

# Corrections

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We list here typos and errors that have been found in the published version of the book. We welcome any additional corrections you may find! We indicate page and line numbers for the published version of the book; negative line numbers are counted from the bottom of the page. Most typos have already been corrected in the version available on our web page (the address is given in the footer). When this has not been done, or if you downloaded an earlier version, the relevant pages are indicated in brackets.

- **p. 20, l. –1 [p. 18, l. –1]:** litterature should be literature.
- **p. 31, l. 11 [p. 28, l. 12]:** see **the** exercise below.
- **p. 83, l. –2 [p. 80, l. –5]:**  $\sum_{r=1}^d |(i_r - j_r) \bmod n|$  should be  $\sum_{r=1}^d (|i_r - j_r| \bmod n - 2)$ .
- **p. 90, l. 3 [p. 86, l. 7]:**  $k_0$  is actually independent of  $h$ .
- **p. 91, l. 8 [p. 87, l. 11]:** the notation  $\mathcal{H}_\Lambda^n$  has never been introduced (although its meaning should be pretty clear from the context).
- **p. 92, l. 1 [p. 87, l. –2]:**  $\Lambda \subset \mathbb{Z}^d$  should be  $\Lambda \subseteq \mathbb{Z}^d$ .
- **p. 93, l. –10 [p. 89, l. 14]:** magnetization should be magnetization density.
- **p. 103, l. –1 [p. 98, l. –16]:** increasing should be nondecreasing.
- **p. 108, l. 10 [p. 102, l. 15]:**  $\Delta_{2k-1} \subset \{\Lambda_n^1 : n \geq 1\}$ ,  $\Delta_{2k} \subset \{\Lambda_n^2 : n \geq 1\}$  should be  $\Delta_{2k-1} \in \{\Lambda_n^1 : n \geq 1\}$ ,  $\Delta_{2k} \in \{\Lambda_n^2 : n \geq 1\}$ .
- **p. 112, l. –5 [p. 106, l. –8]:**  $|\langle \sigma_0 \rangle_{\Lambda_n; \beta, h}^+| \leq 1$  should be  $|\langle \sigma_i \rangle_{\Lambda_n; \beta, h}^+| \leq 1$ .
- **p. 116 [p. 110]:** The code for Figure 3.10 has some bug that make it looks rather poor.
- **p. 121, l. 1 [p. 114, l. 17]:** “There exists” should be “There exist”.
- **p. 129, l. –3:** “with the free boundary condition” should be “with free boundary condition”.

- **p. 130, l. 8 [p. 122, l. –7]:** “Observing that” should be “Since”.
- **p. 136, l. –7 [p. 128, l. 14]:** “For any  $A, B \subset \Lambda$ ” should be “For any  $A \subset \Lambda$ ”. Moreover, the claim in this exercise can be strengthened, see the additional exercises for Chapter 3 below.
- **p. 147, l. –5 [p. 138, l. –3]:**  $\tilde{\omega}_j(n)$  should be  $\tilde{\omega}_j$ .
- **p. 148, l. 4 [p. 139, l. 6]:**  $\tilde{\omega}_j(n)$  should be  $\tilde{\omega}_j$  and  $\omega_j(n)$  should be  $\omega_j$ .
- **p. 151, l. 18 [p. 142, l. 9]:**  $2^{N_\Lambda^w(E)}$  should be  $2^{N_\Lambda^w(E)-1}$ . (Thanks to the students in Ben Lee’s class for pointing this out.)
- **p. 155:** In the equation following (3.82),  $\{i, j\}$  should be  $\{i\}$  (three times) and  $i \xrightarrow{\mathbf{m}} j$  should be  $i \xrightarrow{\mathbf{m}} \partial^{\text{ex}} \Lambda$  (thanks to Vedran Sohinger for pointing this out).
- **p. 156, l. 6 [p. 146, l. 15]:**  $\mathbb{P}_{\Lambda; \beta}$  should be  $\mathbb{P}_{\Lambda; \beta}^+$  (6 times).
- **p. 161, l. 11:** Theorem 3.59 should be Theorem 3.58. A couple of lines below,  $[\beta_c(d-1), \beta_c(d)]$  should be  $(\beta_c(d), \beta_c(d-1))$  and  $[\beta_c(2), \beta_c(3)]$  should be  $(\beta_c(3), \beta_c(2))$ .
- **p. 164, l. 4:**  $e^{-c_1 n}$  should be  $e^{-c_1 R}$ .
- **p. 184, l. –4 [p. 174, l. 5]:** In (4.23),  $\beta N_\Lambda(\eta)$  should be  $\mu N_\Lambda(\eta)$ . (Thanks to Sébastien Ott for pointing this out.)
- **p. 262, l. 5:** the semicolon should be a comma.
- **p. 264, l. 15 [p. 247, l. 19]:** decribe should be describe.
- **p. 265 [p. 248]:** In the second part of Notation 6.2,  $\Lambda$  is a subset of  $S$ .
- **p. 266 [p. 249]:** In Exercise 6.2, countably should be at most countably.
- **p. 277, l. –11 [p. 259, l. 7]:** the comma should be a colon.
- **p. 279, l. 14 and 16:** the sign before the term  $\mathcal{H}_\Delta(\eta_\Delta \tau''_{\Lambda \setminus \Delta} \omega_{\Lambda^c})$  is incorrect (twice). (Thanks to Arved Niklas Fanta for pointing this out.)
- **p. 280 [p. 261]:** The two lines before Definition 6.18 should belong to a new paragraph.
- **p. 284 [p. 265]:** In the caption of Fig. 6.1, one can assume, for simplicity, that  $A$  has its support in  $\Lambda$ .
- **p. 287, l. –1 [p. 268, l. 14]:** it would be better to use  $\eta_i$  rather than  $\omega_i$  in the right-hand side.
- **p. 301, l. 8:** One should replace the assumption that  $\mathcal{G}(\pi) \neq \emptyset$  by the assumption that  $\pi$  is quasilocal. This guarantees that  $\mathcal{G}(\pi)$  is a closed subset of  $\mathcal{M}_1(\Omega)$  (Lemma 6.27). (The condition  $\mathcal{G}(\pi) \neq \emptyset$  is not needed anymore by Theorem 6.26.) (Thanks to Andrew Yuan for pointing this out.)
- **p. 308, l. 12 [p. 286, l. –9]:** Corollary 3.60 should be Theorem 3.60.

- **p. 434, l. –4 [p. 402, l. 2]:** Replace “Let  $\epsilon > 0$  be such that  $e^\epsilon/(1+m^2) < 1$  and  $n$  be large enough to ensure that  $|\eta_i| \leq e^{\epsilon n}$  for all  $i \in \partial^{\text{ext}}\mathbb{B}(n)$ . Then,” by “Let  $\epsilon > 0$  be such that  $e^\epsilon/(1+m^2) < 1$  and, for all  $n$  large enough,  $|\eta_i| \leq e^{\epsilon n}$  for all  $i \in \partial^{\text{ext}}\mathbb{B}(n)$ . Then, for all such  $n$ ,”.
- **p. 462, l. –15 [p. 427, l. –10]:** the exterior-most expectation is missing the proper indices  $\langle \cdot \rangle_{\mathbb{B}(N)}^\eta$  (twice).
- **p. 462, l. –7:**  $\tilde{J}_{oj}, J_{oj}$  should be  $\tilde{J}_{0j}, J_{0j}$ .
- **p. 470, l. 17 [p. 434, l. –2]:** the comma before Namely should be a colon.
- **p. 480, l. 2 [p. 444, l. 8]:**  $J^m(i, \Theta(i))$  should be  $J_{i, \Theta(i)}^m$ .
- **p. 499-500 [p. 462]:** In (10.41) and in the statement of Theorem 10.25,  $\frac{\nu}{4d}$  should be  $\frac{\nu}{4d(2\pi)^d}$  (three occurrences). Moreover, in (10.41),  $\frac{1}{|\mathbb{T}_L|}$  should be  $\frac{(2\pi)^d}{|\mathbb{T}_L|}$ .
- **p.521, l. -2 [p. 484, l. –3]:** The functions  $(f_n)_{n \geq 1}$  should be uniformly strongly convex, in the sense that  $\inf_{n \geq 1} \inf_{x \in \mathbb{R}} f_n''(x) > 0$ .
- **p. 579, l. –17 [p. 532, l. –20]:** remove “a sequence  $(\delta_n)_{n \geq 1}$  decreasing to 0”.
- **p. 582, l. 6:** Theorem 6.27 should be Lemma 6.27.
- **General:** Roman Kotecký pointed out that we incorrectly write Marc Kac’s name as Kać in the book. This is indeed a mistake (and we’ve no idea why we did it).



## Additional exercises

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Here, we suggest a variety of additional relevant exercises. To facilitate using the latter for teaching, we do not plan on providing solutions.

### Chapter 3

► The following two exercises are complements to Section 3.3.

**Exercise.** Consider the one-dimensional Ising model on the torus  $\mathbb{T}_n$ . Show that

$$\langle \sigma_0 \rangle_{V_n; \beta, h}^{\text{per}} = \frac{\partial}{\partial h} \psi_{V_n}^{\text{per}}(\beta, h)$$

and deduce an expression for  $\langle \sigma_0 \rangle_{\beta, h} = \lim_{n \rightarrow \infty} \langle \sigma_0 \rangle_{V_n; \beta, h}^{\text{per}}$ .

**Exercise.** Let  $\Lambda_n = \{-n, \dots, n\}$ . Using the transfer matrix, compute  $\psi_{\Lambda_n}^+(\beta, h)$ ,  $\langle \sigma_0 \rangle_{\Lambda_n; \beta, h}^+$  and  $\langle \sigma_0 \sigma_i \rangle_{\Lambda_n; \beta, h}^+$ , as well as the corresponding limits as  $n \rightarrow \infty$ .

► The following exercises and remark are complements to Lemma 3.31:

**Exercise.** Prove that, in an dimension  $d$ , the function  $h \mapsto \langle \sigma_0 \rangle_{\beta, h}^+$  is strictly increasing on  $\mathbb{R}$  and that, for all  $\beta \geq 0$  and all  $h$ ,

$$\tanh(h) \leq \langle \sigma_0 \rangle_{\beta, h}^+ \leq \tanh(h + 4d\beta).$$

Hint: Use FKG for the upper bound. Then, show that  $\frac{\partial}{\partial h} \langle \sigma_0 \rangle_{\beta, h}^+ \geq 1 - (\langle \sigma_0 \rangle_{\beta, h}^+)^2$ . Deduce from that both the lower bound and the strict monotonicity.

**Remark.** An improved upper bound is provided in Theorem 3.53. When  $h$  is large, improved estimates follow from the expansion in Theorem 5.10. Moreover, related results are established in Theorem 4.12 and Proposition 10.16.  $\diamond$

**Exercise.** Prove that, in any dimension  $d$ , the function  $\beta \mapsto \langle \sigma_0 \rangle_{\beta, h}^+$  is strictly increasing when  $h > 0$  or when  $h = 0$  and  $\beta > \beta_c(d)$ .

► The following exercise is a complement to Section 3.7.2. Remember that the  $q$ -state Potts model ( $q \geq 2$ ) has spins taking value in the set  $\{0, \dots, q-1\}$  and a Hamiltonian given by  $-\beta \sum_{\{i,j\} \in \mathcal{E}_{\mathbb{B}(n)}^{\text{cb}}} \delta_{\sigma_i \sigma_j}$ . The corresponding Gibbs measure in  $\mathbb{B}(n)$  with boundary condition identically equal to 0 is denoted by  $\mu_{\mathbb{B}(n); \beta}^{q\text{-Potts}, 0}$ .

**Exercise.** 1. Check that the 2-state Potts model reduces to the Ising model at inverse temperature  $\beta/2$ .

2. Check that the Hamiltonian of the Potts model is invariant under permutation of the spin values.

3. Adapt Peierls' argument to the 2-dimensional Potts model, in order to show that, for all  $\beta$  large enough,

$$\liminf_{n \rightarrow \infty} \mu_{\mathbb{B}(n); \beta}^{q\text{-Potts}, 0}(\sigma_0 = 0) > \frac{1}{q},$$

which shows that the symmetry under permutation is broken at low temperatures. Hint: One way to proceed is to use contours that keep some information on the values of the spins on each side.

► The following exercise is a complement to Exercise 3.31.

**Exercise.** Deduce from Exercise 3.31, under the same assumptions, that

$$|\langle \sigma_A \rangle_{\Lambda; \mathbf{K}}| \leq \langle \sigma_A \rangle_{\Lambda; \mathbf{K}'}$$

## Chapter 9

► The following exercise is a complement to Theorem 9.14.

**Exercise.** The goal of this exercise is to provide an alternative proof of exponential decay of the 2-point function of  $O(N)$  models at sufficiently high temperature. It relies on a suitable high-temperature expansion. Fix  $0, k \in \mathbb{Z}^d$  and take  $n$  large enough to ensure that  $0, k \in B(n)$ .

1. Writing  $e^{\beta \mathbf{S}_i \cdot \mathbf{S}_j} = e^{-\beta} (e^{\beta (\mathbf{S}_i \cdot \mathbf{S}_j + 1)} - 1 + 1)$  and expanding the product over the edges of  $B(n)$ , show that

$$|\langle \mathbf{S}_0 \cdot \mathbf{S}_k \rangle_{B(n); \beta}^{\infty}| \leq \mathbb{Q}_{B(n); \beta}(0 \longleftrightarrow k),$$

where  $\mathbb{Q}_{B(n); \beta}$  is the probability measure on subsets of  $\mathcal{E}_{B(n)}$  defined by

$$\mathbb{Q}_{B(n); \beta}(E) \propto \int_{\Omega_{B(n)}} \prod_{\{u, v\} \in E} (e^{\beta (\mathbf{S}_u(\omega) \cdot \mathbf{S}_v(\omega) + 1)} - 1) \prod_{i \in B(n)} d\omega_i$$

and  $0 \longleftrightarrow k$  denotes the event that 0 and  $k$  are connected.

2. Let  $\mathbf{E}$  be a random subset of  $\mathcal{E}_{B(n)}$  distributed according to  $\mathbb{Q}_{B(n); \beta}$ . Show that, for any  $e \in \mathcal{E}_{B(n)}$  and  $E \subset \mathcal{E}_{B(n)} \setminus \{e\}$ ,  $\mathbb{Q}_{\Lambda; \beta}(e \in \mathbf{E} \mid \mathbf{E} \setminus \{e\} = E) \leq e^{2\beta} - 1$ .
3. Deduce from this that there exists  $\beta_0 > 0$  such that, for any  $\beta < \beta_0$ ,

$$\mathbb{Q}_{B(n); \beta}(0 \longleftrightarrow k) \leq e^{-c \|k\|_2}$$

for some  $c = c(\beta) > 0$  uniformly in  $n$ .