k,j) = \mathbb{E}_i[G(X_{\tau_{\Lambda^c}},j)]\,.$$

On the other hand, since φ has mean η and since η is solution of the Dirichlet problem in Λ with boundary condition η, we have

$$\mathbb{E}_i\big[E_0^\eta[\varphi_{X_{\tau_{\Lambda^c}}}]E_0^\eta[\varphi_j]\big] = \mathbb{E}_i[\eta_{X_{\tau_{\Lambda^c}}}\eta_j] = \mathbb{E}_i[\eta_{X_{\tau_{\Lambda^c}}}]\eta_j = \eta_i\eta_j\,.$$

This shows that $a_i(\omega) = \mathbb{E}_i[\omega_{X_{\tau_{\Lambda^c}}}]$. In particular, the latter is a linear combination of the ω_js:

$$a_i(\omega) = \sum_{k\in\partial^{\mathrm{ex}}\Lambda} \omega_k \mathbb{P}_i(X_{\tau_{\Lambda^c}} = k)\,,$$

which implies that $(a_i)_{i\in\mathbb{Z}^d}$ is also a Gaussian field. □

Corollary 8.25. *Under μ_0^η, the random vector $(\varphi_i - a_i)_{i\in\Lambda}$ is independent of $\mathscr{F}_{\Lambda^c}$.*

Proof. We know that the variables $\varphi_i - a_i$, $i\in\Lambda$, and φ_j, $j\in\Lambda^c$, form a Gaussian field. Therefore, a classical result (Proposition (B.58)) implies that $(\varphi_i - a_i)_{i\in\Lambda}$, which is centered, is independent of $\mathscr{F}_{\Lambda^c}$ if and only if each pair $\varphi_i - a_i$ ($i\in\Lambda$) and φ_j ($j\in\Lambda^c$) is uncorrelated. But this follows from (8.40). □

Let $a_\Lambda = (a_i)_{i\in\Lambda}$. By Corollary 8.25 and since a_Λ is $\mathscr{F}_{\Lambda^c}$-measurable,

$$E_0^\eta\big[e^{it_\Lambda\cdot\varphi_\Lambda}\;\big|\;\mathscr{F}_{\Lambda^c}\big] = e^{\mathrm{i}t_\Lambda\cdot a_\Lambda}E_0^\eta\big[e^{it_\Lambda\cdot(\varphi_\Lambda - a_\Lambda)}\;\big|\;\mathscr{F}_{\Lambda^c}\big] = e^{\mathrm{i}t_\Lambda\cdot a_\Lambda}E_0^\eta\big[e^{it_\Lambda\cdot(\varphi_\Lambda - a_\Lambda)}\big]\,.$$

We know that the variables $\varphi_i - a_i$, $i\in\Lambda$, form a Gaussian vector under μ_0^η. Since it is centered, we need only compute its covariance. For $i,j\in\Lambda$, write

$$(\varphi_i - a_i)(\varphi_j - a_j) = \varphi_i\varphi_j - (\varphi_i - a_i)a_j - (\varphi_j - a_j)a_i - a_i a_j\,.$$

Using Corollary 8.25 again, we see that $E_0^\eta\big[(\varphi_i - a_i)a_j\big] = 0$ and $E_0^\eta\big((\varphi_j - a_j)a_i\big) = 0$ (since a_i and a_j are $\mathscr{F}_{\Lambda^c}$-measurable). Therefore,

$$\begin{aligned}\mathrm{Cov}_0^\eta\big((\varphi_i - a_i),(\varphi_j - a_j)\big) &= E_0^\eta[\varphi_i\varphi_j] - E_0^\eta(a_i a_j)\\ &= G(i,j) + \eta_i\eta_j - E_0^\eta[a_i a_j]\,.\end{aligned}$$

Proceeding as in (8.41),

$$E_0^\eta[a_i a_j] = \mathbb{E}_{i,j}\big[G(X_{\tau_{\Lambda^c}}, X'_{\tau_{\Lambda^c}})\big] + \mathbb{E}_{i,j}\big[E_0^\eta[\varphi_{X_{\tau_{\Lambda^c}}}]E_0^\eta[\varphi_{X'_{\tau'_{\Lambda^c}}}]\big]\,, \tag{8.42}$$

where X and X' are two independent symmetric simple random walks, starting respectively at i and j, $\mathbb{P}_{i,j}$ denotes their joint distribution, and τ'_{Λ^c} is the first exit time of X' from Λ. As was done earlier,

$$\mathbb{E}_{i,j}\big[E_0^\eta[\varphi_{X_{\tau_{\Lambda^c}}}]E_0^\eta[\varphi_{X'_{\tau'_{\Lambda^c}}}]\big] = \mathbb{E}_{i,j}[\eta_{X_{\tau_{\Lambda^c}}}\eta_{X'_{\tau'_{\Lambda^c}}}] = \mathbb{E}_i[\eta_{X_{\tau_{\Lambda^c}}}]\mathbb{E}_j[\eta_{X_{\tau_{\Lambda^c}}}] = \eta_i\eta_j\,.$$

Let us then define the modified Green function

$$K_\Lambda(i,j) \stackrel{\mathrm{def}}{=} \mathbb{E}_i\Big[\sum_{n\geq\tau_{\Lambda^c}} \mathbf{1}_{\{X_n = j\}}\Big] = G(i,j) - G_\Lambda(i,j)\,.$$

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Observe that $K_\Lambda(i,j) = K_\Lambda(j,i)$, since G and G_Λ are both symmetric; moreover, $K_\Lambda(i,j) = G(i,j)$ if $i \in \Lambda^c$. We can thus write

$$\begin{aligned}\mathbb{E}_{i,j}\big[G(X_{\tau_{\Lambda^c}}, X'_{\tau'_{\Lambda^c}})\big] &= \sum_{k,l\in\partial^{\text{ext}}\Lambda} \mathbb{P}_i(X_{\tau_{\Lambda^c}} = k)\mathbb{P}_j(X_{\tau_{\Lambda^c}} = l)G(k,l)\\ &= \sum_{l\in\partial^{\text{ext}}\Lambda} \mathbb{P}_j(X_{\tau_{\Lambda^c}} = l)K_\Lambda(i,l)\\ &= \sum_{l\in\partial^{\text{ext}}\Lambda} \mathbb{P}_j(X_{\tau_{\Lambda^c}} = l)K_\Lambda(l,i)\\ &= \sum_{l\in\partial^{\text{ext}}\Lambda} \mathbb{P}_j(X_{\tau_{\Lambda^c}} = l)G(l,i)\\ &= K_\Lambda(j,i) = G(i,j) - G_\Lambda(i,j)\,.\end{aligned}$$

We have thus shown that $\mathrm{Cov}_0^\eta\big((\varphi_i - a_i),(\varphi_j - a_j)\big) = G_\Lambda(i,j)$, which implies that

$$E_0^\eta\big[e^{i t_\Lambda\cdot\varphi_\Lambda}\,|\,\mathscr{F}_{\Lambda^c}\big] = e^{\mathrm{i}t_\Lambda\cdot a_\Lambda}e^{-\frac12 t_\Lambda\cdot G_\Lambda t_\Lambda}\,.$$

This shows that, under $\mu_0^\eta(\cdot\,|\,\mathscr{F}_{\Lambda^c})$, φ_Λ is Gaussian with distribution given by $\mu_{\Lambda;0}^\eta(\cdot)$. We have thereby proved (8.39) and Theorem 8.21. □

The proof given above that the limiting Gaussian field belongs to $\mathscr{G}(0)$ only depends on having a convergent expression for the Green function of the associated random walk; it will be used again in the massive case.

8.5 The massive case

A similar analysis, based on a Gaussian description of the finite-volume Gibbs distribution, holds in the massive case $m > 0$. Nevertheless, the presence of a mass term in the Hamiltonian leads to a change in the probabilistic interpretation, which eventually leads to a completely different behavior.

Consider the Hamiltonian $\mathscr{H}_{\Lambda;m}$, which contains the term $\frac{m^2}{2}\sum_{i\in\Lambda}\varphi_i^2$. To express $\mathscr{H}_{\Lambda;m}$ as a scalar product involving the inverse of a covariance matrix, we use (8.19), but this time including the mass term. After rearrangement, this yields

$$\begin{aligned}\mathscr{H}_{\Lambda;m} &= \tfrac12(\varphi-u)\cdot\big(-\tfrac{1}{2d}\Delta_\Lambda + m^2\big)(\varphi-u)\\ &\quad+\sum_{i\in\Lambda}\varphi_i\underbrace{\big((-\tfrac{1}{2d}\Delta+m^2)u\big)_i}_{\text{(i)}} + \tfrac{1}{2d}\sum_{i\in\Lambda}\sum_{j\in\Lambda^c, j\sim i}\varphi_i\underbrace{(u_j-\varphi_j)}_{\text{(ii)}} + B_\Lambda\,.\end{aligned}\tag{8.43}$$

As before, we choose u so as to cancel the extra terms on the second line. The mean $u = (u_i)_{i\in\mathbb{Z}^d}$ we are after must solve a modified Dirichlet problem. Let us say that u is ***m*-harmonic on Λ (resp. $\mathbb{Z}^d$)** if

$$\big((-\tfrac{1}{2d}\Delta + m^2)u\big)_i = 0\,,\quad \forall i\in\Lambda \text{ (resp. } i\in\mathbb{Z}^d)\,.$$

We say that u **solves the massive Dirichlet problem in Λ** if

$$\begin{cases} u \text{ is } m\text{-harmonic on } \Lambda\,,\\ u_j = \eta_j \text{ for all } j\in\Lambda^c\,.\end{cases}\tag{8.44}$$

We will give a probabilistic solution of the massive Dirichlet problem, by again representing the scalar product in the Hamiltonian, using a different random walk.

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Exercise 8.9. *Verify that the solution to* (8.44) *(whose existence will be proved below) is unique.*

When $d = 1$, one can determine all m-harmonic functions explicitly:

Exercise 8.10. *Show that all m-harmonic functions on $\mathbb{Z}$ are of the following type: $u_k = Ae^{\alpha k} + Be^{-\alpha k}$, where $\alpha \stackrel{\text{def}}{=} \log(1 + m^2 + \sqrt{2m^2 + m^4})$.*

8.5.1 Random walk representation

Consider a random walker on $\mathbb{Z}^d$ which, as before, only jumps to nearest neighbors but which, at each step, has a probability $\frac{m^2}{1+m^2} > 0$ of *dying*. That is, assume that, before taking each new step, the walker flips a coin with probability $P(\text{head}) = \frac{1}{1+m^2}$, $P(\text{tail}) = \frac{m^2}{1+m^2}$. If the outcome is head, the walker survives and jumps to a nearest neighbor on $\mathbb{Z}^d$ uniformly, with probability $\frac{1}{2d}$. If the outcome is tail, the walker dies (and remains dead for all subsequent times).

This process can be defined by considering

$$\mathbb{Z}^d_\star \stackrel{\text{def}}{=} \mathbb{Z}^d \cup \{\star\},$$

where $\star \notin \mathbb{Z}^d$ is a new vertex which we call the **graveyard**. We define the following transition probabilities on $\mathbb{Z}^d_\star$:

$$P_m(i,j) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{1+m^2}\frac{1}{2d} & \text{if } i,j \in \mathbb{Z}^d,\ i \sim j,\\ 1 - \frac{1}{1+m^2} & \text{if } i \in \mathbb{Z}^d \text{ and } j = \star,\\ 1 & \text{if } i = j = \star,\\ 0 & \text{otherwise.} \end{cases}$$

Let $Z = (Z_k)_{k\geq 0}$ denote the **killed random walk**, that is the Markov chain on $\mathbb{Z}^d_\star$ associated to the transition matrix P_m.

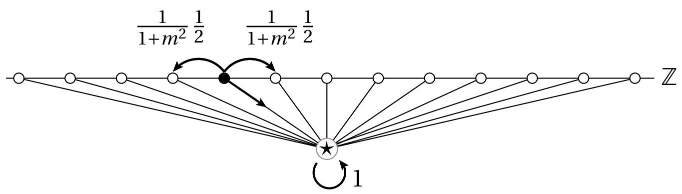

Figure 8.5: The one-dimensional symmetric simple random walk which has a probability $\frac{m^2}{1+m^2}$ of dying at each step.

We denote by $\mathbb{P}^m_i$ the distribution of the process Z starting at $i \in \mathbb{Z}^d$. By definition, $\star$ is an absorbing state for Z:

$$\forall n \geq 0, \quad \mathbb{P}^m_i(Z_{n+1} = \star \,|\, Z_n = \star) = 1.$$

Moreover, when $m > 0$, at all time $n \geq 0$ (up to which the walk has not yet entered the graveyard), the walk has a positive probability of dying:

$$\forall k \in \mathbb{Z}^d, \quad \mathbb{P}^m_i(Z_{n+1} = \star \,|\, Z_n = k) = \frac{m^2}{1+m^2} > 0.$$

Let

$$\tau_\star \stackrel{\text{def}}{=} \inf\{n \geq 0 : Z_n = \star\}$$

be the time at which the walker dies. Since $\mathbb{P}_i^m(\tau_\star > n) = (1+m^2)^{-n}$, $\tau_\star$ is $\mathbb{P}_i^m$-almost surely finite.

Notice that, when $m = 0$, Z reduces to the symmetric simple walk considered in the previous section. The processes X and Z are in fact related by

$$\mathbb{P}_i^m(Z_n = j) = \mathbb{P}_i^m(\tau_\star > n)\mathbb{P}_i(X_n = j) = (1+m^2)^{-n}\mathbb{P}_i(X_n = j)\,, \tag{8.45}$$

for all $i, j \in \mathbb{Z}^d$.

The process Z underlies the probabilistic representation of the mean and covariance of the finite-volume Gibbs distribution of the massive GFF:

Theorem 8.26. *Let $\Lambda \Subset \mathbb{Z}^d$, $d \geq 1$, and η be any boundary condition. Define $\eta_\star \stackrel{\text{def}}{=} 0$. Then, under $\mu^\eta_{\Lambda;m}$, $\varphi_\Lambda = (\varphi_i)_{i\in\Lambda}$ is Gaussian, with mean $u^m_\Lambda = (u^m_i)_{i\in\Lambda}$ given by*

$$u_i^m \stackrel{\text{def}}{=} \mathbb{E}_i^m[\eta_{Z_{\tau_{\Lambda^c}}}]\,, \quad \forall i \in \Lambda\,, \tag{8.46}$$

where $\tau_{\Lambda^c} \stackrel{\text{def}}{=} \inf\{k \geq 0 : Z_k \notin \Lambda\}$, and covariance matrix $G_{m;\Lambda} = (G_{m;\Lambda}(i,j))_{i,j\in\Lambda}$ given by

$$G_{m;\Lambda}(i,j) = \frac{1}{1+m^2}\,\mathbb{E}_i^m\Big[\sum_{n=0}^{\tau_{\Lambda^c}-1} \mathbf{1}_{\{Z_n = j\}}\Big]\,. \tag{8.47}$$

Exercise 8.11. *Returning to the original Hamiltonian* (8.2) *with β not necessarily equal to* 1, *check that the mean and covariance matrix are given by*

$$u_i^{\beta,m} = \beta^{-1/2}\,u_i^{m\beta^{-1/2}}\,, \qquad G_{\beta,m;\Lambda}(i,j) = \beta^{-1}\,G_{m\beta^{-1/2};\Lambda}(i,j)\,.$$

Observe that there are now two ways for the process Z to reach the exit-time τ_{Λ^c}: either by stepping on a vertex $j \notin \Lambda$ or by dying.

Proof. We proceed as in the massless case. First, it is easy to verify that u_i^m defined in (8.46) provides a solution to the massive Dirichlet problem (8.44). Then, we use (8.43), in which only the term involving $-\frac{1}{2d}\Delta_\Lambda + m^2$ remains. By introducing the restriction $P_{m;\Lambda} = (P_m(i,j))_{i,j\in\Lambda}$, we write

$$-\tfrac{1}{2d}\Delta_\Lambda + m^2 = (1+m^2)I_\Lambda - P_\Lambda = (1+m^2)\big\{I_\Lambda - P_{m;\Lambda}\big\}\,.$$

Since $P^k_{m;\Lambda}(i,j) = \mathbb{P}_i^m(Z_k = j, \tau_{\Lambda^c} > k)$ and, by (8.45),

$$\mathbb{P}_i^m(Z_k = j) \leq (1+m^2)^{-k}\,, \tag{8.48}$$

we conclude, as before, that the matrix $I_\Lambda - P_{m;\Lambda}$ is invertible and that its inverse is given by the convergent series

$$G_{m;\Lambda} = \frac{1}{1+m^2}\big(I_\Lambda + P_{m;\Lambda} + P^2_{m;\Lambda} + \cdots\big)\,.$$

Clearly, the entries of $G_{m;\Lambda}$ are exactly those given in (8.47). □

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Example 8.27. Consider the one-dimensional massive GFF in $\{-n,\dots,n\}$, with a boundary condition η. Using Exercises 8.9 and 8.10, we easily check that the solution to the Dirichlet problem with boundary condition η is given by $u_k^m = Ae^{\alpha k} + Be^{-\alpha k}$, where $\alpha = \log(1+m^2+\sqrt{2m^2+m^4})$ and

$$A = \frac{\eta_{n+1}e^{\alpha(n+1)} - \eta_{-n-1}e^{-\alpha(n+1)}}{e^{2\alpha(n+1)} - e^{-2\alpha(n+1)}}, \quad B = \frac{\eta_{-n-1}e^{\alpha(n+1)} - \eta_{n+1}e^{-\alpha(n+1)}}{e^{2\alpha(n+1)} - e^{-2\alpha(n+1)}}. \qquad \diamond$$

Exercise 8.12. *Let $(p(i))_{i\in\mathbb{Z}^d}$ be nonnegative real numbers such that $\sum_i p(i) = 1$. Consider the generalization of the GFF in which the Hamiltonian is given by*

$$\frac{\beta}{2} \sum_{\substack{\{i,j\}\subset\mathbb{Z}^d \\ \{i,j\}\cap\Lambda\neq\varnothing}} p(j-i)(\omega_i - \omega_j)^2 + \frac{m^2}{2}\sum_{i\in\Lambda}\omega_i^2, \quad \omega\in\Omega.$$

Show that the random walk representation derived above extends to this more general situation, provided that one replaces the simple random walk on $\mathbb{Z}^d$ by the random walk on $\mathbb{Z}^d$ with transition probabilities $p(\cdot)$.

8.5.2 The thermodynamic limit

We can easily show that the massive GFF always has at least one infinite-volume Gibbs measure, in any dimension $d \ge 1$. Namely, the boundary condition $\eta \equiv 0$ is m-harmonic, so $u \equiv 0$ is the solution of the corresponding massive Dirichlet problem (8.44). Moreover,

$$G_m(i,j) \stackrel{\text{def}}{=} \lim_{n\to\infty} G_{m;\mathrm{B}(n)}(i,j) = \frac{1}{1+m^2}\sum_{n\ge 0}\mathbb{P}_i^m(Z_n = j). \tag{8.49}$$

In view of (8.48), this series always converges when $m > 0$. By Theorem 8.6, this yields the existence of the Gaussian field with mean zero and covariance matrix G_m, whose distribution we denote by μ_m^0. As in the proof of Theorem 8.21, one then shows that its distribution μ_m^0 belongs to $\mathscr{G}(m)$. Of course, the same argument can be used starting with any m-harmonic function η on $\mathbb{Z}^d$; observe that Exercise 8.8 extends readily to the massive case, providing a description of all m-harmonic functions. We have therefore proved the following result:

Theorem 8.28. *In any dimension $d \ge 1$, the massive Gaussian Free Field possesses infinitely many infinite-volume Gibbs measures: $|\mathscr{G}(m)| = \infty$. More precisely, given any m-harmonic function η on $\mathbb{Z}^d$, there exists a Gaussian Gibbs measure μ_m^η with mean η and covariance matrix $G_m = (G_m(i,j))_{i,j\in\mathbb{Z}^d}$ given in* (8.49).

Remark 8.29. As in the massless case, it can be shown that m-harmonic functions parametrize extremal Gibbs measures: $\mathrm{ex}\mathscr{G}(m) = \{\mu_m^\eta : \eta \text{ is } m\text{-harmonic}\}$. $\diamond$

In contrast to the massless case in dimension $d \ge 3$, in which $G(0,i)$ decreases algebraically when $\|i\|_2 \to \infty$, we will now see that the decay in the massive case is always exponential. Let us thus define the rate

$$\xi_m(i) \stackrel{\text{def}}{=} \lim_{\ell\to\infty} -\frac{1}{\ell}\log G_m(0,\ell i).$$

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Proposition 8.30. *Let $d \geq 1$. For any $i \in \mathbb{Z}^d$, $\xi_m(i)$ exists and*

$$G_m(0,i) \leq G_m(0,0)e^{-\xi_m(i)}.$$

Moreover,

$$\log(1+m^2) \leq \frac{\xi_m(i)}{\|i\|_1} \leq \log(2d) + \log(1+m^2). \tag{8.50}$$

Proof. Let, for all $j \in \mathbb{Z}^d$, $\tau_j \stackrel{\text{def}}{=} \min\{n \geq 0 : Z_n = j\}$. Observe that

$$G_m(0,\ell i) = \mathbb{P}_0^m(\tau_{\ell i} < \tau_\star) G_m(\ell i, \ell i).$$

Therefore, since $G_m(\ell i, \ell i) = G_m(0,0) < \infty$ for any $m > 0$,

$$\lim_{\ell\to\infty} -\frac{1}{\ell}\log G_m(0,\ell i) = \lim_{\ell\to\infty} -\frac{1}{\ell}\log \mathbb{P}_0^m(\tau_{\ell i} < \tau_\star).$$

Now, for all $\ell_1, \ell_2 \in \mathbb{N}$, it follows from the strong Markov property that

$$\mathbb{P}_0^m(\tau_{(\ell_1+\ell_2)i} < \tau_\star) \geq \mathbb{P}_0^m(\tau_{\ell_1 i} < \tau_{(\ell_1+\ell_2)i} < \tau_\star) = \mathbb{P}_0^m(\tau_{\ell_1 i} < \tau_\star)\mathbb{P}_0^m(\tau_{\ell_2 i} < \tau_\star).$$

This implies that the sequence $\bigl(-\log \mathbb{P}_0^m(\tau_{\ell i} < \tau_\star)\bigr)_{\ell\geq 1}$ is subadditive; Lemma B.5 then guarantees the existence of $\xi_m(i)$, and provides the desired upper bound on $G_m(0,i)$, after taking $\ell = 1$.

Let us now turn to the bounds on $\xi_m(i)/\|i\|_1$. For the lower bound, we use (8.48):

$$\begin{aligned}(1+m^2)G_m(i,j) &= \sum_{n\geq 0} \mathbb{P}_i^m(Z_n = j) \\ &\leq \sum_{n\geq \|j-i\|_1} (1+m^2)^{-n} \leq \frac{1+m^2}{m^2}(1+m^2)^{-\|j-i\|_1}.\end{aligned}$$

For the upper bound, we can use

$$(1+m^2)G_m(i,j) \geq \mathbb{P}_i^m(\tau_j < \tau_\star) \geq (2d(1+m^2))^{-\|j-i\|_1},$$

where the second inequality is obtained by fixing an arbitrary shortest path from i to j and then forcing the walk to follow it. □

Using m-harmonic functions as a boundary condition allows one to construct infinitely many distinct Gibbs measures. It turns out, however, that if we only consider boundary conditions growing not too fast, then the corresponding Gaussian field is unique:

Theorem 8.31. *Let $d \geq 1$. For any boundary condition η satisfying*

$$\limsup_{k\to\infty} \max_{i:\|i\|_1=k} \frac{\log|\eta_i|}{k} < \log(1+m^2), \tag{8.51}$$

the Gaussian Gibbs measure μ_m^η constructed in Theorem 8.28 is the same as the one obtained with the boundary condition $\eta \equiv 0$: $\mu_m^\eta = \mu_m^0$.

Since each m-harmonic function leads to a distinct infinite-volume Gibbs measure, Theorem 8.31 shows that the only m-harmonic function with subexponential growth is $\eta \equiv 0$. This is in sharp contrast with the massless case, for which distinct Gibbs measures can be constructed using boundary conditions of the form (8.36), in which η_i diverges linearly in $\|i\|_1$.

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Proof of Theorem 8.31: It suffices to prove that $\lim_{n\to\infty} \mathbb{E}_i^m[\eta_{Z_{\tau_{\mathsf{B}(n)^c}}}] = 0$ whenever η satisfies (8.51). Let $\epsilon > 0$ be such that $e^{\epsilon}/(1+m^2) < 1$ and, for all n large enough, $|\eta_i| \le e^{\epsilon n}$ for all $i \in \partial^{\mathrm{ext}}\mathsf{B}(n)$. Then, for all such n,

$$\left|\mathbb{E}_i^m[\eta_{Z_{\tau_{\mathsf{B}(n)^c}}}]\right| \le e^{\epsilon n}\mathbb{P}_i^m\big(\tau_{\mathsf{B}(n)^c} > d_1(i,\mathsf{B}(n)^c)\big) \le e^{\epsilon n}(1+m^2)^{-n+\|i\|_1}.$$

which tends to 0 as $n \to \infty$. □

8.5.3 The limit $m \downarrow 0$

We have seen that, when $d = 1$ or $d = 2$, the large fluctuations of the field prevent the existence of any infinite-volume Gibbs measure for the massless GFF. It is thus natural to study how these large fluctuations build up as $m \downarrow 0$. One way to quantify the change in behavior as $m \downarrow 0$ (in dimensions 1 and 2) is to consider how fast the variance $\mathrm{Var}_m(\varphi_0)$ diverges and how the rate of exponential decay of the covariance $\mathrm{Cov}_m(\varphi_i, \varphi_j)$ decays to zero.

Divergence of the variance in $d = 1, 2$

We first study the variance of the field in the limit $m \downarrow 0$, when $d = 1$ or 2.

Proposition 8.32. *Let φ be any massive Gaussian Free Field on $\mathbb{Z}^d$. Then, as $m \downarrow 0$,*

$$\mathrm{Var}_m(\varphi_0) \simeq \begin{cases} \frac{1}{\sqrt{2m}} & \text{in } d = 1, \\ \frac{2}{\pi}|\log m| & \text{in } d = 2. \end{cases} \tag{8.52}$$

Proof. Let $e^{\lambda} = 1 + m^2$, and remember that

$$\mathrm{Var}_m(\varphi_0) = G_m(0,0) = (1+m^2)^{-1}\sum_{n\ge 0} e^{-\lambda n}\mathbb{P}_0(X_n = 0).$$

We first consider the case $d = 1$. From the local limit theorem (Theorem B.70), for all $\epsilon > 0$, there exists K_0 such that

$$\frac{1-\epsilon}{\sqrt{\pi k}} \le \mathbb{P}_0(X_{2k} = 0) \le \frac{1+\epsilon}{\sqrt{\pi k}}, \quad \forall k \ge K_0. \tag{8.53}$$

This leads to the lower bound

$$\sum_{n\ge 0} e^{-\lambda n}\mathbb{P}_0(X_n = 0) \ge \frac{1-\epsilon}{\sqrt{\pi}}\sum_{k\ge K_0}\frac{e^{-2\lambda k}}{\sqrt{k}} \ge \frac{1-\epsilon}{\sqrt{\pi}}\int_{K_0}^{\infty}\frac{e^{-2\lambda x}}{\sqrt{x}}\,\mathrm{d}x = \frac{1-\epsilon}{\sqrt{2\lambda}}\big(1 - O(\sqrt{\lambda})\big),$$

where we used the change of variable $2\lambda x \equiv y^2/2$. For the upper bound, we bound the first K_0 terms of the series by 1, and obtain

$$\sum_{k\ge 0} e^{-2\lambda k}\mathbb{P}_0(X_{2k} = 0) \le K_0 + 1 + \frac{1+\epsilon}{\sqrt{\pi}}\sum_{k=K_0+1}^{\infty}\frac{e^{-2\lambda k}}{\sqrt{k}} \le K_0 + 1 + \frac{1+\epsilon}{\sqrt{\pi}}\int_{K_0}^{\infty}\frac{e^{-2\lambda x}}{\sqrt{x}}\,\mathrm{d}x.$$

The case $d = 2$ is similar and is left as an exercise; the main difference is that the integral obtained cannot be computed explicitly. □

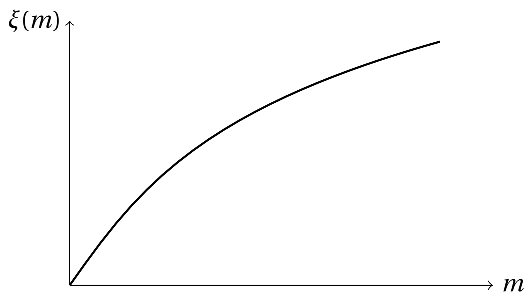

Figure 8.6: The rate of exponential decay ξ_m of the massive Green function in dimension 1.

The rate of decay for small masses

Proposition 8.30 shows that, as $m \to \infty$,

$$G_m(i,j) = e^{-2\log m(1+o(1))\|j-i\|_1} \qquad \forall i \neq j \in \mathbb{Z}^d.$$

It turns out that the rate of decay for small values of m has a very different behavior. We first consider the one-dimensional case, in which an exact computation can be made, valid for all $m > 0$:

Theorem 8.33. *Let $d = 1$ and $m > 0$. For all $i, j \in \mathbb{Z}^d$,*

$$G_m(i,j) = A_m \exp\bigl(-\xi_m |j-i|\bigr), \tag{8.54}$$

where $A_m, \xi_m > 0$ are given in (8.55). *In particular,* $\lim_{m\downarrow 0} \frac{\xi_m}{m} = \sqrt{2}$.

Proof. Since $G_m(i,j) = G_m(0, j-i)$, it suffices to consider $i = 0$. Let $\lambda > 0$ be such that $e^\lambda = 1 + m^2$ and use (8.45) to write

$$(1+m^2)G_m(0,j) = \sum_{n\geq 0} e^{-\lambda n}\mathbb{P}_0(X_n = j) = \mathbb{E}_0\Bigl[\sum_{n\geq 0} e^{-\lambda n}\mathbf{1}_{\{X_n=j\}}\Bigr].$$

We then use a Fourier representation for the indicator: for all $j \in \mathbb{Z}$,

$$\mathbf{1}_{\{X_n=j\}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\mathrm{i}k(X_n-j)}\,\mathrm{d}k.$$

The position of the symmetric simple random walk after n steps, X_n, can be expressed as a sum of independent identically distributed increments: $X_n = \xi_1 + \cdots + \xi_n$, with $\mathbb{P}_0(\xi_1 = \pm 1) = \frac{1}{2}$. Let $\phi(k) \stackrel{\text{def}}{=} \mathbb{E}_0[e^{\mathrm{i}k\xi_1}] = \cos(k)$ denote the characteristic function of the increment. Since the increments are independent, $E[e^{\mathrm{i}kX_n}] = \phi(k)^n$. Since $\lambda > 0$, we can interchange the sum and the integral and get

$$\begin{aligned}(1+m^2)G_m(0,j) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-\mathrm{i}kj}\sum_{n\geq 0}(e^{-\lambda}\phi(k))^n\,\mathrm{d}k\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{-\mathrm{i}kj}}{1-e^{-\lambda}\phi(k)}\,\mathrm{d}k.\end{aligned}$$

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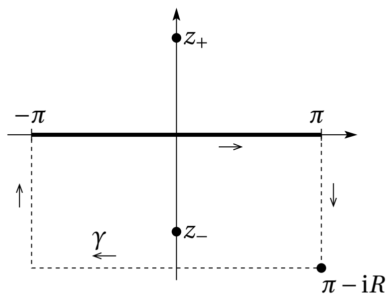
We will study the behavior of this last integral using the residue theorem. To start, we look for the singularities of $z \mapsto \frac{e^{-izj}}{1-e^{-\lambda}\cos z}$ in the complex plane. Solving $\cos z = e^{\lambda}$, we find $z_\pm = it_\pm$, with $t_\pm = t_\pm(\lambda) = -\log(e^\lambda \mp \sqrt{e^{2\lambda}-1})$. Observe that $t_-(\lambda) < 0 < t_+(\lambda)$. Let γ denote the closed clockwise-oriented path in $\mathbb{C}$ depicted on the figure below:



We decompose

$$\oint_\gamma = \int_{-\pi}^{\pi} + \int_{\pi}^{\pi - iR} + \int_{\pi-iR}^{-\pi-iR} + \int_{-\pi-iR}^{-\pi}.$$

Uniformly for all z on the path of integration from $\pi - iR$ to $-\pi - iR$, when R is large enough, $|1-e^{-\lambda}\cos(z)| \geq e^{R-\lambda}/3$. Therefore, as $R \to \infty$,

$$\Big| \int_{\pi-iR}^{-\pi-iR} \frac{e^{-izj}}{1-e^{-\lambda}\phi(z)} \, dz \Big| \to 0.$$

On the other hand, since the integrand is periodic, the integrals $\int_\pi^{\pi-iR}$ and $\int_{-\pi-iR}^{-\pi}$ cancel each other. By the residue theorem (since the path is oriented clockwise),

$$\begin{aligned} -\oint_\gamma \frac{e^{-izj}}{1-e^{-\lambda}\cos(z)} \, dz &= 2\pi i \operatorname{Res}\Big(\frac{e^{-izj}}{1-e^{-\lambda}\cos(z)}; z_- \Big) \\ &= 2\pi i \lim_{z \to z_-} (z - z_-) \frac{e^{-izj}}{1-e^{-\lambda}\cos(z)}. \end{aligned}$$

This yields

$$G_m(0,j) = \frac{e^{t_-(\lambda) j}}{\sinh|t_-(\lambda)|} \equiv A_m e^{-\xi_m j}, \tag{8.55}$$

with $\xi_m = \log(1 + m^2 + \sqrt{2m^2 + m^4})$. □

The previous result shows in particular that the rate of decay $\xi_m(i)/\|i\|_2$ behaves linearly in m as $m \downarrow 0$. We now extend this to all dimensions, using a more probabilistic approach, which has the additional benefit of shedding more light on the underlying mechanism.

Theorem 8.34. *There exist $m_0 > 0$ and constants $0 < \alpha \leq \delta$ such that, for all $0 < m < m_0$ and all $i \in \mathbb{Z}^d$,*

$$\alpha m \|i\|_2 \leq \xi_m(i) \leq \delta m \|i\|_2. \tag{8.56}$$

Let us explain why this behavior should be expected. Let $j \in \mathbb{Z}^d$ (with $\|j\|_2$ large) and let τ_j (resp. $\tau_\star$) be the time at which the walk first reaches j (resp. dies). As we already observed earlier,

$$G_m(0,j) = \mathbb{P}_0^m(\tau_j < \tau_\star) G_m(j,j) = \mathbb{P}_0^m(\tau_j < \tau_\star) G_m(0,0) . \tag{8.57}$$

On the one hand, it is unlikely that the walker survives for a time much larger than $1/m^2$. Indeed, for all $r > 0$ for which r/m^2 is an integer,

$$\mathbb{P}_0^m(\tau_\star > r/m^2) = (1+m^2)^{-r/m^2} \le e^{-r/2} , \tag{8.58}$$

for all sufficiently small m. On the other hand, in a time at most r/m^2, the walker typically cannot get to a distance further than r/m:

$$\begin{aligned} \mathbb{P}_0^m(\|Z_{r/m^2}\|_2 \ge r/m) &\le \mathbb{P}_0(\|X_{r/m^2}\|_2 \ge r/m) \\ &\le \frac{\mathbb{E}_0[\|X_{r/m^2}\|_2^2]}{r^2/m^2} = \frac{r/m^2}{r^2/m^2} = \frac{1}{r} . \end{aligned} \tag{8.59}$$

However, in order for a random walk started at 0 to reach j, such an event has to occur at least $\|j\|_2/(r/m)$ times. Therefore, the probability that the random walk reaches j should decay exponentially with $\|j\|_2/(r/m) = (m/r)\|j\|_2$. The proof below makes this argument precise. ◇

Proof. Lower bound. Let $r \ge 8$ be such that r/m^2 is a positive integer and $m/r < 1$. Set $M \overset{\text{def}}{=} \lfloor \frac{m}{r} \|j\|_2 \rfloor$. Let us introduce the following sequence of random times: $T_0 \overset{\text{def}}{=} 0$ and, for $k > 0$,

$$T_k \overset{\text{def}}{=} \inf\{n > T_{k-1} : \|Z_n - Z_{T_{k-1}}\|_2 \ge r/m\} .$$

Note that, by definition, $T_M \le \tau_j$. Applying the strong Markov Property at times $T_1, T_2, \dots, T_{M-1}$,

$$\mathbb{P}_0^m(\tau_j < \tau_\star) \le \prod_{k=0}^{M-1} \mathbb{P}_0^m(T_1 < \tau_\star) = \mathbb{P}_0^m(T_1 < \tau_\star)^M .$$

Following the heuristics described before the proof, we use the decomposition

$$\begin{aligned} \mathbb{P}_0^m(T_1 < \tau_\star) &= \mathbb{P}_0^m(T_1 < \tau_\star, T_1 \le r/m^2) + \mathbb{P}_0^m(T_1 < \tau_\star, T_1 > r/m^2) \\ &\le \mathbb{P}_0^m(T_1 \le r/m^2) + \mathbb{P}_0^m(\tau_* > r/m^2) . \end{aligned}$$

Now, on the one hand, it follows from (8.58) that $\mathbb{P}_0^m(\tau_* > r/m^2) \le e^{-r/2}$, which is smaller than $\frac{1}{4}$ by our choice of r. On the other hand,

$$\begin{aligned} \mathbb{P}_0^m\big(\|Z_{r/m^2}\|_2 \ge r/m\big) &\ge \mathbb{P}_0^m\big(\|Z_{r/m^2}\|_2 \ge r/m \,\big|\, T_1 \le r/m^2\big) \mathbb{P}_0^m\big(T_1 \le r/m^2\big) \\ &\ge \tfrac{1}{2} \mathbb{P}_0^m\big(T_1 \le r/m^2\big) , \end{aligned}$$

since, by symmetry,

$$\mathbb{P}_0^m\big(\|Z_\ell\|_2 \ge r/m \,\big|\, \|Z_k\|_2 \ge r/m\big) \ge \tfrac{1}{2} ,$$

for all $\ell \ge k$. Therefore, it follows from (8.59) that

$$\mathbb{P}_0^m\big(T_1 \le r/m^2\big) \le 2\mathbb{P}_0^m\big(\|Z_{r/m^2}\|_2 \ge r/m\big) \le \frac{2}{r} \le \frac{1}{4} ,$$

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again by our choice of r. We conclude that

$$G_m(0,j) \le 2G_m(0,0)\, e^{-(\log 2/r)m\|j\|_2}.$$

Upper bound. In (8.57), we write

$$\mathbb{P}_0^m(\tau_j < \tau_*) \ge \mathbb{P}_0(X_{[\|j\|_2/m]} = j)\,\mathbb{P}_0^m(\tau_* > \|j\|_2/m),$$

where we assume $[\|j\|_2/m]$ to be either $\lfloor \|j\|_2/m \rfloor$ or $\lfloor \|j\|_2/m \rfloor + 1$ in such a way that $\{Z_{[\|j\|_2/m]} = j\} \neq \varnothing$. The first factor in the right-hand side can then be estimated using the local limit theorem, Theorem B.70. Namely, provided that m sufficiently small, Theorem B.70 implies the existence of constants c_1, c_2 such that

$$\mathbb{P}_0(X_{[\|j\|_2/m]} = j) \ge \frac{e^{-c_1 m\|j\|_2}}{c_1(\|j\|_2/m)^{d/2}},$$

for all $j \in \mathbb{Z}^d$ with $\|j\|_2 > c_2$. Since

$$\mathbb{P}_0^m(\tau_* > \|j\|_2/m) = (1+m^2)^{-\lfloor \|j\|_2/m \rfloor} \ge e^{-m\|j\|_2},$$

the conclusion follows easily. □

8.6 Bibliographical references

The study of the Gaussian Free Field (often also called *harmonic crystal* in the literature) was initiated in the 1970s. More details can be found in Chapter 13 of Georgii's book [134], in particular proofs of the facts mentioned in Remarks 8.22 and 8.29, as well as an extensive bibliography. Some parts of Section 8.4.2 were inspired by the lecture notes of Spitzer [320].

8.7 Complements and further reading

8.7.1 Random walk representations

The random walk representation presented in this chapter (Theorems 8.17 and 8.26 and Exercise 8.12) can be extended in (at least) two directions.

In the first generalization, one replaces φ_i^2 in the mass term by a more general smooth function $U_i(\varphi_i)$ with a sufficiently fast growth. Building on earlier work by Symanzik [324], a generalization of the random walk representation to this context was first derived by Brydges, Fröhlich and Spencer in [56], which is still a nice place to learn about this material. Another source we recommend is the book [102] by Fernández, Fröhlich and Sokal, which also contains several important applications of this representation. In fact, as explained in these references, the spins φ_i themselves can be allowed to take values in $\mathbb{R}^\nu$, $\nu \ge 1$. Considering suitable sequences of functions $U_i^{(n)}$, this makes it possible to obtain random walk representations for the types of continuous spin models discussed in Chapters 9 and 10.

In the second generalization, it is the quadratic interaction $(\varphi_i - \varphi_j)^2$ that is replaced by a more general function $V(\varphi_i - \varphi_j)$ of the gradients. (Models of this type will be briefly considered in Section 9.3.) In this case, a generalization of the random walk representation was obtained by Helffer and Sjöstrand [158]. A good account can be found in Section 2 of the article [76] by Deuschel, Giacomin and Ioffe.

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8.7.2 Gradient Gibbs states

As discussed in this chapter, the massless GFF delocalizes in dimensions 1 and 2, which leads to the absence of any Gibbs measure in the thermodynamic limit in those two cases.

Nevertheless, we have seen in Exercise 8.7 that, under $\mu^0_{\Lambda;0}$, the random vector $(\widetilde{\varphi}_i)_{i\in\Lambda}$, where $\widetilde{\varphi}_i \stackrel{\text{def}}{=} \varphi_i - \varphi_0$, is centered Gaussian with covariance matrix given by

$$\tilde{G}_\Lambda(i,j) \stackrel{\text{def}}{=} G_\Lambda(i,j) - G_\Lambda(i,0) - G_\Lambda(0,j) + G_\Lambda(0,0)\,.$$

It can be shown [209] that, as $\Lambda \uparrow \mathbb{Z}^d$, the limit of this quantity is given by the convergent series

$$\widetilde{G}(i,j) \stackrel{\text{def}}{=} \sum_{n\geq 0} \mathbb{P}_i(X_n = j, \tau_0 > n)\,, \tag{8.60}$$

where $\tau_0 \stackrel{\text{def}}{=} \min\{n \geq 0 : X_n = 0\}$. In particular, the limiting Gaussian field is always well defined.

More generally, the joint distribution of the *gradients* $\varphi_i - \varphi_j$, $\{i,j\} \in \mathscr{E}_{\mathbb{Z}^d}$, remains well defined in all dimensions. It is thus possible to define Gibbs measures for this collection of random variables, instead of the original random variables φ_i, $i \in \mathbb{Z}^d$. This approach was pursued in a systematic way by Funaki and Spohn in [124], where the reader can find much more information. Other good source are Funaki's lecture notes [125] and Sheffield's thesis [302].

8.7.3 Effective interface models

As mentioned in the text, the Gaussian Free Field on $\mathbb{Z}^d$, as well as the more general class of gradient models, are often used as caricatures of the interfaces in more realistic lattice systems, such as the 3-dimensional Ising model. Such caricatures are known as *effective interface models*. They are much simpler to analyze than the objects they approximate and their analysis yields valuable insights into the properties of the latter. In particular, they are used to study the effect of various external potentials or constraints on interfaces. More information on these topics can be found in the review article [46] by Bricmont, El Mellouki and Fröhlich, and in the lecture notes by Giacomin [136], Funaki [125] and Velenik [347]. In addition, the reader would probably also enjoy the older, but classical, review paper [107] by Fisher, although it only covers one-dimensional effective interface models.

8.7.4 Continuum GFF

In this chapter, we only considered the Gaussian Free Field on the lattice $\mathbb{Z}^d$. It turns out that it is possible to define an analogous model on $\mathbb{R}^d$. The latter object plays a crucial role in the analysis of the scaling limit of critical systems in two dimensions. Good introductions to this topic can be found in the review [303] by Sheffield and the lecture notes [348] by Werner.

8.7.5 A link to discrete spin systems

We saw in Section 2.5.2 how the Hubbard–Stratonovich transformation can be used to compute the pressure of the Curie–Weiss model. Let us use the same idea and explain how discrete spin systems can sometimes be expressed in terms of the GFF.

The approach is very general but, for simplicity, we only consider an Ising ferromagnet with periodic boundary conditions. That is, we work on the torus $\mathbb{T}_n$, whose set of vertices is denoted V_n, as in Chapter 3.

Let us thus consider the Ising ferromagnet on $\mathbb{T}_n$, with the following Hamiltonian:

$$\mathscr{H}_{V_n;\mathbf{J},h} \stackrel{\text{def}}{=} -\tfrac{1}{2}\beta \sum_{i,j\in V_n} J_{ij}\sigma_i\sigma_j - h\sum_{i\in V_n}\sigma_i\,.$$

We will see that an interesting link can be made between the partition function $\mathbf{Z}^{\text{per}}_{V_n;\beta,\mathbf{J},h}$ and the GFF, provided that the coupling constants $\mathbf{J} = (J_{ij})_{i,j\in V_n}$ are well chosen.

The starting point is the following generalization of (2.20): for any positive definite matrix $\Sigma = (\Sigma(i,j))_{1\le i,j\le N}$ and any vector $x = (x_1,\dots,x_N)\in\mathbb{R}^N$,

$$\begin{aligned}\exp\bigl[\tfrac{1}{2}\sum_{i,j=1}^{N}\Sigma(i,j)x_ix_j\bigr] &= \bigl((2\pi)^N\det\Sigma\bigr)^{-1/2}\\ &\times\int_{-\infty}^{\infty}\mathrm{d}y_1\cdots\int_{-\infty}^{\infty}\mathrm{d}y_N\exp\bigl[-\tfrac{1}{2}\sum_{i,j=1}^{N}\Sigma^{-1}(i,j)y_iy_j\bigr]\exp\bigl[\sum_{i=1}^{N}x_iy_i\bigr]\,.\end{aligned}\tag{8.61}$$

(See Exercise B.22.) We will apply this identity to the quadratic part of the Boltzmann weight of the ferromagnet introduced above, with $x_i \stackrel{\text{def}}{=} \sqrt{\beta}\sigma_i$, $\Sigma(i,j) \stackrel{\text{def}}{=} J_{ij}$. To establish a correspondence with the GFF, we choose $\mathbf{J}$ so that the inverse Σ^{-1} can be related to the GFF. Let us therefore take

$$J_{ij} \stackrel{\text{def}}{=} G_{m;\mathbb{T}_n}(i,j)\,,$$

where $G_{m;\mathbb{T}_n}(i,j)$ denotes is the massive Green function of the symmetric simple random walk $(X_n)_{n\ge0}$ on $\mathbb{T}_n$, given by

$$G_{m;\mathbb{T}_n}(i,j) \stackrel{\text{def}}{=} \sum_{n\ge0}(1+m^2)^{-n-1}\mathbb{P}_i(X_n=j)\,.\tag{8.62}$$

A straightforward adaptation of the proof of Theorem 8.26 shows that

$$(G_{m;\mathbb{T}_n})^{-1} = -\tfrac{1}{2d}\Delta + m^2\,.$$

where $\Delta = (\Delta_{ij})_{i,j\in\mathbb{T}_n}$ denotes the discrete Laplacian on $\mathbb{T}_n$, defined as in (8.16).

Notice that, even though the coupling constants J_{ij} defined above depend on n and involve long-range interactions, they converge as $n\to\infty$ and decay exponentially fast in $\|j-i\|_1$, uniformly in n, as can be seen from (8.62).

With this choice of coupling constants, (8.61) can be written as

$$\begin{aligned}\exp\bigl[\tfrac{1}{2}\beta\sum_{i,j\in V_n}J_{ij}\sigma_i\sigma_j\bigr] &= \bigl((2\pi)^{|V_n|}\det G_{m;\mathbb{T}_n}\bigr)^{-1/2}\times\\ &\times\int\exp\bigl[-\tfrac{1}{2}\sum_{i,j\in V_n}y_i(-\tfrac{1}{2d}\Delta_{ij}+m^2)y_j\bigr]\exp\bigl[\beta^{1/2}\sum_{i\in V_n}y_i\sigma_i\bigr]\prod_{i\in V_n}\mathrm{d}y_i\,,\end{aligned}$$

where each $y_i, i\in V_n$, is integrated over $\mathbb{R}$. Since we recognize the Boltzmann weight of the massive centered GFF on $\mathbb{T}_n$, we get

$$\exp\bigl[\tfrac{1}{2}\beta\sum_{i,j\in V_n}J_{ij}\sigma_i\sigma_j\bigr] = \Bigl\langle\exp\bigl[\beta^{1/2}\sum_{i\in V_n}\varphi_i\sigma_i\bigr]\Bigr\rangle^{\text{GFF,per}}_{V_n;\beta,m}.$$

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We can now perform the summation over configurations in the partition function of the ferromagnet, which yields

$$\mathbf{Z}^{\mathrm{per}}_{V_n;\beta,\mathbf{J},h} = 2^{|V_n|}\Big\langle \prod_{i\in V_n} \cosh\big(\beta^{1/2}\varphi_i + h\big)\Big\rangle^{\mathrm{GFF,per}}_{V_n;m}.$$

Note that the numerator in the right-hand side corresponds to a massless GFF with an additional term $\sum_{i\in V_n} W(\varphi_i)$ in the Hamiltonian, where $W(\cdot)$ is an **external potential** defined by (see Figure 8.7)

$$W(x) \stackrel{\mathrm{def}}{=} \frac{m^2}{2}x^2 - \log\cosh\big(\beta^{1/2}x + h\big). \tag{8.63}$$

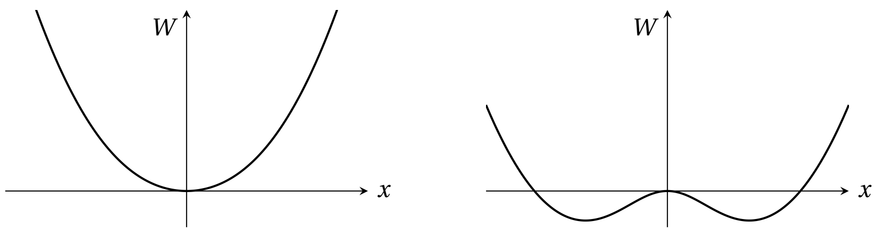

Figure 8.7: The external potential W with $m = 1$ and $h = 0$. Left: $\beta = 0.5$. Right: $\beta = 2$.

More generally, the same argument leads to a similar representation for any correlation function:

$$\langle\sigma_A\rangle^{\mathrm{per}}_{V_n;\beta,h} = \Big\langle \prod_{i\in A} \tanh\big(\beta^{1/2}\varphi_i + h\big)\Big\rangle^{\mathrm{GFF,per}}_{V_n;W},$$

where the latter measure is that of the massless GFF in the external potential W.

In a sense, the above transformation (sometimes called the **sine-Gordon** transformation) allows us to replace the discrete ± 1 spins of the Ising model by the continuous (and unbounded) spins of a Gaussian Free Field. A trace of the two values can still be seen in the resulting double-well potential (8.63) to which this field is submitted when β is sufficiently large; see Figure 8.7. Even though we will not make use of this in the present book, this continuous settings turns out to be very convenient when implementing rigorously the renormalization group approach. We refer to [57] for more information.

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