C Solutions to Exercises

In this appendix are regrouped the solutions to many of the exercises stated in the main body of the book. Some solutions are given with full details, while others are only sketched. In all cases, we recommend that the reader at least spends some time thinking about these problems before reading the solutions.

Solutions of Chapter 1

Exercise 1.1: Fix $n \in \mathbb{Z}_{>0}$ and observe first that our system Σ, with parameters U, V, N can be seen as a system Σ' composed of two subsystems Σ_1, Σ_2 with parameters $\frac{1}{n}U, \frac{1}{n}V, \frac{1}{n}N$ and $\frac{n-1}{n}U, \frac{n-1}{n}V, \frac{n-1}{n}N$. Then, by additivity,

$$S^{\Sigma}(U,V,N) = S^{\Sigma'}(\tfrac{1}{n}U, \tfrac{1}{n}V, \tfrac{1}{n}N, \tfrac{n-1}{n}U, \tfrac{n-1}{n}V, \tfrac{n-1}{n}N) = S^{\Sigma}(\tfrac{1}{n}U, \tfrac{1}{n}V, \tfrac{1}{n}N) + S^{\Sigma}(\tfrac{n-1}{n}U, \tfrac{n-1}{n}V, \tfrac{n-1}{n}N),$$

where we used the fact that each of the two subsystems is of the same type as the original system and is therefore associated to the same entropy function. Iterating this, we get

$$S^{\Sigma}(U,V,N) = nS^{\Sigma}(\tfrac{1}{n}U, \tfrac{1}{n}V, \tfrac{1}{n}N).$$

Using this relation twice, we conclude that, for any $m, n \in \mathbb{Z}_{>0}$,

$$S^{\Sigma}(\tfrac{m}{n}U, \tfrac{m}{n}V, \tfrac{m}{n}N) = mS^{\Sigma}(\tfrac{1}{n}U, \tfrac{1}{n}V, \tfrac{1}{n}N) = \tfrac{m}{n}S^{\Sigma}(U,V,N).$$

This proves (1.7) for $\lambda \in \mathbb{Q}$. Since S^{Σ} is assumed to be differentiable, it is also continuous. We can therefore approximate any real $\lambda > 0$ by a sequence $(\lambda_n)_{n\geq 1} \subset \mathbb{Q}$, $\lambda_n \to \lambda$, and get

$$S^{\Sigma}(\lambda U, \lambda V, \lambda N) = \lim_{n\to\infty} S^{\Sigma}(\lambda_n U, \lambda_n V, \lambda_n N) = \lim_{n\to\infty} \lambda_n S^{\Sigma}(U,V,N) = \lambda S^{\Sigma}(U,V,N).$$

Exercise 1.2: Decompose the system into two subsystems Σ_1, Σ_2. By the postulate, $S^{\Sigma}(U,V,N)$ maximizes $S^{\Sigma}(\widetilde{U}_1, \widetilde{V}_1, \widetilde{N}_1) + S^{\Sigma}(\widetilde{U}_2, \widetilde{V}_2, \widetilde{N}_2)$ over all possible ways of partitioning U, V, N into $\widetilde{U}_1 + \widetilde{U}_2$, $\widetilde{V}_1 + \widetilde{V}_2$ and $\widetilde{N}_1 + \widetilde{N}_2$. This implies in particular that

$$\begin{aligned} S^{\Sigma}(U,V,N) &\geq S^{\Sigma}\big(\alpha U_1, \alpha V_1, \alpha N_1\big) + S^{\Sigma}\big((1-\alpha)U_2, (1-\alpha)V_2, (1-\alpha)N_2\big) \\ &= \alpha S^{\Sigma}(U_1, V_1, N_1) + (1-\alpha) S^{\Sigma}(U_2, V_2, N_2), \end{aligned}$$

where the equality is a consequence of (1.7).

Exercise 1.3: Fix V, N, β_1, β_2 and $\alpha \in [0,1]$. For all U,

$$\{\alpha\beta_1 + (1-\alpha)\beta_2\}U - S(U,V,N) = \alpha\{\underbrace{\beta_1 U - S(U,V,N)}_{\geq \hat{F}(\beta_1,V,N)}\} + (1-\alpha)\{\underbrace{\beta_2 U - S(U,V,N)}_{\geq \hat{F}(\beta_2,V,N)}\}.$$

Taking the infimum over U on the left-hand side,

$$\hat{F}(\alpha\beta_1 + (1-\alpha)\beta_2, V, N) \geq \alpha \hat{F}(\beta_1, V, N) + (1-\alpha)\hat{F}(\beta_2, V, N),$$

so $\hat{F}$ is concave in β. A similar argument, exploiting the concavity of S, shows that $\hat{F}$ is convex in V, N.

Exercise 1.4: The extremum principle follows from the one postulated for S. Indeed, suppose that we keep our system isolated, with a total energy U (and the subsystems can exchange energy, which can always be assumed in the present setting, since they can do that through the reservoir). Then, the equilibrium values are those maximizing

$$S(U^1, V^1, N^1) + S(U^2, V^2, N^2),$$

among all values satisfying the constraints on V^1, N^1, V^2, N^2 as well as $U^1 + U^2 = U$. Therefore, the same values minimize

$$\beta U - \bigl(S(U^1, V^1, N^1) + S(U^2, V^2, N^2)\bigr) = \bigl(\beta U^1 - S(U^1, V^1, N^1)\bigr) + \bigl(\beta U^2 - S(U^2, V^2, N^2)\bigr),$$

under the same conditions. Taking now the infimum over U yields the desired result, since this removes the constraint $U^1 + U^2 = U$.

Exercise 1.5: The critical points of the function $v \mapsto p(v)$ are given by the solutions of the equation $RTv^3 = 2a(v-b)^2$, which is of the form $f(v) = g(v)$. When $v > b$, this equation has zero, one or two solutions depending on the value of T. The critical case corresponds to when there is exactly one solution (at which $f(v) = g(v)$ and $f'(v) = g'(v)$). This happens when $T = \frac{8a}{27Rb}$.

Exercise 1.6: Writing $\mathsf{S}_{\mathrm{Sh}}(\mu) = \sum_{\omega\in\Omega} \psi(\mu(\omega))$, where $\psi(x) \stackrel{\mathrm{def}}{=} -x\log x$, we see that S_{Sh} is concave.

Exercise 1.8: The desired probabilities are given by $\mu(i) = e^{-\beta i}/\mathbf{Z}_\beta$, where $\mathbf{Z}_\beta = \sum_{i=1}^{6} e^{-\beta i}$ and β must be chosen such that $\sum_i i\mu(i) = 4$. Numerically, one finds that

$$\mu(1) \cong 0.10, \mu(2) \cong 0.12, \mu(3) \cong 0.15, \mu(4) \cong 0.17, \mu(5) \cong 0.21, \mu(6) \cong 0.25.$$

Exercise 1.9: Letting $V' = V - \frac{N}{2}$ and writing $N_1 = \frac{N}{2} + m$, $N_2 = \frac{N}{2} - m$, we need to show that

$$m \mapsto (\tfrac{N}{2}+m)!(\tfrac{N}{2}-m)!(V'+m)!(V'-m)!$$

is minimal when $m = 0$. But this follows by simple termwise comparison. For the second part, expressing the desired probability using Stirling's formula (Lemma B.3) shows that there exist constants $c_- < c_+$ such that if V and N are both large, with $\frac{N}{2V}$ bounded away from 0 and 1, then

$$\frac{c_-}{\sqrt{N}} \le \frac{\binom{V}{\frac{N}{2}}\binom{V}{\frac{N}{2}}}{\binom{2V}{N}} \le \frac{c_+}{\sqrt{N}}.$$

Exercise 1.10: Note that the second derivative of $\log \mathbf{Q}_{\Lambda;\beta,N}$ with respect to β yields the variance of $\mathscr{H}$ under the canonical distribution and is thus nonnegative. We conclude that $\beta \mapsto -\log \mathbf{Q}_{\Lambda;\beta,N}$ is concave. Moreover, since the limit of a sequence of concave functions is concave (see Exercise B.3), this implies that $\hat{f}$ is concave in β.

Exercise 1.12: Plugging $\mu_{\Lambda;\beta(U),N}$ in the definition of $\mathsf{S}_{\mathrm{Sh}}(\cdot)$ gives

$$\mathsf{S}_{\mathrm{Sh}}(\mu_{\Lambda;\beta(U),N}) = \beta(U)\langle\mathscr{H}\rangle_{\mu_{\Lambda;\beta(U),N}} + \log \mathbf{Z}_{\Lambda;\beta(U),N} = \beta(U)U + \log \mathbf{Z}_{\Lambda;\beta(U),N}. \tag{C.1}$$

By the Implicit Function theorem, $U \mapsto \beta(U)$ is differentiable. So, differentiating with respect to U,

$$\frac{\partial \mathsf{S}_{\mathrm{Sh}}(\mu_{\Lambda;\beta(U),N})}{\partial U} = \frac{\partial \beta(U)}{\partial U} U + \beta(U) + \underbrace{\frac{\partial}{\partial \beta}\log \mathbf{Z}_{\Lambda;\beta,N}\Big|_{\beta=\beta(U)}}_{=-U} \frac{\partial \beta(U)}{\partial U} = \beta(U),$$

as one expects from the definition of the inverse temperature in (1.3). Then,

$$U - T_U \mathsf{S}_{\mathrm{Sh}}(\mu_{\Lambda;\beta(U),N}) = U - T_U\{\beta(U)U + \log \mathbf{Z}_{\Lambda;\beta(U),N}\} = -\tfrac{1}{\beta(U)}\log \mathbf{Z}_{\Lambda;\beta(U),N},$$

in accordance with the definition of free energy given earlier.

Exercise 1.13: Since $M_\Lambda(-\omega) = -M_\Lambda(\omega)$,

$$\langle M_\Lambda\rangle_{\Lambda;\beta,0} = \sum_{\omega\in\Omega_\Lambda} M_\Lambda(\omega)\mu_{\Lambda;\beta,0}(\omega) = \tfrac{1}{2}\sum_{\omega\in\Omega_\Lambda} M_\Lambda(\omega)\underbrace{\{\mu_{\Lambda;\beta,0}(\omega) - \mu_{\Lambda;\beta,0}(-\omega)\}}_{=0} = 0.$$

Solutions of Chapter 2

Exercise 2.2: If one writes $\widetilde{\mathscr{H}}_{N;\beta,0} = \mathscr{H}^{\mathrm{CW}}_{N;\beta(N),0}$, where $\beta(N) \stackrel{\mathrm{def}}{=} N\beta/\zeta(N)$, then either $\beta(N) \uparrow +\infty$ or $\beta(N) \downarrow 0$. The conclusion now follows from our previous analysis.

Exercise 2.3: The analyticity of $h \mapsto m^{\mathrm{CW}}_{\beta}(h)$ follows from the implicit function theorem (Theorem B.28).

Exercise 2.6: Let us write $\varphi(y) \equiv \varphi_{\beta,h}(y)$. Notice that, since $\beta > 0$, $\varphi(y) \uparrow +\infty$ as $y \to \pm\infty$, sufficiently fast to ensure that $\int_{-\infty}^{+\infty} e^{-c\varphi(y)}\,\mathrm{d}y < \infty$ for all $c > 0$. Depending on β, φ has either one or two global minima. For simplicity, consider the case in which there is a unique global minimum y_*. Let $\widetilde{\varphi}(y) \stackrel{\mathrm{def}}{=} \varphi(y) - \varphi(y_*)$, $B_\epsilon(y_*) \stackrel{\mathrm{def}}{=} [y_* - \epsilon, y_* + \epsilon]$ and write

$$\int_{-\infty}^{\infty} e^{-N(\varphi_{\beta,h}(y) - \min_y \varphi_{\beta,h}(y))}\,\mathrm{d}y \geq \int_{B_\epsilon(y_*)} e^{-N\widetilde{\varphi}(y)}\,\mathrm{d}y .$$

Let $c > 0$ be such that $\widetilde{\varphi}(y) \leq c(y - y_*)^2$ for all $y \in B_\epsilon(y_*)$. Then

$$\sqrt{N}\int_{B_\epsilon(y_*)} e^{-N\widetilde{\varphi}(y)}\,\mathrm{d}y \geq \sqrt{N}\int_{B_\epsilon(y_*)} e^{-cN(y-y_*)^2}\,\mathrm{d}y = \frac{1}{\sqrt{2c}}\int_{-\epsilon\sqrt{2cN}}^{+\epsilon\sqrt{2cN}} e^{-x^2/2}\,\mathrm{d}x ,$$

and this last expression converges to $\sqrt{\pi/c}$ when $N \to \infty$.

Solutions of Chapter 3

Exercise 3.1: Notice that $|\mathsf{B}(n)| = (2n+1)^d$ and that

$$|\partial^{\mathrm{in}}\mathsf{B}(n)| = |\mathsf{B}(n) \setminus \mathsf{B}(n-1)| = (2n+1)^d - (2n-1)^d \leq d(2n+1)^{d-1} ,$$

which shows that $\frac{|\partial^{\mathrm{in}}\mathsf{B}(n)|}{|\mathsf{B}(n)|} \to 0$. Any sequence $\Lambda_n \uparrow \mathbb{Z}^d$ whose boundary grows as fast as its volume, such as $\Lambda_n = \mathsf{B}(n) \cup \{(i,0,\ldots,0) \in \mathbb{Z}^d : 0 \leq i \leq e^n\}$, will not converge in the sense of van Hove.

Exercise 3.5: By a straightforward computation, $m_\beta(h) = \sinh(h)/\sqrt{\sinh^2(h) + e^{-4\beta}}$.

Exercise 3.6: *1.* The partition function with free boundary condition can be expressed as

$$\mathbf{Z}^{\varnothing}_{\mathsf{B}(n);\beta,h} = \sum_{\substack{\omega_i=\pm1\\ i\in\mathsf{B}(n)}} \prod_{i=-n}^{n-1} e^{\beta\omega_i\omega_{i+1}} = e^{2\beta n} \sum_{\substack{\omega_i=\pm1\\ i\in\mathsf{B}(n)}} \prod_{i=-n}^{n-1} e^{\beta(\omega_i\omega_{i+1}-1)} .$$

Each factor in the last product is either equal to 1 (if $\omega_i = \omega_{i+1}$) or to $e^{-2\beta}$. Therefore,

$$\mathbf{Z}^{\varnothing}_{\mathsf{B}(n);\beta,h} = 2\,e^{2\beta n} \sum_{k=0}^{2n} \binom{2n}{k} (e^{-2\beta})^k = 2\,e^{2\beta n}\,(1 + e^{-2\beta})^{2n} .$$

This yields $\psi(\beta) = \log\cosh(\beta) + \log 2$, which of course coincides with (3.10).

2. In terms of the variables τ_i,

$$\mathbf{Z}^{\varnothing}_{\mathsf{B}(n);\beta,h} = \sum_{\omega_{-n}=\pm1} \sum_{\substack{\tau_i=\pm1\\ i=-n+1,\ldots,n}} \prod_{i=-n+1}^{n} e^{\beta\tau_i} = 2\big(e^{\beta} + e^{-\beta}\big)^{2n} .$$

Exercise 3.8: Notice that any local function can be expressed as a finite linear combination of **cylinder functions**, which are of the following form: $f(\omega) = 1$ if ω coincides, on a finite region Λ, with some configuration τ, and zero otherwise. Since each spin ω_i takes only two values, there are countably many cylinder functions, we denote them by $f_1, f_2, \ldots$. Since, for each j, the sequence $(\langle f_j\rangle^{\eta_n}_{\Lambda_n;\beta,h})_{n\geq1}$ is bounded, a standard diagonalization argument (this type of argument will be explained in more detail later, for instance in the proof of Proposition 6.20) allows one to extract a subsequence $(n_k)_{k\geq1}$ such that $\lim_{k\to\infty}\langle f_j\rangle^{\eta_{n_k}}_{\Lambda_{n_k};\beta,h}$ exists for all j. The existence of $\langle f\rangle \stackrel{\mathrm{def}}{=} \lim_{k\to\infty}\langle f\rangle^{\eta_{n_k}}_{\Lambda_{n_k};\beta,h}$ for all local functions f follows by linearity and defines a Gibbs state.

Exercise 3.9: Simply differentiate $\langle\sigma_A\rangle^{+}_{\Lambda;\mathbf{J},\mathbf{h}}$ with respect to J_{ij} or h_i and use (3.22).

Exercise 3.10: Observe that, to show that f is nondecreasing, it suffices to show that $f(\omega) \le f(\omega')$ whenever there exists $i \in \mathbb{Z}^d$ such that $\omega_i = -1$, $\omega'_i = 1$ and $\omega_j = \omega'_j$ for all $j \neq i$. The exercise is then straightforward.

Exercise 3.11: We will come back to this important property in Chapter 6 and prove it in a more general setting (see Lemma 6.7). For simplicity, assume $h = 0$. The numerator appearing in $\mu^\eta_{\Lambda;\beta,h}\big(\omega \mid \sigma_i = \omega'_i, \forall i \in \Lambda\setminus\Delta\big)$ contains the term

$$\exp\big(\beta \sum_{\{i,j\}\in\mathscr{E}^{\mathrm{b}}_\Lambda} \omega_i\omega_j\big) = \exp\big(\beta \sum_{\{i,j\}\in\mathscr{E}^{\mathrm{b}}_\Delta} \omega_i\omega_j\big)\exp\big(\beta \sum_{\substack{\{i,j\}\in\mathscr{E}^{\mathrm{b}}_\Lambda\\ \{i,j\}\cap\Delta=\varnothing}} \omega_i\omega_j\big).$$

The first term, containing the sum over $\{i,j\}\in\mathscr{E}^{\mathrm{b}}_\Delta$, is used to form $\mu^{\omega'}_{\Delta;\beta,h}(\omega)$. The same decomposition can be used for the partition functions; the second factor then cancels out.

Exercise 3.12: Let $D \subset \mathscr{E}_{\Lambda_2}$ be the set of edges $\{i,j\}$ with $i \in \Lambda_2\setminus\Lambda_1$, $j \in \Lambda_1$. Consider, for $s \in [0,1]$, the Hamiltonian

$$\mathscr{H}^s_{\Lambda_2;\beta,h} \stackrel{\mathrm{def}}{=} -\beta \sum_{\substack{\{i,j\}\in\mathscr{E}_{\Lambda_2}\\ \{i,j\}\notin D}} \sigma_i\sigma_j - s\beta \sum_{\{i,j\}\in D} \sigma_i\sigma_j - h\sum_{i\in\Lambda_2}\sigma_i\,.$$

Let $\langle\cdot\rangle^s_{\Lambda_2;\beta,h}$ denote the corresponding Gibbs distribution. Observe that, when $A \subset \Lambda_1$, $\langle\sigma_A\rangle^{\varnothing}_{\Lambda_2;\beta,h} = \langle\sigma_A\rangle^{s=1}_{\Lambda_2;\beta,h}$ and $\langle\sigma_A\rangle^{\varnothing}_{\Lambda_1;\beta,h} = \langle\sigma_A\rangle^{s=0}_{\Lambda_2;\beta,h}$. The conclusion follows since, by Exercise 3.9, $\langle\sigma_A\rangle^{s=0}_{\Lambda_2;\beta,h} \le \langle\sigma_A\rangle^{s=1}_{\Lambda_2;\beta,h}$.

For the other claim, add a magnetic field h' acting on the spins in $\Lambda_2\setminus\Lambda_1$ and let $h'\to\infty$.

Exercise 3.15: First, the FKG inequality and translation invariance yield, for any i,

$$\langle n_A n_{B+i}\rangle^+_{\beta,h} \ge \langle n_A\rangle^+_{\beta,h}\langle n_B\rangle^+_{\beta,h}\,.$$

Fix L large enough to ensure that $A, B \subset \mathsf{B}(L)$. Taking $\|i\|_1$ sufficiently large, we can guarantee that $\mathsf{B}(L+1)\cap(i+\mathsf{B}(L)) = \varnothing$. Fixing all the spins on $\partial^{\mathrm{ex}}\mathsf{B}(L)\cup\partial^{\mathrm{ex}}(i+\mathsf{B}(L))$ to $+1$, it follows from the FKG inequality that

$$\langle n_A n_{B+i}\rangle^+_{\beta,h} \le \langle n_A\rangle^+_{\mathsf{B}(L);\beta,h}\langle n_{B+i}\rangle^+_{i+\mathsf{B}(L);\beta,h} = \langle n_A\rangle^+_{\mathsf{B}(L);\beta,h}\langle n_B\rangle^+_{\mathsf{B}(L);\beta,h}\,.$$

We conclude that

$$\begin{aligned}\langle n_A\rangle^+_{\beta,h}\langle n_B\rangle^+_{\beta,h} &\le \liminf_{\|i\|_1\to\infty}\langle n_A n_{B+i}\rangle^+_{\beta,h}\\ &\le \limsup_{\|i\|_1\to\infty}\langle n_A n_{B+i}\rangle^+_{\beta,h} \le \langle n_A\rangle^+_{\mathsf{B}(L);\beta,h}\langle n_B\rangle^+_{\mathsf{B}(L);\beta,h}\,.\end{aligned}$$

The desired conclusion follows by letting $L\to\infty$ in the right-hand side. The case of general local functions f and g follows from Lemma 3.19.

Exercise 3.16: Follow the steps of the proof of Theorem 3.17, using Exercise 3.12 for the existence of the thermodynamic limit (use $\langle\sigma_A\rangle^{\varnothing}_{\Lambda_n;\beta,h} = (-1)^{|A|}\langle\sigma_A\rangle^{\varnothing}_{\Lambda_n;\beta,-h}$ when dealing with $h<0$).

Exercise 3.17: Proceed as in the proof of Lemma 3.31, using the monotonicity results established in Exercises 3.9 and 3.12.

Exercise 3.18: *1.* This is a consequence of (3.34). Indeed, let us denote by $\mathscr{A}_\ell$ the set of all contours γ (in $\mathsf{B}(n)$) with length ℓ. Then,

$$\mu^+_{\mathsf{B}(n);\beta,0}\big(\exists\gamma\in\Gamma \text{ with } |\gamma|\ge K\log n\big) \le \sum_{\ell\ge K\log n}|\mathscr{A}_\ell|e^{-2\beta\ell}\,.$$

Now, the number of contours of length ℓ passing through a given point is bounded above by 4^ℓ and the number of translates of such a contour entirely contained inside $\mathsf{B}(n)$ is bounded above by $4n^2$. Therefore, the probability we are interested in is bounded above by

$$4n^2\sum_{\ell\ge K\log n}(4e^{-2\beta})^\ell \le 8n^{2-K(2\beta-\log 4)}\,,$$

for all $\beta \ge \log 3$, say. This bound can be made smaller than n^{-c}, for any fixed $c > 0$, by taking K sufficiently large (uniformly in $\beta \ge \log 3$).

2. Partition each row of $\mathsf{B}(n)$ into intervals of length $K \log n$ (and, possibly, a remaining shorter interval that we ignore). We denote by I_k, $k = 1, \ldots, N$, these intervals and consider the event

$$\mathscr{I}_k = \{\sigma_i = -1 \ \forall i \in I_k\}.$$

Of course, there exists $C = C(\beta)$ such that $\mu^+_{I_k;\beta,0}(\mathscr{I}_k) \ge e^{-CK \log n} = n^{-CK}$. Now,

$$\mu^+_{\mathsf{B}(n);\beta,0}\big(\exists \gamma \in \Gamma \text{ with } |\gamma| \ge K \log n\big) \ge \mu^+_{\mathsf{B}(n);\beta,0}\big(\bigcup_{k=1}^N \mathscr{I}_k\big) = 1 - \mu^+_{\mathsf{B}(n);\beta,0}\big(\bigcap_{k=1}^N \mathscr{I}_k^{\mathrm{c}}\big).$$

Notice that

$$\mu^+_{\mathsf{B}(n);\beta,0}\big(\bigcap_{k=1}^N \mathscr{I}_k^{\mathrm{c}}\big) = \prod_{m=1}^N \mu^+_{\mathsf{B}(n);\beta,0}\big(\mathscr{I}_m^{\mathrm{c}} \mid \bigcap_{k=1}^{m-1} \mathscr{I}_k^{\mathrm{c}}\big) = \prod_{m=1}^N \big\{1 - \mu^+_{\mathsf{B}(n);\beta,0}\big(\mathscr{I}_m \mid \bigcap_{k=1}^{m-1} \mathscr{I}_k^{\mathrm{c}}\big)\big\}.$$

By the FKG inequality,

$$\mu^+_{\mathsf{B}(n);\beta,0}\big(\mathscr{I}_m \mid \bigcap_{k=1}^{m-1} \mathscr{I}_k^{\mathrm{c}}\big) \ge \mu^+_{I_m;\beta,0}(\mathscr{I}_m) \ge n^{-CK},$$

so that

$$\mu^+_{\mathsf{B}(n);\beta,0}\big(\exists \gamma \in \Gamma \text{ with } |\gamma| \ge K \log n\big) \ge 1 - (1 - n^{-CK})^N \ge 1 - e^{-n^{-CK} N}.$$

The conclusion follows since $N = (2n+1)\lfloor (2n+1)/K \log n \rfloor \ge n^{2-c/2}/K$ for $n > n_0(c)$ and $n^{-CK}/K \ge n^{-c/2}$ if $K \le K_1(\beta, c)$.

Exercise 3.20: In higher dimensions, the deformation operation leading to contours is less convenient, so we will avoid it. For the sake of concreteness, we consider the case $d = 3$. The bounds we give below are very rough and can be improved. The 3-dimensional analogue of the contours described above are sets of **plaquettes**, which are the squares that form the boundary of the cubic cells of $\mathbb{Z}^3$. For a given configuration ω, the set $\partial \mathscr{M}(\omega)$ can be defined as before and decomposed into maximal connected sets of plaquettes: $\partial \mathscr{M}(\omega) = \hat{\gamma}_1 \cup \cdots \cup \hat{\gamma}_n$. The analogue of (3.38) then becomes

$$\mu^+_{\mathsf{B}(n);\beta,0}(\sigma_0 = -1) \le \sum_{k \ge 6} e^{-2\beta k} \, \#\big\{\hat{\gamma}^* : \mathrm{dist}(\hat{\gamma}^*, 0) \le k, |\hat{\gamma}^*| = k\big\}.$$

To each $\hat{\gamma}^*$ in the latter set, we associate a connected graph G^* whose set of vertices V^* is formed by all the centers of the plaquettes of $\hat{\gamma}^*$ and in which two vertices $u, v \in V^*$ are connected by an edge if the corresponding plaquettes share a common edge. The above sum is then bounded by (observe that a vertex of V^* has at most 12 neighbors and that each edge is shared by two vertices, so that $|E^*| \le 6k$)

$$\sum_{k \ge 6} e^{-2\beta k} \, \#\big\{G^* : |V^*| = k\big\} \le \sum_{k \ge 6} e^{-2\beta k} \cdot k^3 \cdot 12^{6k}.$$

This last inequality was obtained using Lemma 3.38. As in the two-dimensional case, the series is smaller than $\frac{1}{2}$ once β is large enough.

Exercise 3.21: Define τ by $e^{-\tau} \stackrel{\text{def}}{=} 3e^{-2\beta}$. Notice that (3.40) can be written $(4e^{-4\tau} - 3e^{-5\tau})/(1 - e^{-\tau})^2 < \frac{3}{4}$. The first point follows by verifying that this holds once $\beta > 0.88$.

Let us turn to the second point. Write $\mu^\pm_{\mathsf{B}(n);\beta,0}(A) = \frac{\mathbf{Z}^\pm_{\mathsf{B}(n);\beta,0}[A]}{\mathbf{Z}^\pm_{\mathsf{B}(n);\beta,0}[\Omega^\pm_{\mathsf{B}(n)}]}$, where

$$\mathbf{Z}^\pm_{\mathsf{B}(n);\beta,0}[A] \stackrel{\text{def}}{=} \sum_{\omega \in \Omega^\pm_{\mathsf{B}(n)}} \mathbf{1}_A(\omega) \prod_{\gamma \in \Gamma(\omega)} e^{-2\beta|\gamma|}.$$

Let $A^\pm \stackrel{\text{def}}{=} \{\sigma_i = \pm 1 \ \forall i \in \mathsf{B}(R)\}$. Under $\mu^+_{\mathsf{B}(n);\beta,0}$, the occurrence of A^- forces the presence of at least one self-avoiding closed path $\pi^* \subset \partial \mathscr{M}$ surrounding $\mathsf{B}(R)$. Therefore, by flipping all the spins located inside the region delimited by π^*, one gets

$$\mathbf{Z}^+_{\mathsf{B}(n);\beta,0}[A^-] \le \sum_{\pi^*} \mathbf{Z}^+_{\mathsf{B}(n);\beta,0}[\pi^* \subset \partial \mathscr{M}, A^-] \le \sum_{\pi^*} e^{-2\beta|\pi^*|} \mathbf{Z}^+_{\mathsf{B}(n);\beta,0}[A^+].$$

But

$$\sum_{\pi^*} e^{-2\beta|\pi^*|} \le \sum_{k \ge 8R} k e^{-2\beta k} C_k.$$

Now, for all $\epsilon > 0$, $C_k \le (\mu+\epsilon)^k$ for all large enough k. Therefore,

$$\frac{\mu^{+}_{\mathsf{B}(n);\beta,0}(\sigma_i = -1\,\forall i \in \mathsf{B}(R))}{\mu^{-}_{\mathsf{B}(n);\beta,0}(\sigma_i = -1\,\forall i \in \mathsf{B}(R))} = \frac{\mathbf{Z}^{+}_{\mathsf{B}(n);\beta,0}[A^-]}{\mathbf{Z}^{-}_{\mathsf{B}(n);\beta,0}[A^-]} = \frac{\mathbf{Z}^{+}_{\mathsf{B}(n);\beta,0}[A^-]}{\mathbf{Z}^{+}_{\mathsf{B}(n);\beta,0}[A^+]} \le \sum_{k\ge 8R} k e^{-2\beta k}(\mu+\epsilon)^k\,.$$

If $e^{-2\beta}\mu < 1$, ϵ can be chosen such that the last series converges. Taking R sufficiently large allows to make the whole sum < 1.

Exercise 3.23: We only provide the answer for the free boundary condition.

Let $\mathfrak{E}^{\text{even}}_{\Lambda} \overset{\text{def}}{=} \{E \subset \mathscr{E}_\Lambda : I(i,E) \text{ is even for all } i \in \Lambda\}$. Then,

$$\mathbf{Z}^{\varnothing}_{\Lambda;\beta,0} = 2^{|\Lambda|}\cosh(\beta)^{|\mathscr{E}_\Lambda|}\sum_{E\in\mathfrak{E}^{\text{even}}_{\Lambda}}\tanh(\beta)^{|E|}\,.$$

Moreover,

$$\langle\sigma_i\sigma_j\rangle^{\varnothing}_{\Lambda;\beta,0} = \sum_{\substack{E_0\in\mathfrak{E}^{i,j}_{\Lambda}\\ \text{connected},E_0\ni i}} \tanh(\beta)^{|E_0|}\,\frac{\sum_{E'\in\mathfrak{E}^{\text{even}}_{\Lambda}:E'\subset\Delta(E_0)}\tanh(\beta)^{|E'|}}{\sum_{E\in\mathfrak{E}^{\text{even}}_{\Lambda}}\tanh(\beta)^{|E|}}\,,$$

where

$$\mathfrak{E}^{i,j}_{\Lambda} \overset{\text{def}}{=} \{E\subset\mathscr{E}_\Lambda : I(k,E) \text{ is even for all } k\in\mathsf{B}(n)\setminus\{i,j\}, \text{ but } I(i,E) \text{ and } I(j,E) \text{ are odd}\}\,.$$

Exercise 3.24: Since the Gibbs state is unique, we can consider the free boundary condition. Proceeding as we did for the representation of $\langle\sigma_0\rangle^{+}_{\Lambda;\beta,h}$ in terms of a sum over graphs in (3.47), we get for $i, j \in \mathsf{B}(n)$,

$$\langle\sigma_i\sigma_j\rangle^{\varnothing}_{\mathsf{B}(n);\beta,0} \le \sum_{\substack{E\in\mathfrak{E}^{i,j}_{\mathsf{B}(n)}\\ \text{connected},E_0\ni i}}\tanh(\beta)^{|E|}\,.$$

All graphs $E \in \mathfrak{E}^{i,j}_{\mathsf{B}(n)}$ have at least $\|i-j\|_1$ edges. Proceeding as in (3.49), we derive the exponential decay once β is sufficiently small.

Exercise 3.25: Fix a shortest path $\pi = (i = i_1, i_2, \dots, i_m = j)$ from i to j and introduce $\mathscr{E}_\pi \overset{\text{def}}{=} \{\{i_k, i_{k+1}\} : 1 \le k < m\}$. For $s \in [0,1]$, set

$$J_{uv} = \begin{cases}\beta & \text{if } \{u,v\}\in\mathscr{E}_\pi,\\ s\beta & \text{otherwise.}\end{cases}$$

Denote by $\mu^{\varnothing,s}_{\mathsf{B}(n);\beta,h}$ the distribution of the Ising model in $\mathsf{B}(n) \subset \mathbb{Z}^d$ with these coupling constants and free boundary condition. Check that

$$\langle\sigma_i\sigma_j\rangle^{\varnothing,s=1}_{\mathsf{B}(n);\beta,0} = \langle\sigma_i\sigma_j\rangle^{\varnothing}_{\mathsf{B}(n);\beta,0} \quad\text{and}\quad \langle\sigma_i\sigma_j\rangle^{\varnothing,s=0}_{\mathsf{B}(n);\beta,0} = \langle\sigma_0\sigma_{\|j-i\|_1}\rangle^{d=1}_{\Lambda_{ij};\beta,0}\,.$$

Conclude, using the fact that, by GKS inequalities, $\langle\sigma_i\sigma_j\rangle^{\varnothing,s=1}_{\mathsf{B}(n);\beta,0} \ge \langle\sigma_i\sigma_j\rangle^{\varnothing,s=0}_{\mathsf{B}(n);\beta,0}$.

Exercise 3.26:

$$\begin{aligned}\mathbf{Z}^{+}_{\mathsf{B}(n);\beta,0} &= 2^{2n+1}(\cosh\beta)^{2n+2}\big(1+(\tanh\beta)^{2n+2}\big)\,,\\ \mathbf{Z}^{\varnothing}_{\mathsf{B}(n);\beta,0} &= 2^{2n+1}(\cosh\beta)^{2n}\,,\\ \mathbf{Z}^{\text{per}}_{\mathsf{B}(n);\beta,0} &= 2^{2n+1}(\cosh\beta)^{2n+1}\big(1+(\tanh\beta)^{2n+1}\big)\,.\end{aligned}$$

Exercise 3.27: Notice that, by a straightforward computation,

$$|\alpha z+1|^2 - |\alpha+z|^2 = (1-|z|^2)(1-\alpha^2)\,.$$

Since $1-\alpha^2 > 0$, all the claims can be deduced from this identity. For example, $|z| < 1$ implies $1-|z|^2 > 0$ and, therefore, $|\alpha z+1|^2 - |\alpha+z|^2 > 0$, that is, $|\varphi(z)| = |(\alpha z+1)/(\alpha+z)| > 1$.

Exercise 3.28: Since the argument of the logarithm in (3.10) is always larger than 1, only the square root can be responsible for the singularities of the pressure. But the square root vanishes at the values $h \in \mathbb{C}$ at which $e^{\beta}\cosh(h) = 2\sinh(2\beta)$. Since we know that all singularities lie on the imaginary axis, they can be expressed as $h = \mathrm{i}(\pm t + k2\pi)$, where $t = \arccos\sqrt{1-e^{-4\beta}}$, $k \in \mathbb{Z}$. Observe that, as $\beta \to \infty$, the two singularities at $\pm\mathrm{i}t$ converge from above and from below to $h = 0$. This is compatible with the fact that, in that limit, a singularity appears at $h = 0$. Namely, using (3.10),

$$\lim_{\beta\to\infty} \frac{\psi_\beta(\beta h)}{\beta} = |h| + 1\,,$$

which is non-analytic at $h = 0$.

Exercise 3.29: Duplicating the system, we can write

$$|\mathbf{Z}^{\varnothing}_{\Lambda;\beta,h}|^2 = \sum_{\omega,\omega'} e^{\beta\sum_{\{i,j\}\in\mathscr{E}_\Lambda}(\omega_i\omega_j+\omega'_i\omega'_j)+\sum_{i\in\Lambda}(h\omega_i+\bar{h}\omega'_i)}\,.$$

Define the variables $\theta_i \in \{0,\pi/2,\pi,3\pi/2\}$, $i \in \Lambda$, by $\cos\theta_i = \frac{1}{2}(\omega_i+\omega'_i)$ and $\sin\theta_i = \frac{1}{2}(\omega_i-\omega'_i)$. It is easy to check that

$$\omega_i\omega_j + \omega'_i\omega'_j = 2\cos(\theta_i-\theta_j) = e^{\mathrm{i}(\theta_i-\theta_j)} + e^{-\mathrm{i}(\theta_i-\theta_j)}\,,$$

$$h\omega_i + \bar{h}\omega'_i = 2\,\mathfrak{Re}\,h\cos(\theta_i) + 2\mathrm{i}\,\mathfrak{Im}\,h\sin(\theta_i) = (\mathfrak{Re}\,h + \mathfrak{Im}\,h)e^{\mathrm{i}\theta_i} + (\mathfrak{Re}\,h - \mathfrak{Im}\,h)e^{-\mathrm{i}\theta_i}\,.$$

Substituting these expressions yields

$$|\mathbf{Z}^{\varnothing}_{\Lambda;\beta,h}|^2 = \sum_{(\theta_i)_{i\in\Lambda}} \exp\Big\{\sum_{\substack{\mathbf{m}=(m_i)_{i\in\Lambda}\\ m_i\in\{0,1,2,3\}}} \alpha_{\mathbf{m}} e^{\mathrm{i}\sum_{i\in\Lambda} m_i\theta_i}\Big\}\,,$$

for some nonnegative coefficients $\alpha_{\mathbf{m}}$ which are nondecreasing both in $\mathfrak{Re}\,h + \mathfrak{Im}\,h$ and $\mathfrak{Re}\,h - \mathfrak{Im}\,h$. Consequently, expanding the exponential gives

$$|\mathbf{Z}^{\varnothing}_{\Lambda;\beta,h}|^2 = \sum_{(\theta_i)_{i\in\Lambda}} \sum_{\substack{\mathbf{m}=(m_i)_{i\in\Lambda}\\ m_i\in\{0,1,2,3\}}} \widehat{\alpha}_{\mathbf{m}} e^{\mathrm{i}\sum_{i\in\Lambda} m_i\theta_i}\,,$$

where the coefficients $\widehat{\alpha}_{\mathbf{m}}$ are still nonnegative and nondecreasing both in $\mathfrak{Re}\,h+\mathfrak{Im}\,h$ and $\mathfrak{Re}\,h-\mathfrak{Im}\,h$. Now, observe that

$$\sum_{(\theta_i)_{i\in\Lambda}} e^{\mathrm{i}\sum_{i\in\Lambda} m_i\theta_i} = \prod_{i\in\Lambda}\sum_{\theta_i} e^{\mathrm{i}m_i\theta_i} = \begin{cases} 4^{|\Lambda|} & \text{if } m_i = 0,\ \forall i \in \Lambda,\\ 0 & \text{otherwise.}\end{cases}$$

We deduce that $|\mathbf{Z}^{\varnothing}_{\Lambda;\beta,h}|^2 = 4^{|\Lambda|}\,\widehat{\alpha}_{(0,0,\ldots,0)}$ and, thus, that $|\mathbf{Z}^{\varnothing}_{\Lambda;\beta,h}|^2$ is nondecreasing in both $\mathfrak{Re}\,h+\mathfrak{Im}\,h$ and $\mathfrak{Re}\,h-\mathfrak{Im}\,h$. Since $\mathfrak{Re}\,h - |\mathfrak{Im}\,h| = \min(\mathfrak{Re}\,h+\mathfrak{Im}\,h, \mathfrak{Re}\,h-\mathfrak{Im}\,h)$, this proves that

$$|\mathbf{Z}^{\varnothing}_{\Lambda;\beta,h}| \geq \mathbf{Z}^{\varnothing}_{\Lambda;\beta,\mathfrak{Re}\,h-|\mathfrak{Im}\,h|} > 0\,.$$

Exercise 3.31: We write

$$\begin{aligned}(\mathbf{Z}_{\Lambda;\mathbf{K}}\mathbf{Z}_{\Lambda;\mathbf{K}'})\langle\sigma_A - \sigma'_A\rangle_{\nu_{\Lambda;\mathbf{K}}\otimes\nu_{\Lambda;\mathbf{K}'}} &= \sum_{\omega,\omega'}(\omega_A - \omega'_A)\prod_{C\subset\Lambda} e^{K_C\omega_C + K'_C\omega'_C}\\ &= \sum_{\omega''}(1-\omega''_A)\sum_{\omega}\omega_A\prod_{C\subset\Lambda} e^{(K_C+K'_C\omega''_C)\omega_C}\,,\end{aligned}$$

and we can conclude as in the proof of (3.55), since $K_C + K'_C\omega''_C \geq 0$ by assumption.

Exercise 3.35: The only delicate part is showing that, for all $E, E' \subset \mathscr{E}^{\mathrm{b}}_\Lambda$,

$$N^{\mathrm{w}}_\Lambda(E) + N^{\mathrm{w}}_\Lambda(E') \leq N^{\mathrm{w}}_\Lambda(E\cup E') + N^{\mathrm{w}}_\Lambda(E\cap E')\,. \tag{C.2}$$

In order to establish (C.2), it is sufficient to prove that

$$E' \mapsto N^{\mathrm{w}}_\Lambda(E\cup E') - N^{\mathrm{w}}_\Lambda(E') \text{ is nondecreasing.} \tag{C.3}$$

Indeed, (C.3) implies that

$$N^{\mathrm{w}}_\Lambda(E\cup E') - N^{\mathrm{w}}_\Lambda(E') \geq N^{\mathrm{w}}_\Lambda(E\cup(E'\cap E)) - N^{\mathrm{w}}_\Lambda(E'\cap E) = N^{\mathrm{w}}_\Lambda(E) - N^{\mathrm{w}}_\Lambda(E'\cap E),$$

which is equivalent to (C.2). Let $E = \{e_1, \dots, e_n\} \subset \mathscr{E}^{\mathrm{b}}_\Lambda$. Since

$$N^{\mathrm{w}}_\Lambda(E \cup E') - N^{\mathrm{w}}_\Lambda(E') = \sum_{k=1}^n \bigl\{N^{\mathrm{w}}_\Lambda(\{e_1,\dots,e_k\} \cup E') - N^{\mathrm{w}}_\Lambda(\{e_1,\dots,e_{k-1}\} \cup E')\bigr\},$$

it is sufficient to show that each summand in the right-hand side verifies (C.3). But this is immediate, since, if $e_k = \{i, j\}$,

$$N^{\mathrm{w}}_\Lambda(\{e_1,\dots,e_k\} \cup E') - N^{\mathrm{w}}_\Lambda(\{e_1,\dots,e_{k-1}\} \cup E') = \begin{cases} 0 & \text{if } i \leftrightarrow j \text{ in } \{e_1,\dots,e_{k-1}\} \cup E', \\ -1 & \text{otherwise.} \end{cases}$$

Exercise 3.37: Since $\lim_{\Lambda \uparrow \mathbb{Z}^d} \langle \sigma_0 \rangle^+_{\Lambda;\beta,0} = \langle \sigma_0 \rangle^+_{\beta,0}$, it follows from Exercise 3.34 that

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \nu^{\mathrm{FK,w}}_{\Lambda;p_\beta,2}(0 \leftrightarrow \partial^{\mathrm{ex}}\Lambda) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \langle \sigma_0 \rangle^+_{\Lambda;\beta,0} = \langle \sigma_0 \rangle^+_{\beta,0}.$$

Therefore, we only have to check that

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \nu^{\mathrm{FK,w}}_{\Lambda;p_\beta,2}(0 \leftrightarrow \partial^{\mathrm{ex}}\Lambda) = \nu^{\mathrm{FK,w}}_{p_\beta,2}(0 \leftrightarrow \infty).$$

Observe that, for all $0 \in \Delta \subset \Lambda \Subset \mathbb{Z}^d$,

$$\nu^{\mathrm{FK,w}}_{p_\beta,2}(0 \leftrightarrow \partial^{\mathrm{ex}}\Lambda) \le \nu^{\mathrm{FK,w}}_{\Lambda;p_\beta,2}(0 \leftrightarrow \partial^{\mathrm{ex}}\Lambda) \le \nu^{\mathrm{FK,w}}_{\Lambda;p_\beta,2}(0 \leftrightarrow \partial^{\mathrm{ex}}\Delta),$$

the first inequality resulting from the FKG inequality (as can be checked by the reader) and the second one from the inclusion $\{0 \leftrightarrow \partial^{\mathrm{ex}}\Lambda\} \subset \{0 \leftrightarrow \partial^{\mathrm{ex}}\Delta\}$. The desired result follows by taking the limit $\Lambda \uparrow \mathbb{Z}^d$ and then the limit $\Delta \uparrow \mathbb{Z}^d$.

Solutions of Chapter 4

Exercise 4.2: Let $\epsilon > 0$ and let ℓ be such that $\sum_{j \notin \mathsf{B}(\ell)} K(0, j) \le \epsilon$. Let $\Lambda_* \subset \Lambda$ be a parallelepiped, large enough to contain $\lceil \rho|\Lambda| \rceil$ particles, but such that if either of its sides is reduced by 1, then it becomes too small to contain those $\lceil \rho|\Lambda| \rceil$ particles. Then $|\Lambda_*| = \rho|\Lambda| + O(|\partial^{\mathrm{in}}\Lambda|)$. If η_* denotes the configuration obtained by filling densely Λ_* with particles (except possibly along its boundary), we get

$$-\mathscr{H}_{\Lambda;K}(\eta_*) = \frac{1}{2} \sum_{i \in \Lambda_*} \sum_{\substack{j \in \Lambda_* \\ j \neq i}} K(i, j) + O(|\partial^{\mathrm{in}}\Lambda|).$$

Let then Λ^-_* denote the set of vertices $i \in \Lambda_*$ for which $\mathsf{B}(\ell) + i \subset \Lambda_*$. Note that, whenever $i \in \Lambda^-_*$, we have $|\sum_{j \in \Lambda_*, j \neq i} K(i, j) - \kappa| \le \epsilon$ and thus, since $|\Lambda_* \setminus \Lambda^-_*| \le \ell |\partial^{\mathrm{in}}\Lambda|$,

$$|\mathscr{H}_{\Lambda;K}(\eta_*) - \bigl(-\tfrac{1}{2}\kappa\rho|\Lambda|\bigr)| \le \epsilon|\Lambda| + O(|\partial^{\mathrm{in}}\Lambda|).$$

We conclude that $\lim_{\Lambda \uparrow \mathbb{Z}^d} |\mathscr{H}_{\Lambda;K}(\eta_*) - \bigl(-\tfrac{1}{2}\kappa\rho|\Lambda|\bigr)| \,/\, |\Lambda| \le \epsilon$. Since ϵ is arbitrary, the claim follows.

Exercise 4.4: The proof is similar to the one for the free energy: if Λ_1 and Λ_2 are two adjacent parallelepipeds, ignoring the interactions between pairs composed of one particle in Λ_1 and one in Λ_2 gives

$$\Theta_{\Lambda_1 \cup \Lambda_2;\beta,\mu} \ge \Theta_{\Lambda_1;\beta,\mu}\Theta_{\Lambda_2;\beta,\mu}.$$

We conclude, as before, that the thermodynamic limit exists along any increasing sequence of parallelepipeds.

Exercise 4.5: Let us denote by η^1 (resp. η^0) the configuration in which $\eta_j = m_j$ for each $j \neq i$ and $\eta_i = 1$ (resp. $\eta_i = 0$). The difference

$$\{\mathscr{H}_{\Lambda;K}(\eta^1) - \mu N_\Lambda(\eta^1)\} - \{\mathscr{H}_{\Lambda;K}(\eta^0) - \mu N_\Lambda(\eta^0)\} = -\sum_{j \in \Lambda, j \neq i} K(i, j) m_j - \mu$$

belongs to the interval $(-\kappa - \mu, -\mu)$. Therefore, $\nu_{\Lambda;\beta,\mu}\bigl(\eta_i = 1 \,|\, \eta_j = m_j, \forall j \in \Lambda \setminus \{i\}\bigr)$ belongs to the interval $\bigl(1/(1 + e^{-\beta\mu}), 1/(1 + e^{-\beta(\kappa+\mu)})\bigr)$.

Exercise 4.6: Since

$$1 \le \Theta_{\Lambda;\beta,\mu} \le \sum_{N=0}^{|\Lambda|} \binom{|\Lambda|}{N} e^{\beta(\frac{\kappa}{2}+\mu)N} = \big(1+e^{\beta(\frac{\kappa}{2}+\mu)}\big)^{|\Lambda|},$$

we have $0 \le p_\beta(\mu) \le \beta^{-1}\log(1+e^{\beta(\frac{\kappa}{2}+\mu)})$. To bound $\Theta_{\Lambda;\beta,\mu}$ from below, we keep only the configuration in which $\eta_i = 1$ for each $i \in \Lambda$. This leads to $p_\beta(\mu) \ge \frac{\kappa}{2} + \mu$. The first two claims follow. The last two claims about ρ_β follow from the convexity of p_β.

Exercise 4.7: As we did earlier, let $\epsilon > 0$ and take ℓ such that $\sum_{j \notin \mathsf{B}(\ell)} K(i,j) \le \epsilon$. Then

$$\sum_{\substack{i\in\Lambda':\\ i+\mathsf{B}(\ell)\subset\Lambda'}} \sum_{j\in\Lambda''} K(i,j) \le \epsilon|\Lambda'|$$

and, since

$$\sum_{\substack{i\in\Lambda':\\ i+\mathsf{B}(\ell)\not\subset\Lambda'}} \sum_{j\in\Lambda''} K(i,j) \le \kappa\ell|\partial^{\mathrm{in}}\Lambda'|,$$

the conclusion follows easily.

Exercise 4.9: Consider the gas branch: $\rho < \rho_g$. By the strict convexity of the pressure and the equivalence of ensembles, there exists a unique $\mu(\rho)$ such that

$$f_\beta(\rho) = \mu(\rho)\rho - p_\beta(\mu(\rho)).$$

Since $\rho < \rho_g$, we have $\mu(\rho) < \mu_*$ and $\mu(\rho)$ is solution of $\rho = \frac{\partial p_\beta}{\partial \mu}$. Then, we use (i) the analyticity of the pressure, which implies in particular that its first and second derivatives exist, outside μ_*, (ii) the fact, proved in Theorem 4.12, that $\frac{\partial^2 p_{\Lambda;\beta}}{\partial\mu^2} \ge \beta c(1-c) > 0$, which implies that $\frac{\partial^2 p_\beta}{\partial \mu^2} > 0$ whenever it exists, (iii) the implicit function theorem (Section B.28), to conclude that $\mu(\cdot)$ is also analytic in a neighborhood of ρ. Since the composition of analytic maps is also analytic, this shows that $f_\beta(\cdot)$ is analytic in a neighborhood of ρ.

Exercise 4.13: We only consider the case $d = 1$; the general case can be treated in the same way. Let us identify each $\Lambda^{(\alpha)} \subset \mathbb{Z}$ with the interval $J^{(\alpha)} = \{x \in \mathbb{R} : \mathrm{dist}(x, \Lambda^{(\alpha)}) \le \frac{1}{2}\}$, whose length equals $|J^{(\alpha)}| = \ell$, and let $J_\gamma^{(\alpha)} \stackrel{\text{def}}{=} \{\gamma x : x \in J^{(\alpha)}\}$. We have (up to terms that vanish in the van der Waals limit)

$$\sum_{\alpha'>1} |J_\gamma^{(\alpha')}| \inf_{x\in J_\gamma^{(\alpha')}} \varphi(x) \le |\Lambda^{(1)}| \sum_{\alpha'>1} \underline{K}_\gamma(1,\alpha') \le |\Lambda^{(1)}| \sum_{\alpha'>1} \overline{K}_\gamma(1,\alpha') \le \sum_{\alpha'>1} |J_\gamma^{(\alpha')}| \sup_{x\in J_\gamma^{(\alpha')}} \varphi(x). \tag{C.4}$$

The conclusion follows, since the first and last sums of this last display are Darboux sums that converge to $\int \varphi(x)\,\mathrm{d}x$ as $|J_\gamma^{(\alpha')}| = \gamma\ell \downarrow 0$.

Exercise 4.14: Let $N = \lceil \rho|\Lambda| \rceil$. Since $\mathcal{N}(N;M)$ counts the number of ways N identical balls can be distributed in M boxes, with at most $|\Lambda^{(1)}|$ balls per box, this number is obviously smaller than the number of ways of putting N identical balls in M boxes, without restrictions on the number of balls per box. The latter equals

$$\binom{N+M-1}{M-1}.$$

Since

$$M = \frac{|\Lambda|}{|\Lambda^{(1)}|} \stackrel{\text{def}}{=} \delta_{\Lambda,\ell} N,$$

and $\lim_{\Lambda\Uparrow\mathbb{Z}^d} \delta_{\Lambda,\ell} = \frac{1}{\rho|\Lambda^{(1)}|} \stackrel{\text{def}}{=} \delta_\ell$, Stirling's formula gives

$$\lim_{\ell\to\infty}\lim_{N\to\infty} \frac{1}{N}\log\binom{N+M-1}{M-1} = \lim_{\ell\to\infty} \big\{(1+\delta_\ell)\log(1+\delta_\ell) + \delta_\ell\log\delta_\ell\big\} = 0.$$

Exercise 4.15: Let $\epsilon > 0$ and n be large enough to ensure that $f(x) - \epsilon \le f_n(x) \le f(x) + \epsilon$ for all $x \in [a,b]$. Since $\mathrm{CE}\, g \le \mathrm{CE}\, h$ whenever $g \le h$, this implies $\mathrm{CE}\, f(x) - \epsilon \le \mathrm{CE}\, f_n(x) \le \mathrm{CE}\, f(x) + \epsilon$ for all $x \in [a,b]$, which gives the result.

Exercise 4.16: Let $a_n \stackrel{\text{def}}{=} e^{-\beta 2dn^{(d-1)/d}}$. For all compact $K \subset H^+$,

$$\sup_{h\in K}\Big|\sum_{n\geq 1} a_n e^{-hn} - \sum_{n=1}^{N} a_n e^{-hn}\Big| \leq \sup_{h\in K}\sum_{n>N} a_n e^{-\mathfrak{Re}\, hn} \leq \sum_{n>N} a_n e^{-x_0 n},$$

where $x_0 \stackrel{\text{def}}{=} \inf\{\mathfrak{Re}\, h : h \in K\} > 0$, and this last series goes to zero when $N \to \infty$. This implies that the series defining ψ_β converges uniformly on compacts. Since $h \mapsto e^{-hn}$ is analytic on H^+, Theorem B.27 implies that ψ_β is analytic on H^+. Moreover, it can be differentiated term by term an arbitrary number of times, yielding, when $h \in \mathbb{R}_{>0}$,

$$\Big|\lim_{h\downarrow 0}\frac{\mathrm{d}^k \psi_\beta}{\mathrm{d}h^k}\Big| = \Big|(-1)^k \lim_{h\downarrow 0}\sum_{n\geq 1} n^k a_n e^{-hn}\Big| = \sum_{n\geq 1} n^k a_n.$$

A lower bound on the sum is obtained by keeping only its largest term. Notice that $x \mapsto x^k e^{-2d\beta x^{(d-1)/d}}$ is maximal at

$$x_* = x_*(k,\beta,d) \stackrel{\text{def}}{=} \Big(\frac{k}{2(d-1)\beta}\Big)^{d/(d-1)}.$$

Keeping the term $n_* \stackrel{\text{def}}{=} \lfloor x_* \rfloor$, reorganizing the terms and using Stirling's formula, we get

$$\sum_{n\geq 1} n^k a_n \geq n_*^k a_{n_*} \geq C_-^k k!^{d/(d-1)},$$

for some $C_- = C_-(\beta,d) > 0$. The reader may check that an upper bound of the same kind holds, with a constant $C_+ < \infty$.

Solutions of Chapter 5

Exercise 5.1: In (5.9), just distinguish the case $k = 1$ from $k \geq 2$.

Exercise 5.2: We proceed by induction. The case $n = 1$ is trivial. Now if the claim holds for n, it can be shown to hold for $n+1$ too, by writing

$$\Big(\prod_{k=1}^{n+1}(1+\alpha_k)\Big) - 1 = (1+\alpha_{n+1})\Big(\prod_{k=1}^{n}(1+\alpha_k) - 1\Big) + \alpha_{n+1}.$$

Exercise 5.5: When using more general boundary conditions, the same sets S_i can be used, but the surface term $e^{-2\beta|\partial_e S_i|}$ in their weights might have to be modified if $S_i \cap \partial^{\text{in}}\Lambda \neq \varnothing$. The condition (5.26) can nevertheless be seen to hold since the surface term was ignored in our analysis. Then, the contributions to $\log \Xi^{\text{LF}}_{\Lambda;\beta,h}$ coming from clusters containing sets S_i that intersect $\partial^{\text{in}}\Lambda$ is a surface contribution that vanishes in the thermodynamic limit, yielding the same expression for the pressure.

Exercise 5.6: First,

$$\begin{aligned}\phi(\phi^{-1}(z)) &= \sum_{n\geq 1}\tilde a_n\Big(\sum_{k\geq 1}c_k z^k\Big)^n = \sum_{n\geq 1}\tilde a_n\sum_{k_1,\dots,k_n\geq 1}\prod_{i=1}^{n}c_{k_i}z^{k_i} = \sum_{n\geq 1}\tilde a_n\sum_{m\geq n}z^m\sum_{\substack{k_1,\dots,k_n\geq 1\\ k_1+\dots+k_n=m}}\prod_{i=1}^{n}c_{k_i}\\ &= \sum_{m\geq 1}\Big\{\sum_{n=1}^{m}\tilde a_n\sum_{\substack{k_1,\dots,k_n\geq 1\\ k_1+\dots+k_n=m}}\prod_{i=1}^{n}c_{k_i}\Big\}z^m.\end{aligned}$$

However, since $\phi(\phi^{-1}(z)) = z$ by definition, we conclude that the coefficient of z in the last sum, which is $\tilde a_1 c_1$, must be equal to 1, while the coefficient of z^m, $m \geq 2$ must vanish. The claim follows.

Exercise 5.7: The procedure is identical to that used in the proof of Lemma 5.10. The expansion, up to the second nontrivial order, is given by

$$\psi_\beta(0) - d\log(\cosh\beta) - \log 2 = \tfrac{1}{2}d(d-1)(\tanh\beta)^4 + \tfrac{1}{3}d(d-1)(8d-13)(\tanh\beta)^6 + O(\tanh\beta)^8. \quad \text{(C.5)}$$

These two terms correspond, respectively, to sets of 4 and 6 edges. In the terminology used in the proof of Lemma 5.10, one has: $A = 4, B = 1, C = 1$ for the first term and $A = 6, B = 1, C = 1$ for the second. Therefore, the only thing left to do in order to derive (C.5) is to determine the number of such sets containing the origin, which is a purely combinatorial task left to the reader.

Exercise 5.8: First, the high-temperature representation (5.38) needs to be adapted to the presence of a magnetic field. Indeed, (3.44) must be replaced by

$$\sum_{\omega_i=\pm1} \omega_i^{I(i,E)} e^{h\omega_i} = \begin{cases} 2\cosh(h) & \text{if } I(i,E) \text{ is even,} \\ 2\sinh(h) & \text{if } I(i,E) \text{ is odd.} \end{cases}$$

Then, the class of sets E that contribute to the partition function is larger (the incidence numbers $I(i,E)$ are allowed to be odd), giving

$$\mathbf{Z}^{\varnothing}_{\Lambda;\beta,0} = (2\cosh h)^{|\Lambda|}(\cosh\beta)^{|\mathscr{E}_\Lambda|} \sum_{E\subset\mathscr{E}_\Lambda} (\tanh\beta)^{|E|}(\tanh h)^{|\partial E|},$$

where $\partial E \stackrel{\text{def}}{=} \{i \in \mathbb{Z}^d : I(i,E) \text{ is odd}\}$. Notice that $|\tanh h| \le 1$ when $|h|$ is small enough. Then, the weights of the components are bounded by the same weight as the one used above, $(\tanh\beta)^{|E|}$, and the rest of the analysis is essentially the same (keeping in mind that the class of objects is larger).

Exercise 5.9: It is convenient to use the notion of interior of a contour depicted in Figure 5.2. Then, given a collection $\Gamma' = \{\gamma_1,\dots,\gamma_n\} \subset \Gamma_\Lambda$ of pairwise disjoint contours in Λ, consider the configuration

$$\omega_i = (-1)^{\#\{\gamma\in\Gamma' : i\in\operatorname{Int}\gamma\}}.$$

Since $\Lambda^c \cap \bigcup_{\gamma\in\Gamma'} \operatorname{Int}\gamma = \varnothing$ when Λ is c-connected, it follows that $\omega \in \Omega^+_\Lambda$. It is also easy to verify that $\Gamma'(\omega) = \Gamma'$. This shows that the collection is admissible.

When Λ is not c-connected, this implication is not true anymore. For example, consider the set $\Lambda = \mathrm{B}(2n) \setminus \mathrm{B}(n)$. Because of the + boundary condition outside $\mathrm{B}(2n)$ *and inside* $\mathrm{B}(n)$, in any configuration $\omega \in \Omega^+_\Lambda$, the number of contours $\gamma \in \Gamma'(\omega)$ such that $\mathrm{B}(n) \subset \operatorname{Int}\gamma$ *has to be even.* Observe that the latter is a *global* constraint on the family of contours.

Exercise 5.10: See Exercise 3.20.

Exercise 5.11: As was done earlier, one can write for example

$$\sum_{\substack{X\sim A:\\ \overline{X}\subset\Lambda}} \Psi^A_\beta(X) = \sum_{X\sim A} \Psi^A_\beta(X) - \sum_{\substack{X\sim A:\\ \overline{X}\not\subset\Lambda}} \Psi^A_\beta(X).$$

The clusters that satisfy at the same time $X \sim A$ and $\overline{X} \not\subset \Lambda$ have a support of size at least $d(A,\Lambda^c)$. As before, one can show that their contribution vanishes when $\Lambda \uparrow \mathbb{Z}^d$.

Solutions of Chapter 6

Exercise 6.2: Clearly, the family of subsets $\Lambda \Subset \mathbb{Z}^d$ is at most countable. Since each $\mathscr{C}(\Lambda)$ is finite and since a countable union of finite sets is countable, $\mathscr{C}_S$ is countable. To show that $\mathscr{C}_S$ is an algebra, observe that, whenever $A \in \mathscr{C}_S$, there exists some $\Lambda \Subset S$ and some $B \in \Omega_\Lambda$ such that $A = \Pi_\Lambda^{-1}(B)$. But, since $A^c = \Pi_\Lambda^{-1}(B^c)$, we also have $A^c \in \mathscr{C}_S$. Moreover, if $A, A' \in \mathscr{C}_S$, of the form $A = \Pi_\Lambda^{-1}(B)$, $A' = \Pi_{\Lambda'}^{-1}(B')$, then one can find some $\Lambda'' \Subset S$ containing Λ and Λ' (for example $\Lambda'' = \Lambda \cup \Lambda'$), use the hint to express $A = \Pi_{\Lambda''}^{-1}(B_1)$, $A' = \Pi_{\Lambda''}^{-1}(B_2)$, and write $A \cup A' = \Pi_{\Lambda''}^{-1}(B_1 \cup B_2)$. This implies $A \cup A' \in \mathscr{C}_S$.

Exercise 6.4: For example, consider $\Delta = \{0,1\} \times \{0\}$ and $\Lambda = \{0,1\}^2$. It then immediately follows from the high-temperature representation that

$$\langle \sigma_{(0,0)}\sigma_{(1,0)}\rangle^{\varnothing}_\Delta = \tanh\beta,$$

while

$$\langle \sigma_{(0,0)}\sigma_{(1,0)}\rangle^{\varnothing}_\Lambda = (\tanh\beta + \tanh^3\beta)/(1+\tanh^4\beta).$$

Since these two expressions do not coincide when $\beta > 0$, it follows that $\mu^{\varnothing}_\Lambda \circ (\Pi^\Lambda_\Delta)^{-1} \neq \mu^{\varnothing}_\Delta$.

Exercise 6.5: By definition,

$$\pi_\Lambda\pi_\Delta(A\,|\,\eta) = \sum_{\omega_\Lambda\in\Omega_\Lambda} \pi_\Lambda(\omega_\Lambda\,|\,\eta)\,\pi_\Delta(A\,|\,\omega_\Lambda\eta_{\Lambda^c}),$$

which only depends on η_{Λ^c}. This also immediately implies that $\pi_\Lambda\pi_\Delta$ is proper.

Exercise 6.6: If $f = \mathbf{1}_A$,

$$\mu\pi_\Lambda(\mathbf{1}_A) = \int \pi_\Lambda(A\,|\,\omega)\mu(\mathrm{d}\omega) = \int \pi_\Lambda \mathbf{1}_A(\omega)\mu(\mathrm{d}\omega) = \mu(\pi_\Lambda \mathbf{1}_A)\,.$$

For the general case, just approximate f by a sequence of simple functions of the form $\sum_i a_i \mathbf{1}_{A_i}$.

Exercise 6.7: The proof of the first claim is left to the reader. For the second, observe that, for any $A \in \mathscr{F}$,

$$\rho\pi_\Lambda(A) = \int \pi_\Lambda(A\,|\,\omega)\rho(\mathrm{d}\omega) = \int \mathbf{1}_A(\tau_\Lambda \omega_{\Lambda^c})\rho^\Lambda(\mathrm{d}\tau_\Lambda)\rho(\mathrm{d}\omega_{\Lambda^c}) = \rho(A)\,,$$

so that $\rho \in \mathscr{G}(\pi)$. To prove uniqueness, let $\mu \in \mathscr{G}(\pi)$ and consider an arbitrary cylinder $C = \Pi_\Lambda^{-1}(E)$ with base Λ. Then, one must have

$$\mu(C) = \mu\pi_\Lambda(C) = \int \pi_\Lambda(\Pi_\Lambda^{-1}(E)\,|\,\omega)\mu(\mathrm{d}\omega) = \int \rho^\Lambda(E)\mu(\mathrm{d}\omega) = \rho^\Lambda(E) = \rho(C)\,,$$

and therefore μ must coincide with ρ on all cylinders, which implies that $\mu = \rho$.

Exercise 6.8: To show absolute summability, it suffices to prove that

$$\sum_{i \in \mathbb{Z}^d \setminus \{0\}} J_{0i} = \sum_{r \ge 1} |\partial^{\mathrm{in}} \mathsf{B}(r)| r^{-\alpha}$$

is bounded. Since $|\partial^{\mathrm{in}}\mathsf{B}(r)|$ is of order r^{d-1}, the potential is absolutely summable if and only if $\alpha > d$.

Exercise 6.9: Clearly, $d(\omega,\eta) \ge 0$ with equality if and only if $\omega = \eta$. Since $\mathbf{1}_{\{\omega_i \ne \eta_i\}} \le \mathbf{1}_{\{\omega_i \ne \tau_i\}} + \mathbf{1}_{\{\tau_i \ne \eta_i\}}$ for all $i \in \mathbb{Z}^d$, we have $d(\omega,\eta) \le d(\omega,\tau) + d(\tau,\eta)$, so $d(\cdot,\cdot)$ is a distance.

Notice that if $\omega_{\mathsf{B}(r)} = \eta_{\mathsf{B}(r)}$, then $d(\omega,\eta) \le 2d\sum_{k\ge r} k^{d-1}2^{-k} \stackrel{\text{def}}{=} \epsilon(r)$, with $\epsilon(r) \to 0$ as $r \to \infty$.

Suppose that $\omega^{(n)} \to \omega^*$. In this case, for any $r \ge 1$, there exists n_0 such that $\omega^{(n)}_{\mathsf{B}(r)} = \omega^*_{\mathsf{B}(r)}$ for all $n \ge n_0$. This implies that $d(\omega^{(n)},\omega^*) \to 0$ as $n \to \infty$.

Assume now that $d(\omega^{(n)},\omega^*) \to 0$. In that case, for any $k \ge 1$, one can find n_1 such that $d(\omega^{(n)},\omega^*) < 2^{-k}$ for all $n \ge n_1$. But this implies that $\mathbf{1}_{\{\omega_i^{(n)} \ne \omega_i^*\}} = 0$ each time $\|i\|_\infty \le k$. This implies that $\omega^{(n)}_{\mathsf{B}(k)} = \omega^*_{\mathsf{B}(k)}$ for all $n \ge n_1$. Therefore, $\omega^{(n)} \to \omega^*$.

Exercise 6.10: Let $C = \Pi_\Lambda^{-1}(A)$ be a cylinder. If $\omega \in C$, then any configuration ω' which coincides with ω on Λ^{c} is also in C, which implies that C is open. Now let $G \subset \Omega$ be open. For each $\omega \in G$, one can find a cylinder C_ω such that $G \supset C_\omega \ni \omega$. Therefore, $G = \bigcup_{\omega\in G} C_\omega$. But, since $\mathscr{C}$ is countable (Exercise 6.2), that union is countable. This shows that $G \in \mathscr{F}$.

Exercise 6.11: Assume $f : \Omega \to \mathbb{R}$ is continuous but not uniformly continuous. There exist some $\epsilon > 0$ and two sequences $(\omega^{(n)})_{n\ge1}$, $(\eta^{(n)})_{n\ge1} \subset \Omega$ such that $d(\omega^{(n)},\eta^{(n)}) \to 0$ and $|f(\omega^{(n)}) - f(\eta^{(n)})| \ge \epsilon$ for all n. By Proposition 6.20, there exists a subsequence $(\omega^{(n_k)})_{k\ge1}$ and some ω_* such that $\omega^{(n_k)} \to \omega_*$. This implies also $d(\eta^{(n_k)},\omega_*) \le d(\eta^{(n_k)},\omega^{(n_k)}) + d(\omega^{(n_k)},\omega_*) \to 0$. But, since $\epsilon \le |f(\omega^{(n_k)}) - f(\omega_*)| + |f(\eta^{(n_k)}) - f(\omega_*)|$, at least one of the sequences $(|f(\omega^{(n_k)}) - f(\omega_*)|)_{k\ge1}$, $(|f(\eta^{(n_k)}) - f(\omega_*)|)_{k\ge1}$ cannot converge to zero. This implies that f is not continuous at ω_*, a contradiction. The other two facts are proved in a similar way.

Exercise 6.12: 1⇒2 is immediate since local functions can be expressed as finite linear combinations of indicators of cylinders.

2⇒3: Let $f \in C(\Omega)$. Fix $\epsilon > 0$, and let g be a local function such that $\|g - f\|_\infty \le \epsilon$. Then $|\mu_n(f) - \mu(f)| \le |\mu_n(g) - \mu(g)| + 2\epsilon$, and thus $\limsup_n |\mu_n(f) - \mu(f)| \le 2\epsilon$. This implies that $\mu_n(f) \to \mu(f)$.

3⇒1 is immediate, since for each $C \in \mathscr{C}$, $f = \mathbf{1}_C$ is continuous.

1⇒4: Let $m_n(k) \stackrel{\text{def}}{=} \max_{C\in\mathscr{C}(\mathsf{B}(k))} |\mu_n(C) - \mu(C)|$. Notice that $m_n(k) \le 1$. Fix $\epsilon > 0$. Let $k_0 \stackrel{\text{def}}{=} \epsilon^{-1}$, Clearly, as $n \to \infty$,

$$\max_{1\le k\le k_0} m_n(k) \to 0\,.$$

On the other hand, if $k > k_0$, then $\frac{m_n(k)}{k} \le \frac{1}{k} < \epsilon$. Therefore,

$$\limsup_{n\to\infty} \rho(\mu_n,\mu) = \limsup_{n\to\infty} \sup_{k\ge1} \frac{m_n(k)}{k} \le \epsilon\,.$$

4⇒1: Let $C \in \mathscr{C}$ and fix some $\epsilon > 0$. Let k be large enough so that $C \in \mathscr{C}(\mathsf{B}(k))$, and let n_0 be such that $\rho(\mu_n,\mu) \le \frac{\epsilon}{k}$ for all $n \ge n_0$. For those values of n, we also have

$$|\mu_n(C) - \mu(C)| \le \max_{C'\in\mathscr{C}(\mathsf{B}(k))} |\mu_n(C') - \mu(C')| \le \epsilon\,.$$

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Exercise 6.13: Writing $\pi_\Lambda f(\omega) = \sum_{\tau_\Lambda} f(\tau_\Lambda \omega_{\Lambda^c})\pi_\Lambda(\tau_\Lambda\,|\,\omega)$ makes the statement obvious.

Exercise 6.14: The construction of $\mu^{\varnothing}_{\beta,h}$, using Exercise 3.16 and Theorem 6.5, is straightforward. We check that $\mu^{\varnothing}_{\beta,h} \in \mathscr{G}(\beta,h)$. Let f be some local function and take $\Delta \Subset \mathbb{Z}^d$ sufficiently large to contain the support of f. Lemma 6.7 (whose proof extends verbatim to the case of free boundary condition) implies that, for any $\Lambda \Subset \mathbb{Z}^d$ containing Δ, $\langle f\rangle^{\varnothing}_{\Lambda;\beta,h} = \langle\langle f\rangle^{\cdot}_{\Delta;\beta,h}\rangle^{\varnothing}_{\Lambda;\beta,h}$. Again, since $\omega \mapsto \langle f\rangle^{\omega}_{\Delta;\beta,h}$ is local, one can let $\Lambda \uparrow \mathbb{Z}^d$, and obtain $\langle f\rangle^{\varnothing}_{\beta,h} = \langle\langle f\rangle^{\cdot}_{\Delta;\beta,h}\rangle^{\varnothing}_{\beta,h}$, from which the claim follows.

Exercise 6.15: Assume there exists $\mu \in \mathscr{G}(\pi)$. Notice that

$$\mu(N^+ = 0) = \mu(\{\eta^-\}) = \mu\pi_\Lambda(\{\eta^-\}) = \int \pi_\Lambda(\{\eta^-\}\,|\,\omega)\mu(\mathrm{d}\omega) = 0\,.$$

Then, $\mu(N^+ = 1) = \sum_{i\in\mathbb{Z}^d} \mu(\{\eta^{-,i}\})$. However, for all $\Lambda \Subset \mathbb{Z}^d$ containing i,

$$\mu(\{\eta^{-,i}\}) = \mu\pi_\Lambda(\{\eta^{-,i}\}) \le \frac{1}{|\Lambda|}\,,$$

so that $\mu(\{\eta^{-,i}\}) = 0$. We conclude that $\mu(N^+ = 1) = 0$. Finally, $\mu(N^+ \ge 2) \le \sum_{i\neq j} \mu(\{\omega_i = \omega_j = +1\}) = 0$, since $\mu(\{\omega_i = \omega_j = +1\}) = \mu\pi_{\{i,j\}}(\{\omega_i = \omega_j = +1\}) = 0$ for all $i \neq j$. All this implies that $\mu(\Omega) = 0$, which contradicts the assumption that μ is a probability measure.

Exercise 6.16: For example,

$$f(\omega) = \limsup_{n\to\infty} \frac{1}{|\mathsf{B}(n)|} \sum_{i\in\mathsf{B}(n)} \omega_i$$

has $\Delta(f) = 0$, but it is not continuous (see Exercise 6.23). In dimension $d = 2$, take $\epsilon > 0$ and consider, for example,

$$g(\omega) = \sum_{k\ge1} \frac{1}{k^{1+\epsilon}}\Big(\max_{j\in\mathsf{B}(k)} \omega_j - \min_{j\in\mathsf{B}(k)} \omega_j\Big).$$

Then $g \in C(\Omega)$, but $\Delta(g) = \infty$.

Exercise 6.17: By the FKG inequality, for any $\omega \in \Omega$,

$$1 \ge \mu^{\omega}_{\{i\};\beta,h}(\sigma_i = 1) \ge \mu^{-}_{\{i\};\beta,h}(\sigma_i = 1) = \big\{1 + e^{-2h+4d\beta}\big\}^{-1}.$$

Therefore,

$$\sum_{\omega_i=\pm1} |\pi_i(\omega_i\,|\,\omega) - \pi_i(\omega_i\,|\,\omega')| \le \frac{2}{1+e^{2h-4d\beta}}\,.$$

Since the expression in the left-hand side is actually equal to 0 when $\omega_j = \omega'_j$ for all $j \sim i$, we obtain

$$c(\pi) \le \frac{4d}{1+e^{2h-4d\beta}}\,,$$

which is indeed smaller than 1 as soon as $h > 2d\beta + \frac{1}{2}\log(4d-1)$.

Exercise 6.19: Clearly, $c_{ij}(\pi) = 0$ whenever $j \not\sim i$. Let $j \sim i$ and consider two configurations ω,η such that $\omega_k = \eta_k$ for all $k \neq j$. When $s \in \{0,\dots,q-1\} \setminus \{\omega_j,\eta_j\}$, $\pi_i(\sigma_i = s\,|\,\eta) = \pi_i(\sigma_i = s\,|\,\omega)$. Let us therefore assume that $\omega_j = s \neq \eta_j$. In this case,

$$\pi_i(\sigma_i = s\,|\,\eta) - \pi_i(\sigma_i = s\,|\,\omega) = \frac{e^{-\beta\#\{k\sim i:\eta_k=s\}}}{\mathbf{Z}^{\eta}_{\{i\}}}\Big\{1 - \frac{\mathbf{Z}^{\eta}_{\{i\}}}{\mathbf{Z}^{\omega}_{\{i\}}}e^{-\beta}\Big\}.$$

Now, observe that

$$\frac{\mathbf{Z}^{\eta}_{\{i\}}}{\mathbf{Z}^{\omega}_{\{i\}}}e^{-\beta} = \Big\langle e^{-\beta(\delta_{\sigma_i,\eta_j} - \delta_{\sigma_i,\omega_j}+1)}\Big\rangle^{\omega}_{\{i\}} \in [e^{-2\beta},1]\,.$$

Therefore, $|\pi_i(\sigma_i = s\,|\,\eta) - \pi_i(\sigma_i = s\,|\,\omega)| \le 1/\mathbf{Z}^{\eta}_{\{i\}} \le 1/(q-2d)$. This yields $c_{ij}(\pi) \le 2/(q-2d)$ and thus $c(\pi) \le 4d/(q-2d)$, which is indeed smaller than 1 as soon as $q > 6d$.

Exercise 6.20: We have seen in Exercise 6.8 that $\alpha > d$ is necessary for the potential to be absolutely summable. Then,

$$b = \sup_{i\in\mathbb{Z}^d} \sum_{B\ni i} (|B|-1)\|\Phi_B\|_\infty = \sum_{k\ge1} \frac{1}{k^\alpha}\#\big\{j \in \mathbb{Z}^d : j \neq 0, \|j\|_\infty = k\big\} \le 2d \sum_{k\ge1} \frac{1}{k^{1+(\alpha-d)}} \stackrel{\text{def}}{=} b_0(\alpha,d)\,.$$

For all $\alpha > d$, we have uniqueness as soon as $\beta < \beta_0 \stackrel{\text{def}}{=} \frac{1}{2b_0}$. Observe that $\beta_0 \downarrow 0$ when $\alpha \downarrow d$.

Exercise 6.21: Using the invariance of π_Λ in the second equality,

$$(\theta_j\mu)\pi_\Lambda(A) = \int \pi_\Lambda(A\,|\,\theta_j\omega)\mu(\mathrm{d}\omega) = \int \pi_{\theta_j^{-1}\Lambda}(\theta_j^{-1}A\,|\,\omega)\mu(\mathrm{d}\omega) = \mu\pi_{\theta_j^{-1}\Lambda}(\theta_j^{-1}A) = \mu(\theta_j^{-1}A) = \theta_j\mu(A)\,.$$

Exercise 6.22: Let $\mu \in \mathscr{G}(\pi)$ (which is not empty by Theorem 6.26) By Theorem 6.24, we can consider a subsequence along which μ_n converges: $\mu_{n_k} \Rightarrow \mu_*$. To see that μ_* is translation invariant, $\theta_i\mu_* = \mu_*$ for all $i \in \mathbb{Z}^d$, it suffices to observe that, for any local function f,

$$\Big|\sum_{j\in\mathsf{B}(n_k)} \theta_{j+i}\mu(f) - \sum_{j\in\mathsf{B}(n_k)} \theta_j\mu(f)\Big| \le C\|f\|_\infty\|i\|_\infty^d\,|\partial^{\mathrm{ex}}\mathsf{B}(n_k)|\,.$$

Then, $\mu \in \mathscr{G}(\pi)$ and Exercise 6.21 imply that $\mu_{n_k} \in \mathscr{G}(\pi)$ for all k. Since $\mathscr{G}(\pi)$ is closed (Lemma 6.27), this implies that $\mu_* \in \mathscr{G}(\pi)$.

Exercise 6.23: Let $\omega \in \Omega$. Since f is non-constant, there exists ω' such that $f(\omega) \neq f(\omega')$. Let $\omega^{(n)} = \omega_{\mathsf{B}(n)}\omega'_{\mathsf{B}(n)^c}$. Then $\omega^{(n)} \to \omega$. However, $f(\omega^{(n)}) = f(\omega')$ for all n and therefore $f(\omega^{(n)}) \not\to f(\omega)$.

Exercise 6.24: Let g be $\mathscr{F}_{\Lambda^c}$-measurable. We first assume that g is a finite linear combination $\sum_j \alpha_j \mathbf{1}_{A_j}$, with $A_j \in \mathscr{F}_{\Lambda^c}$. On the one hand,

$$(g\nu)\pi_\Lambda(A) = \int \pi_\Lambda(A\,|\,\omega)g(\omega)\nu(\mathrm{d}\omega) = \sum_j \alpha_j \int_{A_j} \pi_\Lambda(A\,|\,\omega)\nu(\mathrm{d}\omega)\,.$$

On the other hand,

$$g(\nu\pi_\Lambda)(A) = \int_A g(\omega')\nu\pi_\Lambda(\mathrm{d}\omega') = \sum_j \alpha_j\nu\pi_\Lambda(A\cap A_j) = \sum_j \alpha_j \int \pi_\Lambda(A\cap A_j\,|\,\omega)\nu(\mathrm{d}\omega)\,.$$

By Lemma 6.13, we have $\pi_\Lambda(A\cap A_j\,|\,\omega) = \pi_\Lambda(A\,|\,\omega)\mathbf{1}_{A_j}(\omega)$. This implies that $(g\nu)\pi_\Lambda = g(\nu\pi_\Lambda)$. In the general case, it suffices to consider a sequence of approximations g_n (each being a finite linear combination of the above type) with $\|g_n - g\|_\infty \to 0$, and use twice dominated convergence to compute

$$(g\nu)\pi_\Lambda(A) = \lim_{n\to\infty}(g_n\nu)\pi_\Lambda(A) = \lim_{n\to\infty} g_n(\nu\pi_\Lambda)(A) = g(\nu\pi_\Lambda)(A)\,.$$

The reader can find counterexamples that show that (6.63) does not hold in general when g is not $\mathscr{F}_{\Lambda^c}$-measurable.

Exercise 6.25: Since $\mathbf{1}_A = (1+\sigma_0)/2$ and $\mathbf{1}_{B_i} = (1+\sigma_i)/2$, $\mu(A\cap B_i)-\mu(A)\mu(B_i) = \frac{1}{4}\big(\mu(\sigma_0\sigma_i)-\mu(\sigma_0)\mu(\sigma_i)\big)$. By symmetry, $\mu^+_{\beta,0}(\sigma_0\sigma_i) = \mu^-_{\beta,0}(\sigma_0\sigma_i)$ and $\mu(\sigma_0) = (2\lambda-1)\mu^+_{\beta,0}(\sigma_0)$, $\mu(\sigma_i) = (2\lambda-1)\mu^+_{\beta,0}(\sigma_i)$. By the FKG inequality, $\mu^+_{\beta,0}(\sigma_0\sigma_i) \ge \mu^+_{\beta,0}(\sigma_0)\mu^+_{\beta,0}(\sigma_i)$. We therefore conclude that $\mu(\sigma_0\sigma_i) - \mu(\sigma_0)\mu(\sigma_i) \ge \big(1-(2\lambda-1)^2\big)\big(\mu^+_{\beta,0}(\sigma_0)\big)^2$, which is positive for all $\beta > \beta_c(2)$ and all $\lambda \in (0,1)$.

Exercise 6.26: Extremality of $\mu^+_{\beta,h}$ implies that, for any $\epsilon > 0$, there exists r such that $0 \le \langle\sigma_i;\sigma_j\rangle^+_{\beta,h} \le \epsilon$ for all $j \not\in i+\mathsf{B}(r)$. Therefore,

$$\begin{aligned}\mathrm{Var}_{\mu^+_{\beta,h}}(m_{\mathsf{B}(n)}) &= |\mathsf{B}(n)|^{-2}\sum_{i,j\in\mathsf{B}(n)}\langle\sigma_i;\sigma_j\rangle^+_{\beta,h}\\ &\le |\mathsf{B}(n)|^{-2}\sum_{i\in\mathsf{B}(n)}\Big\{\sum_{j\in i+\mathsf{B}(r)}\underbrace{\langle\sigma_i;\sigma_j\rangle^+_{\beta,h}}_{\le 1} + \sum_{\substack{j\in\mathsf{B}(n)\\ j\not\in i+\mathsf{B}(r)}}\underbrace{\langle\sigma_i;\sigma_j\rangle^+_{\beta,h}}_{\le\epsilon}\Big\} \le \frac{|\mathsf{B}(r)|}{|\mathsf{B}(n)|}+\epsilon\,.\end{aligned}$$

Letting $n\to\infty$ and then $\epsilon\to 0$ shows that $\lim_{n\to\infty}\mathrm{Var}_{\mu^+_{\beta,h}}(m_{\mathsf{B}(n)}) = 0$. The conclusion follows from Chebyshev's inequality (B.18).

Exercise 6.27: On the one hand, if ν is trivial on $\mathscr{T}_\infty$, then

$$\int_A \nu(B)\nu(\mathrm{d}\omega) = \nu(B)\nu(A) = \nu(A\cap B)\,,$$

for all $B \in \mathscr{F}$ and all $A \in \mathscr{T}_\infty$, since $\nu(A)$ is either 1 or 0. This shows that $\nu(B) = \nu(B\,|\,\mathscr{T}_\infty)$ ν-almost surely. On the other hand, if the latter condition holds, then, for any $A \in \mathscr{T}_\infty$,

$$\nu(A) = \int_A \mathbf{1}_A\mathrm{d}\nu = \int_A \nu(A\,|\,\mathscr{T}_\infty)\mathrm{d}\nu = \int_A \nu(A)\mathrm{d}\nu = \nu(A)^2\,,$$

which implies that $\nu(A) \in \{0,1\}$.

Exercise 6.28: A simple computation yields

$$\mathscr{W}_{V_n}(\mu_{V_n}) = |V_n|\Big\{d\beta m^2 + hm - \frac{1+m}{2}\log\frac{1+m}{2} - \frac{1-m}{2}\log\frac{1-m}{2}\Big\},$$

where we have introduced $m \stackrel{\text{def}}{=} \langle\sigma_i\rangle_{\rho_i}$. It is now a matter of straightforward calculus to show that the unique maximum is attained when m satisfies $m = \tanh(2d\beta m + h)$.

Exercise 6.29: By (6.93), we have, for any $n \ge 0$,

$$\limsup_{k\to\infty} s(\mu_k) = \limsup_{k\to\infty}\inf_{\Lambda\in\mathscr{R}} \frac{S_\Lambda(\mu_k)}{|\Lambda|} \le \limsup_{k\to\infty}\frac{S_{\mathsf{B}(n)}(\mu_k)}{|\mathsf{B}(n)|} = \frac{S_{\mathsf{B}(n)}(\mu)}{|\mathsf{B}(n)|}.$$

Letting $n \to \infty$ yields the desired result.

Exercise 6.32: We show that $\mu\pi_\Lambda^\Phi = \mu$ for all $\Lambda \Subset \mathbb{Z}^d$. For each local function f, we write

$$\mu\pi_\Lambda^\Phi(f) = \{\mu\pi_\Lambda^\Phi(f) - \mu^k\pi_\Lambda^\Phi(f)\} + \{\mu^k\pi_\Lambda^\Phi(f) - \mu^k\pi_\Lambda^{\Phi^k}(f)\} + \mu^k\pi_\Lambda^{\Phi^k}(f).$$

Since Φ has finite range, $\omega \mapsto \pi_\Lambda^\Phi(f|\omega)$ is local. Therefore, $\mu^k \Rightarrow \mu$ implies that $\mu^k\pi_\Lambda^\Phi(f) \to \mu\pi_\Lambda^\Phi(f)$ as $k\to\infty$. For the second term, proceeding as in (6.32) gives

$$|\mu^k\pi_\Lambda^\Phi(f) - \mu^k\pi_\Lambda^{\Phi^k}(f)| \le \int |\pi_\Lambda^\Phi(f\,|\,\omega) - \pi_\Lambda^{\Phi^k}(f\,|\,\omega)|\mu^k(\mathrm{d}\omega) \le 2|\Lambda|\|f\|_\infty \sum_{B\ni 0}\|\Phi_B - \Phi_B^k\|_\infty,$$

which tends to zero when $k\to\infty$. Finally, since $\mu^k \in \mathscr{G}(\Phi^k)$, $\mu^k\pi_\Lambda^{\Phi^k}(f) = \mu^k(f)$, and $\mu^k(f)\to\mu(f)$.

Exercise 6.33: Assume that there is a unique Gibbs measure at (β_0, h_0). Observe that, setting $g = \frac{1}{2d}\sum_{i\sim 0}\sigma_0\sigma_i$ and $\lambda = \beta - \beta_0$, we have

$$\psi(\lambda) \stackrel{\text{def}}{=} \lim_{\Lambda\Uparrow\mathbb{Z}^d} \frac{1}{|\Lambda(g)|}\log\Big\langle \exp\Big\{\lambda\sum_{j\in\Lambda(g)} g\circ\theta_j\Big\}\Big\rangle^+_{\Lambda;\beta_0,h_0} = \psi^{\text{Ising}}(\beta,h_0) - \psi^{\text{Ising}}(\beta_0,h_0).$$

We deduce that

$$\frac{\partial\psi}{\partial\lambda^-}\Big|_{\lambda=0} = \frac{\partial\psi^{\text{Ising}}(\beta,h_0)}{\partial\beta^-}\Big|_{\beta=\beta_0}, \qquad \frac{\partial\psi}{\partial\lambda^+}\Big|_{\lambda=0} = \frac{\partial\psi^{\text{Ising}}(\beta,h_0)}{\partial\beta^+}\Big|_{\beta=\beta_0}.$$

Therefore, if $\psi^{\text{Ising}}(\beta,h_0)$ was not differentiable at β_0, then the same would be true of ψ and Proposition 6.91 would imply the existence of multiple Gibbs measures at (β_0,h_0), which would contradict our assumption.

Solutions of Chapter 7

Exercise 7.2: It suffices to show that η enjoys the following property. For each $k\ge 1$, η is a minimizer (possibly not unique) of $\mathscr{H}_{\mathsf{B}(k);\Phi^0}$ among all configurations of $\Omega^\eta_{\mathsf{B}(k)}$. To prove this, observe that the configuration η possesses a unique Peierls contour γ and check that the length of $\gamma\cap\{x\in\mathbb{R}^2 : \|x\|_\infty \le k\}$ cannot be decreased by flipping spins in $\mathsf{B}(k-1)$.

Exercise 7.3: The following construction relies on a diagonalization argument, as already done earlier in the book. Fix some arbitrary configuration $\eta\in\Omega$. For each $n\ge 0$, let $\omega^{(n)}$ be a configuration coinciding with η outside $\mathsf{B}(n)$ and minimizing $\mathscr{H}_{\mathsf{B}(n);\Phi}$. Order the vertices of $\mathbb{Z}^d$: $i_1, i_2, \dots$. Let $(n_{1,k})_{k\ge1}$ be a sequence such that $\omega_{i_1}^{(n_{1,k})}$ converges as $k\to\infty$. Let then $(n_{2,k})_{k\ge2}$ be a subsequence of $(n_{1,k})_{k\ge1}$ such that $\omega_{i_2}^{(n_{2,k})}$ converges. We proceed in the same way for all vertices of $\mathbb{Z}^d$: for each $m\ge1$, the sequences $(\omega_{i_m}^{(n_{m,k})})_{k\ge1}$ converges as $k\to\infty$. We claim that the configuration ω defined by

$$\omega_i \stackrel{\text{def}}{=} \lim_{m\to\infty}\omega_i^{(n_{m,m})}, \qquad \forall i\in\mathbb{Z}^d,$$

is a ground state. Indeed, let $\omega' \stackrel{\infty}{=} \omega$ and choose n so large that ω and ω' coincide outside $\mathsf{B}(n)$. Let N be so large that ω coincides with $\omega^{(N)}$ on $\mathsf{B}(n+r(\Phi))$. Then, by our choice of $\omega^{(N)}$,

$$\begin{aligned}\mathscr{H}_\Phi(\omega'\,|\,\omega) &= \sum_{B\cap\mathsf{B}(n)\neq\varnothing}\{\Phi_B(\omega') - \Phi_B(\omega)\}\\ &= \sum_{B\cap\mathsf{B}(n)\neq\varnothing}\{\Phi_B(\omega'_{\mathsf{B}(N)}\eta_{\mathsf{B}(N)^c}) - \Phi_B(\omega^{(N)})\} = \mathscr{H}_\Phi(\omega'_{\mathsf{B}(N)}\eta_{\mathsf{B}(N)^c}\,|\,\omega^{(N)}) \ge 0.\end{aligned}$$

Exercise 7.4: *1.* Consider the pressure constructed using a boundary condition $\eta \in g^{\mathrm{per}}(\Phi)$. On the one hand, $\mathbf{Z}^{\eta}_{\Phi}(\Lambda) \geq e^{-\beta\mathscr{H}_{\Lambda;\Phi}(\eta)}$, which gives $\psi(\Phi) \geq -\mathsf{e}_{\Phi}(\eta)$. On the other hand, for any $\omega \in \Omega^{\eta}_{\Lambda}$,

$$\mathscr{H}_{\Lambda;\Phi}(\omega) = \mathscr{H}_{\Lambda;\Phi}(\eta) + \{\mathscr{H}_{\Lambda;\Phi}(\omega) - \mathscr{H}_{\Lambda;\Phi}(\eta)\} = \mathscr{H}_{\Lambda;\Phi}(\eta) + \mathscr{H}_{\Phi}(\omega\,|\,\eta) \geq \mathscr{H}_{\Lambda;\Phi}(\eta).$$

This gives

$$\mathbf{Z}^{\eta}_{\Phi}(\Lambda) \leq e^{-\beta\mathscr{H}_{\Lambda;\Phi}(\eta)}|\Omega^{\eta}_{\Lambda}|.$$

Since $|\Omega^{\eta}_{\Lambda}| = |\Omega_0|^{|\Lambda|}$, this yields $\psi(\Phi) \leq -\mathsf{e}_{\Phi}(\eta) + \beta^{-1}\log|\Omega_0|$.
2. Observe that a configuration $\omega \in \Omega^{\eta}_{\Lambda}$ is completely characterized by its restriction to $\mathscr{B}(\omega)$. Therefore,

$$\begin{aligned}\mathbf{Z}^{\eta}_{\Phi}(\Lambda) &= \sum_{\omega\in\Omega^{\eta}_{\Lambda}} e^{-\beta\mathscr{H}_{\Lambda;\Phi}(\omega)} \leq e^{-\beta\mathscr{H}_{\Lambda;\Phi}(\eta)}\sum_{\omega\in\Omega^{\eta}_{\Lambda}} e^{-\beta\rho|\mathscr{B}(\omega)|} = e^{-\beta\mathscr{H}_{\Lambda;\Phi}(\eta)}\sum_{B\subset\Lambda}\ \sum_{\substack{\omega\in\Omega^{\eta}_{\Lambda}:\\ \mathscr{B}(\omega)\cap\Lambda=B}} e^{-\beta\rho|B|}\\ &\leq e^{-\beta\mathscr{H}_{\Lambda;\Phi}(\eta)}\sum_{n=0}^{|\Lambda|}\binom{|\Lambda|}{n}\bigl(|\Omega_0|e^{-\beta\rho}\bigr)^n = e^{-\beta\mathscr{H}_{\Lambda;\Phi}(\eta)}\bigl(1+|\Omega_0|e^{-\beta\rho}\bigr)^{|\Lambda|}.\end{aligned}$$

This gives $\psi(\Phi) \leq -\underline{\mathsf{e}}_{\Phi} + \beta^{-1}\log\bigl(1+|\Omega_0|e^{-\beta\rho}\bigr) \leq -\underline{\mathsf{e}}_{\Phi} + \beta^{-1}|\Omega_0|e^{-\beta\rho}$.

Exercise 7.5: It is convenient to work with the following equivalent potential:

$$\tilde{\Phi}_B(\omega) \stackrel{\text{def}}{=} \begin{cases}\omega_i\omega_j - \frac{h}{2d}(\omega_i+\omega_j) & \text{if } B=\{i,j\},\ i\sim j,\\ 0 & \text{otherwise.}\end{cases}$$

We are going to determine $g_m(\Phi)$. For any pair $i \sim j$,

$$\phi_{\{i,j\}} = \min_{\omega}\tilde{\Phi}_{\{i,j\}}(\omega) = \begin{cases}\ \ 1-(h/d) & \text{if } h \geq +2d,\\ -1 & \text{if } |h| \leq 2d,\\ \ \ 1+(h/d) & \text{if } h \leq -2d.\end{cases}$$

(The three cases correspond to $\omega_i = \omega_j = 1$, $\omega_i = \omega_j = -1$ and $\omega_i \neq \omega_j$ respectively.) The cases $h = \pm 2d$ are discussed below; for all other cases:

$$g_m(\Phi) = \begin{cases}\{\eta^+\} & \text{if } h > +2d,\\ \{\eta^{\pm},\eta^{\mp}\} & \text{if } |h| < 2d,\\ \{\eta^-\} & \text{if } h < -2d,\end{cases}$$

where $\eta^{\pm},\eta^{\mp}$ are the two chessboard configurations defined by $\eta^{\pm}_i \stackrel{\text{def}}{=} (-1)^{\sum_{k=1}^d i_k}$ and $\eta^{\mp} \stackrel{\text{def}}{=} -\eta^{\pm}$. When $h = \pm 2d$, $g_m(\Phi)$ contains infinitely many ground states. For example, if $h = +2d$,

$$g_m(\Phi) = \{\omega \in \Omega : \nexists i,j \in \mathbb{Z}^d, i \sim j, \text{ such that } \omega_i = \omega_j = -1\}.$$

Exercise 7.7: For all $\{i,j\} \in \mathscr{T}$, $\omega_i\omega_j$ is minimal if and only if $\omega_i \neq \omega_j$, and this cannot be realized simultaneously for all three pairs of spins living at the vertices of any given triangle. This implies that Φ is not an m-potential.

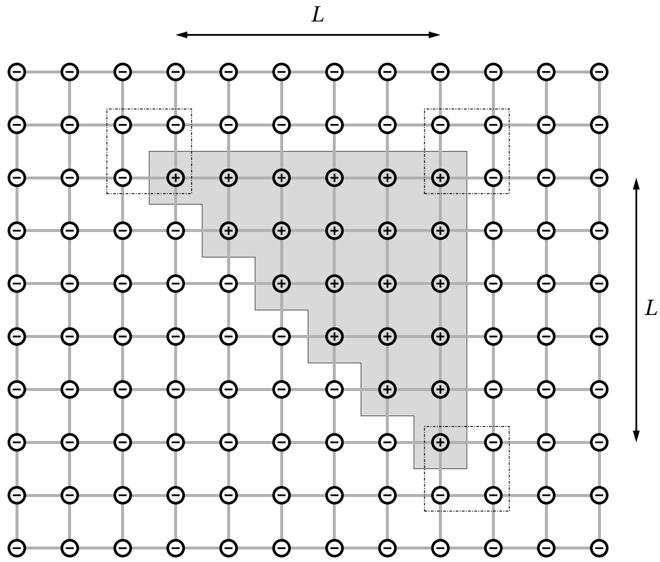
For the triangle $T = \{(0,0),(0,1),(1,1)\}$, let

$$\widetilde{\Phi}_T(\omega) = \omega_{(0,0)}\omega_{(0,1)} + \omega_{(0,1)}\omega_{(1,1)} + \omega_{(0,0)}\omega_{(1,1)}.$$

Define $\widetilde{\Phi}$ similarly on all translates of T. Then $\widetilde{\Phi}$ is clearly equivalent to Φ, and can be easily seen to be an m-potential; each $\widetilde{\Phi}_T$ being minimized if the configuration on T contains at least one spin of each sign. This allows to construct infinitely many periodic ground states for $\widetilde{\Phi}$. For example, any configuration obtained by alternating the spin values along every column necessarily belongs to $g_m(\widetilde{\Phi})$. Since this yields two possible configurations for each column, one can alternate them in order to construct configurations on $\mathbb{Z}^2$ of arbitrarily large period.

Exercise 7.8: Clearly, the constant configurations η^+ and η^- are periodic ground states, and their energy density equals $\underline{\mathsf{e}}_{\Phi} = \alpha$. Then, any other periodic configuration will necessarily contain (infinitely many) plaquettes whose energy is $\delta > \alpha$. By Lemma 7.13, this implies that $g^{\mathrm{per}}(\Phi) = \{\eta^+,\eta^-\}$. Examples of non-periodic ground states are obtained easily, by patching plaquettes with minimal energy.

To see that Peierls' condition is not satisfied, consider a configuration $\omega \stackrel{\infty}{=} \eta^-$, which coincides everywhere with η^- except on a triangular region of the following type, with L large:



Notice that all points along the boundary of the triangle are incorrect, which implies that $|\mathscr{B}(\omega)|$ (and therefore $|\Gamma(\omega)|$) grows linearly with L. Nevertheless, for each L, there are exactly three plaquettes with a non-zero contribution to $\mathscr{H}_\Phi(\omega\,|\,\eta)$, indicated at the three corners of the triangle. This means that $\mathscr{H}_\Phi(\omega\,|\,\eta)$ is bounded, uniformly in L: Peierls' condition is not satisfied.

Exercise 7.9: Let

$$W_B^i(\omega) \stackrel{\text{def}}{=} \begin{cases} |\mathsf{B}(r_*)|^{-1}\mathbf{1}_{\{\omega_B=\eta_B^i\}} & \text{if } B \text{ is a translate of } \mathsf{B}(r_*),\\ 0 & \text{otherwise.}\end{cases}$$

Suppose first that $m \in I$. In that case, setting $\lambda^i = 0$ for all $i \in I$ and $\lambda^j = \lambda > 0$ for all $j \in \{1,\dots,m\}\setminus I$, we obtain, for each $k \in \{1,2,\dots,m\}$,

$$\mathsf{e}_{\Phi^0+\sum_{i=1}^{m-1}\lambda^i W^i}(\eta^k) = \begin{cases} \underline{\mathsf{e}}_{\Phi^0} + \lambda & \text{if } k \in \{1,\dots,m\}\setminus I,\\ \underline{\mathsf{e}}_{\Phi^0} & \text{if } k \in I.\end{cases}$$

When $m \notin I$, we proceed similarly by setting $\lambda^i = 0$ for all $i \in \{1,\dots,m\}\setminus I$ and $\lambda^j = \lambda < 0$ for all $j \in I$.

The reader can check that these potentials do not create new ground states.

Exercise 7.10: By construction,

$$\mathscr{H}_{\hat\Phi}(\hat\omega\,|\,\hat\eta) = \mathscr{H}_\Phi(\omega\,|\,\eta) \ge \rho|\Gamma(\omega)| \ge \rho|\mathscr{B}(\omega)|,$$

where we used Peierls' condition for Φ. Now, observe that if a vertex $\hat\imath$ of the renormalized lattice is not $\hat\#$-correct, there must exist a vertex of the original lattice such that $j \in \hat\imath r_* + \mathsf{B}(3r_*)$ and j is not $\#$-correct. Therefore, $|\mathscr{B}(\hat\omega)| \ge |\mathsf{B}(3r_*)|^{-1}|\mathscr{B}(\omega)| \ge 3^{-d}|\mathsf{B}(r_*)|^{-1}|\mathscr{B}(\omega)|$. Thus, since $|\Gamma(\hat\omega)| \le 3^d|\mathscr{B}(\hat\omega)|$,

$$\rho|\mathscr{B}(\omega)| \ge \rho 3^{-d}|\mathsf{B}(r_*)|^{-1}|\mathscr{B}(\hat\omega)| \ge \rho 3^{-2d}|\mathsf{B}(r_*)|^{-1}|\Gamma(\hat\omega)|.$$

We conclude that Peierls' condition holds for $\hat\Phi$ with a constant $\rho 3^{-2d}|\mathsf{B}(r_*)|^{-1}$.

Exercise 7.11: Let $i, j \in A_\ell^{\mathrm{c}}$ and consider a path $\pi = (i_1 = i, i_2, \dots, i_{n-1}, i_n = j)$, with $d_\infty(i_k, i_{k+1}) = 1$. If π exits A_ℓ^{c}, let $s_1 \stackrel{\text{def}}{=} \min\{k : i_k \in A_\ell\} - 1$ and $s_2 \stackrel{\text{def}}{=} \max\{k : i_k \in A_\ell\} + 1$. By construction, $i_{s_1}, i_{s_2} \in \overline{\gamma}$. Since $\overline{\gamma}$ is connected, there exists a path from i_{s_1} to i_{s_2} entirely contained inside $\overline{\gamma}$. But this allows us to deform π so that it is entirely contained in A_ℓ^{c}.

Exercise 7.12: For ease of notation, we treat the case of a single contour; the same argument applies in the general situation. Proceeding exactly as we did to arrive at (7.34), treating separately the numerator and the denominator, we arrive at the following representation:

$$\mu^{\#}_{\Lambda;\Phi}(\Gamma' \ni \gamma') = \frac{\sum_{\Gamma\in\mathscr{A}_1}\prod_{\gamma\in\Gamma}\mathsf{w}^{\#}(\gamma)}{\sum_{\Gamma\in\mathscr{A}_0}\prod_{\gamma\in\Gamma}\mathsf{w}^{\#}(\gamma)} = \mathsf{w}^{\#}(\gamma')\frac{\sum_{\Gamma\in\mathscr{A}_2}\prod_{\gamma\in\Gamma}\mathsf{w}^{\#}(\gamma)}{\sum_{\Gamma\in\mathscr{A}_0}\prod_{\gamma\in\Gamma}\mathsf{w}^{\#}(\gamma)} \le \mathsf{w}^{\#}(\gamma'),$$

where we have introduced the following families of contours:

$$\mathscr{A}_0 \stackrel{\text{def}}{=} \{\Gamma \text{ compatible}\}, \qquad \mathscr{A}_1 \stackrel{\text{def}}{=} \{\Gamma \in \mathscr{A}_0 : \gamma' \text{ is an external contour of } \Gamma\},$$

while $\mathscr{A}_2$ is the set of all $\Gamma \in \mathscr{A}_0$ such that each $\gamma \in \Gamma$ is compatible with γ' and there does not exist $\gamma \in \Gamma$ such that $\overline{\gamma}' \subset \operatorname{int}\gamma$.

Exercise 7.13: Notice that if R_1, R_2 are two parallelepipeds such that $R_1 \cup R_2$ is also a parallelepiped, then $\Xi(R_1 \cup R_2) \geq \Xi(R_1)\Xi(R_2)$. Namely, the union of two compatible families contributing to $\Xi(R_1)$ and $\Xi(R_2)$ is always a family contributing to $\Xi(R_1 \cup R_2)$. One can then use Lemma B.6.

Solutions of Chapter 8

Exercise 8.1: One could use a Gaussian integration formula, but we prefer to provide an argument that also works for more general gradient models. We can assume that Λ is connected. Consider a spanning tree [1] T of the graph $(\Lambda, \mathscr{E}_\Lambda)$ and denote by T_0 the tree obtained by adding to T one vertex $i_0 \in \partial^{\text{ex}}\Lambda$ and an edge of $\mathscr{E}^{\text{b}}_\Lambda$ between i_0 and one of the vertices of T; we consider i_0 to be the root of the tree $T_0 = (V_0, E_0)$. Clearly,

$$\mathscr{H}_{\Lambda;\beta,m}(\omega) \geq \frac{\beta}{4d} \sum_{\{i,j\} \in E_0} (\omega_i - \omega_j)^2 \stackrel{\text{def}}{=} \tilde{\mathscr{H}}_{T_0;\beta}(\omega).$$

Of course,

$$\mathbf{Z}^\eta_{\Lambda;\beta,m} \leq \int e^{-\tilde{\mathscr{H}}_{T_0;\beta}(\omega_{V_0}\eta_{i_0})} \prod_{i \in \Lambda} \mathrm{d}\omega_i .$$

Let $i \in \Lambda$ be a leaf [2] of the tree T_0. Then, denoting by j the unique neighbor of i in T_0,

$$\int_{-\infty}^{\infty} e^{-\frac{\beta}{4d}\left(\omega_i - \omega_j\right)^2} \mathrm{d}\omega_i = \int_{-\infty}^{\infty} e^{-\frac{\beta}{4d}x^2} \mathrm{d}x = \sqrt{4\pi d/\beta}.$$

We can thus integrate over each variable in Λ, removing one leaf at a time. The end result is the upper bound

$$\mathbf{Z}^\eta_{\Lambda;\beta,m} \leq \{4\pi d/\beta\}^{|\Lambda|/2} .$$

Exercise 8.2: The problem arises from the fact that, when no spins are fixed on the boundary, all spins inside Λ can be shifted by the same amount without changing the energy. This can already be seen in the simple case where $\Lambda = \{0,1\} \subset \mathbb{Z}$, with free boundary condition:

$$\mathbf{Z}^\varnothing_{\Lambda;\beta,0} = \int \Big\{ \int e^{-\frac{\beta}{4d}(\omega_0 - \omega_1)^2} \mathrm{d}\omega_1 \Big\} \mathrm{d}\omega_0 = \sqrt{\frac{4d\pi}{\beta}} \int \mathrm{d}\omega_0 = +\infty .$$

Exercise 8.3: The first claim follows from the fact $\varphi_{i_1} \cdots \varphi_{i_{2n+1}}$ is an odd function, so its integral with respect to the density (8.10) (with $a_\Lambda = 0$) vanishes.

Let us turn to the second claim. First, observe that one can assume that all vertices $i_1, \ldots, i_{2n}$ are distinct; otherwise for each vertex j appearing $r_j > 1$ times, introduce $r_j - 1$ new random variables, perfectly correlated with φ_j.

The desired expectation can be obtained from the moment generating function by differentiation:

$$E_\Lambda[\varphi_{i_1} \ldots \varphi_{i_{2n}}] = \frac{\partial^{2n}}{\partial t_{i_1} \cdots \partial t_{i_{2n}}} E_\Lambda[e^{t_\Lambda \cdot \varphi_\Lambda}]\Big|_{t_\Lambda \equiv 0} .$$

The identity (8.9) allows one to perform this computation in another way. First,

$$\exp\{\tfrac{1}{2} t_\Lambda \cdot \Sigma_\Lambda t_\Lambda\} = \sum_{n \geq 0} \tfrac{1}{n!} 2^{-n} \Big\{ \sum_{j,k \in \Lambda} \Sigma_\Lambda(j,k) t_j t_k \Big\}^n .$$

Therefore,

$$\frac{\partial^{2n}}{\partial t_{i_1} \cdots \partial t_{i_{2n}}} \exp\{\tfrac{1}{2} t_\Lambda \cdot \Sigma_\Lambda t_\Lambda\}\Big|_{t_\Lambda \equiv 0} = \tfrac{1}{n!} \sum_{\substack{\{j_1,k_1\},\ldots,\{j_n,k_n\} \subset \Lambda \\ \cup_{m=1}^n \{j_m,k_m\} = \{i_1,\ldots,i_{2n}\}}} \prod_{m=1}^n \Sigma_\Lambda(j_m, k_m),$$

where the factor 2^{-n} was canceled by the factor 2^n accounting for the possible interchange of j_m and k_m for each $m = 1, \ldots, n$. Now, we can rewrite the latter sum in terms of pairings as in the claim. Note that, doing so, we lose the ordering of the n pairs $\{j_m, k_m\}$, so that we have to introduce an additional factor of $n!$, canceling the factor $1/n!$. The claim follows.

[1] A **spanning tree** of a graph $G = (V, E)$ is a connected subgraph of G which is a tree and contains all vertices of G.

[2] A **leaf** of a tree if a vertex of degree 1 distinct from the root.

Exercise 8.4: The procedure is very similar to the one used in the previous exercise. We write $C_{ij} = E(\varphi_i\varphi_j)$. Then, using (8.11) and (8.12),

$$\begin{aligned}
E\big[e^{t_\Lambda\cdot\varphi_\Lambda}\big] &= \sum_{n\geq 0}\sum_{\substack{(r_i)_{i\in\Lambda}\subset\mathbb{Z}_{\geq 0}:\\ \sum_i r_i=2n}} E\Big[\prod_i \varphi_i^{r_i}\Big]\prod_i t_i^{r_i}\\
&= \sum_{n\geq 0}\sum_{\substack{(r_i)_{i\in\Lambda}\subset\mathbb{Z}_{\geq 0}:\\ \sum_i r_i=2n}} \Big\{\sum_{\mathscr{P}}\prod_{\{\ell,\ell'\}\in\mathscr{P}} C_{\ell\ell'}\Big\}\prod_i t_i^{r_i}\\
&= \sum_{n\geq 0}\sum_{\substack{(r_i)_{i\in\Lambda}\subset\mathbb{Z}_{\geq 0}:\\ \sum_i r_i=2n}} \tfrac{1}{n!}2^{-n}\Big\{\sum_{\substack{\{j_1,k_1\},\dots,\{j_n,k_n\}\subset\Lambda\\ \sum_{m=1}^n(\mathbf{1}_{\{j_m=i\}}+\mathbf{1}_{\{k_m=i\}})=r_i,\forall i\in\Lambda}} \prod_{m=1}^n C_{i_m j_m}\Big\}\prod_i t_i^{r_i}\\
&= \sum_{n\geq 0}\tfrac{1}{n!}2^{-n}\Big\{\sum_{\{j_1,k_1\},\dots,\{j_n,k_n\}\subset\Lambda}\prod_{m=1}^n C_{i_m j_m} t_{i_m} t_{j_m}\Big\}\\
&= \sum_{n\geq 0}\tfrac{1}{n!}2^{-n}\Big\{\sum_{i,j\in\Lambda} C_{ij}t_i t_j\Big\}^n = \exp\Big\{\sum_{i,j\in\Lambda}\tfrac{1}{2}C_{ij}t_i t_j\Big\}.
\end{aligned}$$

Exercise 8.6: Let $(\xi_i)_{i=-n-1,\dots,n}$ be i.i.d. random variables with distribution $\xi_i\sim\mathscr{N}(0,2)$. Let $L_n \stackrel{\text{def}}{=} \xi_{-n-1}+\cdots+\xi_{-1}$ and $R_n \stackrel{\text{def}}{=} \xi_0+\cdots+\xi_n$. The density of φ_0 coincides with the conditional probability density of L_n given that $L_n+R_n=0$, which is equal to

$$\frac{f_{L_n}(x)f_{R_n}(-x)}{f_{L_n+R_n}(0)} = \frac{\Big\{\frac{1}{\sqrt{4\pi(n+1)}}e^{-\frac12\frac{x^2}{2(n+1)}}\Big\}^2}{\frac{1}{\sqrt{8\pi(n+1)}}} = \frac{1}{\sqrt{2\pi(n+1)}}e^{-\frac12\frac{x^2}{(n+1)}},$$

so that $\varphi_0\sim\mathscr{N}(0,n+1)$.

Exercise 8.7: Since φ is centered, $\tilde\varphi$ also is. Then, observe that

$$E^0_{\Lambda;0}\big[e^{\mathrm{i}t_\Lambda\cdot\tilde\varphi_\Lambda}\big] = E^0_{\Lambda;0}\big[e^{\mathrm{i}\tilde t_\Lambda\cdot\varphi_\Lambda}\big],$$

where $\tilde t_i \stackrel{\text{def}}{=} t_i$ for all $i\neq 0$ and $\tilde t_0 = -\sum_{j\in\Lambda\setminus\{0\}} t_j$. From (8.8), we get

$$E^0_{\Lambda;0}\big[e^{\mathrm{i}\tilde t_\Lambda\cdot\varphi_\Lambda}\big] = \exp\{-\tfrac12\sum_{i,j\in\Lambda}G_\Lambda(i,j)\tilde t_i\tilde t_j\} = \exp\{-\tfrac12\sum_{i,j\in\Lambda\setminus\{0\}}\tilde G_\Lambda(i,j)t_i t_j\},$$

with $\tilde G_\Lambda(i,j)$ given in (8.34).

Exercise 8.10: When $d=1$, $(-\frac{1}{2d}\Delta+m^2)u=0$ becomes

$$u_{k+1} = 2(1+m^2)u_k - u_{k-1}.$$

For any pair of initial values $u_0,u_1\in\mathbb{R}$, we can then easily verify that u_k, $k\geq 2$, is of the form

$$u_k = Az_+^k + Bz_-^k,$$

where $z_\pm = 1+m^2\pm\sqrt{2m^2+m^4}$ and A,B are functions of u_0,u_1. The conclusion follows, since $z_- = 1/z_+$.

Solutions of Chapter 9

Exercise 9.1: Since $\langle\mathbf{S}_i\cdot\mathbf{S}_j\rangle_\mu\to 0$ uniformly as $\|j-i\|_2\to\infty$, we can find, for any $\epsilon>0$, a number R such that $\langle\mathbf{S}_i\cdot\mathbf{S}_j\rangle_\mu\leq\epsilon$ for all i,j such that $\|j-i\|_2>R$. Consequently, for any $i\in\mathsf{B}(n)$,

$$\sum_{j\in\mathsf{B}(n)}\langle\mathbf{S}_i\cdot\mathbf{S}_j\rangle_\mu \leq |\mathsf{B}(R)| + \epsilon|\mathsf{B}(n)|.$$

It follows that

$$\limsup_{n\to\infty}\langle\|m_{\mathsf{B}(n)}\|_2^2\rangle_\mu = \limsup_{n\to\infty}|\mathsf{B}(n)|^{-2}\sum_{i,j\in\mathsf{B}(n)}\langle\mathbf{S}_i\cdot\mathbf{S}_j\rangle_\mu \leq \limsup_{n\to\infty}|\mathsf{B}(n)|^{-1}\max_{i\in\mathsf{B}(n)}\sum_{j\in\mathsf{B}(n)}\langle\mathbf{S}_i\cdot\mathbf{S}_j\rangle_\mu\leq\epsilon.$$

Since ϵ was arbitrary, the conclusion follows.

Exercise 9.3: Simply take $\theta_i^{\text{SW}} = \big(1 - (\|i\|_\infty/n)\big)\pi$.

Exercise 9.4: First, observe that

$$\sum_{\{i,j\}\in\mathscr{E}^{\mathrm{b}}_{\mathsf{B}(n)}} \big(f(\|i\|_\infty) - f(\|j\|_\infty)\big)^2 \geq \sum_{k=\ell}^{n-1} k^{d-1}\big(f(k) - f(k+1)\big)^2 .$$

By the Cauchy–Schwarz inequality,

$$\Big\{\sum_{k=\ell}^{n-1}\big(f(k) - f(k+1)\big)\Big\}^2 \leq \Big\{\sum_{k=\ell}^{n-1} k^{d-1}\big(f(k) - f(k+1)\big)^2\Big\}\Big\{\sum_{k=\ell}^{n-1} k^{-(d-1)}\Big\}.$$

Therefore,

$$\begin{aligned}\sum_{\{i,j\}\in\mathscr{E}^{\mathrm{b}}_{\mathsf{B}(n)}} \big(f(\|i\|_\infty) - f(\|j\|_\infty)\big)^2 &\geq \Big\{\sum_{k=\ell}^{n-1}\big(f(k) - f(k+1)\big)\Big\}^2\Big\{\sum_{k=\ell}^{n-1} k^{-(d-1)}\Big\}^{-1}\\ &\geq \big(f(\ell) - f(n)\big)^2\Big\{\sum_{k\geq\ell} k^{-(d-1)}\Big\}^{-1} = \Big\{\sum_{k\geq\ell} k^{-(d-1)}\Big\}^{-1}.\end{aligned}$$

Exercise 9.5: Fix $M \geq 2$ and partition $(-\pi,\pi]$ into intervals $I_1,\dots,I_M$ of length $2\pi/M$. Write $\nu_r \stackrel{\text{def}}{=} \mu^\eta_{\mathsf{B}(n);\beta}(\vartheta_0 \in I_r)$. Then, (9.9) implies that $|\nu_r - \nu_s| \leq c/T_n(d)$ for all $1 \leq r,s \leq M$ and thus $|\nu_r - \frac{1}{M}| = \frac{1}{M}|\sum_{s=1}^M(\nu_r - \nu_s)| \leq c/T_n(d)$. The first claim follows. The second claim is an immediate consequence of the first one.

Exercise 9.7: Writing $\mathbf{S}_i = (\cos\vartheta_i, \sin\vartheta_i)$ gives $\mathbf{S}_i\cdot\mathbf{S}_j = \cos(\vartheta_i - \vartheta_j)$, so the partition function with free boundary condition can be written as

$$\mathbf{Z}^{\varnothing}_{\mathsf{B}(n);\beta} = \int_{-\pi}^{\pi}\mathrm{d}\theta_{-n}\cdots\int_{-\pi}^{\pi}\mathrm{d}\theta_n \prod_{i=-n+1}^{n} e^{\beta\cos(\theta_i - \theta_{i-1})}.$$

Now, observe that

$$\int_{-\pi}^{\pi}\mathrm{d}\theta_n e^{\beta\cos(\theta_n-\theta_{n-1})} = \int_{-\pi}^{\pi}\mathrm{d}\tau e^{\beta\cos\tau} = 2\pi I_0(\beta).$$

One can then continue integrating successively over $\theta_{n-1},\dots,\theta_{-n+1}$, getting each time a factor $2\pi I_0(\beta)$, with a final factor 2π for the last integration (over θ_{-n}). Therefore,

$$\mathbf{Z}^{\varnothing}_{\mathsf{B}(n);\beta} = (2\pi)^{|\mathsf{B}(n)|} I_0(\beta)^{|\mathsf{B}(n)|-1},$$

and, thus, $\psi(\beta) = \lim_{n\to\infty}|\mathsf{B}(n)|^{-1}\log\mathbf{Z}^{\varnothing}_{\mathsf{B}(n);\beta} = \log(2\pi) + \log I_0(\beta)$. The computation of the numerator of the correlation function $\langle\mathbf{S}_0\cdot\mathbf{S}_i\rangle^{\varnothing}_{\mathsf{B}(n);\beta}$ is similar. We assume that $i > 0$. Integration over $\theta_n,\dots,\theta_{i+1}$ is carried out as before and yields $I_0(\beta)^{n-i}$. The integration over θ_i yields, using the identity $\cos(s+t) = \cos s\cos t + \sin s\sin t$,

$$\begin{aligned}\int_{-\pi}^{\pi}\mathrm{d}\theta_i e^{\beta\cos(\theta_i-\theta_{i-1})}\cos(\theta_i - \theta_0) &= \int_{-\pi}^{\pi}\mathrm{d}\tau e^{\beta\cos\tau}\cos(\tau + \theta_{i-1} - \theta_0)\\ &= \cos(\theta_{i-1} - \theta_0)\int_{-\pi}^{\pi}\mathrm{d}\tau e^{\beta\cos\tau}\cos(\tau) = 2\pi I_1(\beta)\cos(\theta_{i-1} - \theta_0).\end{aligned}$$

The integration over $\theta_{i-1},\dots,\theta_1$ is performed identically. Then, the integration over $\theta_0,\dots,\theta_{-n}$ is done as for the partition function. We thus get, after simplification,

$$\langle\mathbf{S}_0\cdot\mathbf{S}_i\rangle^{\varnothing}_{\mathsf{B}(n);\beta} = \frac{2\pi\big(2\pi I_0(\beta)\big)^n\big(2\pi I_1(\beta)\big)^i\big(2\pi I_0(\beta)\big)^{n-i}}{2\pi\big(2\pi I_0(\beta)\big)^{2n}} = \Big(\frac{I_1(\beta)}{I_0(\beta)}\Big)^{|i|}.$$

Letting $n\to\infty$ yields $\langle\mathbf{S}_0\cdot\mathbf{S}_i\rangle_\mu = (I_1(\beta)/I_0(\beta))^i$.

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Solutions of Chapter 10

Exercise 10.2: Suppose first that $\Theta(\mu)=\mu$. In this case, for any $f,g\in\mathfrak{A}_+(\Theta)$,

$$\langle f\Theta(g)\rangle_\mu = \langle\Theta(f\Theta(g))\rangle_\mu = \langle\Theta(f)g\rangle_\mu .$$

Conversely, suppose that $\langle f\,\Theta(g)\rangle_\mu = \langle g\,\Theta(f)\rangle_\mu$, for all $f,g\in\mathfrak{A}_+(\Theta)$. Let $A\subset\Omega_0^{\mathbb{T}_{L,+}(\Theta)}$, $B\subset\Omega_0^{\mathbb{T}_{L,-}(\Theta)}$ be arbitrary measurable sets and set $\bar A \stackrel{\text{def}}{=} A\times\Omega_0^{\mathbb{T}_{L,-}(\Theta)}$, $\bar B\stackrel{\text{def}}{=}\Omega_0^{\mathbb{T}_{L,+}(\Theta)}\times B$, $f=\mathbf{1}_{\bar A}$ and $g=\mathbf{1}_{\Theta(\bar B)}$. We then have

$$\mu(\bar A\cap\bar B) = \langle f\Theta(g)\rangle_\mu = \langle\Theta(f)g\rangle_\mu = \mu(\Theta(\bar A\cap\bar B)).$$

Since events of the form $\bar A\cap\bar B$ generate the product σ-algebra, we have shown that $\mu=\Theta(\mu)$.

Exercise 10.3: Consider $\Omega_0=\{\pm1\}$. Let $\omega',\omega''\in\Omega_L$ be defined as follows: $\omega'_i=(-1)^{\sum_{k=1}^d i_k}$, $\omega''_i=-\omega''_i$. Let $\mu\stackrel{\text{def}}{=}\frac12(\delta_{\omega'}+\delta_{\omega''})$. Then μ is translation invariant but not reflection positive. Namely, let Θ be any reflection through the edges and let $e=\{i,j\}\in\mathscr{E}_L$ be such that $j=\Theta(i)$. Let $f(\omega)\stackrel{\text{def}}{=}\omega_i$. Then $\langle f\Theta(f)\rangle_\mu=-1<0$.

Exercise 10.4: As a counterexample to such an identity, one can consider, for example, the Ising model on $\mathbb{T}_8$ with blocks of length $B=2$ and the four Λ_2-local functions given by $f_0=\mathbf{1}_{++}, f_1=\mathbf{1}_{+-}, f_2=\mathbf{1}_{--}, f_3=\mathbf{1}_{-+}$, where $\mathbf{1}_{ss'}\stackrel{\text{def}}{=}\mathbf{1}_{\{\sigma_0=s,\sigma_1=s'\}}$.

Exercise 10.5: Write $f(\mathbf{S}_i,\mathbf{S}_j)=-\alpha\mathbf{S}_i\cdot\mathbf{S}_j-(1-\alpha)S_i^1S_j^1$. The first term is minimal if and only if $\mathbf{S}_i=\mathbf{S}_j$. The second term is minimal if and only if $S_i^1=S_j^1=\pm1$. The claim follows.

Exercise 10.6: This is immediate using the following elementary identities:

$$|\mathbb{T}_L^\star|^{-1}\sum_{p\in\mathbb{T}_L^\star} e^{\mathrm{i}(j-k)\cdot p}=\delta_{j,k}\quad\text{for all } j,k\in\mathbb{T}_L,\quad\text{and}\quad |\mathbb{T}_L|^{-1}\sum_{j\in\mathbb{T}_L} e^{\mathrm{i}(p-p')\cdot j}=\delta_{p,p'}\quad\text{for all } p,p'\in\mathbb{T}_L^\star.$$

Exercise 10.7: We will use twice an adaptation of the discrete Green identity (8.14) on the torus. First, since (using $\sum_{i\in\mathbb{T}_L}(\Delta h)_i=0$ for the last identity)

$$\begin{aligned}\frac{\mathbf{Z}_{L;\beta}(h)}{\mathbf{Z}_{L;\beta}(0)} &= \Big\langle \exp\Big\{-2\beta\sum_{\{i,j\}\in\mathscr{E}_L}(\mathbf{S}_i-\mathbf{S}_j)\cdot(h_i-h_j)-\beta\sum_{\{i,j\}\in\mathscr{E}_L}\|h_i-h_j\|_2^2\Big\}\Big\rangle_{L;\beta}\\ &= \Big\langle \exp\Big\{2\beta\sum_{i\in\mathbb{T}_L}\mathbf{S}_i\cdot(\Delta h)_i-\beta\sum_{\{i,j\}\in\mathscr{E}_L}\|h_i-h_j\|_2^2\Big\}\Big\rangle_{L;\beta}\\ &= \Big\langle \exp\Big\{2\beta\sum_{i\in\mathbb{T}_L}(\mathbf{S}_i-\mathbf{S}_0)\cdot(\Delta h)_i-\beta\sum_{\{i,j\}\in\mathscr{E}_L}\|h_i-h_j\|_2^2\Big\}\Big\rangle_{L;\beta},\end{aligned}$$

the equivalence of (10.43) and (10.45) follows.

Now, observe that the Boltzmann weight appearing in $\nu_{L;\beta}$ can be written as a product:

$$\exp\Big(-\beta\sum_{\{i,j\}\in\mathscr{E}_L}\|\mathbf{S}_i-\mathbf{S}_j\|_2^2\Big)=\prod_{k=1}^{\nu}\exp\Big(-\tfrac{1}{4d}\sum_{\{i,j\}\in\mathscr{E}_L}(\varphi_i^k-\varphi_j^k)^2\Big),$$

where we defined the collections $\varphi_i^k\stackrel{\text{def}}{=}\sqrt{4d\beta}S_i^k$. Therefore, the families $(\varphi_i^k)_{i\in\mathbb{T}_L}$ and $(\varphi_i^\ell)_{i\in\mathbb{T}_L}$ are independent of each other if $k\neq\ell$ and each is distributed as a massless Gaussian Free Field on $\mathbb{T}_L$. Of course, the latter is ill-defined on the torus. However, notice that the expectation we are interested in only involves the field $\tilde\varphi_i^k\stackrel{\text{def}}{=}\varphi_i^k-\varphi_0^k$. Adapting the arguments of Chapter 8 (working on $\mathbb{T}_L$ with 0 boundary condition at the vertex $\{0\}$), the reader can check that the latter is a well-defined centered Gaussian field with covariance matrix

$$G_{\mathbb{T}_L\setminus\{0\}}(i,j)\stackrel{\text{def}}{=}\sum_{n\ge0}\mathbb{P}_i(X_n=j,\tau_0>n),$$

that is, the Green function of the simple random walk on $\mathbb{T}_L$, killed at its first visit at 0. Moreover, this Green function is the inverse of the discrete Laplacian $-\frac{1}{2d}\Delta$ on $\mathbb{T}_L$.

We can thus fix $k\in\{1,2,\dots,\nu\}$, define

$$h^k=(h_i^k)_{i\in\mathbb{T}_L},\quad t_L^k\stackrel{\text{def}}{=}\big(\sqrt{\beta/d}(\Delta h^k)_i\big)_{i\in\mathbb{T}_L},\quad \varphi_L^k\stackrel{\text{def}}{=}(\varphi_i^k)_{i\in\mathbb{T}_L},$$

and use (8.9):

$$\Big\langle \exp\Big\{2\beta \sum_{i\in\mathbb{T}_L} (\Delta h^k)_i\,(S_i^k - S_0^k)\Big\}\Big\rangle_{\nu_{L;\beta}} = \Big\langle e^{t_L^k\cdot\tilde{\varphi}_L^k}\Big\rangle_{\nu_{L;\beta}} = \exp\big(\tfrac12\, t_L^k\cdot G_{\mathbb{T}_L\setminus\{0\}}\, t_L^k\big).$$

Changing back to the original variables and using the fact that the Green function is the inverse of the discrete Laplacian, the conclusion follows:

$$\tfrac12\, t_L^k\cdot G_{\mathbb{T}_L\setminus\{0\}}\, t_L^k = -\beta\Delta h^k\cdot\big\{G_{\mathbb{T}_L\setminus\{0\}}\big(-\tfrac{1}{2d}\Delta\big)h^k\big\} = -\beta\Delta h^k\cdot h^k.$$

Solutions of Appendix B

Exercise B.1: For the first inequality, it suffices to write y as $y=\alpha x+(1-\alpha)z$, with $\alpha=(z-y)/(z-x)$. The second follows by subtracting $f(x)$ on both sides of (B.5).

Exercise B.4: Assume first that $f''(x)\geq 0$ for all $x\in I$. Then, for all $x,y\in I$,

$$f(y)=f(x)+\int_x^y f'(u)\mathrm{d}u = f(x)+\int_x^y\Big\{f'(x)+\int_x^u f''(v)\mathrm{d}v\Big\}\mathrm{d}u \geq f(x)+f'(x)(y-x).$$

This implies that f has a supporting line at each point of I and is thus convex by Theorem B.13. Now if f is convex, then, for all $x\in I$ and all $h\neq 0$ (small enough), $(f'(x+h)-f'(x))/h\geq 0$, since f' is increasing. By letting $h\to 0$, it follows that $f''(x)\geq 0$.

Exercise B.5: Assume f is affine on some interval $I=[a,b]$, and consider $a<a_0<b_0<b$. On the one hand, by Theorem B.12 and since each f_n is differentiable, $0=f'(b_0)-f'(a_0)=\lim_n(f_n'(b_0)-f_n'(a_0))$. On the other hand, $f_n''\geq c$ and the Mean Value Theorem implies that, uniformly in n, $f_n'(b_0)-f_n'(a_0)\geq c(b_0-a_0)>0$, a contradiction.

Exercise B.6: It suffices to write, for all x, y_1, y_2 and $\alpha\in[0,1]$,

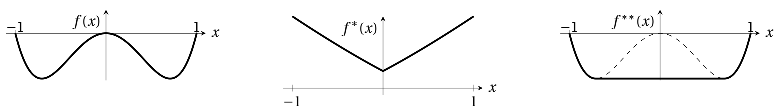
$$x(\alpha y_1+(1-\alpha)y_2)-f(x)=\alpha\big\{xy_1-f(x)\big\}+(1-\alpha)\big\{xy_2-f(x)\big\}\leq \alpha f^*(y_1)+(1-\alpha)f^*(y_2).$$

Exercise B.7: By explicit computation: $f_1^*(y)=\frac12 y^2$, $f_2^*(y)=\frac{3}{4^{4/3}}y^{4/3}$, $f_3^*(y)=|y|$, which are all convex. Furthermore, $f_1^{**}=f_1$, $f_2^{**}=f_2$ but $f_3^{**}\neq f_3$ since $f_3^{**}(x)=0$ if $|x|\leq 1$, $+\infty$ otherwise.

Exercise B.8:

$$f^{**}(x)=\sup_y\{xy-\underbrace{\sup_z\big(yz-f(z)\big)}_{\geq yx-f(x)}\}\leq f(x).$$

Exercise B.10:



Exercise B.11: Let $x_n\to x$. Then, for any $z\in I$,

$$\liminf_{n\to\infty} f^*(x_n)=\liminf_{n\to\infty}\sup_{y\in I}\big\{x_n y-f(y)\big\}\geq \liminf_{n\to\infty}\big\{x_n z-f(z)\big\}=xz-f(z).$$

Therefore, $\liminf_{n\to\infty}f^*(x_n)\geq\sup_{z\in I}\big\{xz-f(z)\big\}=f^*(x)$.

Exercise B.12: Since $f(x)\geq f(x_0)+m(x-x_0)$ for all x, we have $f^*(m)=x_0m-f(x_0)$. By definition, $f^*(y)=\sup_x\{yx-f(x)\}$, and so

$$f^*(y)\geq x_0y-f(x_0)=x_0(y-m)+(x_0m-f(x_0))=x_0(y-m)+f^*(m).$$

Exercise B.14: Since $\mathrm{epi}(g)$ is convex, closed and contains $\mathrm{epi}(f)$ (since $g\leq f$), we have

$$\mathrm{CE}\,f(x)=\inf\big\{y:(x,y)\in C\big\}\geq\inf\big\{y:(x,y)\in\mathrm{epi}(g)\big\}=g(x).$$

Exercise B.15: Let μ be the counting measure on $(\mathbb{N},\mathscr{P}(\mathbb{N}))$: $\mu(n)\stackrel{\text{def}}{=}1$ for all $n\in\mathbb{N}$. Let $(x_n)_{n\geq 1}\subset I$ be any sequence converging to x_0 and consider the sequence $(f_n)_{n\geq1}$ of functions $f_n:\mathbb{N}\to\mathbb{R}$ defined by $f_n(k)\stackrel{\text{def}}{=}\xi_k(x_n)$. Then $\sum_k\xi_k(x_n)=\int f_n\,\mathrm{d}\mu$, so the result follows from Theorem B.40.

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