Phase Separation as a Large Deviations Problem

A Microscopic Derivation of Surface Thermodynamics for some 2D Spin Systems

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Version abrégée

L'organisation spatiale de plusieurs phases en coexistence, dans un réservoir dont les différentes parois adsorbent préférentiellement certaines d'entre elles, peut être déterminée, au niveau de la thermodynamique, à l'aide d'un principe variationnel: l'état d'équilibre est celui pour lequel la tension superficielle totale des interfaces entre les phases est minimale. Le but de cette thèse est de dériver ce principe variationnel à partir d'une description microscopique pour certains modèles de physique statistique classique sur réseau.

Plus précisément, nous développons un ensemble d'outils permettant une étude détaillée du phénomène de coexistence des phases, ainsi que de l'effet des conditions aux bords, dans certains modèles de spins en 2 dimensions. Dans la première partie de ce travail, nous considérons le modèle d'Ising pour lequel les résultats les plus satisfaisants sont obtenus; l'analyse est entièrement non-perturbative. Dans la seconde partie, certains des outils développés pour le modèle d'Ising sont étendus au modèle d'Ashkin-Teller; les résultats sont en partie perturbatifs.

Cette analyse permet de comprendre précisément comment une interface déterministe peut apparaître à un niveau macroscopique, ainsi que son statut pour un système de taille finie (mais grand), c'est-à-dire d'obtenir des informations sur les corrections à la thermodynamique. En particulier, nous expliquons pourquoi notre approche, basée sur une limite du continu, est plus adaptée que le formalisme usuel de la "limite thermodynamique" (plus précisément le formalisme de DLR) en ce qui concerne la description de phénomènes de coexistence.

Techniquement, nous obtenons des résultats de concentration de la mesure des lignes de séparation des phases (microscopiques) sur certains domaines que nous identifions aux interfaces. Notre analyse se base sur l'étude des grandes déviations de ces lignes de séparation des phases, et fait apparaître clairement le rôle de la tension superficielle comme fonction vitesse associée à ces grandes déviations.

Nous mettons ensuite en pratique ces techniques afin d'étudier deux exemples précis.

Le premier cas considéré est celui de l'accrochage d'une interface. Nous considérons un modèle d'Ising en 2 dimensions, avec des conditions aux bords induisant la présence d'un contour ouvert reliant deux points fixés sur des parois verticales opposées; la paroi inférieure est soumise à l'action d'un champ magnétique de surface. Nous démontrons que les configurations typiques de ce système possèdent deux régions macroscopiques contenant chacune des phases d'équilibre, séparées par une interface qui est solution du problème variationnel correspondant. Le diagramme de phase est obtenu explicitement et présente un phénomène très subtil connu sous le nom de réentrance.

Le second cas est celui du mouillage dans un système dont la quantité de chaque phase présente est fixée. Le modèle microscopique correspondant est le modèle d'Ising en

2 dimensions, avec aimantation totale fixée. On impose des conditions aux bords $+$, et un champ magnétique de surface (de signe arbitraire) agit sur la paroi inférieure. Nous démontrons que les configurations typiques du système sont composées d'une "goutte" de phase $-$ plongée dans la phase $+$, dont la forme est donnée par le problème variationnel correspondant, c'est-à-dire qu'elle peut être obtenue soit à l'aide de la "contruction de Wulff", soit à l'aide de la "construction de Winterbottom", suivant les valeurs du champ magnétique et de la température. Le diagramme de phase est obtenu explicitement, et vérifie en particulier le "critère de Cahn".

Finalement, nous considérons le problème des grandes déviations de l'aimantation dans chacun des deux plans du modèle d'Ashkin-Teller. La fonction vitesse associée à ces grandes déviations est obtenue explicitement dans certains cas.

Abstract

The spatial arrangement of several phases in coexistence, in a container the different sides of which adsorb preferentially some of them, can be determined, at the level of thermodynamics, using a variational problem: the equilibrium state is such that the overall surface tension of the interfaces between the phases is minimal. The aim of this thesis is to derive this variational principle from a microscopic description for some lattice models of classical equilibrium statistical physics.

More precisely, we develop a set of tools allowing a detailed study of the phase coexistence phenomenon, as well as the effect of boundary conditions, in some 2 dimensional spin systems. In the first part of this work, we consider the Ising model for which the most satisfactory results are obtained; the analysis is entirely non-perturbative. In the second part, some of the tools developed for the Ising model are extended to the Ashkin–Teller model; the results are partly perturbative.

This analysis allows us to understand precisely how a deterministic interface can appear at a macroscopic scale, and what is its nature in the case of finite (but large) systems, i.e. to obtain informations about the corrections to thermodynamics. In particular, we explain why our approach, based on some continuum limit, is better suited than the formalism of "thermodynamic limit" (more precisely, DLR formalism) to describe coexistence phenomena.

Technically, we obtain concentration properties of the measure describing the set of (microscopic) phase separation lines on some domains which we identify to interfaces. Our analysis is based on the study of the large deviations of these phase separation lines, and shows clearly that the surface tension plays the role of the rate function associated to these large deviations.

We then apply these techniques to study two specific examples.

The first considered case concerns the pinning of an interface. We consider a 2 dimensional Ising model, with boundary condition implying the presence of an open contour linking two fixed points on opposite vertical walls; the bottom wall is subject to a boundary magnetic field. We prove that the typical configurations of this system possess two macroscopic regions each of them containing one of the equilibrium phases, separated by an interface which is solution to the corresponding variational problem. The phase diagram is obtained explicitly and exhibits a highly non-trivial behaviour known as reentrance.

The second case is about the phenomenon of wetting in a system in which the total amount of each present phase is fixed. The corresponding microscopic model is the 2 dimensional Ising model with fixed total magnetization. We impose +-boundary condition, and a boundary magnetic field (with arbitrary sign) acts on the bottom wall. We prove that the typical configurations of this system are composed of one "droplet" of $-$ phase

immersed in the + phase, the shape of which is given by the corresponding variational problem, i.e. it can be obtained either by the "Wulff construction", or the "Winterbottom construction", depending on the values of the temperature and boundary magnetic field. The phase diagram is obtained explicitly, and confirms in particular "Cahn's criterion".

Finally, we consider the problem of large deviations of the magnetization in each of the two planes of the Ashkin–Teller model. The rate function is obtained explicitly in some cases.

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Contents

Chapter 0

Introduction

Looking for the fundamental laws of Nature is one of the main tasks of Physics; to make progress in understanding the universe, one needs to know the rules of the game. However, this is not the only goal of Physics; of as much interest is the development of techniques allowing one to derive informations from such microscopic laws relevant to macroscopic phenomena.

The analysis of large systems is extremely complicated and physicists have devised two theories to deal with such problems. The first one, developed during the 19th Century by people like Carnot, Clausius, Kelvin, Joule and others is Thermodynamics, an extension of previously known theories of macroscopic systems (Mechanics, Electrodynamics) introducing the new concept of heat. This general theory of macroscopic phenomena can be divided into two main parts: Thermostatics which describes systems at equilibrium (or evolving in such a way as to remain close to equilibrium all the time), and Thermokinetics which describes the evolution of systems far from equilibrium. Both are phenomenological: Their purpose is to find relations between the behaviour of different systems, or of a given system in different situations, while ignoring the details of the microscopic evolution, relying only on some set of *principles*. This makes Thermodynamics an incredibly versatile tool which can be used to study any macroscopic physical system (from a glass of freezing water to imploding stars, including superconductors, and possibly even black holes). The fact that Thermodynamics is independent of a precise understanding of the microscopic phenomenon allows people to apply it even to systems for which such a description is still lacking and therefore makes it a very useful theory in Chemistry or Biology for example. However, from a physicist's point of view, it is very important to be able to make the connection between the laws governing microscopic evolutions and those describing macroscopic phenomena. Statistical Mechanics, which has been introduced at the end of 19th Century by Maxwell, Boltzmann, and Gibbs is such a theory.

This theory can again be decomposed into two main parts: Equilibrium Statistical Mechanics which describes systems at equilibrium, and Non-Equilibrium Statistical Mechanics which (would/should) describe systems far from equilibrium[1]. Non-Equilibrium Statistical Mechanics is the subject of much work nowadays but is still far from being conceptually understood. On the other hand Equilibrium Statistical Mechanics has be-

[1]In fact a further distinction can be done in both cases, between the Classical and Quantum versions of the theory.

come a very well established theory, even though a satisfactory derivation of its basic postulates from microscopic laws is still missing. Contrarily to Thermodynamics, it relies on a description of the microscopic system, combining the knowledge of the microscopic dynamics with techniques of Probability Theory to obtain precise statistical informations on sufficiently "nice" observables.

No system out of equilibrium will be considered in this thesis and therefore the rest of this introductory chapter will be dedicated to Thermostatics and Equilibrium Statistical Mechanics only. In Section 0.1, we describe the philosophy and the results of this thesis in non-technical terms. Section 0.2 is dedicated to a basic exposition of Thermostatics and the formalism of infinite volume Statistical Mechanics; some explanations about how the former can be understood starting from the latter is given. In Section 0.3, the structure of the thesis and a description of the content of each chapter is given. Finally, a comparison between the techniques developed here and other ones, as well as a brief overview of possible extensions and open problems are given in Section 0.4.

0.1 Non-technical description of the content of this work

0.1.1 What is this work about?

As discussed above, physicists have introduced two theories of macroscopic phenomena, with very different physical status. While Thermostatics is purely empirical, Equilibrium Statistical Mechanics, on the contrary, should be derivable from the laws of microscopic dynamics. This has not yet been achieved, and we are not going to discuss these issues here. However, assuming the validity of Equilibrium Statistical Mechanics, it is important to understand how it is related to Thermostatics. More precisely, in what sense is it possible to deduce the basic principles of Thermostatics from those of Equilibrium Statistical Mechanics? To answer this question, we first sketch what Thermostatics has to say regarding the description of physical systems.

Thermostatics

Let us first consider some part of some homogeneous substance deep inside the bulk, in such a way as to ensure that it is possible to neglect the effect of the boundaries. In this situation, the theory says that there exists a small number of "variables" (pressure, temperature, density,...) providing a complete description of the system, in the sense that it is not possible to distinguish between two macroscopic states corresponding to the same values of these variables. Moreover, there exists some function of these variables, the *thermodynamic potential*, which contains all the relevant informations about the system. Knowing this function, we can obtain all quantities of interest by differentiating it with respect to its variables. The equilibrium values of the parameters are those extremizing the thermodynamic potential under some constraints.

If the thermodynamic potential satisfies some strict convexity (or strict concavity) property (it is always concave or convex in each of the variables[2]), then there is only one

[2]The theory is sometimes extended to consider also non convex (or concave) function, thus allowing the description of metastability; however, this scenario for metastability does not seem to be correct (or at least not always), see next subsection.

solution to this variational problem and therefore a unique equilibrium phase. When the potential is not strictly convex (or concave), then there can be more than one solution to the variational problem and therefore more than one equilibrium state (if this is the case, then we say that a phase transition occurs). In such a case, it is possible to distinguish some particular solutions, the *extremal states*, which correspond to a situation where the whole system is in one of the equilibrium phases. The other solutions, called *mixed states* (or mixtures), correspond to phase coexistence: Various parts of the system are in various extremal states. It is possible, in some circumstances, to determine the amount of each of the extremal phases. However, the theory does not give any information about the spatial organization of the system.

To determine the shapes of the different regions, one has to add to the basic theory some new quantity, the *surface tension*, which describes the "cost" of an interface between two different phases, or between one phase and the boundary of the container, and a new principle which states that the geometry of the system can be obtained by solving the following variational problem: Find the interface, or the family of interfaces, which minimizes the surface tension, under suitable constraints (notice that the solution cannot be described using only a "small" set of variables, since we have to describe the shape and location of spatial regions together with the values of the variables in each of these regions). We refer to this extension of Thermostatics as *Surface Thermostatics*[3]. It provides an accurate (phenomenological) description of the macroscopic organization of the equilibrium phases of the system, taking also into account the effect of the boundaries.

Equilibrium Statistical Mechanics

The usual formalism of Equilibrium Statistical Mechanics makes use of the so-called *thermodynamic limit.* In this limit, the size of the system becomes infinite. It is then possible to describe the state of the system by some distribution on the set of all possible microscopic states. The structure of these distributions is very well known (see Subsection 0.2.2) and provides a nice justification of Thermostatics: It is possible to understand why a few variables are sufficient to describe the system, where the thermodynamic potential comes from, what are phase transitions. In this sense, Thermostatics is pretty well understood in terms of infinite-volume statistical mechanics. However, there is a price to pay; once the thermodynamic limit has been taken, the boundaries are sent to infinity, and in fact all information about the non-local behaviour of the system is lost. It is still possible to describe phase transitions in a similar way as in Thermostatics, i.e. the set of all possible equilibrium distributions for some fixed parameters can contain more than one element, and therefore there can be more than one equilibrium state; the structure of this set is such that it is possible to distinguish between *extremal* and *mixture* states. But any information about the spatial organization of the system is lost. In particular, it is not possible, using this approach, to justify *Surface* Thermostatics. In fact, this formalism provides an adequate description of homogeneous systems only[4].

The aim of this work is to develop some tools allowing one to understand Surface

[3] There is no really axiomatic theory of such phenomena, but rather a set of particular models. The terminology "surface thermodynamics" should therefore not be taken too seriously.

[4] It is nevertheless possible to have non-translation invariant states in this formalism; we discuss this point later in this chapter.

Thermostatics, starting from the statistical mechanical description of one of the simplest non-trivial systems, the 2D Ising model. We try to advocate that to study these phenomena, it is useful to replace the usual Thermodynamic limit (and especially the DLR formalism, see Subsection 0.2.2) by some continuum limit, in which interfaces appear as natural objects and their typical spatial disposition is given by the variational problem of Surface Thermostatics. In this sense, it is possible to deduce Surface Thermostatics from Equilibrium Statistical Mechanics for this model (at least in many situations).

0.1.2 What are the results?

We first develop a set of tools to deal with such questions. These are very versatile and can be used in many different situations. We illustrate their use with two typical problems, which correspond to two natural ways of imposing the presence of interfaces in the system: The first one by some well-chosen boundary conditions implying the presence of an interface connecting to opposite walls of a box; the second by fixing the amount of each of the two phases present. We discuss now the physical results obtained for these two cases.

Reentrant pinning transition

Consider a 2D system in some rectangular box B. We suppose that the parameters are chosen in such a way as to have phase coexistence: There are two equilibrium states, $+$ and $-$. Suppose, moreover, that the boundary of B acts in such a way as to impose the presence of the $+$ phase in the upper part of the box and the $-$ phase in the bottom part. There is an interface between the two phases, the extremities of which are at fixed heights on the left-hand and the right-hand vertical walls of the box. The bottom wall of B adsorbs preferentially one of the two phases.

To obtain the equilibrium states from Thermostatics, one has to give the equations for the surface tensions for an interface between the two phases at arbitrary angles, and for the surface free energy associated to the contact of each of the two phases with the bottom wall. With these informations, the rules of Surface Thermostatics state that the equilibrium state of the system is given by the interface (with fixed extremities) which minimizes the overall surface tension. It is not difficult, using convexity of the surface tension, to see that, depending on the value of the temperature, the surface tension of the bottom wall and the heights of the endpoints of the interface, there are two kinds of solutions: Either the interface is the straight line between the two fixed points, or it is a broken line which touches the bottom wall, such that the angles it makes with the wall are given by the so-called *Herring-Young* formula.

In chapter 6, we study the statistical mechanical equivalent of this problem for the 2D Ising model. We show that the typical configurations of the system indeed possess two macroscopic regions corresponding to the two equilibrium phases of the Ising model, which are separated by a well-defined interface whose shape is given precisely by the variational problem of Surface Thermostatics. Moreover, when the size of the system is finite (but large), we are able to discuss in which sense this remains true, i.e. to show that the interface becomes "diffuse", but that its "width" remains very small (it is possible to give a precise meaning to this).

Not only does the statistical mechanical derivation justify the Surface Thermostatics variational problem, but it also provides analytic expressions for the surface tensions. Using these explicit informations, it is possible to compute the exact phase diagram for this problem, which happens to exhibit a very interesting non-perturbative behaviour. For well-chosen values of the heights and of the bottom wall free energy, there exist $0 < T_1 < T_2 < T_3 < T_c$ such that when the temperature is between 0 and T_1, the interface is pinned to the wall; when it is between T_1 and T_2, it is unpinned; between T_2 and T_3 it is pinned again; finally, between T_3 and T_c it is once more unpinned (above T_c, the critical temperature, there is only one equilibrium phase and the interface disappears.). Such a phenomenon is known as *reentrant pinning-depinning transition*; there is no simple way to explain it, since it is a consequence of the fact that several quantities have a non-trivial dependence on the temperature in this problem: the value of the surface tension, its anisotropy and the wall free energy.

Wetting transition and Wulff shape

The other natural way to induce phase separation in some system is to impose the total amount of each phase present in the system. Consider, for example, the case of some gas in a container which prevents the atoms to enter or leave the container. In the liquid-vapor phase coexistence region the system has at its disposal only two equilibrium densities. To obtain these two densities with the constraint that the mean density in the box is fixed (since the volume and the number of atoms do not change), several regions in which one of these two densities is realized will appear, resulting thus in the separation of the gas and liquid phases. For example, we may have a large component of gas phase, with droplets of liquid inside it, or a large component of liquid phase with bubbles of gas inside it (this will depend on what phase is preferentially adsorb at the boundary, or on the presence of a possible gravitational field). The case we have studied corresponds to a two dimensional container with three walls adsorbing preferentially one of the phases, while the fourth one (let's say, the bottom one) has a wall free energy which we can choose as we want; there is no gravitational field. We are interested in the shape of the interfaces separating the two phases. In particular, what happens when the bottom wall free energy is changed?

Using the convexity of the surface tension, it is possible to show that the solution of the variational problem of Surface Thermodynamics corresponds to a single droplet of one phase, which we call $-$, immersed inside the other, which we call $+$. The shape of the droplet is given by the so-called *Wulff construction* and it remains "in the middle of the box" when the bottom wall adsorbs preferentially the $+$ phase. When the bottom wall free energy becomes sufficiently favorable to the $-$ phase, then the droplet becomes sticked to the wall, this is known as the *wetting transition*[5]. The shape of the droplet in this case is given by the *Winterbottom construction*; in particular, the angles between the droplet and the wall are solution to the Herring-Young equation.

In chapter 7, we study the statistical mechanical equivalent of this problem for the 2D Ising model. As before, we show that the typical configurations of the system indeed possess two macroscopic regions corresponding to the two equilibrium phases of the Ising model, which are separated by a well-defined interface whose shape is given precisely by the variational problem of Surface Thermostatics. Again the effect of a finite size of the

[5]A simple physical discussion of this phenomenon can be found in [EC].

system can be understood precisely. The variational problem of Surface Thermostatics is therefore obtained again from Equilibrium Statistical Mechanics.

Since the surface tensions are explicitly known, it is possible to compute the phase diagram associated to this problem. In particular, the so-called *Cahn's criterion* is verified, and the phase diagram obtained is precisely the same one as in the case of the wetting transition in a system in which the total amount of each phase is *not* fixed. In such a system, the transition manifests itself by a qualitative change in the width of the film of $-$ on the bottom wall. When the wall adsorbs preferentially the $+$ phase, nothing happens at the bottom wall. When it becomes sufficiently favorable to the $-$ phase, a microscopic film of $-$ phase is created; if we make the wall adsorb even more the $-$ phase then there is a second transition, where the film becomes very thick (its thichness diverging when the system becomes large). The second wetting transition can also be seen in a system with fixed amount of phases, if the corresponding droplet is not "macroscopic" (see Chapter 7 for details). In this case, the second transition corresponds to a degenerate droplet which completely covers the wall.

Extension of the techniques to other models

In the second part of this work, we study a generalization of the Ising model, known as the Ashkin–Teller model, which possesses four equilibrium phases at sufficiently low temperature. Several basic tools can be straightforwardly extended to this case. However, some new problems appear which we were able to solve only by using perturbative techniques. The main purpose of this second part is to explain the difficulties that have to be dealt with when considering more complicated systems. We study as an example the problem of large deviations in this model, for which results can be obtained in some situations.

0.2 Description of macroscopic systems

0.2.1 Thermostatics

It is not our aim here to give a self-contained exposition of the basics of Thermostatics which can be found in many good books (for example [Ca]; see also the recent paper [LiYn]), but rather to state the postulates and some related results[6] which will enable us to discuss the relevance of the hypotheses of Thermostatics from the point of view of Equilibrium Statistical Mechanics.

Equilibrium states and the Entropy Representation

The fundamental observation is that it is possible to describe a macroscopic system with a very small number of parameters. Indeed, consider a glass of water; to describe this system from a microscopic[7] point of view it is necessary to specify the position, orientation and momentum of each molecule in the glass, i.e. roughly 10^{24} variables. However, under normal circumstances, anything you may do to or with the glass of water will only depend on a small set of quantities: Volume, temperature and density for example.

[6] This section follows [Gru].

[7] We will always think in classical terms. Clearly this distinction is irrelevant for the discussion given here.

This phenomenon is known as *reduction of variables*. Of course, this is true only when the system has reached equilibrium; for example if you shake the glass, then you will need more variables to describe the macroscopic behaviour of the water (until its return to equilibrium). Thermostatics' purpose is to study the properties of such equilibrium states, which we now define.

Definition.

(D1) *An* **equilibrium state** *is a stationary, stable state of an isolated system. The state must not be modified if we add further constraints (for example by the introduction of a set of walls permeable to some quantities and not to other ones). Moreover, if two systems Σ_1 and Σ_2 are in equilibrium with a third system Σ_3, then they are also in equilibrium with each other.*

Experiment shows that it is possible to describe an equilibrium system with a small number of variables. More precisely, the set of equilibrium states is a convex subset $\mathbb{X}$ of $\mathbb{R}^m$, for some $m \ll N$, independent of N, N being the number of variables necessary to give a complete microscopic description of the system. A state function is a function from $\mathbb{X}$ to $\mathbb{R}$. The first principle states the existence of a state function corresponding to the energy of the system.

Postulate 1 (First Principle)**.** *For any system, there exists a scalar, extensive, conserved state function E associated to time homogeneity, which is called the* energy*.*

A system whose equilibrium states are homogeneous and isotropic is called a *simple system.* It is a macroscopic system (sufficiently large so that the boundary effects are negligible) for which there is a one-to-one correspondence between its equilibrium states and a set of extensive, conserved quantities, $\underline{X} = (X_0 \doteqdot E, X_1, \ldots, X_n) \in \mathbb{X}$, $X_n > 0$. The systems studied in Thermostatics are composed of a union of simple systems in contact with one another.

We need a method to determine the set of equilibrium states of a system under some constraints. A fundamental idea in Thermostatics, which is due to Gibbs, is to postulate the existence of some fundamental quantity $\Phi(\underline{X})$, the *thermodynamic potential*, associated to the set of variables $(X_0, \ldots, X_n)$ with which we describe the system, which gives a complete description of the system (from the point of view of Thermostatics). The space $\mathbb{R}^{n+2}$ of the variables $(\Phi, \underline{X})$ is the *Gibbs space*. The manifold in the Gibbs space given by the *fundamental relation*

$$\Phi = \Phi(\underline{X}) \tag{1}$$

describes the equilibrium states of the corresponding system. In particular, all the thermodynamic quantities can then be obtained by partial differentiation of the function Φ with respect to its various arguments. One such thermodynamic potential is the *entropy.*

Postulate 2 (Second Principle)**.** *For any system Σ, there exists a scalar, extensive state function S^Σ, which is called the* entropy*. It has the following property: If the system is composed of K simple subsystems, then its equilibrium state $(\overline{X}_1^A = \overline{U}^A, \ldots, \overline{X}_n^A)$, $A = 1, \ldots, K$, satisfies*

$$S^\Sigma = \sum_{A=1}^{K} S^A(\overline{\underline{X}}^A) = \max_{\substack{\underline{X}^A \\ \text{comp. with constraints}}} \sum_{A=1}^{K} S^A(\underline{X}^A)\,.$$

The equation $S = S(\underline{X})$ of a simple system is the *fundamental relation in the Entropy Representation*. From the Second Principle, it is easy to derive the following properties of the entropy[8]:

- The entropy of a simple system is homogeneous of degree one, i.e., for any α,

$$S(\underline{X}) = \alpha^{-1} S(\alpha \underline{X}) \,. \tag{2}$$

- The entropy of a simple system is necessarily concave.

The proof of the above properties is elementary. The first one follows from the fact that any system which is in a homogeneous isotropic equilibrium state can be thought of as being build of any number of identical (simple) subsystems in the same state. To prove the second statement, consider a simple system being composed of two subsystems with no walls, not necessarily in the same state; then the concavity follows from the Second Principle applied to this composed system.

As a consequence of the first property, we can define the *entropy density*,

$$\begin{aligned} s(u, x_1, \ldots, x_{n-1}) &\doteqdot \frac{1}{X_n} S(U, X_1, \ldots, X_n) \\ &= S(u, x_1, \ldots, x_{n-1}, 1) \,, \end{aligned} \tag{3}$$

where $u \doteqdot U/X_n$, $x_i = X_i/X_n$, $i = 1, \ldots, n-1$. Of course, s is concave if and only if S is concave.

Let us study a system Σ composed of K identical subsystems. The second principle gives

$$S^{\Sigma}(X_0, \ldots, X_n) = \max_{\substack{X_i^A \in \mathbb{X} \\ \sum_A X_i^A = X_i}} \sum_{A=1}^{K} S(X_0^A, \ldots, X_n^A) \,. \tag{4}$$

It is not difficult to study this equation under various assumptions on the properties of S.

- S is concave, and it is strictly concave at $\underline{X}^A = \frac{1}{K}\underline{X}$, $A = 1, \ldots, K$.
  In this case, there is a unique solution $\overline{\underline{X}}$ to the above equation, given by $\overline{X}_i = X_i/K$, $i = 0, \ldots, k$, i.e. the equilibrium state is homogeneous. That this is a solution follows from the fact that, by concavity,

$$\sum_{A=1}^{K} S(\frac{1}{K}\underline{X}) = KS(\sum_{A=1}^{K} \frac{1}{K}\underline{X}^A) \geqslant K \sum_{A=1}^{K} \frac{1}{K} S(\underline{X}^A) \,, \tag{5}$$

  for any decomposition $(\underline{X}^A)$ such that $\sum_A \underline{X}^A = \underline{X}$. The uniqueness is a consequence of strict concavity.

[8]Thermostatics is sometimes extended to allow for non-concave entropy function. The values of the parameters for which the function is locally convex represent *unstable* states, while the values of the parameters for which the function is locally concave, but is different from its concave envelope represent *metastable* states. However, this description of metastability of a system by analytic continuation of the thermodynamic potential does not seem to be correct, at least in several cases, as has been shown in Equilibrium Statistical Mechanics in the case of lattice gases, see [Is1, Is2, Is3] where it is proved that the phase transition point corresponds to an essential singularity of the thermodynamic potential.

- S is concave, but it is affine on an open set $\mathcal{P}$ containing the point $\underline{X}^A = \frac{1}{K}\underline{X}$, $A = 1, \ldots, K$.
  $\mathcal{P}$ is supposed to be the largest such set. Since S is concave, the homogeneous state is still solution, but it is no more unique. Indeed, we show now that every decomposition with $\sum_A \underline{X}^A = \underline{X}$ and $\underline{X}^A \in \mathcal{P}$, $A = 1, \ldots, K$ is also solution. For any such decomposition, we have since S is affine on $\mathcal{P}$,

$$\sum_A S(\underline{X}^A) = KS(\sum_A \frac{1}{K}\underline{X}^A) = KS(\frac{1}{K}\underline{X}) \,. \tag{6}$$

That there are no other solutions follows from maximality of $\mathcal{P}$ and concavity of S. The second case corresponds to a *first-order phase transition*. In such a case, it is always possible to write the solutions $\underline{X}^A$ as a convex combination of points on the boundary of $\mathcal{P}$,

$$\underline{X}^A = \int_{\partial\mathcal{P}} \underline{Y} \, \mathrm{d}\mu^A(\underline{Y}) \,, \tag{7}$$

with μ^A a probability measure on $\partial\mathcal{P}$ (notice that $\mathcal{P}$ is necessarily convex). In such a case we can write

$$S^\Sigma(\underline{X}) = \int_{\partial\mathcal{P}} S(\underline{Y}) \, \mathrm{d}\mu(\underline{Y}) \,, \tag{8}$$

with $\mu(\cdot) \doteq \frac{1}{K}\sum_A \mu^A(\cdot)$. Therefore the system behaves as if its constituent subsystems were distributed[9] on $\partial\mathcal{P}$ with the measure μ. The corresponding extremal points are called *pure phases*, while all other decompositions are called *mixtures*. In Nature, $\mathcal{P}$ is usually a simplex and therefore this convex decomposition is unique.

The Energy Representation

There exist equivalent formulations of equation (4), in terms of other thermodynamic potentials which depend on different sets of variables. The first, simplest such reformulation is the *Energy Representation*, where the thermodynamic potential is the energy E, seen as a function of the variables $S, X_1, \ldots, X_n$. To define this function we have to invert S; this is possible in the regions where $\partial S/\partial E > 0$, or $\partial S/\partial E < 0$.

Definition.

(D2) *The* temperature T *is defined by* $T \doteq \dfrac{1}{\partial S/\partial E}$.

Remark. Observe that, when S is differentiable, all simple subsystems Σ^A of a given system Σ satisfy $T^A = T$, where T^A is the temperature of the system Σ^A, while T is the temperature of the system Σ. Thus, at equilibrium, the temperatures are equal.

The equation

$$E = E(S, X_1, \ldots, X_n) \tag{9}$$

[9] Of course, since the number of subsystems is finite, it is not always possible to distribute them exactly in this way; however as K becomes large the approximation will be better and better. In particular, in the continuum case considered below, the number of subsystems will be unbounded.

is the *fundamental relation in the Energy Representation.* It is easy to derive informations on the function E from the corresponding properties of the function S. In particular, the following is true:

- $E(S, X_1, \ldots, X_n)$ is homogeneous of degree one.
- If $T > 0$, then $E(S, X_1, \ldots, X_n)$ is convex;
- If $T < 0$, then $E(S, X_1, \ldots, X_n)$ is concave;
- If $T > 0$, then

$$E(S, X_1, \ldots, X_n) = \min_{\substack{S^A, X_1^A, \ldots, X_n^A \\ \sum_A S^A = S, \sum_A X_i^A = X_i}} \sum_{A=1}^{K} E(S^A, X_1^A, \ldots, X_n^A)\,.$$

- If $T < 0$, then

$$E(S, X_1, \ldots, X_n) = \max_{\substack{S^A, X_1^A, \ldots, X_n^A \\ \sum_A S^A = S, \sum_A X_i^A = X_i}} \sum_{A=1}^{K} E(S^A, X_1^A, \ldots, X_n^A)\,.$$

Similarly, it is possible to define other representations, which are constructed using Legendre transforms of these two representations with respect to some subset of their variables. Before proceeding to the discussion of these other representations, we have to introduce the conjugate intensive quantities.

Intensive quantities and state equations

Definition.

(D3) *The quantity conjugate to* X_i $(i = 0, \ldots, n)$ *in the Entropy Representation is defined by*

$$F_i(X_0 = E, \ldots, X_n) \doteqdot \frac{\partial S(X_0, X_1, \ldots, X_n)}{\partial X_i}\,.$$

(D4) *The quantity conjugate to* X_i $(i = 1, \ldots, n)$ *in the Energy Representation is defined by*

$$P_i(S, \ldots, X_n) \doteqdot \frac{\partial E(S, X_1, \ldots, X_n)}{\partial X_i}\,.$$

The quantity conjugate to S *in the Energy Representation is the temperature.*

It is not difficult to prove the following properties of these conjugate quantities,

- At all points of differentiability of S and E,

$$F_0(E, X_1, \ldots, X_n) = \frac{1}{T(S(E, X_1, \ldots, X_n), X_1, \ldots, X_n)}\,, \tag{10}$$

$$F_i(E, X_1, \ldots, X_n) = -\frac{P_i(S(E, X_1, \ldots, X_n), X_1, \ldots, X_n)}{T(S(E, X_1, \ldots, X_n), X_1, \ldots, X_n)}\,; \tag{11}$$

- The quantities $F_0, \ldots, F_n$, $T, P_1, \ldots, P_n$ are homogeneous functions of degree 0 in the corresponding extensive variables;

- The function S can be written as

$$S(X_0 = E, X_1, \dots, X_n) = \sum_{i=0}^{n} F_i(X_0, \dots, X_n) X_i \, ; \tag{12}$$

- The function E can be written as

$$E(S, X_1, \dots, X_n) = T(S, X_1, \dots, X_n) S + \sum_{i=1}^{n} P_i(S, X_1, \dots, X_n) X_i \, . \tag{13}$$

The proofs of the two last statements follow from the homogeneity of S and E (write $S(\lambda \underline{X}) = \lambda S(\underline{X})$ and derive with respect to λ).

Definition.

(D5) *Homogeneous functions of degree 0 in their extensive parameters are called* intensive.

The two equations (12) and (13) are the *Euler relations*. The equations

$$F_i = F_i(S, X_1, \dots, X_n) \, , \quad i = 0, \dots, n \, , \tag{14}$$

and

$$T = T(S, X_1, \dots, X_n) \tag{15}$$

$$P_i = P_i(E, X_1, \dots, X_n) \, , \quad i = 1, \dots, n \, , \tag{16}$$

are the *state equations* in the Entropy and Energy Representations. Clearly the knowledge of all state equations in one representation is equivalent to the knowledge of the corresponding fundamental relation.

Equivalent representations

We want to construct other thermodynamic potentials, whose variables are a combination of extensive and intensive quantities[10]. More precisely, let $I \subset \{1, \dots, n+1\}$, and $J = \{1, \dots, n+1\} \setminus I$, where we used the notation $X_{n+1} \doteqdot S$. We consider the variables $X_I \doteqdot \{X_i \, : \, i \in I\}$ and $P_J \doteqdot \{P_i \, : \, i \in J\}$.

Definition.

(D6) *The* thermodynamic potential *for the set of variables (X_I, P_J) is defined as the Legendre transform of U with respect to the variables X_i, $i \in J$, i.e.*

$$U^\star_J(X_I, P_J) \doteqdot \inf_{X_J} \Big[U(S, X_1, \dots, X_n) - \sum_{i \in J} X_i P_i \Big] \, .$$

[10] We start from the Energy Representation, but we could also consider the thermodynamic potentials obtained from the Entropy Representation.

The equation

$$U_J^\star = U_J^\star(X_I, P_J) \tag{17}$$

is the fundamental relation in the representation $U_J^\star$. The following properties follow from the definition

- The quantities conjugate to the variables X_I, P_J are

$$\frac{\partial U_J^\star}{\partial X_i} = P_i\,, \qquad \frac{\partial U_J^\star}{\partial P_i} = -X_i\,; \tag{18}$$

- $U_J^\star(X_I, P_J)$ is homogeneous of degree one in the extensive variables X_I;
- If $T > 0$, $U_J^\star(X_I, P_J)$ is convex as a function of X_I, otherwise it concave as a function of X_I;
- If $T < 0$, $U_J^\star(X_I, P_J)$ is concave as a function of P_J, otherwise it convex as a function of P_J;
- If $T > 0$, the thermodynamic potential $U_J^\star(X_I, P_J)$ satisfies

$$U_J^\star(X_I, P_J) = \min_{\substack{X_I^A \\ \sum_A X_I^A = X_I}} \sum_{A=1}^{K} U_J^\star(X_I^A, P_J)\,. \tag{19}$$

- If $T < 0$, the thermodynamic potential $U_J^\star(X_I, P_J)$ satisfies

$$U_J^\star(X_I, P_J) = \max_{\substack{X_I^A \\ \sum_A X_I^A = X_I}} \sum_{A=1}^{K} U_J^\star(X_I^A, P_J)\,. \tag{20}$$

The continuum limit

In general, a system Σ cannot be decomposed into a finite number of simple subsystems. However, it is always possible to approximate Σ by an increasing number of simple subsystems. We construct in this way a sequence of systems Σ_K which converges to the system Σ when K goes to infinity. We have to partition the system Σ into a number of subsystems. We only consider the Entropy Representation, although a similar construction can be performed in other cases. To do this, we suppose that the volume V of the box Λ confining Σ is one of the variables describing the system (any extensive quantity would do the trick, but this one is the more natural). We then divide the box Λ into K boxes of volume V/K (of reasonable shape, so that boundary effects can be neglected). The concavity of the entropy density implies that

$$S^\Sigma(X_0 = E, \ldots, X_n = V) = \max_{\substack{X_i^A \\ \sum_A X_i^A = X_i}} \sum_{A=1}^{K} \frac{V}{K} s(x_0^A, \ldots, x_{n-1}^A)\,. \tag{21}$$

Taking the limit, we obtain

$$S^\Sigma(X_0 = E, \ldots, X_n = V) = \max_{\substack{x_i(r)\,,\, r\in\Lambda \\ \int \mathrm{d}V(r) x_i(r) = X_i}} \int_\Lambda \mathrm{d}V(r) s(x_0(r), \ldots, x_{n-1}(r))\,. \tag{22}$$

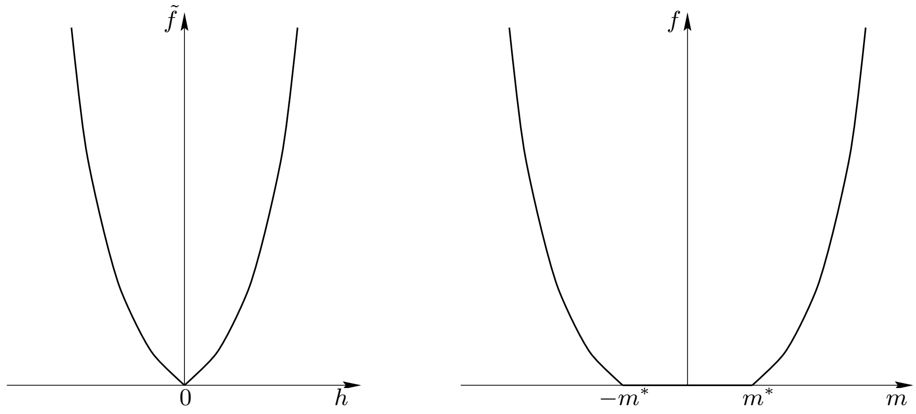

FIGURE 1. Schematic plots of the densities of the Helmholtz free energy $\tilde{f}$ and free energy f of a ferromagnet.

This limit is referred to as the *continuum limit*[11]. Observe that the homogeneous states (i.e. the constant functions, $x_i(r) = X_i/V$) are always solutions of this variational problem.

The case of a magnetic system

We briefly discuss the Thermostatics description of a magnetic system corresponding to the microscopic system considered in the first Part of this work.

A macroscopic sample of some magnetic system, which we suppose to have a preferred axis, can be described by three extensive variables: the internal energy E, the magnetization M and the volume; when we apply to it some magnetic field, it is in the direction of the axis. Since we will only consider systems which have a crystalline structure, we can replace the volume by the number of "atoms", N. According to the Second Principle, we can write

$$S(U,M,N) = \max_{\substack{u(r),m(r)\,,\,r\in\Lambda \\ \int \mathrm{d}V(r)u(r)=U\,,\,\int \mathrm{d}V(r)m(r)=M}} \int \rho\,\mathrm{d}V(r)s(u(r),m(r))\,, \tag{23}$$

where $\rho = N/V$ is the density of the sample. Two thermodynamic potentials play an important role for these systems:

Definition.

(D7) *The* **free energy** F *is the thermodynamic potential associated to the set of variables* (T,M,N)*.*

(D8) *The* **Helmholtz free energy** $\tilde{F}$ *is the thermodynamic potential associated to the set of variables* (T,h,N)*, where* $h \doteqdot T\partial S/\partial M$ *is the* **magnetic field**.

We suppose that $\tilde{F}$ is an even function of the magnetic field (reflecting the fact that the system is invariant under reflection). If the magnetization and the magnetic field have the same sign then the system is *paramagnetic*, otherwise it is *diamagnetic*. The quantity

$$M^* \doteqdot \lim_{h\to 0^+} \frac{\partial \tilde{F}}{\partial h}$$

[11] A similar construction will be done for Gibbs measure in Chapter 7. However these two notions must be distinguished.

is called the *spontaneous magnetization*. If it is non-zero, then the system is said to be *ferromagnetic*. Clearly, in such a case the system undergoes a phase transition. Indeed, in such a case the Helmholtz free energy is not differentiable at 0 and, correspondingly, the free energy is constant between $-M^*$ and M^*.

0.2.2 Equilibrium Statistical Mechanics

The purpose of Equilibrium Statistical Mechanics is to describe macroscopic systems starting from the microscopic interactions. In principle, to obtain information on a system, one should integrate the corresponding equations of motion for a given initial condition. Of course, it is impossible to know precisely the microscopic state of a macroscopic system, and therefore it is reasonable to replace the initial condition by some measure on the phase space. Then the problem amounts to computing the evolution of this measure. Under suitable hypotheses, it should be possible to prove that any "reasonable" measure should converge after some time to a stationary measure; such a stationary measure would describe an equilibrium state of the system. The usual "justification" of Equilibrium Statistical Mechanics starting from Classical Mechanics amounts to prove that there is one and only one such stationary measure, the *microcanonical measure*, for a Hamiltonian flow if the evolution is ergodic, and that any measure absolutely continuous with respect to this one will eventually converge to the stationary measure if the evolution is mixing. Such a justification is far from being satisfactory from a physicist's point of view for several reasons: First, these restrictions on the dynamics are too strong (application of Equilibrium Statistical Mechanics to various non-ergodic systems provides correct predictions), and, moreover, these conditions involve every measurable functions although only some very particular quantities are relevant as macroscopic observables; Second, it does not explain why convergence to equilibrium only requires a finite amount of time; Third, we would like to interpret the stationary measure as a probability measure (after suitable normalization)[12], however this requires the further assumption that the typical duration of any measurement on the system is far larger than the relaxation time, but this is not satisfactory since there are systems for which the relaxation time is long (that's why we can observe systems out of equilibrium!) and, moreover, there is no criterion allowing us to decide if a given system will satisfy this condition or not.

Since no really convincing foundations of Equilibrium Statistical Mechanics have been developed up to now, we will introduce it in an axiomatic way, without trying to give any justification; we start with the canonical measure and discuss at the end of this subsection its relation to the microcanonical measure. Moreover, since we are not interested in the dynamical aspects, we will restrict our attention to lattice systems, avoiding thus some technicalities.[13,14]

[12] This is a physical theory and for this reason one has to explain in what sense the measure of an event is related to a measurement done on the system: For a physicist, probability is not an *a priori* notion but something that has an empirical meaning.

[13] The ultimate reference for the first part of this subsection is Georgii's book [Ge1]. A nice paper to read in parallel to this book is [EFS] where these notions are explained in physically more intuitive terms. In this subsection, we will ignore most of this beautiful theory, so we can only encourage the reader who has not looked at this book to do so.

[14] Some of the notations introduced in this chapter will be used in later chapters with slightly different meanings; however they will always be redefined accordingly in the beginning of Part I and II.

As discussed above, the relevant quantity in Statistical Mechanics is not the actual microscopic state of the system, but statistical properties of the set Ω of all microscopic states sharing the same macroscopic description, i.e. described by the same set of macroscopic parameters (density, ...). Therefore a *state* in Equilibrium Statistical Mechanics will be a probability measure on the set Ω; it contains all the (statistical) information on the system which is accessible. The basic axiom of Equilibrium Statistical Mechanics is in fact an Ansatz for this probability measure.

We consider a system composed of a large number of identical elementary units located on a lattice. We first give the basic objects and terminology of its mathematical modelization.

- A countably infinite set $\mathbb{L}$ called the *lattice*; elements $t \in \mathbb{L}$ are called *sites*.
  (Each site represents the location of one of the elementary units belonging to the system. For example, we could choose $\mathbb{L} \doteqdot \mathbb{Z}^d$ to modelize some simple crystalline structure.)
- A probability space $(S, \mathcal{S}, \lambda)$ called the *single spin space*.
  (This is the set of all possible states of a given elementary unit. For example, the set $S = \{-1, 1\}$ could describe the z-component of a $\frac{1}{2}$-spin, or the coexistence of two different species in binary alloys. λ is an *a priori* measure on the set S (for example if S is discrete λ is usually the counting measure, if it is some bounded Borel subset of $\mathbb{R}^d$ then λ is usually the Lebesgue measure, if S is a locally compact group then λ is usually the Haar measure, ...))
- The product space $(\Omega, \mathcal{F})$, where $\Omega \doteqdot S^{\mathbb{L}}$ and $\mathcal{F} \doteqdot \mathcal{S}^{\mathbb{L}}$, called the *configuration space* of the system; the elements ω of Ω are called the *configurations*.
  (This is the set of all possible microscopic states of the system, specifying the state of every elementary units.)
- The restriction of a configuration ω on some subset $\Lambda \subset \mathbb{L}$, $\omega_\Lambda \doteqdot (\omega(t))_{t\in\Lambda}$. If $\Lambda_1 \cap \Lambda_2 = \varnothing$ and $\Lambda_1 \cup \Lambda_2 = \mathbb{L}$ then $\omega_{\Lambda_1}\omega_{\Lambda_2}$ is the configuration whose restriction to Λ_1 and Λ_2 are ω_{Λ_1} and ω_{Λ_2}, respectively.
- The family $(\sigma(t))_{t\in\mathbb{L}}$ of random variables $\sigma(t) : \Omega \to S$, $\omega \mapsto \omega(t)$; $\sigma(t)$ is called the *spin at* t.
  (The state of the elementary unit located at t in the configuration ω is given by $\sigma(t)(\omega)$.)
- The set $\mathcal{L} \doteqdot \{\Lambda \subset \mathbb{L} : 0 < |\Lambda| < \infty\}$ of all finite subsets of the lattice ($|\cdot|$ denotes the cardinality of a set).
- For each subset Λ of $\mathbb{L}$, we define the σ-algebra $\mathcal{F}_\Lambda$ generated by the functions $\sigma(t)$, $t \in \Lambda$. We also define $\mathcal{F}^0 \doteqdot \bigcup_{\Lambda\in\mathcal{L}} \mathcal{F}_\Lambda$.
  (Functions $\mathcal{F}_\Lambda$-measurable depend only on what happens inside the set Λ and $\mathcal{F}^0$ contains all local events.)
- A family $\Phi = (\Phi_A)_{A\in\mathcal{L}}$ of functions $\Phi_A : \Omega \to \mathbb{R}$ such that
  1. For each $A \in \mathcal{L}$, Φ_A is $\mathcal{F}_A$-measurable;
  2. For all $\Lambda \in \mathcal{L}$ and $\omega \in \Omega$, the series

$$H_\Lambda^\Phi(\omega) \doteqdot \sum_{\substack{A\in\mathcal{L}\\ A\cap\Lambda\neq\varnothing}} \Phi_A(\omega)$$

exists; the family Φ is called the *potential*.

(This is the set of all interaction potentials between any finite number of elementary units.)

- For any $\Lambda \in \mathcal{L}$ and $\overline{\omega} \in \Omega$, we define the *Hamiltonian in* Λ *with* $\overline{\omega}$*-boundary condition* for the potential Φ by

$$H_\Lambda^\Phi(\omega|\overline{\omega}) \doteqdot H_\Lambda^\Phi(\omega_\Lambda \overline{\omega}_{\Lambda^c})$$

where $\Lambda^c \doteqdot \mathbb{L} \setminus \Lambda$.
(This function represents the total energy associated to the elementary units located in Λ whose states are determined by the configuration ω, knowing that the rest of the system is in a state specified by $\overline{\omega}$. It is necessary to define the energy with respect to some finite part of the lattice since otherwise it would generally be infinite.)

- A potential Φ is λ-admissible if

$$Z_\Lambda^\Phi(\overline{\omega}) \doteqdot \int \lambda^\Lambda(\,\mathrm{d}\omega) \exp[-\beta H_\Lambda^\Phi(\omega|\overline{\omega})]$$

is finite for all $\Lambda \in \mathcal{L}$, $\overline{\omega} \in \Omega$ and $\beta > 0$. $Z_\Lambda^\Phi(\overline{\omega})$ is called the *partition function* in Λ with boundary condition $\overline{\omega}$ at inverse temperature β (for Φ and λ).

- For any $\Lambda \in \mathcal{L}$, any configuration $\overline{\omega} \in \Omega$, any λ-admissible potential Φ, we define the *Gibbs measure in* Λ *with* $\overline{\omega}$*-boundary condition at inverse temperature* β for the potential Φ by

$$\mu_\Lambda^{\Phi,\overline{\omega}}(A) \doteqdot Z_\Lambda^\Phi(\overline{\omega})^{-1} \int \lambda^\Lambda(\,\mathrm{d}\omega) \exp[-\beta H_\Lambda^\Phi(\omega|\overline{\omega})]\, 1_A(\omega_\Lambda \overline{\omega}_{\Lambda^c})\,.$$

The basic postulate of Equilibrium Statistical Mechanics is that the equilibrium states of the system at temperature T inside Λ and in interaction with an external system described by the configuration $\overline{\omega}$ is given by the Gibbs measure in Λ with $\overline{\omega}$-boundary condition at inverse temperature $\beta = 1/T$ [15]. That is, expectation values of all observable quantities on the system can be obtained using this measure. It is also possible to have non-deterministic boundary conditions, distributed with some probability measure; the generalization to such a case is immediate. From now on we absorb the parameter β in the potential Φ.

Since all physical systems are finite, this formalism should be sufficient for all practical purposes. However, it appears to be very useful to consider the case of infinite systems whose properties are much easier to study. Since the size of macroscopic systems is huge, their properties will generally be very well approximated by such infinite systems. The traditional way to construct infinite volume Gibbs measures is to consider a sequence[16] of finite sets $\Lambda_n \in \mathcal{L}$ converging to $\mathbb{L}$, together with a sequence of boundary conditions $\overline{\omega}_n$ (possibly stochastic). We then consider the sequence of (finite volume) Gibbs measures $\big(\mu_{\Lambda_n}^{\Phi,\overline{\omega}_n}\big)_n$. Infinite volume Gibbs measures are then defined as cluster points of such

[15] Usually in the Physics literature, β is defined as $1/k_{\mathrm{B}}T$, where k_{B} is Boltzmann's constant. However we can get rid of it by a redefinition of the energy scale .

[16] In fact, the correct notion would be that of a *net*, which is the generalization of the notion of sequences when the index belongs to a set which is only partially ordered (as is the case for the subsets of $\mathbb{L}$).

sequences with respect to the topology of local convergence: a sequence of probability measures (μ_n) on $(\Omega, \mathcal{F})$ *converges locally* to μ if and only if

$$\lim_{n\to\infty} \mu_n(A) = \mu(A)\,, \quad \forall A \in \mathcal{F}^0\,, \tag{24}$$

i.e. if and only if the expectation values of any local event converge. This procedure is known as the *Thermodynamic Limit.*

Such a definition of infinite volume Gibbs measures is quite awkward and makes it difficult to prove general results. For this reason, it would be very useful to have a characterization of infinite volume Gibbs measures among all probability measures on $(\Omega, \mathcal{F})$, which does not use the Thermodynamic Limit. Such a characterization exists; it has been introduced by Dobrushin [Do1] and Lanford and Ruelle [LR]. It is based on the so-called *DLR equations.*

Definition.

(D9) *A probability measure μ on $(\Omega, \mathcal{F})$ is a* **Gibbs measure** *for a λ-admissible potential Φ, if it satisfies the DLR equations:*

$$\int \mathrm{d}\mu(\overline{\omega}) Z_\Lambda^\Phi(\overline{\omega})^{-1} \int \lambda^\Lambda(\,\mathrm{d}\omega) \exp[-H_\Lambda^\Phi(\omega|\overline{\omega})]\, 1_A(\omega_\Lambda \overline{\omega}_{\Lambda^c}) = \mu(A)\,,$$

for all $A \in \mathcal{F}$ and all $\Lambda \in \mathcal{L}$.

The physical meaning of these equation is that any finite part of the system is in equilibrium with the rest of the system (this is even stronger than the definition of equilibrium states in Thermodynamics, since here even microscopic parts of the system (i.e. finite ones) are in equilibrium with each others). The precise relationship between this definition and the previous one using the Thermodynamic Limit is the following,

Proposition 0.2.1. *Any cluster points of the sequence of finite volume Gibbs measures $\big(\mu_{\Lambda_n}^{\Phi,\overline{\omega}_n}\big)_n$, where $(\Lambda_n \in \mathcal{L})_n$ is a sequence of finite sets converging to $\mathbb{L}$ and $\overline{\omega}_n$ a sequence of boundary conditions (possibly stochastic), satisfies the DLR equations (for the potential Φ).*

There is a partial converse to this proposition which we will give later. Let us first study the properties of the set $\mathcal{G}_\lambda(\Phi)$ of Gibbs measures for the λ-admissible potential Φ.

Definition.

(D10) *A potential Φ is said to exhibit a* (first-order) phase transition *if $|\mathcal{G}_\lambda(\Phi)| > 1$.*

Indeed, in such a case for a given potential (which implies in particular that the temperature and all other parameters are fixed) there is more than one equilibrium state.

We will not be interested in the existence problem (i.e. $|\mathcal{G}_\lambda(\Phi)| > 0$) since it is guaranteed to hold in most physically interesting situations[17]. So suppose that $\mathcal{G}_\lambda(\Phi)$ is not empty and that moreover it is not restricted to one point; in such a case, what is the structure of this set? From its definition, it is easy to see that $\mathcal{G}_\lambda(\Phi)$ must be convex. In fact, the following very interesting proposition shows that this set is a simplex, i.e. a convex set such that any point of the set can be decomposed in a *unique* way as a

[17]Notice however that there are some physically very reasonable cases where $\mathcal{G}_\lambda(\Phi)$ is empty: the infinite harmonic crystal in less than three dimensions for example.

convex combination of extreme elements (i.e. those that have a trivial decomposition). We will need some terminology. We first have to define what we mean by macroscopic observables.

Definition.

(D11) *The σ-field*

$$\mathcal{F}^\infty \doteqdot \bigcap_{\Lambda \in \mathcal{L}} \mathcal{F}_{\Lambda^c}$$

is called tail σ-field.

(D12) *The functions measurable with respect to the tail σ-field are called the* observables at infinity.

(D13) *A measure on $(\Omega, \mathcal{F})$ such that $\mu(A) = 0$ or $\mu(A) = 1$ for all $A \in \mathcal{F}^\infty$ is said to be* trivial on the tail σ-field.

The tail σ-field is the set of all events which do not depend on any finite set of elementary units; such events can be thought of as a mathematical model of macroscopic events; the observables at infinity can correspondingly be interpreted as macroscopic observables. A probability measure μ which is trivial on the tail σ-field also satisfies a strong clustering property in that it has *short-range correlations*, i.e.

$$\lim_{\substack{\Lambda \nearrow \mathbb{L} \\ \Lambda \in \mathcal{L}}} \sup_{B \in \mathcal{F}_{\Lambda^c}} |\mu(A \cap B) - \mu(A)\mu(B)| = 0 \,, \quad \forall A \in \mathcal{F} \,. \tag{25}$$

This property shows that two observables become asymptotically independent as the distance between their support is increased (in fact, it is even stronger, since there is uniformity in $B \in \mathcal{F}_{\Lambda^c}$).

Proposition 0.2.2. *The set $\mathcal{G}_\lambda(\Phi)$ is a simplex whose extreme elements are characterized by the following equivalent properties:*

- *μ is an extreme point of $\mathcal{G}_\lambda(\Phi)$;*
- *μ is trivial on the tail σ-field;*
- *μ has short-range correlations.*

Moreover, the following statement holds:

- *Distinct extreme elements μ and ν of $\mathcal{G}_\lambda(\Phi)$ are mutually singular on $\mathcal{F}^\infty$, i.e. there exists $A \in \mathcal{F}^\infty$ with $\mu(A) = 1$ and $\nu(A) = 0$.*

The preceding proposition has important physical consequences. We want to describe the state of a system by a probability measure on the configuration space. On the other hand, the description of the system provided by Thermostatics only refers to deterministic observables. The proposition shows that this is not incompatible, since for any extreme Gibbs measure μ, all macroscopic observables are constant μ-almost surely (but the local observables still have local fluctuations). The extreme Gibbs measures are therefore good candidates for the role of macroscopic pure states of the system.

How can the non-extreme Gibbs measures be interpreted? Since they can be decomposed in a unique way as a convex combination of extreme Gibbs measures, it is possible to think of the coefficients of this decomposition as being the probabilities that the system is actually in the macroscopic state described by the corresponding extreme Gibbs measure.

The last statement of the proposition gives a nice explanation of the phenomenon of reduction of variables: It is indeed possible to find a "small"[18] set of macroscopic observables whose measurement determine uniquely into which macroscopic state the system actually is[19]. The next proposition, the promised partial converse to Proposition 0.2.1, shows how an experimenter can prepare a system in a given macroscopic state; moreover, it shows how a non-extreme Gibbs state can be interpreted as a state of partial knowledge of the actual macroscopic state of the system by the experimenter.

Proposition 0.2.3. *Let Ω be a compact metric space and let ν be an extreme point of $\mathcal{G}_\lambda(\Phi)$. Then, for ν-almost every configuration $\overline{\omega}$,*

$$\lim_{\substack{\Lambda \nearrow \mathbb{L} \\ \Lambda \in \mathcal{L}}} \mu_\Lambda^{\Phi,\overline{\omega}} = \nu$$

in the topology of local convergence. Moreover, if ν is a Gibbs measure (not necessarily extreme), then for ν-almost every configuration $\overline{\omega}$,

$$\lim_{\substack{\Lambda \nearrow \mathbb{L} \\ \Lambda \in \mathcal{L}}} \mu_\Lambda^{\Phi,\overline{\omega}} = \tilde{\nu}$$

where $\tilde{\nu}$ is one of the extreme Gibbs measures in the decomposition of ν.

The first part of the proposition thus says that if a system is put into contact with a typical configuration of some macroscopic state (i.e. some extreme Gibbs measure), then when the system becomes large, it will be in that state. This shows how an (ideal) experimenter can prepare a system in a given macroscopic state. The second part shows that if we look at a system whose state is described by a non-extreme Gibbs measure, then we will see that its configuration will be typical of one of the extreme Gibbs measure of its decomposition; this strengthens the interpretation of such a measure as describing a state of partial knowledge.

We emphasize that non-extreme Gibbs measures *do not* represent phase coexistence. Phase coexistence can result in non-extreme Gibbs measures in some cases, but the following remarks show that this is neither a sufficient nor a necessary criterion.

- Phase coexistence can result in *extreme* Gibbs measures. This is the case, for example, in the Ising model in more than 2 dimensions at sufficiently low temperature; the states describing the coexistence of the plus and minus phases separated by a horizontal interface are extreme Gibbs states [vB, Do3] (although we do not think that such a description of phase coexistence at the scale of the lattice is really relevant for Thermodynamics, even if it is very interesting from the point of view of Statistical Mechanics).
- The Gibbs measure obtained by imposing free b.c. in the Ising model is not extreme, but in the continuum limit of the system, there will be no interface: With probability 1/2 the system will be in the + phase, otherwise it will be in the − one.

[18] It may be necessary to have an infinity of such observables, since there can be a continuum of extreme Gibbs measures; for example, when a continuous symmetry is broken as in the XY model. This would also be the case in Thermodynamics in case of systems with continuous symmetry.

[19] Of course, it is possible that the observables in that set, although they are tail measurable, happen to be quite "pathological" from a physical point of view. Since a precise definition of physically reasonable observables is lacking (and the question itself probably does not make sense) to obtain good observables can only be done on a case by case basis.

Moreover, the absence of non-translation invariant states in the 2D Ising model, for example, does not mean that there is *no* phase coexistence (or no interface). Indeed, at a macroscopic level, which is the relevant scale to study phase separation, there is a well defined interface[20]. All these points will be discussed in more details later in this work, after suitable tools have been introduced. The point is that Thermodynamic limit, and in particular the infinite-volume approach via the DLR equations are not the relevant formalisms to study this type of phenomena, which should not be surprising since the local topology considered destroys all information about global observables describing the macroscopic geometry of the system under consideration; in this work, another kind of limit is considered, the continuum limit[21]. Of course, we do not say that continuum limit must replace DLR formalism, but that these two approaches are complementary and must be both used if we want to obtain a complete description: with the continuum limit, we obtain a global description of the system, while thermodynamic limit allows us to "zoom in" and investigate its local behaviour.

We still have to understand how Thermodynamic Potentials appear in Equilibrium Statistical Mechanics. To do this, we analyze the relationship between the canonical measure introduced above and the microcanonical measure. We follow the analysis of [LP, LPS1, LPS2, LPS3][22], another very nice paper on that kind of problems is [La]. We introduce some more notations to discuss these issues. Let $\mathbb{L} = \mathbb{Z}^d$, and let $(\Lambda_n)_{n\geqslant 1}$ be an increasing sequence of cubes in $\mathbb{Z}^d$. We write $\Omega_n \doteqdot S^{\Lambda_n}$. The *a priori* measure λ on the spin space is supposed to be normalized; we denote by $\rho_n \doteqdot \lambda^{\Lambda_n}$ and $\rho \doteqdot \lambda^{\mathbb{L}}$ the corresponding product measures. There is a natural action of $\mathbb{Z}^d$ on itself, $i \mapsto i+j$ which can be lifted on Ω by $\vartheta_j : \Omega \to \Omega$, $(\vartheta_j\omega)(i) = \omega(i-j)$ and to functions on Ω by $\vartheta_j f(\omega) = f(\vartheta_j\omega)$. We consider a k-dimensional vector $\underline{\Phi} = (\Phi^1, \dots, \Phi^k)$ of translation invariant absolutely summable potentials, i.e. potentials Φ^i such that

$$\sum_{\substack{A\in\mathcal{L}\\ A\ni 0}} \sup_{\omega\in\Omega} |\Phi^i_A(\omega)| < \infty\,,$$

$$\vartheta_j \Phi^i_A = \Phi^i_{A+j}\,. \tag{26}$$

Using these potentials, we define mappings $T_n : \Omega_n \to X$, where $X \subset \mathbb{R}^k$ is convex and compact, by $T_n(\omega)_i = H^{\Phi^i}_\Lambda(\omega|\overline{\omega}^i)/|\Lambda_n|$, for some fixed $\overline{\omega}^i$, $i = 1, \dots, k$. The components of the mappings T_n can be any set of macroscopic observables: For example the energy density, the magnetization density, and so on ....

We consider two measures on Ω_n. Let $C \in \mathfrak{B}(X)$, the set of Borel subsets of X, be such that $\rho_n(T_n^{-1}C) > 0$, for all n sufficiently large; the *microcanonical measure* is defined by

$$\nu_n^C(\,\mathrm{d}\omega) \doteqdot \frac{1_{T_n^{-1}C}(\omega)\rho_n(\,\mathrm{d}\omega)}{\rho_n(T_n^{-1}C)}\,; \tag{27}$$

this is the uniform measure on the set of all configurations ω such that $T_n(\omega) \in C$, i.e. all configuration for which the chosen observables take value in some fixed set. Let $t \in \mathbb{R}^k$

[20]It is possible to define a surface tension in this model and to show that below the critical temperature, it is strictly positive. If there were no interface, to what would that quantity correspond?

[21]We only consider lattice systems; the corresponding continuum limit for systems in $\mathbb{R}^d$ may be more subtle.

[22]We do not discuss the most general settings. We only want to give a flavour of these works.

such that $0 < \int_{\Omega_n} \exp(|\Lambda_n|\langle t|T_n\omega\rangle)\rho_n(\,\mathrm{d}\omega) < \infty$, for all n sufficiently large; the *canonical measure* is defined by[23]

$$\gamma_n^t(\,\mathrm{d}\omega) \doteqdot \frac{\exp(|\Lambda_n|\langle t|T_n\omega\rangle)\rho_n(\,\mathrm{d}\omega)}{\int_{\Omega_n}\exp(|\Lambda_n|\langle t|T_n\omega\rangle)\rho_n(\,\mathrm{d}\omega)}\,; \tag{28}$$

this is exactly the Gibbs measure in Λ for the Hamiltonian $H_{\Lambda_n}^{\langle t|\Phi\rangle}$. We want to compare these two measures as $n \to \infty$. The key observation is that the densities of both these measures with respect to ρ_n are functions of T_n. It is therefore possible to define the corresponding measures on X. This is very useful since X corresponds to the space of densities of extensive thermodynamical parameters (the macroscopic observables); therefore studying the distribution of these quantities as the system becomes large provides information on its thermodynamical description in terms of these parameters. Writing $\mathbb{M}_n \doteqdot \rho_n \circ T_n^{-1}$, we have

$$\begin{aligned}
\nu_n^C \circ T_n^{-1}(\,\cdot\,) &= \frac{\mathbb{M}_n(\,\cdot \cap C)}{\mathbb{M}_n(C)} \;=:\; \mathbb{M}_n(\,\cdot\,|C)\,,\\
\gamma_n^t \circ T_n^{-1}(\,\cdot\,) &= \frac{\mathbb{M}_n^t(\,\cdot\,)}{\mathbb{M}_n^t(X)} \;=:\; \mathbb{M}_n^t(\,\cdot\,|X)\,,
\end{aligned} \tag{29}$$

where $\mathbb{M}_n^t(\,\mathrm{d}x) \doteqdot \exp(|\Lambda_n|\langle t|x\rangle)\mathbb{M}_n(\,\mathrm{d}x)$. We introduce p_n, the *grand-canonical pressure in* Λ_n [24] by

$$\exp(|\Lambda_n|p_n(t)) \doteqdot \mathbb{M}_n^t(X)\,. \tag{30}$$

We need to understand the behaviour of these two measures on X; since they both have a density with respect to $\mathbb{M}_n$, it is natural to analyze the behaviour of this measure. Let us introduce the following set functions,

$$\begin{aligned}
m_n[B] &\doteqdot \frac{1}{|\Lambda_n|}\log \mathbb{M}_n[B]\,,\\
\underline{m}[B] &\doteqdot \liminf_{n\to\infty} m_n[B]\,,\\
\overline{m}[B] &\doteqdot \limsup_{n\to\infty} m_n[B]\,.
\end{aligned} \tag{31}$$

We also define two functions on X by

$$\begin{aligned}
\underline{\mu}(x) &\doteqdot \inf_{G\ni x} \underline{m}[G]\,, \quad G \text{ open},\\
\overline{\mu}(x) &\doteqdot \inf_{G\ni x} \overline{m}[G]\,, \quad G \text{ open}.
\end{aligned} \tag{32}$$

It can be shown that, for all $x \in X$, $\underline{\mu}(x) = \overline{\mu}(x)$; moreover these functions are concave. We write $\mu(x) \doteqdot \underline{\mu}(x) = \overline{\mu}(x)$; this function is called the *entropy* (or Ruelle-Lanford function). It can be shown that the grand-canonical pressure $p(t) \doteqdot \lim_n p_n(t)$ exists and is given by

$$p(t) = \sup_{x\in X}(\langle t|x\rangle + \mu(x))\,, \tag{33}$$

[23] $\langle x|y\rangle$ is the scalar product in $\mathbb{R}^k$.

[24] In the terminology of large deviations, it is called *scaled generating function*.

that is, the grand-canonical pressure is the Legendre transform of minus the entropy, $p(t) = (-\mu)^*(t)$. Since μ is concave, we also have $\mu(x) = -p^*(x)$. It can also be shown that the function p is strictly concave. These two functions play an important role since they characterize the sets on which the microcanonical and canonical measures become concentrated when $n \to \infty$.

Let $X_{\underline{C}} \doteqdot \{x \in \overline{C} \,:\, \mu(x) = \sup_{y\in\overline{C}} \mu(y)\}$ [25] and $X^t \doteqdot \{x \in X \,:\, p(t) = \langle t|x\rangle + \mu(x)\}$; then

Lemma 0.2.1. *Suppose that* $\sup_{x\in C} \mu(x) > -\infty$. X_C *and* X^t *are compact and nonempty. Moreover, the sequence* $\mathbb{M}_n(\,\cdot\,|C)$ *of probability measures is eventually concentrated on the set* $X_{\overline{C}}$*, i.e.*

$$\lim_{n\to\infty} \mathbb{M}_n(G|C) = 1\,, \quad \textit{for each open neighbourhood } G \textit{ of } X_{\overline{C}},$$

and the sequence $\mathbb{M}^t_n(\,\cdot\,|X)$ *of probability measures is eventually concentrated on the set* X^t.

Therefore, the microcanonical measure is eventually concentrated on the set of values maximizing the entropy. It is not difficult to show that $X^t = \partial p(t)$, where $\partial p(t)$ is the subdifferential of p at t. Therefore the equilibrium values of the observables Φ^i (e.g. the energy density, magnetization density,...) can be obtained by differentiating the grand-canonical pressure with respect to conjugate variables t_i.

The measures ν_n^C associated to the cubes Λ_n are not translation invariant; for technical reasons, it is useful to consider translation-averaged versions of them,

$$\overline{\nu}_n^C \doteqdot \frac{1}{|\Lambda_n|} \sum_{j\in\Lambda_n} \nu_n^C \circ \vartheta_j^{-1}\,, \tag{34}$$

where ν_n^C is extended to Ω by imposing some fixed configuration outside Λ_n. It is then possible to prove the following result.

Theorem 0.2.1. *Let* C *be an open neighbourhood of a point at which the entropy is finite. Then there exists* t^* *such that any cluster point of the sequence* $(\overline{\nu}_n^C)_{n\geqslant 1}$ *is a Gibbs measure for the potential* $\langle t^*|\underline{\Phi}\rangle$.

Forgetting about the averaging, this shows that Thermodynamic potentials are related to large deviations properties of the corresponding measures; the concavity of entropy and the maximum entropy principle follow once the identification has been done. This also explains why the entropy representation for which the variables are densities of extensive observables is related at equilibrium by a Legendre transform to the grand-canonical pressure, which is the Thermodynamic Potential depending only on intensive quantities (the variable t), and why it is possible to obtain the equilibrium values of the densities of extensive observables by differentiating the pressure with respect to the intensive variables. In fact, it is not necessary that ρ_n be a product measure; this is interesting, since it is possible to absorb part of the interaction into it. This allows to consider other Thermodynamic potentials with some intensive and some (densities of) extensive variables.

This theorem is also important for Equilibrium Statistical Mechanics, since it shows that the local behaviour of microcanonical and canonical measures are "identical", both being described by some Gibbs measure; this type of equivalence is called *equivalence of ensembles*.

[25] $\overline{C}$ denotes the closure of C.

0.3 Structure of the thesis

We briefly sketch here the content of the various chapters of this work. We use some notations that are introduced in the main body of this thesis without explaining them. This work is composed of two parts and three short appendices, in which some tools used in the main body of the text are briefly described. For the reader's convenience, most of the definitions introduced are gathered at the end, see page 285.

0.3.1 Part I: Ising model

The first part of this work deals with the Ising model.

Chapter 1

This chapter is devoted to the basic definitions and notations relevant to the Ising model: some elementary geometrical terminology, the notion of boundary conditions, Gibbs states, contours,.... Some basic standard results about this model are also given.

Chapter 2

Two geometrical representations of the Ising model in terms of contours are introduced, together with some related terminology. The self–duality of the model is discussed. The high-temperature representation of the 2-point function, which plays an essential role in the sequel, is also given there.

Chapter 3

The notion of surface tension and wall free energies are introduced and some very useful properties are stated. It is also shown how these quantities are related by duality to the decay–rate of some 2-point functions. A discussion of the wetting phenomenon in the Grand Canonical Ensemble is given at the end of the chapter.

Chapter 4

The main tools necessary to study surface effects in the 2D Ising model are developed in this chapter. The main results can be roughly stated in the following way:

- Estimate on the probability of a contour conditioned to go through a given set of sites, in the bulk or near a wall (with a special wall free energy).
- An estimate on the size of typical contours.
- An estimate on the size of excursions of the open contour away from the wall in the partial wetting regime.
- A rough description of the set of typical contours contributing to a given 2-point function, without using the Sharp Triangle Inequality.
- A precise description of the set of typical contour contributing to a given 2-point function, using the Sharp Triangle Inequality.

We also give some results on various properties of the 2-point functions, such as lower bounds (mostly taken from the exact computations) and monotonicity properties.

Chapter 5

The phase of small contours is introduced and several basic results regarding the decay of correlations and the large deviations in this constrained ensemble are proved. These results are used in an essential way in Chapter 7.

Chapter 6

This chapter deals with the first application of the techniques of Chapter 4 to a physical situation: the pinning of an interface. We consider an interface with endpoints fixed at some given heights on opposite vertical walls of a box, while the free energy of the bottom wall of the box can be tuned. Precise concentration properties are obtained for the measure on the set of phase separation lines, showing that, for large systems, typical configurations of the system can be described by some interface between regions of + or − phases. The interface's shape is the solution of the variational problem provided by Surface Thermodynamics.

Chapter 7

The second application of the tools of Chapters 4 and 5 is to the study of the large (and moderately large) deviations of the magnetization in the Ising model, and to the related wetting transition in the canonical ensemble. More precisely, we consider the Ising model in some box, say $\Lambda_L = \{t \in \mathbb{Z}^2 \,:\, \|t\|_\infty \leqslant L\}$, with +-b.c. and some boundary magnetic field h on the bottom wall. we are interested in the following event,

$$\mathcal{A}(m,c) = \{\omega \,:\, |\frac{1}{|\Lambda_L|}\sum_{t\in\Lambda_L}\sigma(t) - m| \leqslant L^{-c}\}\,,$$

where either $-m^*(\beta) < m < m^*(\beta)$ (large deviations), or $m = m^*(\beta) - L^{-\nu}$, for some $\nu > 0$ not too large (moderately large deviations). We obtain the exact asymptotics (at leading order) for the probability of these events, for all subcritical values of the temperature and boundary magnetic field. We also have a control on the rate of convergence. An interesting result is the following one: For moderately large deviations, when the boundary magnetic field satisfies $h \leqslant -h_{\mathrm{w}}(\beta)$ (i.e. in the complete wetting regime), the scale on which these large deviations take place is *not* $\mathcal{O}(L^{1-\nu/2})$ as might have been expected, but $\mathcal{O}(L^{1-2\nu})$. This can be easily understood, and such an unusual behaviour of the deviations of the magnetization should in fact already be felt on the level of *typical* fluctuations, as is discussed in Section 7.4.5.

We then use these informations to study the typical configurations in a fixed magnetization ensemble. Fix the magnetization at some value in the interval $(-m^*(\beta), m^*(\beta))$. What do the typical configurations look like, and how can they be described macroscopically? We show that, in some continuum limit, there is a unique large droplet of − phase inside the + phase. Moreover, the shape of its boundary is given by the solution of the corresponding Thermodynamical variational problem. Roughly stated, we show that, if $h \geqslant h_{\mathrm{w}}(\beta)$, then the droplet "floats" inside the box, and its shape can be constructed with the Wulff construction; on the other hand, if $h < h_{\mathrm{w}}(\beta)$, it is tied to the bottom wall and its shape can be obtained through the Winterbottom construction. In particular, Cahn's criterion and the Herring-Young equation for the angle between the droplet and the wall are recovered. This gives a precise description of the wetting phenomenon in a fixed magnetization ensemble; the phase diagram is the same as in the case of the canonical ensemble, but the manifestation of the transition is quite different. This shows that the problem of equivalence of ensembles for surface phenomena can be rather subtle, different ensembles providing complementary rather than equivalent informations on the system.

0.3.2 Part II: Ashkin–Teller model

This second part deals with the Ashkin–Teller model. It is less detailed than the first one. Its main purpose is to exhibit some of the new difficulties that have to be dealt with when trying to extend the techniques developed for the Ising model to other cases. The proofs are given with less details and some are somewhat sketchy; moreover, we have not always tried to obtain the best possible results, but rather to emphasize the main technical problems.

Chapter 8

Basic terminology, notations and definitions are introduced.

Chapter 9

Low- and high-temperature representations of the model are introduced; moreover, duality is studied. High-temperature representation of the 2-point functions is given; the self-dual manifold is computed.

Chapter 10

A generalized class of Random–Cluster models is introduced, and its main properties are studied; in particular, duality, FKG inequalities and comparison inequalities are shown to hold. It is then proved that one model in this class is equivalent to the Ashkin–Teller model (in the same sense that the usual Random–Cluster and Potts models are equivalent). Moreover, duality in the Random–Cluster model commute with the duality in the Ashkin–Teller model.

Although we do not use this representation (except for some comments at some isolated places), it may be useful in the study of surface phenomena, since the Random–Cluster representation of the Ising model has been used in several approaches to the problem of large deviations.

Chapter 11

The surface tensions of the Ashkin–Teller model are introduced and their basic properties are studied. It is also shown how they are related through duality to the decay–rate of some 2-point functions of the dual model. Some of the properties of the surface tensions are obtained by comparison with the Ising model and are valid only at sufficiently low temperature.

Chapter 12

Several tools of chapter 4 are extended to the Ashkin–Teller model. The main difficulty to extend some of these techniques is that there is no simple way to cut the contours into pieces while preserving nice properties. It is then necessary to show that typical contours have enough cutting-points (i.e. good points at which to cut a contour); we do not know how to do this non-perturbatively. We therefore use cluster expansion methods to analyze the geometry of the typical contours. With this information, it is possible to prove a (slightly weaker and perturbative) form of the "box proposition" of Chapter 4.

Chapter 13

Tools of the preceding chapters are used to study the large deviations of the magnetization in one or the two planes of Ising spins of the Ashkin–Teller model. Lower bounds can be obtained in every cases (they are only valid at low temperature). The corresponding upper bounds are much more complicated to prove and are obtained only in the case of a large deviation in only one plane; the applied technique requires extensive use of perturbative arguments.

0.4 Some remarks

We make now some general remarks about this work: We compare our techniques with other methods available to discuss such questions, and we give a list of possible extensions and interesting open problems related to those treated here.

0.4.1 Comparison with other techniques

There are not a lot of tools to study surface phenomena. Most of the literature on this subject is restricted to some version of the 1D SOS model (which behaves similarly to a 2D Ising interface) which is studied by exact computation, or by using various approximation schemes. A nice discussion of several surface effects using very simple situations (mostly 1D simple random walks) is given in [Fi1, Fi2]. However to study more complicated systems, like the Ising model, very few techniques exist; mainly exact computations using transfer matrix methods, and perturbative arguments involving cluster expansion techniques. We make a brief comparison of these two methods and those used in the present work.

Exact computations

A lot of progress has been made in exact computations of interesting quantities with various kinds of boundary conditions. Even if the procedure cannot always be described as rigorous[26], it is possible to obtain some really remarkable results. Let us point out the main differences between the results obtained using this approach and those yielded by our techniques.

- **Mathematical rigour.** As mentioned above, exact computations cannot always claim to be rigorous, although it might be possible to improve this.

[26]It is necessary to study the asymptotics of some often complicated integrals, and usually only the (hopefully) leading term is computed but no estimate on the remaining terms are given.

- **Boundary conditions.** Exact computations are very sensitive to boundary conditions; often toroidal or cylindrical boundary conditions are used. This is a severe drawback when studying surface phenomena, since it is precisely the effect of the boundary which is interesting. Moreover, changing the boundary conditions (for example in order to add one interface in the system) requires to make the analysis again. Our techniques, on the contrary, are quite insensitive to such a modification: Since our analysis is mostly local, this amounts essentially to study a new variational problem. Our tools are therefore adequate to prove results about general boundary conditions.
- **Coupling constants.** Since our techniques are based on correlation inequalities, it is necessary that the coupling constants be ferromagnetic, although the boundary magnetic field (or fields) does not have to be necessarily positive. Exact computations, on the other hand, are not subject to such restrictions.
- **Observables.** Our methods allow us to work directly with the relevant objects, namely the phase separation lines. Since establishing the validity of the Thermodynamical variational problem essentially amounts to study typical phase separation lines, it is important to have access to these quantities. Through exact computations, however, it is only possible to compute expectation values of correlation functions. To relate the results thus obtained to the behaviour of phase separation lines is certainly not obvious. However, both with our techniques and exact computations, it is possible to obtain, for example, a magnetization profile (although defined quite differently).
- **Variational problem.** Since we work with phase separation lines, we have a natural way to define interfaces, namely as the support of typical such lines. We can then show that the shape of the interface is given by some Thermodynamical variational problem. Since exact computations do not give access to such objects, it is difficult to make the link between the results obtained and the variational problem of Thermodynamics.
  We do not always know how to solve the variational problem. Exact computations, however, yield the value of the functional at the minimum. This may seem to be an advantage in some cases, but we would be quite surprised if there was some case which can be solved using exact computations, while the corresponding variational problem cannot. It rather seems to us that in such cases exact computations should become quite involved, while our techniques still provide some informations.
- **Fluctuations.** This is the main advantage of exact computations over our techniques. We do not know how to obtain informations on the typical fluctuations of the phase separation lines. In other words, we cannot "zoom in" to explore the region occupied by the interface at the lattice scale. With our techniques, we control the large deviations of the phase separation lines (it is however possible to obtain very precise estimates *up to* the scale of typical fluctuations). On the other hand, with exact computations, it is possible, by a suitable change of scale, to obtain precise informations on, for example, the fluctuations of the magnetization.
- **Canonical constraints.** The presence of a global constraint on the magnetization is completely impossible to handle with exact computations (at least with today's techniques). The techniques used in this work, however, provide precise informations

about such problems (see Chapter 7).

- **Understanding.** Exact computations do not necessarily provide an understanding of what is going on. It is sometimes quite difficult to make a link between the results and the underlying physical phenomenon. There are examples in the literature of wrong interpretations of exact results. Since we work with natural objects, it seems to us that our techniques give a better physical understanding of what is happening.

To summarize, our techniques seem better to study macroscopic phenomena, but do not provide informations about typical fluctuations (except some information about the scale at which they occur). With exact computations, on the other hand, it is possible to have access to such fluctuations, but it is more difficult to discuss typical macroscopic configurations. To obtain a precise description of the system at all scales, it may be useful to combine both techniques.

Perturbative techniques

Perturbative techniques, such as cluster expansion, are the best *systematic* ways to analyze complicated problems. Let us compare these techniques with ours.

- **Generality.** In principle, perturbative arguments are much more general. With them neither dimensionality, nor range of interactions, for example, are very important. This is true in *principle*: In *fact*, very few cases have been studied besides the 2D Ising model with nearest-neighbours pair interactions and without magnetic field. For example, in the case of Wulff construction, using only perturbative methods, only this model with periodic boundary conditions have been considered. We do not say that it is not possible to generalize these techniques of course, but that this is not a trivial extension.

- **Range of temperature.** A major drawback of perturbative techniques is precisely their perturbative character. Only low (and often *very* low temperature are accessible). This is annoying since genuinely non-perturbative effects come into play when studying such surface problems (see, for example, the reentrance phenomenon in Chapter 6).

- **Fluctuations.** Perturbative techniques can be used to obtain very precise information on typical fluctuations. It is for example possible to prove the Ornstein-Zernicke behaviour of the 2-point function (see [DKS1]). Moreover, they do not suffer the problems of exact solutions in the sense that they have direct access to the natural objects.

Perturbative techniques are, without any doubt, the way to go if we want to obtain generic results for wide classes of models. It would be very interesting to have some really new phenomena studied with these methods (for example, the coexistence of several droplets in a model with more than two phases). Some new ideas seem however to be needed. Again these techniques are complementary to those used here: The former may be used to prove very general results in some perturbative regime, while the latter give non-perturbative results in some particular cases.

0.4.2 What to do next?

We present here a list of possible extensions and problems related to those studied here. We write them (more or less) in order of increasing complexity (in our opinion).

General results for the 2D Ising model

The first possible extensions of this work would be to show how a general macroscopic boundary condition (i.e. with several boundary magnetic fields which remain constant on macroscopic parts of the boundary) for the 2D Ising model gives rise to a set of interfaces whose shapes are given by the corresponding Thermodynamical variational problem, both in the unconstrained and constrained ensembles. This should be a simple extension (at least in the unconstrained ensemble) if the solution of the variational problem has sufficiently nice properties. This would be interesting since it would give a "general" derivation of Surface Thermodynamics for the 2D Ising model.

More precise description at the level of the lattice

A finer description of the typical configurations at a *microscopic* level would be interesting from the point of view of Statistical Mechanics. For example, in the case of the wetting phenomenon of Chapter 7, we prove unicity of the large droplet only on the macroscopic scale. In fact, we expect that all other contours have sizes at most logarithmic in the volume of the box. Such a precise description of the phase has been obtained perturbatively for periodic b.c. in [DKS1]. Recently, Dima Ioffe and Roberto Schonmann have announced a non-perturbative proof (but without considering boundary effects).

Fluctuations

Our techniques deal with the large deviations of the phase separation lines. It would be very nice to improve them to study *typical* fluctuations of these quantities. This would allow an analysis of the fine structure of the interface, for example. The following problems are related to an understanding of typical fluctuations: A proof of Ornstein-Zernicke behaviour of the 2-point function (not relying on exact computations); a proof of the positive stiffness of the surface tension (not relying on exact computations); a study of the typical open contours in the complete wetting regime (see Section 7.4.5).

Generalization to other models

As the second Part of this work should show, extension to other 2D models is not trivial, even perturbatively. It would be interesting to study such problems, especially those in which new physical phenomena occur. New ideas may be needed.

More than two dimensions

Very few is understood for problems in more than two dimensions, although remarkable progress has been achieved in the study of 2D SOS models with volume constraint, for which the (three dimensional version of) Winterbottom construction has been shown to hold (see [De, BI]). For the 3D Ising model, even simpler problems, like the existence of a roughening transition or even the instability of the $(1, 1, 1)$ interface, have not been solved.

Notice however, that a large part of our techniques can be used to study *correlation functions* in more than two dimensions; for example the high temperature manifestation of the pinning transition in Chapter 6 can be studied in the 3D Ising model. What is specifically two-dimensional is the equivalence between the open contour of high temperature representation of the 2-point functions and the low temperature phase separation line.

Part I

Ising model

Chapter 1

Definitions, notations

This chapter is devoted to the definition of the two-dimensional Ising model on the square lattice, as well as the introduction of some of the basic geometrical concepts needed in subsequent chapters.

1.1 The model

In this work, we are mostly interested in two-dimensional models on the square lattice, although some of the results presented have a larger domain of validity. We therefore restrict our definitions to this case. The underlying physical space is the lattice

$$\mathbb{Z}^2 \doteqdot \{t = (t(1), t(2)) \,:\, t(i) \in \mathbb{Z}\,,\; i = 1, 2\} \tag{1.1}$$

which we consider as being embedded in the plane $\mathbb{R}^2$. We introduce three norms on $\mathbb{R}^2$,

$$\begin{aligned} \|t\|_1 &\doteqdot |t(1)| + |t(2)|\,, \\ \|t\|_2 &\doteqdot [t(1)^2 + t(2)^2]^{\frac{1}{2}}\,, \\ \|t\|_\infty &\doteqdot \max\{|t(1)|, |t(2)|\}\,. \end{aligned} \tag{1.2}$$

The induced norms on $\mathbb{Z}^2$ are denoted in the same way.

The lattice $\mathbb{Z}^2$ is considered as a totally ordered set, with the order

$$t < t' \Leftrightarrow (t(2) = t'(2) \text{ and } t(1) < t'(1)) \text{ or } t(2) < t'(2) \tag{1.3}$$

The techniques used in this work are in a large part of a geometrical nature; we therefore introduce an appropriate terminology.

Definition.

(D14) *An element $t \in \mathbb{Z}^2$ is called a* **site***; it is sometimes identified with the corresponding point of the embedding plane $\mathbb{R}^2$.*

(D15) *Two sites t and t' are called* **nearest-neighbours** *if $\|t - t'\|_1 = 1$.*

(D16) *An unordered pair of nearest-neighbours sites $e \doteqdot \langle t, t' \rangle$ is called an* **edge***; it is sometimes identified with the unit length segment in $\mathbb{R}^2$ with endpoints t and t'. The set of all edges is $\mathcal{E}$.*

(D17) *An edge e is* **adjacent to a site** t *if* $e = \langle t, t' \rangle$*; it is* **adjacent to a set of sites** *if it is adjacent to a site of the set.*

(D18) *Let* $B \subset \mathcal{E}$*; the* **index of a site** t **in** B, $\mathrm{i}(t, B)$, *is the number of edges of* B *which are adjacent to* t.

(D19) *A unit square in* $\mathbb{R}^2$ *whose corners are sites is called a* **plaquette**.

We introduce now two subsets of $\mathbb{Z}^2$ which play an important role in the following.

Definition.

(D20) $\mathbb{L} \doteqdot \{t \in \mathbb{Z}^2 \,:\, t(2) \geqslant 0\}$ *is called the* **half-plane**.

(D21) $\Sigma \doteqdot \{t \in \mathbb{Z}^2 \,:\, t(2) = 0\}$ *is called the* **wall**.

In the Ising model, a random variable (the *spin*) taking value in $\{-1, 1\}$ is tied to each site of the lattice; more precisely, we have the following

Definition.

(D22) $\{-1, 1\}$ *is called the (single)* **spin space**.

(D23) $\Omega \doteqdot \{-1, 1\}^{\mathbb{Z}^2}$ *is called the* **configuration space**. *The elements* $\omega \in \Omega$ *are called* **configurations**.

(D24) *Let* t *be some site; the random variable* $\sigma(t) : \Omega \to \{-1, 1\}, \sigma(t)(\omega) = \omega(t)$*, is called the* **spin at** t.

There is a natural partial order on Ω which is the order induced by the total order on $\{-1, 1\}$.

Definition.

(D25) *Let* ω *and* $\omega' \in \Omega$. $\omega \leqslant \omega'$ *if and only if* $\omega(t) \leqslant \omega'(t)$ *for all* t.

(D26) *A function* $f : \Omega \to \mathbb{R}$ *is* **increasing** *if* $\omega \leqslant \omega'$ *implies that* $f(\omega) \leqslant f(\omega')$.

(D27) *A function* $f : \Omega \to \mathbb{R}$ *is* **decreasing** *if* $-f$ *is increasing.*

We would like now to define some probability measures on the configuration space Ω. To this end, we first have to give Ω the structure of a measurable space $(\Omega, \mathcal{F})$.

Definition.

(D28) *Let* $\Lambda \subset \mathbb{Z}^2$, $\mathcal{F}_\Lambda$ *is the* σ*-algebra generated by the random variables* $\sigma(t)$, $t \in \Lambda$. *We set* $\mathcal{F} \doteqdot \mathcal{F}_{\mathbb{Z}^2}$.

(D29) *A function* $f : \Omega \to \mathbb{R}$ *is* Λ**-local** *if it is* $\mathcal{F}_\Lambda$*-measurable and* Λ *is a finite subset of* $\mathbb{Z}^2$.

We can now define some probability measures on $(\Omega, \mathcal{F})$, which are known as *Gibbs measures*. These measures are characterized by a function on the configuration space, the *Hamiltonian*, which gives the energy of a configuration. However, as the energy of most configurations will be infinite, it is necessary to be careful. One way to solve this problem[1] is to restrict our attention to configurations which are identical outside some finite region

[1] There is an alternative construction of Gibbs measures, directly at infinite volume, through the so-called DLR equation, see the Chapter 0. However, this approach is not adequate for the study of phase separation, see discussion in Chapter 6.

so that the difference of energy between any two such configurations is finite. This is exactly the role of the boundary condition.

Definition.

(D30) *Let* $\Lambda \subset \mathbb{Z}^2$ *be a* finite *subset of* $\mathbb{Z}^2$ *and* $\overline{\omega} \in \Omega$ *be some configuration. A configuration* ω *is said to* **satisfy the** $\overline{\omega}$**-boundary condition in** Λ *, or shortly the* $\Lambda^{\overline{\omega}}$**-b.c.**, *if* $\omega(t) = \overline{\omega}(t)$*, for all* $t \notin \Lambda$.

(D31) *To each edge* $e \in \mathcal{E}$*, we associate a real number* $J(e)$*, the* **coupling constant at** e.

(D32) *Let* $\Lambda \subset \mathbb{Z}^2$ *be a* finite *subset of* $\mathbb{Z}^2$. *The* **energy in** Λ *, or* **Hamiltonian in** Λ *, is the function* $H_\Lambda : \Omega \to \mathbb{R}$ *defined by*

$$H_\Lambda(\omega) \doteq - \sum_{\substack{e=\langle t,t'\rangle \\ e\cap\Lambda\neq\varnothing}} J(e)\sigma(t)(\omega)\sigma(t')(\omega)\,.$$

(D33) *The* **Gibbs measure in** Λ **with** $\overline{\omega}$**-b.c.** *is the probability measure on* $(\Omega, \mathcal{F})$ *given by*[2]

$$\mu_\Lambda^{\overline{\omega}}(\omega) \doteq \begin{cases} \Xi^{\overline{\omega}}(\Lambda)^{-1} \exp(-H_\Lambda(\omega)) & \text{if } \omega \text{ satisfies the } \Lambda^{\overline{\omega}}\text{-b.c.,} \\ 0 & \text{otherwise,} \end{cases}$$

where $\Xi^{\overline{\omega}}(\Lambda)$*, the normalization constant, is called the* **partition function in** Λ **with** $\overline{\omega}$**-b.c.**.

Expectation value with respect to $\mu_\Lambda^{\overline{\omega}}$ is denoted by $P_\Lambda^{\overline{\omega}}[\cdot]$, $P_\Lambda^{\overline{\omega},J}[\cdot]$, $\langle\cdot\rangle_\Lambda^{\overline{\omega}}$ or $\langle\cdot\rangle_\Lambda^{\overline{\omega},J}$.
We finally introduce a last kind of Gibbs measures, corresponding to a different kind of boundary condition.

Definition.

(D34) *Let* $\Lambda \subset \mathbb{Z}^2$. *The* **Gibbs measure in** Λ **with free-b.c.** *is the probability measure on* $(\{-1,1\}^\Lambda, \mathcal{F}_\Lambda)$ *given by*

$$\mu_\Lambda(\omega) \doteq \Xi(\Lambda)^{-1} \prod_{e=\langle t,t'\rangle\subset\Lambda} \exp(J(e)\sigma(t)(\omega)\sigma(t')(\omega))$$

where $\Xi(\Lambda)$*, the normalization constant, is called the* **partition function in** Λ **with free-b.c.**.

Expectation value with respect to μ_Λ is denoted by $P_\Lambda[\cdot]$, $P_\Lambda^J[\cdot]$, $\langle\cdot\rangle_\Lambda$ or $\langle\cdot\rangle_\Lambda^J$.

Besides the free boundary condition, three boundary conditions play a major role in the following. The first two boundary conditions are the + and −-b.c, and correspond respectively to $\overline{\omega}^+ \equiv 1$ and $\overline{\omega}^- \equiv -1$. The last one is the d-b.c. which is the boundary condition corresponding to the configuration $\overline{\omega}^d$ which is defined in the following way. Let d be a straight line in $\mathbb{R}^2$which we suppose to be non-vertical. We then define

$$\overline{\omega}^d(t) \doteq \begin{cases} +1 & \text{if } t \text{ is above or on the line } d, \\ -1 & \text{otherwise,} \end{cases} \tag{1.4}$$

[2] Notice that, contrarily to what is often done in Statistical Physics, we do not introduce explicitly the temperature in the definition of the measure, but absorb it in the coupling constants.

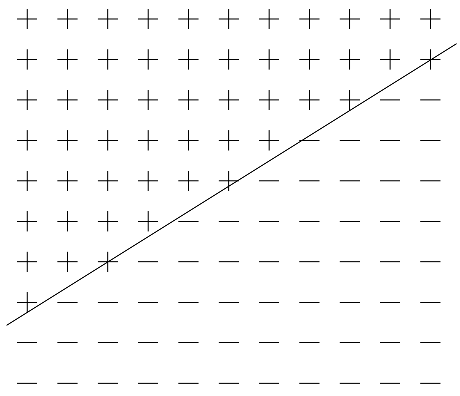

FIGURE 1.1. The d-boundary condition.

Since these boundary conditions play a special role, we introduce convenient notations for their expectation values,

$$\mu^+_\Lambda(\cdot) \doteqdot \mu^{\overline{\omega}^+}_\Lambda(\cdot)\,,\ \ P^+_\Lambda[\cdot] \doteqdot P^{\overline{\omega}^+}_\Lambda[\cdot]\,,\ \ P^{+,J}_\Lambda[\cdot] \doteqdot P^{\overline{\omega}^+,J}_\Lambda[\cdot]\,,\ \ \langle\cdot\rangle^+_\Lambda \doteqdot \langle\cdot\rangle^{\overline{\omega}^+}_\Lambda\,,\ \ \langle\cdot\rangle^{+,J}_\Lambda \doteqdot \langle\cdot\rangle^{\overline{\omega}^+,J}_\Lambda\,, \tag{1.5}$$

and similarly for $-$ and d-b.c..

Definition.

(D35) *Let $A \subset \mathbb{Z}^2$ and let $\Lambda_n \subset A$, $n \geqslant 1$. We say that $(\Lambda_n)_{n\,\geqslant\,1}$* **tends monotonously to** *A, $\Lambda_n \nearrow A$, if $\Lambda_n \subset \Lambda_{n+1}$ and for all $t \in A$ there exists n_0 such that $t \in \Lambda_n$ for all $n \geqslant n_0$.*

Suppose that $J(e) \geqslant 0$, for all edges e. The limits

$$\mu^+ \doteqdot \lim_{\Lambda\nearrow\mathbb{Z}^2} \mu^+_\Lambda\,,\quad \mu^- \doteqdot \lim_{\Lambda\nearrow\mathbb{Z}^2} \mu^-_\Lambda \tag{1.6}$$

exist[3] (see Appendix A). The corresponding expectation values are denoted by

$$P^+[\cdot]\,,\ \ P^{+,J}[\cdot]\,,\ \ \langle\cdot\rangle^+\,,\ \ \langle\cdot\rangle^{+,J} \tag{1.7}$$

and similarly for $-$-b.c.. In fact, it can be proved [Ge1] that the set of all limiting Gibbs states, i.e. the set of all accumulation points of all sequences $(\mu^{\overline{\omega}_n}_{\Lambda_n})_{n\in\mathbb{N}}$ of Gibbs measures $\mu^{\overline{\omega}_n}$ in Λ_n with $\overline{\omega}_n$-b.c., belongs to the convex set[4]

$$\{\tilde\mu\,:\,\tilde\mu = \alpha\mu^- + (1-\alpha)\mu^+\,,\,0 \leqslant \alpha \leqslant 1\}\,. \tag{1.8}$$

The limit

$$\mu \doteqdot \lim_{\Lambda\nearrow\mathbb{Z}^2} \mu_\Lambda \tag{1.9}$$

also exists (see Appendix A) (in fact it is equal to $\frac{1}{2}\mu^+ + \frac{1}{2}\mu^-$). Expectation value with respect to this limiting measure is denoted by

$$P[\cdot]\,,\ \ P^J[\cdot]\,,\ \ \langle\cdot\rangle\,,\ \ \langle\cdot\rangle^J\,. \tag{1.10}$$

[3] With the topology of local convergence, as in the Introduction. We recall that μ_n converges to μ locally, if $\mu_n(A)$ converges to $\mu(A)$, for all $A \in \bigcup_{\Lambda\subset\mathbb{Z}^2,\ \text{finite}} \mathcal{F}_\Lambda$.

[4] This set is exactly the set of all solutions to the DLR equation, see [Ai2, Hi1].

Suppose that $J(e) = \beta$, for every edges e, then μ^+ and μ^- are translation invariant and therefore the same is true for each limiting Gibbs state.

Remark. As discussed in Chapter 6, although it is true from a mathematical point of view that, for example, the measure μ^d is translation invariant, this may be quite unsatisfactory from a physical point of view and another kind of limit, the continuum limit, appears to be more relevant in the study of phenomena involving phase separation.

Remark. The problem of existence of the limit for a given sequence $(\mu_{\Lambda_n}^{\overline{\omega}_n})_{n\in\mathbb{N}}$ is a non-trivial problem (except in the uniqueness region, when $\mu^+ = \mu^-$, see Appendix A). See Figure 1.2 for a simple example of a sequence for which the limit does not exist.

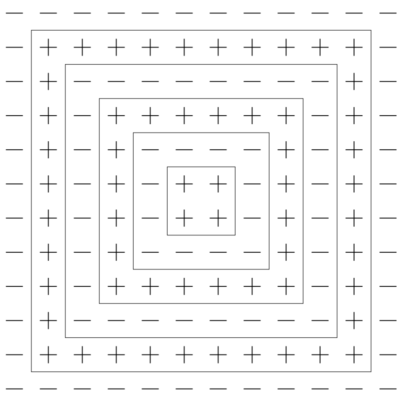

FIGURE 1.2. A boundary condition $\overline{\omega}$ and a sequence of boxes Λ_n. The corresponding sequence of Gibbs measure $(\mu_{\Lambda_n}^{\overline{\omega}})_{n\in\mathbb{N}}$ has two cluster points, μ^+ and μ^-. Therefore the limiting Gibbs state does not exist when $\mu^+ \neq \mu^-$.

Since all limiting Gibbs measures of the $2D$ Ising model have a decomposition as convex combination of the two extreme Gibbs measures μ^+ and μ^-, this set is reduced to a single point as soon as $\mu^+ = \mu^-$. The fact that there exists a critical value of the coupling constant such that this set becomes non-trivial (and therefore that the formalism of Statistical Physics is sufficiently rich to describe phase transitions[5]) is one of the cornerstone of 20th Century Physics. Let us quickly discuss this point.

Definition.
(D36) *The quantity* $m^*(\beta) \doteqdot \langle\sigma(t)\rangle^{+,\beta}$ *is called the* **spontaneous magnetization**.

We recall the following famous result
Theorem 1.1.1. *[O, Y] Suppose* $J(e) = \beta$*, for all* $e \in \mathcal{E}$*. There exists a particular value* $\beta_c \doteqdot \frac{1}{2}\log(1+\sqrt{2})$*, called the* **critical coupling** *such that*

$$\langle\cdot\rangle^{+,\beta} \neq \langle\cdot\rangle^{-,\beta} \Leftrightarrow m^*(\beta) > 0 \Leftrightarrow \beta > \beta_c\,.$$

[5] This was really not obvious at all. In 1937, during a congress held in Amsterdam to commemorate the birth of van der Waals, Kramers, who was chairman, put to a vote the question of whether statistical mechanics (with thermodynamic limit) could explain sharp phase transitions. The result of the vote was inconclusive [D]. Notice that Peierls published his famous paper on the existence of a phase transition in the 2D Ising model in 1936!

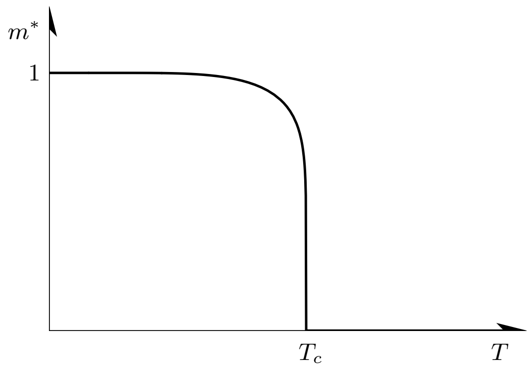

FIGURE 1.3. The spontaneous magnetization of the 2D Ising model as a function of the temperature $T = 1/k\beta$.

1.2 Contours

In many situations, the random variables $\sigma(t)$ are not the most useful, and it is very convenient to introduce other quantities of a more geometrical nature. Among these objects, one kind has become of major importance in Statistical Physics, the *contours*. The original notion has been introduced in 1936 by Peierls in the first "proof" of the existence of a phase transition in the 2D Ising model [Pe]. Since then, through a sequence of generalizations, these objects have become one the main tools in Rigorous Statistical Physics; they are, for example, the basic quantities in the famous Pirogov-Sinai theory [PS1, PS2, Z1, BIm][6] (as a matter of fact, the most abstract notion of contours up to now is given in a new extension of this theory, see [Z2, HZ][7]). However for our need, the original notion of contours is sufficient.

We introduce now the basic definitions and notations we need and refer to the next chapter for applications of these concepts to the Ising model.

Definition.

(D37) *A* **path** *is an ordered sequence of sites and edges,* $t_0, e_0, t_1, e_1, \ldots, t_n$*, such that*
- $t_i \in \mathbb{Z}^2$, $i = 0, \ldots, n$,
- $e_i = \langle t_i, t_{i+1} \rangle \in \mathcal{E}$, $i = 0, \ldots, n-1$,
- $e_i \neq e_j$ *if* $i \neq j$ [8].

(D38) t_0 *is the* **initial point** *of the path,* t_n *the* **final point**.

(D39) e_0 *is the* **initial edge** *of the path,* e_{n-1} *the* **final edge**.

(D40) *A path is* **closed** *if its initial and final points coincide:* $t_0 = t_n$*, otherwise, it is* **open**.

(D41) *A path* $t_0, e_0, \ldots, t_n$ *is a* **subset of** $A \subset \mathbb{Z}^2$ *if* $t_i \in A$, $i = 0, \ldots, n$.

(D42) *A path* $t_0, e_0, \ldots, t_n$ *is a* **subset of** $B \subset \mathcal{E}$ *if* $e_i \in B$, $i = 0, \ldots, n-1$.

[6]For pedagogical introductions to this theory, see [K, Z2]

[7]According to D. Ueltschi, during a seminar in Churaňov in 1995, the following exchange took place:

Borgs: "Miloš, what is now a contour?"

Zahradník: "Well, I would say, it's a matter of personal choice."

[8]Notice that it is possible that $t_i = t_j$, $i \neq j$.

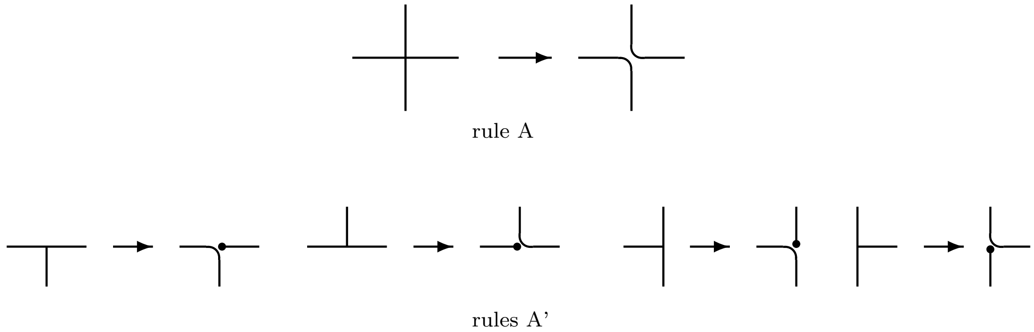

FIGURE 1.4. Deformation rules A and A'. The dots denote initial points of open paths.

Paths may have a rather complicated structure, however it is possible to decompose them further in such a way as to obtain very simple objects[9].

Definition.

(D43) *Let $A \subset \mathbb{Z}^2$; the* **boundary of** *A is the set*

$$\partial A \doteqdot \{t \in A \,:\, \exists t' \in \mathbb{Z}^2 \setminus A\,,\, \|t'-t\|_1 = 1\}\,.$$

(D44) *Let $B \subset \mathcal{E}$; the* **boundary of** *B is the set $\partial B \doteqdot \{t \in \mathbb{Z}^2 \,:\, \mathrm{i}(t,B) \text{is odd}\}$.*

(D45) *The* **boundary of a path** *is the boundary of the set of its edges.*

Let $\varnothing \neq B \subset \mathcal{E}$, finite. We decompose uniquely (up to orientation) this set into a finite number of paths, which are disjoint two by two when considered as sets of edges (but several paths may contain the same site). There are two cases:

1. If $\partial B = \varnothing$, then choose an edge $e = \langle t, t' \rangle \in B$ and set $t_0 \doteqdot t$, $e_0 \doteqdot e$, $t_1 \doteqdot t'$. The path is uniquely continued using rule A specified in Fig. 1.4 and the requirements that it is maximal and that its final point is t_0. This defines a closed path. This construction is repeated with the remaining edges until all edges of B belong to some path.
2. If $\partial B \neq \varnothing$, then first choose a site $t \in \partial B$ and set $t_0 \doteqdot t$. Then choose e_0 among the edges adjacent to t_0 according to rule A' of Fig. 1.4. The path is uniquely continued using rules A and A' and the requirements that it is maximal and that its final point is $t_n \in \partial B$. This defines an open path. Repeat this construction starting from another site of ∂B; when all such sites have been used then the remaining edges (if any) are treated using the procedure described in 1..

Definition.

[9]The fact that this simplification is possible and that the resulting objects have nice properties is a particularity of this model which simplify hugely the analysis, as can be seen in the next chapters; in particular, this is the lack of such a property which makes the case of the Ashkin-Teller model so much more complicated (see Part II).

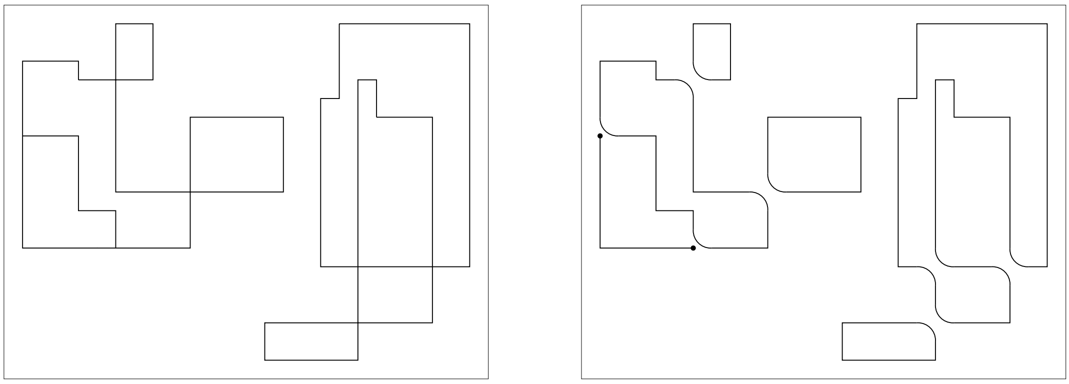

FIGURE 1.5. A set of edges and its decomposition into a family of contours (one open and five closed).

(D46) *The unoriented paths obtained with these procedures are called* **contours**.

(D47) *The closed unoriented paths obtained with these procedures are called* **closed contours**.

(D48) *The open unoriented paths obtained with procedure 2. are called* **open contours**.

Let $\{\gamma_1, \dots, \gamma_n\}$ be a family of contours and let $\mathcal{E}(\gamma_1, \dots, \gamma_n)$ denotes the set of all edges of the family.

Definition.

(D49) $\{\gamma_1, \dots, \gamma_n\}$ *is* **compatible** *if either* $\mathcal{E}(\gamma_1, \dots, \gamma_n) = \varnothing$, *or* $\{\gamma_1, \dots, \gamma_n\}$ *corresponds to the set of contours of the decomposition of* $\mathcal{E}(\gamma_1, \dots, \gamma_n)$. *A compatible family of contours is denoted by* $\underline{\gamma}$.

(D50) *Let* $\Lambda \subset \mathbb{Z}^2$; *the family* $\{\gamma_1, \dots, \gamma_n\}$ *is* **Λ-compatible** *if it is compatible and all edges of* $\mathcal{E}(\gamma_1, \dots, \gamma_n)$ *are pairs of points in* Λ .

(D51) *The set of edges of a family of contour* $\underline{\gamma}$ *is denoted by* $\mathcal{E}(\underline{\gamma})$.

(D52) *The* **length of a contour** γ *is the number of edges in* $\mathcal{E}(\gamma)$; *it is denoted by* $|\gamma|$.

(D53) *The* **length of a compatible family of contours** $\underline{\gamma}$ *is* $|\underline{\gamma}| \doteqdot \sum_{\gamma \in \underline{\gamma}} |\gamma|$

The contours we have just defined are unoriented paths. It is however sometimes useful to choose an orientation and consider a contour γ as a unit-speed parameterized curve in $\mathbb{R}^2$,

$$[0, |\gamma|] \to \mathbb{R}^2 \tag{1.11}$$

$$s \mapsto \gamma(s) \tag{1.12}$$

where $\gamma(0) = \gamma(|\gamma|)$ is the first site of the lattice belonging to γ (according to the total order on $\mathbb{Z}^2$) if $\partial\gamma = \varnothing$, and $\gamma(0) = t_0$, $\gamma(|\gamma|) = t_n$ if $\partial\gamma = \{t_0, t_n\}$.

Chapter 2

Contours representations and duality

In this chapter, we first introduce two of the geometrical representations of the 2D Ising model, known as the low temperature (LT) and high temperature (HT) representations[1]. We then show how these representations establish a duality between the low and high temperature Ising models.

2.1 Low temperature representation

This is the most ancient geometrical representation of the Ising model and the core of the famous Peierls argument [Pe]. The idea is the following: Instead of specifying the value of each spin, we can indicate the boundaries of the regions of constant spin; these sets together with the value of the boundary spins (specified by the boundary condition) provide sufficient information to reconstruct the complete configuration.
Before giving this representation, we introduce another lattice of fundamental importance for our purposes.

Definition.

(D54) *The* **dual lattice** $\mathbb{Z}^{2*}$ *is defined as*

$$\mathbb{Z}^{2*} \doteqdot \{t = (t(1), t(2)) \,:\, t(i) - \tfrac{1}{2} \in \mathbb{Z}\,,\, i = 1, 2\}\,.$$

We will again consider $\mathbb{Z}^{2*}$ as being embedded in $\mathbb{R}^2$. All notions introduced for $\mathbb{Z}^2$ have a natural correspondent for $\mathbb{Z}^{2*}$, for which we use the same notation except that we add a "*" superscript, as in $\mathcal{E}^*$ for example. When we want to emphasize the fact that we work with objects of the dual lattice we use expressions as dual sites, dual edges, and so on...

There is a natural mapping between the basic geometrical quantities of the two lattices $\mathbb{Z}^2$ and $\mathbb{Z}^{2*}$:

- To each site t we associate the dual plaquette $p^*(t)$ having t at its center.

[1]They are also often called LT and HT *expansions*. We do not use this terminology here since it may give the misleading impression that the techniques used are perturbative.

- To each edge e we associate the dual edge $e^*(e)$ which crosses e.
- To each plaquette p we associate the dual site $t^*(p)$ which is at its center.

We would like now to associate to a set $\Lambda \subset \mathbb{Z}^2$ a dual set $\Lambda^* \subset \mathbb{Z}^{2*}$.

Definition.

(D55) *Let $\Lambda \subset \mathbb{Z}^2$. The* **dual set** Λ^* *of Λ is given by*

$$\mathbb{Z}^{2*} \doteqdot \bigcup_{t\in\Lambda} \{t^* \,:\, t^* \text{ is a corner of } p^*(t)\}\,.$$

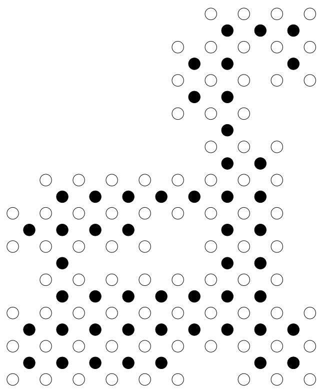

FIGURE 2.1. A set (the black dots) and its dual (the white dots).

2.1.1 The +-boundary condition

Let $\omega \in \Omega$ be some configuration satisfying the Λ^+-b.c., for some finite $\Lambda \subset \mathbb{Z}^2$. Our aim is to give a description of such a configuration in terms of contours.

Definition.

(D56) *The contours obtained by decomposing the set of dual edges*

$$\mathcal{E}^*(\omega) \doteqdot \{e^* \in \mathcal{E}^* \,:\, \omega(t)\omega(t') = -1\,,\ \langle t, t'\rangle \text{ dual to } e^*\}$$

are called **contours of the configuration** ω *and are denoted by* $\underline{\gamma}(\omega)$.

We first state some basic properties of the family $\underline{\gamma}(\omega)$.

Lemma 2.1.1. *Let ω be a configuration satisfying the Λ^+-b.c.. Then $\partial\mathcal{E}^*(\omega) = \varnothing$ and therefore $\underline{\gamma}(\omega)$ is a Λ^*-compatible family of closed contours (see* (D49)*, p. 40).*

Proof. Consider a plaquette p with corners t, t', t'' and $t''' \in \mathbb{Z}^2$. Clearly,

$$(\omega(t)\omega(t'))(\omega(t')\omega(t''))(\omega(t'')\omega(t'''))(\omega(t''')\omega(t)) = 1 \tag{2.1}$$

and consequently there must be an even number of -1 in the four products of (2.1). But that implies that $\mathrm{i}(t^*(p), \mathcal{E}^*(\omega))$ is even. Since this argument can be made for every plaquette of $\mathbb{Z}^2$, we obtain $\partial\mathcal{E}^* = \varnothing$. □

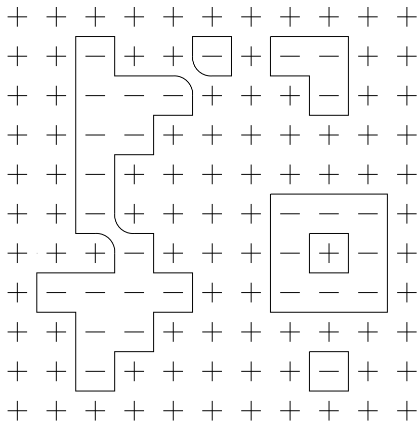

FIGURE 2.2. A configuration and its family of contours.

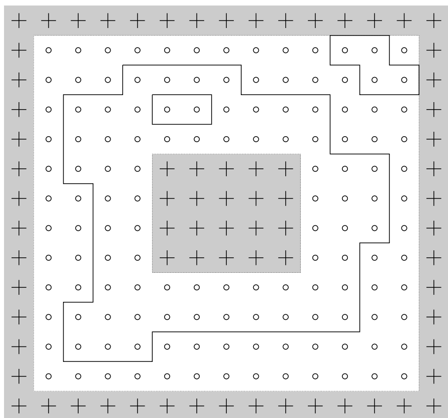

FIGURE 2.3. A Λ^*-compatible family of contours which is not Λ^+-compatible. The dots represent the sites of Λ, the + represent the boundary condition. Observe that Λ has a "hole".

This lemma shows that to any configuration ω satisfying the Λ^+-b.c. it is possible to associate a Λ^*-compatible family of closed contours. In general, the converse is not true (see Fig. 2.3). This motivates the following

Definition.

(D57) *A Λ^*-compatible family $\underline{\gamma}$ of contours is Λ^+-compatible if and only if there exists a configuration ω satisfying Λ^+-b.c. such that $\underline{\gamma}(\omega) = \underline{\gamma}$.*

We want now to find sufficient conditions on Λ to ensure that any Λ^*-compatible family of closed contours is Λ^+-compatible.

Definition.

(D58) *A set $A \subset \mathbb{Z}^2$ is* **simply connected** *if the subset of $\mathbb{R}^2$ given by*

$$\bigcup_{t \in A} p^*(t)$$

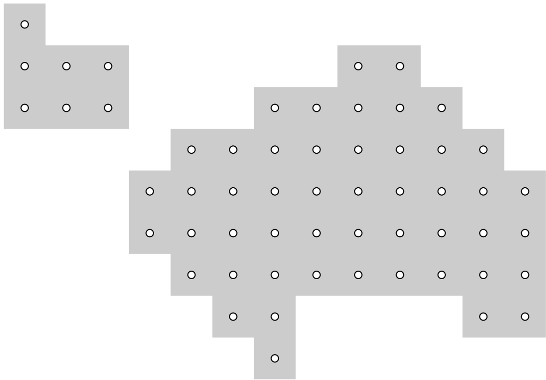

FIGURE 2.4. A simply connected set. The union of the plaquettes corresponds to the shaded region.

is simply connected.

Before proceeding to the proof that this is indeed a sufficient condition, we have to introduce some further terminology related to contours.

Definition.

(D59) *Let* γ *be some (dual) closed contour. We denote by* ω_γ *the unique configuration such that* $\omega_\gamma(t) = 1$ *except for a finite number of sites and such that* γ *is the unique contour of* ω_γ*.*

(D60) *The* **interior** *of* γ *is* $\mathrm{int}\gamma \doteqdot \{t \in \mathbb{Z}^2 : \omega_\gamma(t) = -1\}$*.*

(D61) *The* **exterior** *of* γ *is* $\mathrm{ext}\gamma \doteqdot \mathbb{Z}^2 \setminus \mathrm{int}\gamma$*.*

(D62) *The* **volume** *of* γ *is* $\mathrm{vol}\,\gamma \doteqdot |\mathrm{int}\gamma|$*.*

(D63) *Let* ω *be some configuration. A contour* γ *of* ω *is* **external** *if there is no other contour* γ' *of* ω *with* $\mathrm{int}\gamma \subset \mathrm{int}\gamma'$*.*

(D64) *Let* γ *be a closed* Λ^**-compatible contour. The* **closure in** Λ **of the interior of** γ*,* $\overline{\mathrm{int}}\gamma$*, is the union of* $\mathrm{int}\gamma$ *and the set of all* $t \in \Lambda \setminus \mathrm{int}\gamma$ *such that* $\omega(t) = 1$ *for* **any** *configuration* ω *such that*

- ω *satisfies the* Λ^+*-b.c.;*
- γ *is an external contour of* ω *.*

(D65) *Let* γ *be a closed* Λ^**-compatible contour. The* **closure in** Λ **of the exterior of** γ*,* $\overline{\mathrm{ext}}\gamma$*, is the union of* $\mathrm{ext}\gamma$ *and the set of all* $t \in \Lambda \setminus \mathrm{ext}\gamma$ *such that* $\omega(t) = -1$ *for* **any** *configuration* ω *such that*

- ω *satisfies the* Λ^+*-b.c.;*
- γ *is an external contour of* ω *.*

The sites which have been added when introducing the closure of the interior and exterior of a contour are exactly the sites whose spin has a fixed value whenever the contour is present. Let us now show that simple connectedness is sufficient for our purposes.

Lemma 2.1.2. *If* Λ *is simply connected then* $\underline{\gamma}$ *is a* Λ^**-compatible family of closed contours if and only if* $\underline{\gamma}$ *is a* Λ^+*-compatible family of contours.*

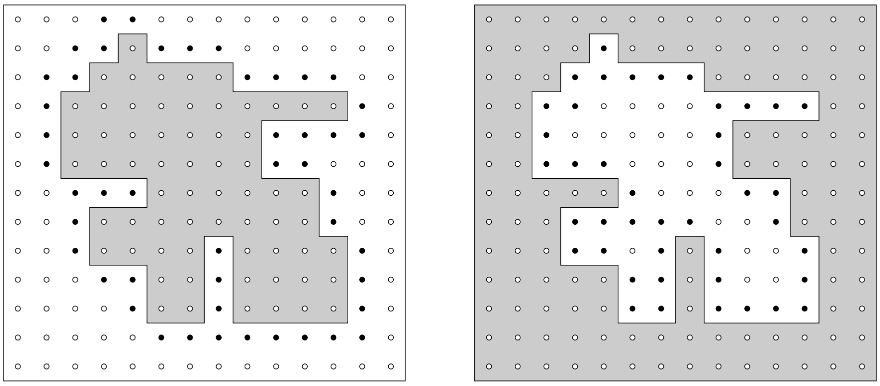

FIGURE 2.5. a) A contour, its interior (the shaded region) and the closure of its interior (the shaded region and the black dots). b) A contour, its exterior (the shaded region) and the closure of its exterior (the shaded region and the black dots).

Proof. Let $\underline{\gamma}$ be a Λ^*-compatible family of closed contours. We construct the following configuration,

$$\omega_{\underline{\gamma}}(t) \doteqdot \prod_{\substack{\gamma\in\underline{\gamma}:\\ t\in\mathrm{int}\gamma}} (-1)\,. \tag{2.2}$$

Since Λ is simply connected, $\omega_{\underline{\gamma}}$ obviously satisfies Λ^+-b.c.; indeed, suppose there exist $\gamma \in \underline{\gamma}$ and $t \notin \Lambda$ with $t \in \mathrm{int}\gamma$. Then the plaquette $p^*(t)$ dual to t does not belong to the set $\mathcal{P}(\Lambda) \doteqdot \bigcup_{t\in\Lambda} p^*(t)$. But this implies that γ is a simple closed curve in $\mathcal{P}(\Lambda)$ which is not nullhomotopic in $\mathcal{P}(\Lambda)$. This contradicts the fact that $\mathcal{P}(\Lambda)$ is simply connected. The conclusion follows since

$$\underline{\gamma}(\omega_{\underline{\gamma}}) = \underline{\gamma}\,. \tag{2.3}$$

□

It is therefore possible to associate to any configuration satisfying Λ^+-b.c. a unique Λ^+-compatible family of contours. Consequently we can try now to work directly at the level of the contours. To do this, it is necessary to obtain expressions for the probability of events described in terms of contours. The first step is to introduce a probability measure on the set of all Λ^+-compatible families of contours.

Definition.

(D66) *Let $\underline{\gamma}$ be a Λ^+-compatible family of contours. The probability of $\underline{\gamma}$ is denoted by*

$$P^+_\Lambda[\underline{\gamma}; J] \doteqdot P^{+,J}_\Lambda[\omega_{\underline{\gamma}}]\,.$$

(D67) *Let $\underline{\gamma}$ be a Λ^*-compatible family of contours. The probability that $\underline{\gamma}$ belong to the set of contours of a configuration is denoted by*

$$q^+_\Lambda(\underline{\gamma}; J) \doteqdot P^+_\Lambda[\{\underline{\gamma}' \,:\, \underline{\gamma} \subset \underline{\gamma}'\}; J]\,.$$

We would like now to obtain an explicit expression for these probabilities in terms of contours.

Definition.

(D68) *The* **weight of a (dual) contour** *is* $w(\gamma) \doteqdot \prod_{e^*\in\gamma} \exp[-2J(e)]$. *The* **weight of a compatible family** *of (dual) contours* $\underline{\gamma}$ *is given by* $w(\underline{\gamma}) \doteqdot \prod_{\gamma\in\underline{\gamma}} w(\gamma)$.

Let Λ be a finite subset of $\mathbb{Z}^2$. In term of the weights we have just defined, we can write the partition function in Λ with +-b.c. as

$$\begin{aligned}\Xi^+(\Lambda) &= C_\Lambda \sum_\omega \prod_{\substack{e=\langle t,t'\rangle:\\ e\cap\Lambda\neq\varnothing}} \exp[J(e)(\sigma(t)(\omega)\sigma(t')(\omega)-1)] \\ &= C_\Lambda \sum_{\underline{\gamma}\ \Lambda^+\text{-comp.}} \prod_{\gamma\in\underline{\gamma}} \prod_{e^*\in\gamma} \exp[-2J(e)] = C_\Lambda \sum_{\underline{\gamma}\ \Lambda^+\text{-comp.}} w(\underline{\gamma})\,, \end{aligned} \tag{2.4}$$

where $C_\Lambda \doteqdot \prod_{\substack{e\in\mathcal{E}\\ e\cap\Lambda\neq\varnothing}} \exp J(e)$.

Definition.

(D69) *Let* $\underline{\gamma}'$ *be some* Λ^**-compatible family of closed contours. We set*

$$Z^+(\Lambda|\underline{\gamma}';J) \doteqdot \sum_{\substack{\underline{\gamma}:\partial\underline{\gamma}=\varnothing\\ \underline{\gamma}\cup\underline{\gamma}'\ \Lambda^+\text{-compatible}}} w(\underline{\gamma})\,.$$

If $\underline{\gamma}'=\varnothing$ *, we write*

$$Z^+(\Lambda;J) \doteqdot Z^+(\Lambda|\varnothing;J)\,. \tag{2.5}$$

This last quantity is called the **normalized partition function in Λ with +-b.c.**.

These quantities, defined in terms of contours, give an explicit expression for the probabilities discussed above.

Lemma 2.1.3. *Let Λ be a finite subset of $\mathbb{Z}^2$. Then*

$$\Xi^+(\Lambda;J) = C_\Lambda Z^+(\Lambda;J)\,.$$

Let $\underline{\gamma}$ *be a* Λ^+*-compatible family of contours; then*

$$P^+_\Lambda[\underline{\gamma};J] = \frac{w(\underline{\gamma})}{Z^+(\Lambda;J)}\,.$$

Let $\underline{\gamma}$ *be a* Λ^**-compatible family of closed contours; then*

$$q^+_\Lambda(\underline{\gamma};J) = w(\underline{\gamma})\frac{Z^+(\Lambda|\underline{\gamma};J)}{Z^+(\Lambda;J)}\,.$$

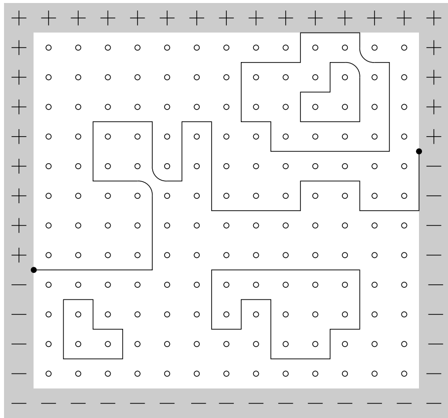

FIGURE 2.6. The family of contours of a configuration satisfying the Λ^d-b.c..

2.1.2 The d-boundary condition

Let $\Lambda_{L,M} \doteqdot \{t \in \mathbb{Z}^2 \, : \, -L < t(1) \leqslant L, -M < t(2) \leqslant M\}$, and d be some straight line in $\mathbb{R}^2$. We suppose that L and M are large enough so that d intersects the set $\{t \in \mathbb{R}^2 \, : \, -L < t(1) \leqslant L, -M < t(2) \leqslant M\}$.

Let ω be any configuration satisfying the $\Lambda^d_{L,M}$-b.c.. We denote by $t^*_l = t^*_l(L,M)$ and $t^*_r = t^*_r(L,M)$ the two dual site of $\Lambda^*_{L,M}$ such that

- $t^*_l(1) = -L + \frac{1}{2}$,
- $t^*_r(1) = L + \frac{1}{2}$,
- $t^*_l \in \mathcal{E}^*(\omega)$,
- $t^*_r \in \mathcal{E}^*(\omega)$.

Definition.

(D70) *The $\Lambda^*_{L,M}$-compatible family of contours obtained by decomposing the set of dual edges $\mathcal{E}^*_{\Lambda^*_{L,M}}(\omega) \doteqdot \{e^* \in \mathcal{E}^*(\omega) \, : \, e^* \subset \Lambda^*_{L,M}\}$ are called* **contours of the configuration** ω *and are denoted by* $\underline{\gamma}(\omega)$.

Lemma 2.1.4. *Let ω be a configuration satisfying the $\Lambda^d_{L,M}$-b.c.. Then $\partial\mathcal{E}^*_{\Lambda^*_{L,M}}(\omega) = \{t^*_l, t^*_r\}$ and therefore the contours of $\gamma(\omega)$ are all closed except for one open contour λ such that $\partial\lambda = \{t^*_l, t^*_r\}$.*

Proof. It is similar to the proof for the +-b.c. case, see Lemma 2.1.1. □

As $\Lambda_{L,M}$ is simply connected, it is easy to check that there is a one-to-one correspondence between configurations satisfying $\Lambda^d_{L,M}$-b.c. and families of $\Lambda^*_{L,M}$-compatible contours as in the previous lemma. In particular, the partition function in $\Lambda_{L,M}$ with d-b.c. can be written (see (2.4))

$$\Xi^d(\Lambda_{L,M}) = C_{\Lambda_{L,M}} Z^d(\Lambda_{L,M}) \tag{2.6}$$

where we have defined

Definition.

(D71) $$Z^d(\Lambda_{L,M}|\lambda) \doteqdot \sum_{\substack{\underline{\gamma}:\partial\underline{\gamma}=\varnothing \\ \underline{\gamma}\cup\lambda\ \Lambda^d\text{-comp.}}} w(\underline{\gamma})$$

(D72) $$Z^d(\Lambda_{L,M}) \doteqdot \sum_{\substack{\lambda:\Lambda^*\text{-comp.} \\ \partial\lambda=\{t_l^*,t_r^*\}}} w(\lambda) Z^d(\Lambda_{L,M}|\lambda).$$

2.2 High-temperature representation

There is another contours representation of the 2D Ising model which is of major importance for our purposes. In this section, we consider the Gibbs measure with free boundary condition on subgraphs of the graph[2] $(\mathbb{Z}^2, \mathcal{E})$.

Definition.

(D73) *Let* $\Lambda \subset \mathbb{Z}^2$*; the* **graph** $\mathcal{G}(\Lambda)$ **specified by the set of sites** Λ *has* Λ *as set of sites, and*

$$\mathcal{E}(\Lambda) \doteqdot \{e = \langle t, t' \rangle \,:\, t, t' \in \Lambda\}$$

as set of edges.

(D74) *Let* $B \subset \mathcal{E}$*; the* **graph** $\mathcal{G}(B)$ **specified by the set of edges B** *has* B *as set of edges, and*

$$\Lambda(B) \doteqdot \{t \in \mathbb{Z}^2 \,:\, \mathrm{i}(t, B) \neq 0\}$$

as set of sites.

(D75) *If* $\mathcal{G}$ *is some graph, then* $\Lambda(\mathcal{G})$ *and* $\mathcal{E}(\mathcal{G})$ *denote respectively the sets of its sites and of its edges.*

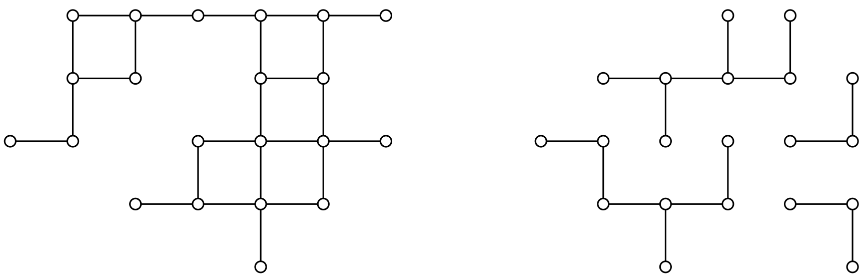

FIGURE 2.7. a) The graph specified by a set of sites. b) The graph specified by a set of edges.

For later purposes (see Chapter 4), it is useful to define partition function on graphs defined by a set of edges.

[2]For our considerations, a graph (V, E) is a set of sites (or dual sites) V and a set of edges (or dual edges) E connecting some of these sites

Definition.

(D76) *Let $B \subset \mathcal{E}$ be finite; the* **partition function with free b.c. on the graph** $\mathcal{G}(B)$ *is defined by*

$$\Xi(\mathcal{G}(B)) \doteqdot \sum_{\omega \in \{-1,1\}^{\Lambda(B)}} \prod_{e=\langle t,t'\rangle \subset B} \exp(J(e)\sigma(t)(\omega)\sigma(t')(\omega)) .$$

Notice that if $B = \mathcal{E}(\Lambda)$, then $\Xi(\mathcal{G}(B)) = \Xi(\Lambda)$.

Let $B \subset \mathcal{E}$ be finite. The basic idea of this representation is to write the partition function in $\mathcal{G}(B)$ with free boundary condition in the following way:

$$\Xi(\mathcal{G}(B)) = \sum_{\omega \in \{-1,1\}^{\Lambda(B)}} \prod_{e=\langle t,t'\rangle \in B} \cosh J(e)(1 + \sigma(t)(\omega)\sigma(t')(\omega) \tanh J(e)) . \tag{2.7}$$

Expanding the product over edges gives a sum of terms, each of which can be indexed by a subset of B,

$$\begin{aligned}\Xi(\mathcal{G}(B)) &= \prod_{e \in B} \cosh J(e) \sum_{\omega \in \{-1,1\}^{\Lambda(B)}} \sum_{E \subset B} \prod_{e=\langle t,t'\rangle \in E} \tanh J(e)\sigma(t)(\omega)\sigma(t')(\omega) \\ &= \prod_{e \in B} \cosh J(e) \sum_{E \subset B} \prod_{e \in E} \tanh J(e) \sum_{\omega \in \{-1,1\}^{\Lambda(B)}} \prod_{\langle t,t'\rangle \in E} \sigma(t)(\omega)\sigma(t')(\omega) . \end{aligned} \tag{2.8}$$

The last sum can be easily evaluated:

$$\sum_{\omega \in \{-1,1\}^{\Lambda(B)}} \prod_{\langle t,t'\rangle \in E} \sigma(t)(\omega)\sigma(t')(\omega) = \prod_{t \in \Lambda(B)} \sum_{\omega(t)=\pm 1} \sigma(t)(\omega)^{\mathrm{i}(t,E)} = 2^{|\Lambda(B)|} 1_{\{\partial E = \varnothing\}} . \tag{2.9}$$

Putting all this together, we obtain

$$\begin{aligned}\Xi(\mathcal{G}(B)) &= 2^{|\Lambda(B)|} \prod_{e \in B} \cosh J(e) \sum_{\substack{E \subset B \\ \partial E = \varnothing}} \prod_{e \in E} \tanh J(e) \\ &= 2^{|\Lambda(B)|} \prod_{e \in B} \cosh J(e) \sum_{\substack{\underline{\gamma} \subset B \\ \text{closed}}} w^*(\underline{\gamma}) , \end{aligned} \tag{2.10}$$

where $\underline{\gamma}$ is the decomposition of E into a compatible family of closed contours and we have introduced

Definition.

(D77) *The* **$*$-weight of a contour** γ *is given by* $w^*(\gamma) \doteqdot \prod_{e \in \gamma} \tanh J(e)$.

(D78) *The* **$*$-weight of a Λ-compatible family of contour** $\underline{\gamma}$ *is given by*

$$w^*(\underline{\gamma}) \doteqdot \prod_{\gamma \in \underline{\gamma}} w^*(\gamma) .$$

Looking at (2.10), it is natural to define

Definition.

(D79) *The* **normalized partition function in $\mathcal{G}(B)$ with free b.c.** *is defined as*

$$Z(\mathcal{G}(B); J) \doteqdot \sum_{\substack{\underline{\gamma} \subset B \\ \text{closed}}} w^*(\underline{\gamma}) .$$

(D80) *The* **normalized partition function in Λ with free b.c.** *is defined as*

$$Z(\Lambda | J) \doteqdot \sum_{\substack{\underline{\gamma} \\ \Lambda\text{-comp., closed}}} w^*(\underline{\gamma}) .$$

Again, if $B = \mathcal{E}(\Lambda)$, then $Z(\mathcal{G}(B)) = Z(\Lambda)$.

Remark. Notice that we can have $Z(\mathcal{G}(B_1)) = Z(\mathcal{G}(B_2))$ even if $B_1 \neq B_2$. Indeed, for this to hold, it is sufficient that these two graphs have the same set of closed contours.

More generally, we introduce some other quantities similar to those defined in the previous section.

Definition.

(D81) *Let $B \subset \mathcal{E}$ finite, and $\underline{\gamma} \subset B$ be a compatible family of contours; we set*

$$Z(\mathcal{G}(B) | \underline{\gamma}; J) \doteqdot \sum_{\substack{\underline{\gamma}' \subset B \colon \partial \underline{\gamma}' = \varnothing \\ \underline{\gamma} \cup \underline{\gamma}' \text{ -comp.}}} w^*(\underline{\gamma}) .$$

(D82) *Let $\underline{\gamma}$ be a Λ-compatible family of contours; we set*

$$Z(\Lambda | \underline{\gamma}; J) \doteqdot \sum_{\substack{\underline{\gamma}' \colon \partial \underline{\gamma}' = \varnothing \\ \underline{\gamma} \cup \underline{\gamma}' \ \Lambda\text{-comp.}}} w^*(\underline{\gamma}) .$$

In particular, $Z(\mathcal{G}; J) = Z(\mathcal{G} | \varnothing; J)$. Notice that, as before, $Z(\mathcal{G}(\mathcal{E}(\Lambda)) | \underline{\gamma}; J) = Z(\Lambda | \underline{\gamma}; J)$. We have therefore obtained an expression of the partition function in terms of contours. The important point is that it is also possible to obtain a similar representation for correlation functions $\langle \sigma_A \rangle_\Lambda$, where we have used the standard notation,

Definition.

(D83) *Let $A \subset \mathbb{Z}^2$; the random variable σ_A is defined by $\sigma_A \doteqdot \prod_{t \in A} \sigma(t)$.*

Indeed, the correlation function can be written as

$$\begin{aligned}\langle \sigma_A \rangle_\Lambda = \Xi(\Lambda)^{-1} \prod_{e \in \mathcal{E}(\Lambda)} \cosh J(e) \sum_{B \subset \mathcal{E}(\Lambda)} \prod_{e \in B} \tanh J(e) \sum_{\omega \in \{-1,1\}^\Lambda} \sigma_A \prod_{\langle t,t' \rangle \in B} \sigma(t)(\omega) \sigma(t')(\omega) \\ = 2^{|\Lambda|} \prod_{e \in \mathcal{E}(\Lambda)} \cosh J(e) \sum_{\substack{\underline{\gamma} \ \Lambda\text{-comp.} \\ \partial \underline{\gamma} = A}} w^*(\underline{\gamma}) , \quad (2.11)\end{aligned}$$

and, therefore,

$$\langle \sigma_A \rangle_\Lambda = Z(\Lambda; J)^{-1} \sum_{\substack{\underline{\gamma} \ \Lambda\text{-comp.} \\ \partial \underline{\gamma} = A}} w^*(\underline{\gamma})\,. \tag{2.12}$$

Remark. If $|A|$ is odd then clearly $\langle \sigma_A \rangle_\Lambda = 0$, since there are no Λ-compatible family of contours with boundary A.

The most important quantity in this work is the 2-point function,

Definition.

(D84) *Let t and t' be two sites. The correlation function on $\{t, t'\}$ is called the* **2-point function**.

This quantity has the representation

$$\langle \sigma(t)\sigma(t') \rangle_\Lambda = Z(\Lambda; J)^{-1} \sum_{\substack{\lambda : \Lambda\text{-comp.} \\ \partial\lambda = \{t,t'\}}} w^*(\lambda) Z(\Lambda|\lambda; J)\,. \tag{2.13}$$

This motivates the introduction of the fundamental quantity

Definition.

(D85) *Let $\underline{\gamma}$ be a Λ-compatible family of contours; we set*

$$q_\Lambda(\underline{\gamma}; J) \doteqdot w^*(\underline{\gamma}) \frac{Z(\Lambda|\underline{\gamma}; J)}{Z(\Lambda; J)}\,.$$

With this notation, the 2-point function can be written very simply as

$$\langle \sigma(t)\sigma(t') \rangle_\Lambda = \sum_{\substack{\lambda : \Lambda\text{-comp.} \\ \partial\lambda = \{t,t'\}}} q_\Lambda(\lambda; J)\,. \tag{2.14}$$

This is the basis of the random line representation of Section 4.4.
As we have done for the low temperature representation, we introduce a probability measure on the set of all Λ-compatible family of *closed* contours.

Definition.

(D86) *Let $\underline{\gamma}$ be a Λ-compatible family of closed contours; we set*

$$P_\Lambda[\underline{\gamma}; J] \doteqdot \frac{w^*(\underline{\gamma})}{Z(\Lambda; J)}\,.$$

Observe that, if $\underline{\gamma}$ is a Λ-compatible family of closed contours, we have the identity

$$q_\Lambda(\underline{\gamma}; J) = P_\Lambda[\{\underline{\gamma}' \,:\, \underline{\gamma} \subset \underline{\gamma}'\}; J]\,. \tag{2.15}$$

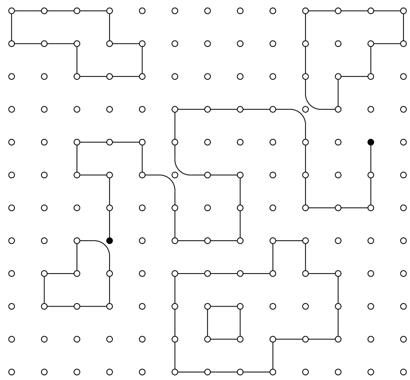

FIGURE 2.8. A family of contours contributing to the high-temperature expansion of the 2-point-function.

2.3 Duality

Suppose Λ is a simply connected finite subset of $\mathbb{Z}^2$. Lemma 2.1.2 states that the set of all Λ^+-compatible families of contours is identical to the set of all Λ^*-compatible family of closed contours. Therefore the normalized partition function in Λ with +-boundary conditions can be interpreted as the normalized partition function in Λ^* with free b.c. if $w(\underline{\gamma}) = w^*(\underline{\gamma})$ (compare (D69), p. 46, and (D80), p. 50).

Definition.

(D87) *Let $J(e) \geqslant 0$, for all edges $e \in \mathcal{E}$. To the set $J(e)$, we associate a set $J^*(e^*)$, $e^* \in \mathcal{E}^*$, of* **dual coupling constants** *defined by*

$$\tanh J^*(e^*) = \exp(-2J(e))\,.$$

Remark. Suppose $J(e) = \beta$, for all $e \in \mathcal{E}$. We denote by β^* the dual coupling constant in this case. The fixed point of the duality relation (D87), known as the *self-dual coupling*, coincide with the critical coupling $\beta_c = \frac{1}{2}\log(1+\sqrt{2})$ [O]. Historically, it is by using this relation that β_c has first been computed [KW]. Notice however that it is a non-trivial fact that the self-dual manifold and the critical manifold coincide in the 2D Ising model; another model in which this is not true is the Ashkin–Teller model, see Section 10.4.2 for a discussion.

Lemma 2.3.1. *Let Λ be a simply connected finite subset of $\mathbb{Z}^2$. Then*

$$Z^+(\Lambda; J) = Z(\Lambda^*; J^*)\,.$$

Moreover, for any Λ^*-compatible family of contours $\underline{\gamma}$,

$$P^+_\Lambda[\underline{\gamma}; J] = P_{\Lambda^*}[\underline{\gamma}; J^*]\,,$$

and, for any Λ^*-compatible family of closed contours,

$$q^+_\Lambda(\underline{\gamma}; J) = q_{\Lambda^*}(\underline{\gamma}; J^*)\,.$$

When the low temperature representation of a model coincides with the high temperature representation of another model, we say that the two models are related by *duality*; this is a very general phenomenon which has been studied in depth, see for example [GHM, DW]. When, as it is the case here, the two representations of a same model have this property, we say that the model is *self-dual*; this is much more exceptional (another such example is the Ashkin–Teller model of Part II)[3].

[3] Both models studied in this work are self-dual; this is done for convenience, however self-duality is not a necessary condition for this approach to hold.

Chapter 3

Surface tension and massgap

In this chapter, we introduce two fundamental quantities: The surface tension and the massgap of the 2-point function, and show how they are related by duality.

3.1 The surface tension and massgap

In this section[1], we suppose that the coupling constants are given by $J(e) = \beta$, for all edges $e \in \mathcal{E}$.

3.1.1 The surface tension

From a physical point of view, the surface tension is the contribution to the free energy due to the coexistence of phases. We first define this quantity and then try to motivate the definition[2].

Let $\Lambda_{L,M}$ be the box

$$\Lambda_{L,M} \doteqdot \{t \in \mathbb{Z}^2 \,:\, -L < t(1) \leqslant L\,,\, -M < t(2) \leqslant M\}. \tag{3.1}$$

Let $\boldsymbol{n}$ be a unit vector in $\mathbb{R}^2$, and $d(\boldsymbol{n})$ be the straight line containing the dual site $(\frac{1}{2}, \frac{1}{2})$ and perpendicular to $\boldsymbol{n}$. We define t_l^* and t_r^* as in Section 2.1.2.

Definition.

(D88) *Let* $\boldsymbol{n}$ *be some unit vector in* $\mathbb{R}^2$*. The* **surface tension in the direction** $\boldsymbol{n}$ *is defined as*

$$\tau(\boldsymbol{n};\beta) \doteqdot \lim_{\substack{L\to\infty \\ M\to\infty}} \tau(\boldsymbol{n}|\Lambda_{L,M};\beta)\,,$$

where we have introduced

$$\tau(\boldsymbol{n}|\Lambda_{L,M};\beta) \doteqdot -\frac{1}{\|t_r^* - t_l^*\|_2} \log \frac{\Xi^{d(\boldsymbol{n})}(\Lambda_{L,M})}{\Xi^{+}(\Lambda_{L,M})}\,.$$

[1] For this section, we follow [Pf1]; other nice introductions and discussions of these concepts can be found, for example, in [Pf3, FG, GW, CF, FC, C].

[2] The motivation given here is heuristic, its purpose is to show the basic physical idea behind the definition. In fact, the results of Chapters 6 and 7 provide an *a posteriori* justification of this definition, since it is proved there that the surface tension introduced in the present chapter plays exactly the role attributed to the surface tension in Thermodynamics.

The existence of the above limit is proved in Subsection 3.1.3; in particular, the limits can be taken in any order.

Let us consider a configuration satisfying the $\Lambda_{L,M}^{d(\boldsymbol{n})}$-boundary condition. We have seen that the family of contours of such a configuration is composed of closed contours and one open contour whose boundary is $\{t_l^*, t_r^*\}$; we call this open contour the *phase separation line*. In Chapter 6, we prove that for all typical configurations, the open contour is contained into some deterministic set of very small width (from a macroscopic point of view, see discussion in that chapter), above which the system is in the + phase (up to very small corrections, going to zero when $|\Lambda|$ goes to infinity), while below it is in the minus phase[3]. This deterministic set is what we call the *interface*, see the above mentioned chapter for a more detailed discussion of this subject. Hence, the free energy of a system satisfying the $\Lambda_{L,M}^{d(\boldsymbol{n})}$-boundary condition should be the same as that of a system in the + or − phase plus a contribution coming from the presence of the interface. Let us make these ideas a little bit more precise.
We first define the bulk and surface free energies for a given boundary condition $\overline{\omega}$.

Definition.

(D89) *The* **bulk free energy** *is defined by*

$$f_{\mathrm{b}}^{\overline{\omega}} \doteqdot - \lim_{\substack{L\to\infty \\ M\to\infty}} \frac{1}{|\Lambda_{L,M}|} \log \Xi^{\overline{\omega}}(\Lambda_{L,M}) .$$

(D90) *The* **surface free energy** *is defined by*

$$f_{\mathrm{s}}^{\overline{\omega}} \doteqdot - \lim_{\substack{L\to\infty \\ M\to\infty}} \frac{1}{|\partial\Lambda_{L,M}|} [\log \Xi^{\overline{\omega}}(\Lambda_{L,M}) + f_{\mathrm{b}}^{\overline{\omega}} |\Lambda_{L,M}|] .$$

It is easy to prove that the bulk free energy does not depend on the boundary condition (indeed the difference in energy for two different boundary conditions is $\mathcal{O}(|\partial\Lambda_{L,M}|)$), so it is useless to indicate the configuration $\overline{\omega}$ (this is of course not the case for the surface free energy). In the same way, it is not difficult to show that the surface free energy is finite.

With these notations, the total free energy for $\overline{\omega}$ boundary condition is given by

$$F^{\overline{\omega}}(\Lambda_{L,M}) \equiv -\log \Xi^{\overline{\omega}}(\Lambda_{L,M}) = f_{\mathrm{b}}|\Lambda_{L,M}| + f_{\mathrm{s}}^{\overline{\omega}}|\partial\Lambda_{L,M}| + f_{\mathrm{corr}}^{\overline{\omega}}(\Lambda_{L,M}) , \tag{3.2}$$

where $f_{\mathrm{corr}}^{\overline{\omega}}(\Lambda_{L,M})$ takes into account the corrections to these leading order terms; it is such that

$$\lim_{\substack{L\to\infty \\ M\to\infty}} \frac{1}{|\partial\Lambda_{L,M}|} f_{\mathrm{corr}}^{\overline{\omega}}(\Lambda_{L,M}) = 0 . \tag{3.3}$$

By the symmetry between + and − boundary conditions, the corresponding free energies are equal

$$F^{+}(\Lambda_{L,M}) = F^{-}(\Lambda_{L,M}) , \tag{3.4}$$

[3] To say that the system is in the + or − phase amounts to state that expectation values of local observables can be computed with the infinite volume Gibbs measure μ^+, or μ^-.

and therefore the contribution to the free energy coming from the presence of the interface for the $\Lambda_{L,M}^{d(\boldsymbol{n})}$-boundary condition should be given by

$$F^{d(\boldsymbol{n})}(\Lambda_{L,M}) - F^{+}(\Lambda_{L,M}) = (f_{\mathrm{s}}^{d(\boldsymbol{n})} - f_{\mathrm{s}}^{+})|\partial\Lambda_{L,M}| + o(|\partial\Lambda_{L,M}|) \tag{3.5}$$

which gives for the surface tension

$$\tau(\boldsymbol{n};\beta) = \lim_{\substack{L\to\infty \\ M\to\infty}} \frac{1}{|\partial\Lambda_{L,M}|}(F^{d(\boldsymbol{n})}(\Lambda_{L,M}) - F^{+}(\Lambda_{L,M})) \tag{3.6}$$

which is exactly (D88).

3.1.2 The massgap of the bulk 2-point function

A large part of this work consists in the study of the properties of the 2-point function $\langle\sigma(t)\sigma(t')\rangle_\Lambda$, or of its limiting value $\langle\sigma(t)\sigma(t')\rangle$, see Chapter 4. A fundamental quantity in this study is the massgap of this 2-point function, which measures the rate of decrease of $\langle\sigma(t)\sigma(t')\rangle$ as $\|t'-t\|_2 \to \infty$.

An immediate application of GKS inequalities imply (see Appendix A)

$$\Lambda \subset \Lambda' \Rightarrow \langle\sigma_A\rangle_\Lambda \leqslant \langle\sigma_A\rangle_{\Lambda'} \,. \tag{3.7}$$

As a consequence, $\langle\sigma_A\rangle \doteqdot \lim_{\Lambda\nearrow\mathbb{Z}^2}\langle\sigma_A\rangle_\Lambda$ is well defined.

Let $\boldsymbol{n}$ be some unit vector in $\mathbb{R}^2$and $d^*(\boldsymbol{n})$ be the straight line through $(\frac{1}{2},\frac{1}{2})$, in the direction $\boldsymbol{n}$. We suppose that $\boldsymbol{n}$ is such that $d^*(\boldsymbol{n})$ contains an infinite number of dual sites.

Let $t_0^* = (\frac{1}{2},\frac{1}{2})$ and let $0 \neq t \in \mathbb{Z}^2$ be such that $t_0^* + t \in d^*(\boldsymbol{n})$ and $\|t\|_2$ is minimal.

Definition.

(D91) *The* **massgap of the 2-point function in the direction $\boldsymbol{n}$** *is defined by*

$$\alpha(\boldsymbol{n};\beta) \doteqdot -\lim_{k\to\infty}\frac{1}{\|kt\|_2}\log\langle\sigma(t_0^*)\sigma(t_0^*+kt)\rangle^\beta \,.$$

Lemma 3.1.1. *The limit in* (D91) *exists and the convergence is uniform in $\boldsymbol{n} \in \mathcal{S}^1$. Moreover,*

$$\langle\sigma(t)\sigma(t')\rangle^\beta \leqslant \exp(-\|t'-t\|_2\alpha(\boldsymbol{n}_{t'-t};\beta)) \,,$$

where $\boldsymbol{n}_{t'-t}$ is the unit vector in $\mathbb{R}^2$given by $\boldsymbol{n}_{t'-t} \doteqdot \dfrac{t'-t}{\|t'-t\|_2}$.

Proof. We write $t_k^* \doteqdot t_0^* + kt$, $k \in \mathbb{N}$. By GKS inequalities and translation invariance,

$$\begin{aligned}\langle\sigma(t_0^*)\sigma(t_{k_1+k_2}^*)\rangle &= \langle\sigma(t_0^*)\sigma(t_{k_1}^*)\sigma(t_{k_1}^*)\sigma(t_{k_1+k_2}^*)\rangle \\ &\geqslant \langle\sigma(t_0^*)\sigma(t_{k_1}^*)\rangle\langle\sigma(t_{k_1}^*)\sigma(t_{k_1+k_2}^*)\rangle = \langle\sigma(t_0^*)\sigma(t_{k_1}^*)\rangle\langle\sigma(t_0^*)\sigma(t_{k_2}^*)\rangle \,.\end{aligned} \tag{3.8}$$

Hence the function $G(k) \doteqdot \log\langle\sigma(t_0^*)\sigma(t_k^*)\rangle$ is subadditive, i.e.

$$G(k_1+k_2) \geqslant G(k_1) + G(k_2) \tag{3.9}$$

and consequently[4]

$$\lim_{k\to\infty} \frac{G(k)}{k} = \sup_k \frac{G(k)}{k} \tag{3.10}$$

exists. Since

$$\alpha(\boldsymbol{n}) = \frac{1}{\|t\|_2} \lim_{k\to\infty} \frac{G(k)}{k}\,, \tag{3.11}$$

the conclusions follow. To prove uniformity, we follow the proof of [I1]. Let $x \mapsto [x]$ be an odd map from $\mathbb{R}^2$ to $\mathbb{Z}^2$ such that $[x]$ is a vertex of a unit plaquette containing x and $[x] = 0$ for some small ball around the origin. We write

$$f_N(x) = -\frac{1}{N} \log \langle \sigma(0)\sigma(Nx)\rangle\,. \tag{3.12}$$

Suppose that there exists a sequence $(\boldsymbol{n}_N)_{N\geqslant 1}$, $\boldsymbol{n}_N \in \mathcal{S}^1$, and $\varepsilon > 0$ such that, for all N,

$$f_N(\boldsymbol{n}_N) > \alpha(\boldsymbol{n}_N) + \varepsilon\,. \tag{3.13}$$

Since the function f_N has the following two properties, which are not difficult to establish (see [I1] for details),

- there exists a constant c such that $|f_N(x) - f_N(y)| \leqslant f_N(x-y) + c/N$,
- for each $\varepsilon > 0$, there exists $k = k(\varepsilon) > 0$ such that $f_N(x) \leqslant k|x| + \varepsilon$,

this implies that

$$\liminf_{N\to\infty} f_N(\boldsymbol{n}) \geqslant \alpha(\boldsymbol{n}) + \varepsilon/2\,, \tag{3.14}$$

where $\boldsymbol{n} = \lim_{N\to\infty} \boldsymbol{n}_N$ (going to a subsequence if necessary). This contradicts the existence of the limit for any fixed vector $\boldsymbol{n}$. □

3.1.3 Existence of the surface tension and relation to the massgap

We want to show the existence of

$$\tau(\boldsymbol{n};\beta) = \lim_{\substack{L\to\infty\\ M\to\infty}} \tau(\boldsymbol{n}|\Lambda_{L,M};\beta) \tag{3.15}$$

with

$$\begin{aligned}
\tau(\boldsymbol{n}|\Lambda_{L,M};\beta) &= -\frac{1}{\|t_r^* - t_l^*\|_2} \log \frac{\Xi^{d(\boldsymbol{n})}(\Lambda_{L,M})}{\Xi^{+}(\Lambda_{L,M})} \\
&= -\frac{1}{\|t_r^* - t_l^*\|_2} \log \frac{Z^{d(\boldsymbol{n})}(\Lambda_{L,M})}{Z^{+}(\Lambda_{L,M})} \\
&= -\frac{1}{\|t_r^* - t_l^*\|_2} \log \frac{1}{Z^{+}(\Lambda_{L,M})} \sum_{\substack{\lambda:\,\Lambda^*_{L,M}\text{-comp.}\\ \partial\lambda=\{t_l^*,t_r^*\}}} w(\lambda) Z^{d(\boldsymbol{n})}(\Lambda_{L,M}|\lambda) \\
&= -\frac{1}{\|t_r^* - t_l^*\|_2} \log \frac{1}{Z(\Lambda^*_{L,M})} \sum_{\substack{\lambda:\,\Lambda^*_{L,M}\text{-comp.}\\ \partial\lambda=\{t_l^*,t_r^*\}}} w^*(\lambda) Z(\Lambda^*_{L,M}|\lambda) \\
&= -\frac{1}{\|t_r^* - t_l^*\|_2} \log \langle \sigma(t_l^*)\sigma(t_r^*)\rangle^{\beta^*}_{\Lambda^*_{L,M}}\,,
\end{aligned} \tag{3.16}$$

[4] We recall the following standard result: Let $(a_n)_{n\in\mathbb{N}}$ be subadditive. Then $\lim_{n\to\infty} \frac{a_n}{n} = \sup_n \frac{a_n}{n}$.

where we have used Lemma 2.1.3, (2.6), (2.14), Lemma 2.3.1 and (D71), p. 48.
We first prove the following lemma.

Lemma 3.1.2. *Suppose $\boldsymbol{n}$ is such that $d(\boldsymbol{n})$ contains an infinite number of dual points. Then, denoting by $\boldsymbol{n}_\perp$ a unit vector normal to $\boldsymbol{n}$,*

$$\tau(\boldsymbol{n};\beta) = \alpha(\boldsymbol{n}_\perp;\beta^*)\,.$$

Proof. Let us consider a sequence of boxes such that t_l^* and $t_r^* \in d(\boldsymbol{n})$. From (3.16), we see that the lemma follows, along this special sequence of boxes, if the following identity is true

$$-\lim_{\substack{L\to\infty\\M\to\infty}} \frac{1}{\|t_r^*-t_l^*\|_2} \log \langle\sigma(t_l^*)\sigma(t_r^*)\rangle_{\Lambda_{L,M}}^{\beta^*} = -\lim_{\substack{L\to\infty\\M\to\infty}} \frac{1}{\|t_r^*-t_l^*\|_2} \log \langle\sigma(t_l^*)\sigma(t_r^*)\rangle^{\beta^*} \tag{3.17}$$

We call the quantity defined by the right-hand side of the above equation the "short" correlation length, while we call the quantity defined by the left-hand side, the massgap, "long" correlation length (see [SML] for a related terminology). It is a non-trivial fact that these two expressions define actually the same quantity, and a very simple counterexample is given in Chapter 6. However, in the case at hands it is true as is shown in the following Lemma. □

Lemma 3.1.3.

$$-\lim_{\substack{L\to\infty\\M\to\infty}} \frac{1}{\|t_r^*-t_l^*\|_2} \log \langle\sigma(t_l^*)\sigma(t_r^*)\rangle_{\Lambda_{L,M}}^{\beta^*} = -\lim_{\substack{L\to\infty\\M\to\infty}} \frac{1}{\|t_r^*-t_l^*\|_2} \log \langle\sigma(t_l^*)\sigma(t_r^*)\rangle^{\beta^*}$$

Proof. We follow the proof of [BLP3].

First, by GKS inequalities,

$$\langle\sigma(t_l^*)\sigma(t_r^*)\rangle_{\Lambda_{L,M}} \leqslant \langle\sigma(t_l^*)\sigma(t_r^*)\rangle \tag{3.18}$$

which implies

$$-\liminf_{\substack{L\to\infty\\M\to\infty}} \frac{1}{\|t_r^*-t_l^*\|_2} \log \langle\sigma(t_l^*)\sigma(t_r^*)\rangle_{\Lambda_{L,M}}^{\beta^*} \geqslant \alpha(\boldsymbol{n}_\perp)\,. \tag{3.19}$$

The other inequality is slightly more difficult to prove.

Let $t_k^* \doteqdot (\frac{1}{2},\frac{1}{2}) + k(t_r^* - (\frac{1}{2},\frac{1}{2}))$; we have $t_r^* = t_1^*$ and $t_l^* = t_{-1}^*$. Moreover, let us use the notation $\Lambda_k \doteqdot \Lambda_{kL,kM}$. Then

$$\langle\sigma(t_{-nk}^*)\sigma(t_{nk}^*)\rangle_{\Lambda_{nk}} \geqslant \prod_{i=-n}^{n-1} \langle\sigma(t_{ik}^*)\sigma(t_{(i+1)k}^*)\rangle_{\Lambda_{nk}}\,. \tag{3.20}$$

For any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\langle\sigma(t_{ik}^*)\sigma(t_{(i+1)k}^*)\rangle_{\Lambda_{nk}} \geqslant \langle\sigma(t_{ik}^*)\sigma(t_{(i+1)k}^*)\rangle - \varepsilon\,, \tag{3.21}$$

as soon as $\|t_{(i+1)k}^* - t_{nk}^*\}\|_1 > \delta$ and $\|t_{ik}^* - t_{-nk}^*\}\|_1 > \delta$, as a consequence of Lemma A.4.1, see Appendix A. Hence, using $\langle\sigma(t_{ik}^*)\sigma(t_{(i+1)k}^*)\rangle_{\Lambda_{nk}} > 0$ [5] and $\langle\sigma(t_{ik}^*)\sigma(t_{(i+1)k}^*)\rangle =$

[5] This is a simple consequence of GKS inequalities and a direct estimate (just remove enough edges to obtain the correlation function of a 1D Ising model).

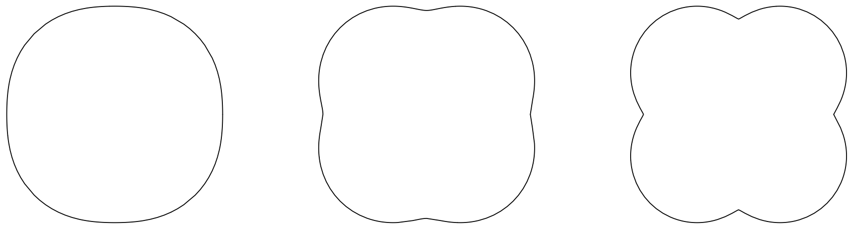

FIGURE 3.1. The surface tension τ as a function of the angle ϑ ($\tau(\vartheta) \doteqdot \tau((\cos\vartheta, \sin\vartheta))$) for decreasing values of the temperature. The scales are different.

$\langle \sigma(t_0^*)\sigma(t_k^*)\rangle$,

$$\frac{1}{\|t_{-nk}^* - t_{nk}^*\|_2} \log \langle \sigma(t_{-nk}^*)\sigma(t_{nk}^*)\rangle_{\Lambda_{nk}}^{\beta^*} \geqslant \frac{1}{\|t_k^* - t_0^*\|_2} \frac{1}{2n} \sum_{i=-n}^{n-1} \log \langle \sigma(t_{ik}^*)\sigma(t_{(i+1)k}^*)\rangle_{\Lambda_{nk}}^{\beta^*}$$
$$\geqslant \frac{1}{\|t_k^* - t_0^*\|_2} \frac{1}{2n} \Big\{ \mathcal{O}(\frac{\delta}{k}) + (2n - \mathcal{O}(\frac{\delta}{k})) \log \langle \sigma(t_0^*)\sigma(t_k^*)\rangle^{\beta^*} - \mathcal{O}(n\varepsilon) \Big\} . \quad (3.22)$$

This implies

$$\limsup_{n\to\infty} -\frac{1}{\|t_{-nk}^* - t_{nk}^*\|_2} \log \langle \sigma(t_{-nk}^*)\sigma(t_{nk}^*)\rangle_{\Lambda_{nk}}^{\beta^*}$$
$$\leqslant -\frac{1}{\|t_k^* - t_0^*\|_2} \log \langle \sigma(t_0^*)\sigma(t_k^*)\rangle^{\beta^*} + \mathcal{O}(\varepsilon) . \quad (3.23)$$

The conclusion follows easily. □

The existence of the limit has been shown only for a dense set of unit vector $\boldsymbol{n}$, however it is shown in Proposition 3.1.1 that τ is continuous in $\boldsymbol{n}$ which implies the existence for any direction.

It is interesting to know that it can be proved that the two limits $L \to \infty$ and $M \to \infty$ in the definition of the surface tension can be taken in any order [FP3]. This is certainly not obvious. Indeed, consider for example the case of an horizontal interface in $\Lambda_{L,M}$. Then the open contour can be shown to have typical fluctuations of $\mathcal{O}(L^{1/2})$ lattice spacings. Suppose we take first the $L \to \infty$ limit, then we have an infinitely long interface inside a cylinder of finite height; this is clearly a strong constraint on the interface whose fluctuations are hugely reduced. It may seem that this should have an effect on the surface tension. In fact, the above mentioned proof shows that the effect is only felt by the corrections to the exponential decay. This is also the case when the interface is constrained to stay in a half plane, see Chapter 4.

3.1.4 Properties of the surface tension and massgap

In this subsection we give some important properties of the surface tension, and consequently of the massgap of the bulk 2-point function as a consequence of Lemma 3.1.2.

Proposition 3.1.1. *The surface tension has the following properties.*

1. *The surface tension $\tau(\boldsymbol{n};\beta)$ can be extended to a positively homogeneous, uniformly Lipschitz, convex function $\tau(x;\beta)$ on $\mathbb{R}^2$.*
2. $\tau(x;\beta) = \tau(-x;\beta) = \tau(x_\perp;\beta) = \tau(-x_\perp;\beta)$*;*
3. *$\tau(x;\beta)$ is a non-negative, increasing function of β; moreover, for all $x \neq 0$,*

$$\tau(x;\beta) > 0 \Leftrightarrow \beta > \beta_c\,.$$

4. *$\tau(x;\beta)$ is a norm on $\mathbb{R}^2$if and only if $\beta > \beta_c$.*
5. *Let ϑ be a parameterization of the unit circle such that $\boldsymbol{n}(\vartheta) = (\cos\vartheta, \sin\vartheta)$. We write $\tau(\vartheta;\beta) \doteqdot \tau(\boldsymbol{n}(\vartheta))$. Then $\tau(\vartheta;\beta)$ satisfies*
   - $\min_\vartheta \tau(\vartheta;\beta) = \tau(0)$*;*
   - $\inf_\theta \tau(\vartheta;\beta) + \tau''(\vartheta;\beta) > 0$. *(Positive stiffness)*
6. *When $\beta > \beta_c$, the norm $\tau(x;\beta)$ satisfies the* **Sharp Triangle Inequality***; let $x_1, x_2 \in \mathbb{R}^2$, then*

$$\tau(x_1;\beta) + \tau(x_2;\beta) - \tau(x_1+x_2;\beta) \geqslant \kappa(\beta)(\|x_1\|_2 + \|x_2\|_2 - \|x_1+x_2\|_2)\,,$$

with $\kappa(\beta) > 0$, for all $\beta > \beta_c$.

Proof. 1. We follow [Pf1]. Let $\Lambda_L \doteqdot \{t \in \mathbb{Z}^2 : |t(i)| \leqslant L\,, i = 1,2\}$. For any $\boldsymbol{n}_1$, $\boldsymbol{n}_2$ we denote by $\theta(\boldsymbol{n}_1,\boldsymbol{n}_2)$ their interior angle. There exists $K = K(\beta) > 0$ such that, for all $\boldsymbol{n}_1$, $\boldsymbol{n}_2$,

$$\Xi^{d(\boldsymbol{n}_1)}(\Lambda_L)\,\mathrm{e}^{-K\theta(\boldsymbol{n}_1,\boldsymbol{n}_2)L} \leqslant \Xi^{d(\boldsymbol{n}_2)}(\Lambda_L) \leqslant \Xi^{d(\boldsymbol{n}_1)}(\Lambda_L)\,\mathrm{e}^{K\theta(\boldsymbol{n}_1,\boldsymbol{n}_2)L}\,. \tag{3.24}$$

Indeed, $|H_\Lambda(\omega_1) - H_\Lambda(\omega_2)| \leqslant -K\theta(\boldsymbol{n}_1,\boldsymbol{n}_2)L$, for all configurations ω_1 satisfying $\Lambda^{d(\boldsymbol{n}_1)}$-b.c., ω_2 satisfying $\Lambda^{d(\boldsymbol{n}_2)}$-b.c. and $\omega_1(t) = \omega_2(t)$, for all $t \in \Lambda$.
Therefore there exists $K'(\beta) > 0$ such that

$$|\tau(\boldsymbol{n}_1;\beta) - \tau(\boldsymbol{n}_2;\beta)| \leqslant K'(\beta)\|\boldsymbol{n}_1 - \boldsymbol{n}_2\|_2\,. \tag{3.25}$$

We then define

$$\begin{cases} \tau(x;\beta) \doteqdot \|x\|_2\,\tau(\frac{x}{\|x\|_2};\beta)\,, & \forall 0 \neq x \in \mathbb{R}^2,\\ \tau(0;\beta) \doteqdot 0\,. \end{cases} \tag{3.26}$$

This shows that $\tau(x;\beta)$ is Lipschitz. Let us prove now that $\tau(x;\beta)$ is a convex function on $\mathbb{R}^2$. By GKS inequalities, for any $x_1, x_2 \in \mathbb{R}^2$,

$$\begin{aligned} \tau(x_1+x_2;\beta) &= -\lim_{k\to\infty}\frac{1}{k}\log\langle\sigma(0)\sigma(k(x_1+x_2))\rangle \\ &\leqslant -\lim_{k\to\infty}\frac{1}{k}\big(\log\langle\sigma(0)\sigma(kx_1)\rangle + \log\langle\sigma(0)\sigma(kx_2)\rangle\big) = \tau(x_1;\beta) + \tau(x_2;\beta)\,, \end{aligned} \tag{3.27}$$

and therefore $\tau(x;\beta)$ is convex.

2. This follows immediately from the definition and the symmetries of $H_\Lambda(\omega)$.

3. By GKS inequalities, for all $k > 0$,

$$0 \leqslant -\frac{1}{k}\log\langle\sigma(0)\sigma(nk)\rangle^{\beta_1^*} \leqslant -\frac{1}{k}\log\langle\sigma(0)\sigma(nk)\rangle^{\beta_2^*}\,,\ \forall\beta_2^* \geqslant \beta_1^*\,, \tag{3.28}$$

which implies

$$0 \leqslant \tau(\boldsymbol{n};\beta_2) \leqslant \tau(\boldsymbol{n};\beta_1)\,,\ \forall\beta_2 \leqslant \beta_1\,. \tag{3.29}$$

The second affirmation is a consequence of [BLP3] and [LP].

4. This is a consequence of (3.26), (3.27) and point 3. of this proposition.

5. The first affirmation is proved in [BMF].

The second affirmation is known from the exact solution [MW, AA]. There is no other non-perturbative proof of this fact up to now.

6. This inequality is equivalent to the positive stiffness of $\tau(x)$ as proven in Appendix B. The fact that it is a consequence of the positive stiffness has already been proved in [I1][6].

□

3.2 The wall free energy and massgap

In Chapters 6 and 7, two phenomena involving phase separation are studied: The pinning of an interface and the wetting of a wall. In both cases, there are large separate regions of + and − phases and the choice of the boundary condition has a strong influence on the behaviour of the system. One of the goal of these chapters is to show how phase transitions can occur in such situations by changing the characteristics of one of the wall of the box. For this reason, it is useful to have a parameter independent of the temperature, so that we can have a fine control on the effect of the boundary. We consider in this section coupling constants given by

$$\begin{cases} J(e) = \beta\,, & \forall e = \langle t_1, t_2\rangle \text{ with } \min_i t_i(2) \geqslant 0\,, \\ J(e) = \beta h\,, & \text{otherwise.} \end{cases} \tag{3.30}$$

[6] In fact he writes it in a slightly different but, as is easy to check, equivalent way.

Definition.

(D92) *The parameter h is the* **boundary magnetic field**.

The three walls with normal coupling constants impose the bulk phase, however by changing the boundary magnetic field, we can choose which one of the two equilibrium phases is adsorbed preferentially at the bottom wall, thus leading at interesting behaviour of the system under consideration, see Chapters 6 and 7 and the end of this section. One physical quantity plays a fundamental role in understanding the effect of the boundary magnetic field on the two phases: The wall free energy. This quantity is analogous to the surface tension, but corresponds to the free energy cost of an interface along the bottom wall; it is introduced in Subsection 3.2.1, and its relation to a dual quantity, the massgap of the boundary 2-point function (see Subsection 3.2.2), is explained in Subsection 3.2.3. Finally basic properties of the wall free energy are given in Subsection 3.2.4, as well as a brief discussion of the wetting phenomenon in the grand canonical ensemble, see Subsection 3.2.5.

3.2.1 The wall free energy

Let $\Lambda_{L,M} := \{t \in \mathbb{Z}^2 : |t(1)| \leqslant L\,,\, 0 \leqslant t(2) \leqslant M\}$ and let d be the horizontal straight line containing 0; we introduce $t_l^* = (-L-\frac{1}{2}, -\frac{1}{2})$, $t_r^* = (L+\frac{1}{2}, -\frac{1}{2})$.

Definition.

(D93) *The* **wall free energy** *is given by*

$$\tau_{\mathrm{bd}}(\beta, h) \doteqdot \lim_{\substack{L\to\infty \\ M\to\infty}} \tau_{\mathrm{bd}}(\beta, h|\Lambda_{L,M})\,,$$

where

$$\tau_{\mathrm{bd}}(\beta, h|\Lambda_{L,M}) \doteqdot -\frac{1}{2L+1}\log\frac{\Xi^d(\Lambda_{L,M};\beta,h)}{\Xi^+(\Lambda_{L,M};\beta,h)}\,.$$

Obviously, the symmetry of the model implies that $\tau_{\mathrm{bd}}(\beta, h)$ is an odd function of h.

Lemma 3.2.1. *The limits in* (D93) *exist and can be taken in any order.*

Proof. We can write $\tau_{\mathrm{bd}}(\beta, h|\Lambda_{L,M})$ in the following way,

$$\tau_{\mathrm{bd}}(\beta, h|\Lambda_{L,M}) = -\frac{1}{2L+1}\log\frac{\Xi^+(\Lambda_{L,M};\beta,-h)}{\Xi^+(\Lambda_{L,M};\beta,h)} = \beta\int_{-h}^{h}\frac{1}{2L+1}\sum_{x=-L}^{L}\langle\sigma((x,0))\rangle_{\Lambda_{L,M}}^{+,\beta,h'}\,\mathrm{d}h'\,. \tag{3.31}$$

FKG inequalities imply that

$$\Lambda \subset \Lambda' \implies \langle\sigma((x,0))\rangle_{\Lambda}^{+,\beta,h'} \geqslant \langle\sigma((x,0))\rangle_{\Lambda'}^{+,\beta,h'}\,. \tag{3.32}$$

Indeed, let us write by $\langle\,\cdot\,\rangle_{\Lambda'}^{+,\beta,h',\rho}$ the expectation value with respect to the Gibbs measure with +-b.c. in Λ' corresponding to the Hamiltonian

$$-\sum_{\substack{e=\langle t,t'\rangle \\ e\cap\Lambda\neq\varnothing}} J(e)\sigma(t)\sigma(t') - \rho\sum_{t\in\Lambda'\setminus\Lambda}\sigma(t)\,, \tag{3.33}$$

where $J(e)$ are given by (3.30). Then $\lim_{\rho\to\infty} \langle\,\cdot\,\rangle_{\Lambda'}^{+,\beta,h',\rho} = \langle\,\cdot\,\rangle_{\Lambda}^{+,\beta,h'}$ from which (3.32) follows. From this,

$$\langle\sigma((x,0))\rangle^{+,\beta,h'} \doteqdot \lim_{\substack{L\to\infty\\ M\to\infty}} \langle\sigma((x,0))\rangle_{\Lambda_{L,M}}^{+,\beta,h'} \tag{3.34}$$

exists. In particular, for any $\varepsilon > 0$, there exists $\Delta \subset \mathbb{L}$ such that

$$\varepsilon \geqslant \langle\sigma((x,0))\rangle_{\Delta}^{+,\beta,h'} - \langle\sigma((x,0))\rangle^{+,\beta,h'} = \langle\sigma((x,0))\rangle_{\Delta}^{+,\beta,h'} - \langle\sigma((0,0))\rangle^{+,\beta,h'} . \tag{3.35}$$

This implies that

$$\varepsilon \geqslant \langle\sigma((x,0))\rangle_{\Lambda_{L,M}}^{+,\beta,h'} - \langle\sigma((0,0))\rangle^{+,\beta,h'} \geqslant 0 , \tag{3.36}$$

for any x such that $\Delta + (x,0) \subset \Lambda_{L,M}$. Therefore,

$$\beta\int_{-h}^{h} \frac{1}{2L+1}\sum_{x=-L}^{L} \langle\sigma((x,0))\rangle_{\Lambda_{L,M}}^{+,\beta,h'}\,\mathrm{d}h' \geqslant \beta\int_{-h}^{h} \langle\sigma((0,0))\rangle_{\Lambda_{L,M}}^{+,\beta,h'}\,\mathrm{d}h' ,$$

and

$$\beta\int_{-h}^{h} \frac{1}{2L+1}\sum_{x=-L}^{L} \langle\sigma((x,0))\rangle_{\Lambda_{L,M}}^{+,\beta,h'}\,\mathrm{d}h' \leqslant \beta\int_{-h}^{h} \langle\sigma((0,0))\rangle_{\Lambda_{L,M}}^{+,\beta,h'}\,\mathrm{d}h' + 2\varepsilon\beta h + \frac{|\Delta|}{2L+1}\beta h . \tag{3.37}$$

Consequently, for any $\varepsilon > 0$, there exists $L_0(\varepsilon)$ and $M_0(\varepsilon)$ such that for all $M \geqslant M_0$ and $N \geqslant N_0$,

$$0 \leqslant \tau_{\mathrm{bd}}(\beta,h|\Lambda_{L,M}) - \beta\int_{-h}^{h} \langle\sigma((0,0))\rangle_{\Lambda_{L,M}}^{+,\beta,h'}\,\mathrm{d}h' \leqslant 3\beta h\varepsilon . \tag{3.38}$$

□

3.2.2 The massgap of the boundary 2-point function

As it was the case for the surface tension, the wall free energy is also dual to an important quantity. In the preceding section, it is shown that the quantity dual to the surface tension was the massgap (or decay–rate) of the corresponding 2-point function of the dual model. We prove in the next subsection that the quantity dual to the wall free energy is again the massgap of a 2-point function of the dual model.

Definition.

(D94) *The set* $\Sigma^* \doteqdot \{t \in \mathbb{Z}^{2*} : t(2) = -\frac{1}{2}\}$ *is called the* **wall of the dual lattice**.

(D95) *A* **boundary 2-point function on the dual lattice** *is a 2-point function* $\langle \sigma(t_1^*)\sigma(t_2^*)\rangle$ *with* $t_1^* \in \Sigma^*$, $t_2^* \in \Sigma^*$.

This is one of the quantities which is studied in detail in Chapter 4.

Definition.

(D96) *The* **massgap of the boundary 2-point function** *is defined by*

$$\alpha_{\mathrm{bd}}(\beta, h) \doteqdot -\lim_{\substack{\|t^*\|_2 \to \infty \\ t^* \in \Sigma^*}} \frac{1}{\|t^*\|_2} \log \langle \sigma((-\tfrac{1}{2}, -\tfrac{1}{2}))\sigma(t^*)\rangle_{\mathbb{L}^*}^{\beta,h}$$

Lemma 3.2.2. *The limit in* (D96) *exists. Moreover, for any* $t, t' \in \Sigma^*$,

$$\langle \sigma(t)\sigma(t')\rangle_{\mathbb{L}^*}^{\beta,h} \leqslant \exp(-\|t' - t\|_2 \alpha_{\mathrm{bd}}(\beta, h)),$$

Proof. This is the same proof as that of lemma 3.1.1. □

3.2.3 Relation between wall free energy and massgap

In Section 3.1.3, the surface tension and the 2-point function of the dual model are shown to be related by a duality relation. The same is true for the wall free energy and the massgap of the boundary 2-point function.

Lemma 3.2.3. *Let* $h \geqslant 0$, *then*

$$\tau_{\mathrm{bd}}(\beta, h) = \alpha_{\mathrm{bd}}(\beta^*, h^*),$$

where h^* *is defined by the duality relation* $\tanh \beta^* h^* = \exp(-2\beta h)$ *(see* (D87)*, p. 52).*

Proof. The proof is analogous to that of Lemma 3.1.2. □

3.2.4 Properties of the wall free energy and massgap

As in Section 3.1.4, we state these properties only for the wall free energy since the corresponding properties of the massgap of the boundary 2-point function are easily obtained from the duality relation. We also restrict our attention to the case $h \geqslant 0$, since $\tau_{\mathrm{bd}}(\beta, h)$ is an odd function of h.

Proposition 3.2.1. *Let* $h \geqslant 0$. *The wall free energy* $\tau_{\mathrm{bd}}(\beta, h)$ *satisfies the following properties.*

1. $\tau_{\mathrm{bd}}(\beta, h)$ *is a non-negative, increasing function of* β *and* h, *concave in* h; *moreover, if* $h \neq 0$,
$$\tau_{\mathrm{bd}}(\beta, h) > 0 \Leftrightarrow \beta > \beta_c .$$
2. *We define a function on* $\mathbb{Z}$ *by*
$$\tau_{\mathrm{bd}}(t; \beta, h) \doteqdot \tau_{\mathrm{bd}}\|t\|_2$$
for all $t \in \Sigma$. *This function satisfies*

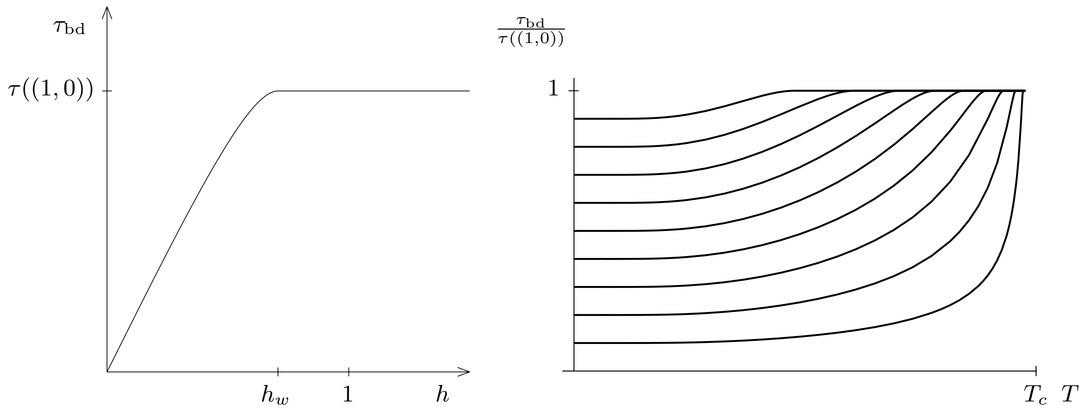

FIGURE 3.2. Left: $\tau_{\rm bd}$ as a function of the magnetic field h, for $\beta = 1.4\beta_c$. Right: $\tau_{\rm bd}/\tau((1,0))$ as a function of the temperature T for values of the magnetic field h ranging from 0.1 to 0.9.

- *For all* $t \in \Sigma$, $\tau_{\rm bd}(t;\beta,h) \leqslant \tau(t;\beta)$*;*
- *If* $\beta > \beta_c$*, then there exists* $h_{\rm w}(\beta) > 0$ *such that, for all* $0 \neq t \in \Sigma$*,*

$$\tau_{\rm bd}(t;\beta,h) < \tau(t;\beta) \Leftrightarrow h < h_{\rm w}(\beta).$$

Proof. 1. The fact that $\tau_{\rm bd}(\beta,h)$ is non-negative, increasing is proved in the same way as the corresponding statement of Proposition 3.1.1. The concavity is proved in [FP4].

2. This is proved in [FP4].

□

The change of behaviour of $\tau_{\rm bd}(\beta,h)$ at $\pm h_{\rm w}(\beta)$ is known as the *wetting transition*. When $h \geqslant h_{\rm w}(\beta)$, we say that we are in the *complete drying*[7] regime, when $|h| < h_{\rm w}(\beta)$ we say that we have *partial wetting*[8] and when $h \leqslant -h_{\rm w}(\beta)$ there is *complete wetting*. See the next subsection and Chapters 6 and 7 for an explanation of this terminology.

3.2.5 The wetting transition in the grand canonical ensemble

The existence of such a phase transition in the Ising model was "discovered" by McCoy and Wu [MW2], however their interpretation in terms of metastable states was incorrect; their error was due to their particular choice of boundary conditions, as was shown by Pfister and Penrose [PP][9]. The same problem was then discussed by Abraham in [A1][10]. We are mostly interested in the study by Fröhlich and Pfister [FP1, FP2, FP3, FP4, Pf2] in the case the Ising model[11]. We describe briefly their results and refer to their work for

[7] Complete drying of the minus phase on the wall. All these notions depend on which phase is taken as reference.

[8] Sometimes the cases $0 < h < h_{\rm w}(\beta)$ and $-h_{\rm w}(\beta) < h < 0$ are distinguished, the first one being called *partial drying* and the second one *partial wetting*.

[9] This shows that exact computations do not necessarily provide an understanding of the underlying phenomena.

[10] Although it was presented as an example of a roughening transition, it was in fact a wetting transition.

[11] The situation considered by these authors goes much beyond that of the previous discussions (through exact computations) of this phenomenon. Their results hold for the Ising and XY models, for any dimensions greater or equal than two and with a possible bulk magnetic field.

details (see also [PV1] for a short review on the results about the wetting transition in the 2D Ising model).
Let

$$\Lambda_L \doteqdot \{t \in \mathbb{Z}^2 \,:\, \|t\|_\infty \leqslant L\}\,, \tag{3.39}$$

and

$$\overline{\omega}_a(t) \doteqdot \begin{cases} a & \text{if } t(2) \geqslant 0\,, \\ 1 & \text{otherwise,} \end{cases} \tag{3.40}$$

where $a = \pm 1$.

We consider an Ising model in Λ_L, with $\overline{\omega}_+$-boundary condition, and coupling constants given by (3.30) with $\beta > \beta_c$. Then the limiting Gibbs state $\mu^{+,\beta}$ is independent of h (which can be positive or negative). Similarly, if we impose the $\overline{\omega}_-$-b.c., then the limiting Gibbs state $\mu^{-,\beta}$ is also independent of h [12]. This is a consequence of Proposition 4.6.1 and Lemma A.4.1. However this does not tell us what happens in the vicinity of the wall. To obtain that kind of information, let us introduce the surface Gibbs states. Let

$$\Lambda'_L \doteqdot \{t \in \mathbb{Z}^2 \,:\, |t_1| \leqslant L\,,\, 0 \leqslant t(2) \leqslant 2L\}\,. \tag{3.41}$$

Definition.

(D97) *The limiting Gibbs states,*

$$\mu_{\mathbb{L}}^{+,\beta,h} \doteqdot \lim_{L\to\infty} \mu_{\Lambda'_L}^{\overline{\omega}_+,\beta,h} \qquad \textit{and} \qquad \mu_{\mathbb{L}}^{-,\beta,h} \doteqdot \lim_{L\to\infty} \mu_{\Lambda'_L}^{\overline{\omega}_-,\beta,h} \tag{3.42}$$

are called **surface Gibbs states**.

The following question is natural: Does the surface Gibbs state depend on the value of a; or, more physically, does the bulk phase have an influence on the phase seen near the bottom wall? Indeed, it may happen that, even if the bulk is in the + phase, there is creation of a film of − phase between the bottom wall and the + phase, preventing the bulk phase to touch the wall. This is the phenomenon of complete wetting.

The main result of Fröhlich and Pfister's study, for our purposes, is summarized in the following theorem.

Theorem 3.2.1. *Suppose that the coupling constants are given by* (3.30)*. Then the following statements are equivalent.*

1. *There is a unique surface Gibbs state (i.e. complete wetting occurs).*
2. *The wall free energy satisfy*

$$|\tau_{\text{bd}}(\beta, h)| = \tau((1,0);\beta) > 0\,.$$

This theorem gives a criterion for complete wetting to occur. This criterion can be established on Thermodynamical bases, using some stability argument. In such a context it is known as Cahn's criterion [C]. Lemma 4.4.8 shows that when this criterion is not satisfied, then the interface is pinned to the bottom wall, in the sense that it must return to the wall very often. This results in a microscopic (i.e. not divergent) thickness of the

[12] In particular $\mu^{+,\beta} \neq \mu^{-,\beta}$, since we can take $h = 1$.

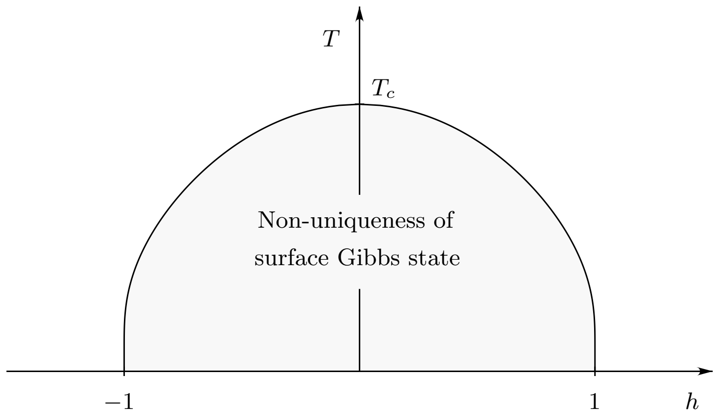

FIGURE 3.3. Phase diagram. The region of non-uniqueness of the surface Gibbs state is shaded. In the other region, there is a single surface Gibbs state.

film (i.e. the expectation value of the height is finite). In such a situation, one says that there is partial wetting. The transition between the situations of partial and complete wetting is called the *wetting transition*. See also the heuristic discussion of Section 7.4.5.

Theorem 3.2.1 gives a precise description of the wetting phenomenon when the amount of − phase is not fixed, i.e. in the grand canonical ensemble. In Chapter 7, another complementary approach to this phenomenon is done in the canonical ensemble, where the total magnetization is (essentially) fixed.

Chapter 4

The 2-point function

The basic quantity in all this study is the 2-point function computed with the Gibbs measure with free boundary condition above the critical temperature. It is proven in Section 2.2 that the 2-point function in $\Lambda \subset \mathbb{Z}^2$ admits the following expression (see (2.14))

$$\langle \sigma(t_1)\sigma(t_2)\rangle_\Lambda^J = \sum_{\substack{\lambda\,:\,\Lambda\text{-comp.}\\ \partial\lambda=\{t_1,t_2\}}} q_\Lambda(\lambda; J)\,, \tag{4.1}$$

where

$$q_\Lambda(\underline{\gamma}; J) = w^*(\underline{\gamma})\frac{Z(\Lambda|\underline{\gamma})}{Z(\Lambda)}\,, \tag{4.2}$$

$$w^*(\underline{\gamma}) = \prod_{\gamma\in\underline{\gamma}} w^*(\gamma)\,, \tag{4.3}$$

$$w^*(\gamma) = \prod_{e\in\gamma} \tanh J(e)\,. \tag{4.4}$$

This chapter is dedicated to the study of the function $q_\Lambda(\,\cdot\,; J)$ and of some consequences. This analysis is done on the lattice $\mathbb{Z}^2$ rather than $\mathbb{Z}^{2*}$ to avoid cumbersome notations. However, in some places where these high temperature results[1] have interesting low temperature counterparts, we switch to the dual lattice to use duality.

Although several results of this section are stated more generally, we are mostly interested in the case when the coupling constants are given by[2]

$$\begin{cases} J(e) = \beta\,, & \forall e = \langle t_1, t_2\rangle \text{ with } \max_i t_i(2) > 0\,,\\ J(e) = \beta h\,, & \text{otherwise,} \end{cases} \tag{4.5}$$

with $\beta \geqslant 0$ and $h \geqslant 0$.

In Section 4.1, we introduce some useful terminology. Basic estimates on $q_\Lambda(\,\cdot\,; J)$ are derived in Section 4.2. These estimates are used in Section 4.3 to obtain an estimate on

[1] Let us make a general comment on this chapter. Many of the results given here are stated in the case $\beta < \beta_c$, but in fact hold also in the case $\beta \geqslant \beta_c$ in which ,however, they become trivial.

[2] Notice that up to a translation to the lattice $\mathbb{Z}^2$, these coupling constants are dual to those defined in (3.30) if the corresponding values of β and h are well-chosen.

FIGURE 4.1. Two edges and their edge-boundary.

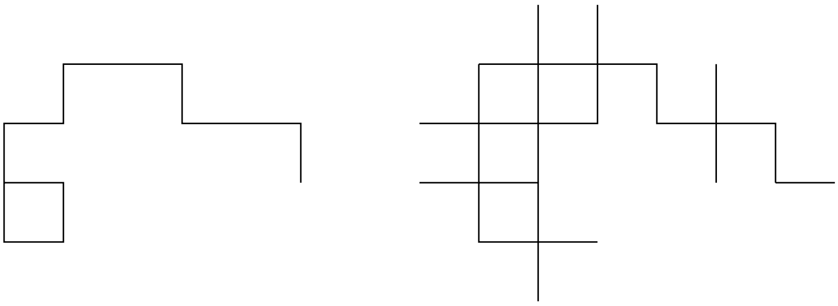

FIGURE 4.2. A set of edges and its edge-boundary.

the size of typical contours in the high- and low-temperature representations. In Section 4.4 we introduce the random-line representation and state some of its properties. Some of these properties related to the determination of the set of typical contours contributing to the 2-point function are greatly improved in Section 4.6 in which an extensive use of the Sharp Triangle Inequality is made. Section 4.5 is devoted to lower bounds on the 2-point functions[3].

4.1 Some terminology

In Section 2 the fundamental notion of (Λ-)compatibility of a family of contours is introduced (see (D49), p. 40). However the definition given there is not very convenient to work with because it is not explicit enough. To obtain an easier, but equivalent, formulation of (Λ-)compatibility it is useful to introduce the notion of edge-boundary of a set of edges.

Definition.

(D98) *Let $e \in \mathcal{E}$; we denote by $\mathcal{A}(e)$ the set of all edges adjacent to e.*

(D99) *The* **edge-boundary of an edge** *e is the contour $\Delta(e) \ni e$ of the decomposition of $\mathcal{A}(e)$ into contours.*

(D100) *Let $A \subset \mathcal{E}$; the* **edge-boundary of** *A is $\Delta(A) \doteqdot \bigcup_{e \in A} \Delta(e)$.*

Remark. Let $t \in \underline{\gamma}$. Then $\mathrm{i}(t, \Delta(\underline{\gamma})) = 2$ or 4, and therefore $\mathrm{i}(t, \mathcal{E} \setminus \Delta(\underline{\gamma})) = 2$ or 0.

We can now state an equivalent characterization of (Λ-)compatibility.

Definition.

[3]Most of the results of this chapter have appeared in [PV3]; similar results as that of Section 4.6 have appeared in [PV4].

(D101) *Let $\underline{\gamma}$ and $\underline{\gamma}'$ be two (Λ-)compatible families of contours. $\underline{\gamma}$* **is (Λ-)compatible with** *$\underline{\gamma}'$ if $\underline{\gamma} \cup \underline{\gamma}'$ is (Λ-)compatible.*

Lemma 4.1.1. *Let $\underline{\gamma}'$ be a (Λ-)compatible family of contours (which may be closed or open). Then a non-empty (Λ-)compatible family of* closed *contours $\underline{\gamma}$ is (Λ-)compatible with $\underline{\gamma}'$ if and only if no edge of $\underline{\gamma}$ is an edge of $\Delta(\underline{\gamma}')$.*

Proof. Suppose $\underline{\gamma}$ is compatible with $\underline{\gamma}'$, and $e \in \mathcal{E}(\gamma_i)$, $\gamma_i \in \underline{\gamma}$. Then $e \notin \mathcal{E}(\underline{\gamma}')$; we show that $e \notin \Delta(\underline{\gamma}')$. Suppose the contrary, then $\mathrm{i}(e, \Delta(\underline{\gamma}') \cup \mathcal{E}(\gamma_i)) \geqslant 3$, since γ_i is closed. This implies that the decomposition of $\mathcal{E}(\underline{\gamma}') \cup \mathcal{E}(\gamma_i)$ is not $(\gamma_i, \underline{\gamma}')$ and therefore γ_i and $\underline{\gamma}'$ are not compatible, which is a contradiction.

Suppose e_1, e_2 are adjacent to a site $t \in \underline{\gamma}'$ such that $\{e_1, e_2\} \cap \Delta(\underline{\gamma}') = \varnothing$. Then the decomposition of $\mathcal{E}(\underline{\gamma}') \cup \{e_1, e_2\}$ into contours is $(\underline{\gamma}', \{e_1, e_2\})$. Consequently, if $\mathcal{E}(\gamma_i) \cap \Delta(\underline{\gamma}') = \varnothing$, then γ_i is compatible with $\underline{\gamma}'$. □

Definition.

(D102) *Let $\Lambda \subset \mathbb{Z}^2$, and let $\underline{\gamma}$ be a given Λ-compatible family of contours. The graph $\mathcal{G}(\underline{\gamma})$ is defined by the set of edges $\mathcal{E}(\Lambda) \setminus \Delta(\underline{\gamma})$.*

We emphasize the fact that $\mathcal{G}(\underline{\gamma}) \neq \mathcal{G}(\mathcal{E}(\underline{\gamma}))$.

The main interest of the above result is that the constrained sum over families of contours appearing in $Z(\Lambda|\underline{\gamma})$ can be replaced by an *unconstrained* sum over families of contours on some subgraph of $\mathcal{G}(\Lambda)$, as is shown in the next lemma.

Lemma 4.1.2.

1. *Let $\Lambda \subset \mathbb{Z}^2$ and let $\underline{\gamma}$ be a Λ-compatible family of contours. Then*

$$Z(\Lambda|\underline{\gamma}) = Z(\mathcal{G}(\underline{\gamma}))\,.$$

2. *Let λ be some open contour and $t \in \lambda$ such that t partition λ into two open contours λ_1 and λ_2. The graph $\mathcal{G}_t(\lambda_2)$, which is defined by the set of edges obtained by adding the edge $\overline{e}$ of $\Delta(\lambda_2) \setminus \mathcal{E}(\lambda_2)$ which is adjacent to t to the set of edges of $\mathcal{G}(\lambda_2)$, is such that*

$$Z(\Lambda|\lambda) = Z(\mathcal{G}_t(\lambda_2)|\lambda_1)\,.$$

Proof. The first statement is a consequence of Lemma 4.1.1. We prove the second one. Let $\underline{\gamma}$ be a compatible family of closed contours.

Suppose first that $\underline{\gamma}$ is Λ-compatible with λ. Then, since $\Delta(\lambda) = \Delta(\lambda_1) \cup \Delta(\lambda_2)$, the following statements hold,

$$\mathcal{E}(\underline{\gamma}) \cap \Delta(\lambda_1) = \varnothing\,, \tag{4.6}$$

$$\underline{\gamma} \subset \mathcal{E}(\mathcal{G}(\lambda_2)) \subset \mathcal{E}(\mathcal{G}_t(\lambda_2))\,. \tag{4.7}$$

But this implies that $\underline{\gamma} \subset \mathcal{E}(\mathcal{G}_t(\lambda_2))$ is compatible with λ_1.

To prove the converse, suppose that $\underline{\gamma} \subset \mathcal{E}(\mathcal{G}_t(\lambda_2))$ is compatible with λ_1. Then

$$\mathcal{E}(\underline{\gamma}) \cap \Delta(\lambda_1) = \varnothing\,. \tag{4.8}$$

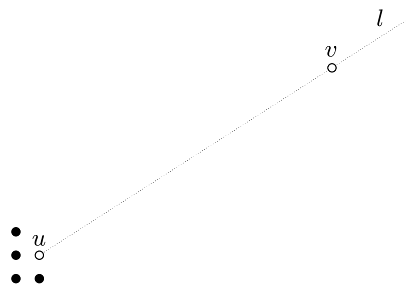

FIGURE 4.3. The set of points $\overline{u}$ (the black dots) for which Lemma 4.2.1 applies.

If $\overline{e} \notin \underline{\gamma}$, then $\underline{\gamma}$ is Λ-compatible with λ. Otherwise, we must have $\mathrm{i}(t_2, \underline{\gamma}) = 2$, since the contours of $\underline{\gamma}$ are closed. This implies that the edge of λ_2 which is adjacent to t belongs to $\Delta(\lambda_1)$ and therefore that $\underline{\gamma}$ is Λ-compatible with λ_1.

□

4.2 Basic properties

In this section, we prove diverse properties of the 2-point function and of the weight $q_\Lambda(\,\cdot\,; J)$. The main results are similar to correlation inequalities for spin observables, but are stated directly in terms of contours thus dispensing us with the use of the spin representation. This section is divided into several subsections where related results are gathered.

It is useful to extend $q_\Lambda(\underline{\gamma}; J)$ to non Λ-compatible families of contours by defining $q_\Lambda(\underline{\gamma}; J) = 0$ if $\underline{\gamma}$ is not Λ-compatible.

4.2.1 Monotonicity in space

We first state a result, which was proved in [MM], showing that the 2-point function $\langle \sigma(0)\sigma(t)\rangle^\beta$ in the thermodynamic limit has some nice monotonicity properties as a function of the site t.

Lemma 4.2.1. *Suppose that $J(e) = \beta$, for all edges e. Let $u, v \in \mathbb{Z}^2$ and let l be the half-line containing u and v with endpoint u. Then, if $\overline{u}$ satisfies one the following conditions*

- $|\overline{u}(1) - u(1)| = 1$, $\overline{u}(2) = u(2)$ *and the vertical line separating u and $\overline{u}$ does not intersect l,*
- $\overline{u}(1) = u(1)$, $|\overline{u}(2) - u(2)| = 1$ *and the horizontal line separating u and $\overline{u}$ does not intersect l,*
- $|\overline{u}(1) - u(1)| = 1$, $|\overline{u}(2) - u(2)| = 1$ *and the diagonal line separating u and $\overline{u}$ does not intersect l,*

the following inequality holds,

$$\langle \sigma(u)\sigma(v)\rangle^\beta \geqslant \langle \sigma(\overline{u})\sigma(v)\rangle^\beta \,.$$

Proof. We prove only the first statement. The proof uses a coupling of the variables *à la Percus* [Per]. Let m be the vertical line separating u and $\overline{u}$ and let Λ be any box containing u and v and invariant under a reflection of axis m. If t is some site of the lattice, let us write $\bar{t}$ for the site obtained by a reflection of axis m. We define $\omega_1(t) \doteqdot \omega(t) + \omega(\bar{t})$ and $\omega_2(t) \doteqdot \omega(t) - \omega(\bar{t})$. The following relations hold: Let t and t' be two nearest neighbours sites.

- If $t' \neq \bar{t}$, then $\omega(t)\omega(t') + \omega(\bar{t})\omega(\bar{t}') = \frac{1}{2}(\omega_1(t)\omega_1(t') + \omega_2(\bar{t})\omega_2(\bar{t}'))$;
- If $t' = \bar{t}$, then $\omega(t)\omega(t') = \frac{1}{2}(\omega_1(t)^2 - 1)$.

Therefore the Gibbs measure of the system with free b.c. in Λ can be written in terms of the variables $\omega_1(t)$, $\omega_2(t)$,

$$\frac{1}{\Xi(\Lambda)} \prod_{\substack{e=\langle t,t'\rangle\subset\Lambda \\ e\cap m=\varnothing}} \exp\{\tfrac{1}{4}\beta(\omega_1(t)\omega_1(t') + \omega_2(t)\omega_2(t'))\} \prod_{\substack{e=\langle t,t'\rangle\subset\Lambda \\ e\cap m\neq\varnothing}} \exp\{\tfrac{1}{2}\beta(\omega_1(t)^2 - 1)\}\,. \quad (4.9)$$

This is again a ferromagnetic model, so the only thing preventing the application of GKS inequalities is the constraint $\omega_1(t) = \pm 2 \Leftrightarrow \omega_2(t) = 0$. However, when computing the numerator of the expectation value, we can split the summation over configurations into sets in which we freeze half of the variables $\omega_1(t)$ and $\omega_2(t)$ to 0. All of these sums can be reinterpreted as sums over the configurations of an Ising model with ferromagnetic couplings to which GKS inequalities apply. For example, we have, $(\sigma_i(t)(\omega_1,\omega_2) \doteqdot \omega_i(t))$

$$\langle \sigma_1(u)\sigma_2(v)\rangle_\Lambda^\beta \geqslant 0\,, \quad (4.10)$$

which is equivalent, using the symmetry of the system, to the desired inequality,

$$\langle \sigma(u)\sigma(v)\rangle_\Lambda^\beta \geqslant \langle \sigma(\overline{u})\sigma(v)\rangle_\Lambda^\beta\,. \quad (4.11)$$

□

4.2.2 Monotonicity in coupling constants

The properties stated in this subsection are consequences of the following important technical lemma.

Lemma 4.2.2. *Let $B_1 \subset B_2$ finite, and let us write $\mathcal{G}_1$ and $\mathcal{G}_2$ for the corresponding graphs. Suppose $0 \leqslant J(e) \leqslant J'(e)$ $\forall e$. Then*

$$\frac{Z(\mathcal{G}_1; J')}{Z(\mathcal{G}_2; J')} \leqslant \frac{Z(\mathcal{G}_1; J)}{Z(\mathcal{G}_2; J)}\,.$$

Proof. Let us first suppose that $e \in B_2 \setminus B_1$. Then we have

$$\frac{\mathrm{d}}{\mathrm{d}J(e)} \frac{Z(\mathcal{G}_1; J)}{Z(\mathcal{G}_2; J)} = -\frac{Z(\mathcal{G}_1; J)}{Z(\mathcal{G}_2; J)^2} \frac{\mathrm{d}}{\mathrm{d}J(e)} Z(\mathcal{G}_2; J) \leqslant 0\,, \quad (4.12)$$

since partition functions are positive and $\frac{\mathrm{d}}{\mathrm{d}J(e)} \tanh J(e) = (1 - \tanh^2 J(e)) \geqslant 0$. Finally, let $e = \langle t, t'\rangle \in B_1$. Then, we can write

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}J(e)} \log \frac{Z(\mathcal{G}_1; J)}{Z(\mathcal{G}_2; J)} &= \frac{\mathrm{d}}{\mathrm{d}J(e)} \log\Big[\frac{\Xi(\Lambda(B_1); J)}{\Xi(\Lambda(B_2); J)} 2^{|\Lambda(B_2)|-|\Lambda(B_1)|} \prod_{e\in B_2\setminus B_1} \cosh J(e)\Big] \\ &= \langle \sigma(t)\sigma(t')\rangle_{\Lambda(B_1)} - \langle \sigma(t)\sigma(t')\rangle_{\Lambda(B_2)} \\ &\leqslant 0\,, \end{aligned} \quad (4.13)$$

where the last inequality is a simple application of GKS inequalities.

□

The following consequence of the previous lemma is the basic result of this subsection,

Lemma 4.2.3. *Let $\underline{\gamma}$ be a Λ-compatible family of contours. Suppose $J'(e) \geqslant J(e) \geqslant 0$, for all edges. Then*

$$\frac{Z(\Lambda|\underline{\gamma};J')}{Z(\Lambda;J')} \leqslant \frac{Z(\Lambda|\underline{\gamma};J)}{Z(\Lambda;J)}\,.$$

Proof. Since $Z(\Lambda|\underline{\gamma}) = Z(\mathcal{G}(\underline{\gamma}))$ with $\mathcal{E}(\mathcal{G}(\underline{\gamma})) \subset \mathcal{E}(\Lambda)$, Lemma 4.2.2 yields the desired result. □

It is now possible to state the first set of properties of the weights $q_\Lambda(\,\cdot\,;J)$.

Lemma 4.2.4. *Suppose $J(e) \geqslant 0$ for all edges e.*

1. *Let $\mathcal{M} \subset \mathcal{E}$ and let $\underline{\gamma}$ be a Λ-compatible family of contours such that $\mathcal{E}(\underline{\gamma}) \cap \mathcal{M} = \varnothing$. Then $q_\Lambda(\underline{\gamma};J)$ is decreasing in $J(e)$, $e \in \mathcal{M}$.*
2. *Let $\underline{\gamma}$ be a Λ-compatible family of contours. Then*[4]*, for all $\Lambda' \supset \Lambda$, $q_{\Lambda'}(\underline{\gamma};J) \leqslant q_\Lambda(\underline{\gamma};J)$.*
3. *$q(\underline{\gamma};J) \doteqdot \lim_{\Lambda\nearrow\mathbb{Z}^2} q_\Lambda(\underline{\gamma};J)$ exists. Moreover if $\underline{\gamma}$ is a Λ-compatible family of contours then $q(\underline{\gamma};J) \leqslant q_\Lambda(\underline{\gamma};J)$.*
4. *$q_{\mathbb{L}}(\underline{\gamma};J) \doteqdot \lim_{\Lambda\nearrow\mathbb{L}} q_\Lambda(\underline{\gamma};J)$ exists. Moreover if $\underline{\gamma}$ is a Λ-compatible family of contours and $\Lambda \subset \mathbb{L}$ then $q_{\mathbb{L}}(\underline{\gamma};J) \leqslant q_\Lambda(\underline{\gamma};J)$.*
5. *Let $Q \subset \Lambda$, with Λ a finite subset of $\mathbb{Z}^2$, and let $\underline{\gamma}$ be a Λ-compatible family of contours such that $\mathrm{i}(t,\mathcal{E}(Q)) = 0$ $\forall t \in \underline{\gamma}$. Then*

$$\frac{q_\Lambda(\underline{\gamma};J)}{q_{\Lambda\setminus Q}(\underline{\gamma};J)} \geqslant \exp\Bigg\{-\sum_{\substack{e=\langle t,t'\rangle\in\Delta(\underline{\gamma})\\ e'=\langle t'',t'''\rangle,\, t''\in\Lambda\setminus Q,\, t'''\in Q}} J(e)J(e')\Big(\langle\sigma(t)\sigma(t'')\rangle^J_{\Lambda\setminus Q} + \langle\sigma(t')\sigma(t'')\rangle^J_{\Lambda\setminus Q}\Big)\Bigg\}\,.$$

6. *Let $J(e)$ be as in (4.5) with $\beta < \beta_c$ and let $\underline{\gamma}$ be a Λ-compatible family of contours. Then*
   - *If $\mathcal{E}(\underline{\gamma}) \cap \mathcal{E}(\Sigma) = \varnothing$ and $0 \leqslant h \leqslant 1$ then $q_\Lambda(\underline{\gamma};\beta,h) \geqslant q(\underline{\gamma};\beta)$.*
   - *If $\mathrm{i}(t,\mathcal{E}(\Sigma)) = 0$ $\forall t \in \underline{\gamma}$ then*

$$q_\Lambda(\underline{\gamma};\beta,h) \geqslant q(\underline{\gamma};\beta)\cdot\exp\Bigg\{-\beta^2 \sum_{e=\langle t,t'\rangle\in\Delta(\underline{\gamma})} \sum_{t'':\, t''(2)=1} \big[\mathrm{e}^{-\tau(t-t'';\beta^*)} + \mathrm{e}^{-\tau(t'-t'';\beta^*)}\big]\Bigg\}\,.$$

7. *Let λ_1, λ_2 be two Λ-compatible open contours such that $\partial\lambda_1 = \{t_1,t_2\}$, $\partial\lambda_2 = \{t_2,t_3\}$, $\mathcal{E}(\lambda_1)\cap\mathcal{E}(\lambda_2) = \varnothing$ and the decomposition of $\mathcal{E}(\lambda_1)\cup\mathcal{E}(\lambda_2)$ into contours gives a single contour λ with $\partial\lambda = \{t_1,t_3\}$. Then*

$$q_\Lambda(\lambda;J) \geqslant q_\Lambda(\lambda_1;J)q_\Lambda(\lambda_2;J)\,.$$

Proof. 1. This is an immediate consequence of Lemma 4.2.3.

---

[4]Notice that this is a non-trivial result, as can be seen from the following inequality (which is also a consequence of GKS inequalities!):

$$\sum_{\lambda\,:\,\delta\lambda=\{t_1,t_2\}} q_{\Lambda'}(\lambda) = \langle\sigma(t_1)\sigma(t_2)\rangle_{\Lambda'} \geqslant \langle\sigma(t_1)\sigma(t_2)\rangle_\Lambda = \sum_{\lambda\,:\,\delta\lambda=\{t_1,t_2\}} q_\Lambda(\lambda)\,,$$

where $t_1,t_2 \in \Lambda \subset \Lambda'$.

2. Let $J'(e) = J(e)$ for all edges in $\mathcal{E}(\Lambda_2)$ and $J'(e) = 0$ otherwise. Then clearly,

$$\frac{Z(\Lambda_2|\underline{\gamma}; J)}{Z(\Lambda; J)} = \frac{Z(\Lambda_1|\underline{\gamma}; J')}{Z(\Lambda; J')} . \tag{4.14}$$

   The conclusion follows from point 1..

3. This is immediate since the sequence is monotonous by point 2..
4. As before.
5. The first step is to establish an integral representation for $\log[Z(\Lambda|\gamma; J)/Z(\Lambda; J)]$. Let us introduce new weights $J_s(e)$ given by

$$J_s(e) \doteqdot \begin{cases} J(e) & \text{if } e \notin \Delta(\underline{\gamma}) , \\ sJ(e) & \text{otherwise.} \end{cases} \tag{4.15}$$

   With these weights, we can write

$$\frac{Z(\Lambda|\underline{\gamma}; J)}{Z(\Lambda; J)} = \frac{Z(\Lambda; J_0)}{Z(\Lambda; J_1)} . \tag{4.16}$$

   where we used Lemma 4.1.2 and the fact that $Z(\mathcal{G}(\underline{\gamma}); J) = Z(\Lambda; J_0)$.
   Therefore we obtain the following representation

$$\begin{aligned} \log \frac{Z(\Lambda|\underline{\gamma}; J)}{Z(\Lambda; J)} &= \log \frac{Z(\Lambda; J_0)}{Z(\Lambda; J_1)} \\ &= \log \frac{\Xi(\Lambda; J_0)}{\Xi(\Lambda; J_1)} + \log \prod_{e \in \Delta(\underline{\gamma})} \cosh J(e) \\ &= -\int_0^1 \mathrm{d}s \, \tfrac{\mathrm{d}}{\mathrm{d}s} \log \Xi(\Lambda; J_s) + \log \prod_{e \in \Delta(\underline{\gamma})} \cosh J(e) \\ &= -\sum_{e=\langle t,t'\rangle \in \Delta(\underline{\gamma})} J(e) \int_0^1 \mathrm{d}s \, \langle \sigma(t)\sigma(t')\rangle_\Lambda^{J_s} + \log \prod_{e \in \Delta(\underline{\gamma})} \cosh J(e) , \end{aligned} \tag{4.17}$$

   Consequently,

$$\frac{q_\Lambda(\underline{\gamma}; J)}{q_{\Lambda\setminus Q}(\underline{\gamma}; J)} = \exp\Big\{ -\sum_{e=\langle t,t'\rangle \in \Delta(\underline{\gamma})} J(e) \int_0^1 \mathrm{d}s \, \Big(\langle \sigma(t)\sigma(t')\rangle_\Lambda^{J_s} - \langle \sigma(t)\sigma(t')\rangle_{\Lambda\setminus Q}^{J_s} \Big) \Big\} . \tag{4.18}$$

   The next step is to estimate this integral. GKS–inequalities give

$$\langle \sigma(t)\sigma(t')\rangle_\Lambda^{J_s} - \langle \sigma(t)\sigma(t')\rangle_{\Lambda\setminus Q}^{J_s} \leqslant \langle \sigma(t)\sigma(t')\rangle_{\Lambda\setminus Q}^{+_Q, J_s} - \langle \sigma(t)\sigma(t')\rangle_{\Lambda\setminus Q}^{J_s} , \tag{4.19}$$

   where the $(\Lambda \setminus Q)^{+_Q}$–boundary condition is obtained by introducing an external field on Q and then letting this field go to ∞. The difference of 2-point functions in the right–hand side can be bounded above by a difference of 1-point functions, using

FKG–inequalities. Indeed, since $\sigma(t)+\sigma(t')-\sigma(t)\sigma(t')$ is an increasing function, these inequalities imply

$$\langle\sigma(t)\sigma(t')\rangle_{\Lambda\backslash Q}^{+_Q,J_s}-\langle\sigma(t)\sigma(t')\rangle_{\Lambda\backslash Q}^{J_s}\leqslant \langle\sigma(t)\rangle_{\Lambda\backslash Q}^{+_Q,J_s}-\langle\sigma(t)\rangle_{\Lambda\backslash Q}^{J_s}+\langle\sigma(t')\rangle_{\Lambda\backslash Q}^{+_Q,J_s}-\langle\sigma(t')\rangle_{\Lambda\backslash Q}^{J_s}\,. \tag{4.20}$$

Define new coupling constants $J''_a(e)$,

$$J''_a(e)\doteqdot\begin{cases}aJ_s(e) & \text{if } e\cap Q\neq\varnothing \text{ and } e\cap\Lambda\setminus Q\neq\varnothing,\\ J_s(e) & \text{otherwise.}\end{cases} \tag{4.21}$$

Observe that $\langle\sigma(t)\rangle_{\Lambda\backslash Q}^{+_Q,J_s}=\langle\sigma(t)\rangle_{\Lambda\backslash Q}^{+_Q,J''_1}$ and $\langle\sigma(t)\rangle_{\Lambda\backslash Q}^{J_s}=\langle\sigma(t)\rangle_{\Lambda\backslash Q}^{+_Q,J''_0}$. Hence

$$\langle\sigma(t)\sigma(t')\rangle_{\Lambda\backslash Q}^{+_Q,J_s}-\langle\sigma(t)\sigma(t')\rangle_{\Lambda\backslash Q}^{J_s}= \sum_{\substack{e=\langle t'',t'''\rangle\\ t''\in\Lambda\backslash Q,\,t'''\in Q}}J(e)\int_0^1 \mathrm{d}a\Big(\langle\sigma(t)\,;\,\sigma(t'')\rangle_{\Lambda\backslash Q}^{+_Q,J''_a}+\langle\sigma(t')\,;\,\sigma(t'')\rangle_{\Lambda\backslash Q}^{+_Q,J''_a}\Big), \tag{4.22}$$

where

$$\langle\sigma(t)\,;\,\sigma(t'')\rangle_{\Lambda\backslash Q}^{+_Q,J''_a}\doteqdot\langle\sigma(t)\,\sigma(t'')\rangle_{\Lambda\backslash Q}^{+_Q,J''_a}-\langle\sigma(t)\rangle_{\Lambda\backslash Q}^{+_Q,J''_a}\cdot\langle\sigma(t'')\rangle_{\Lambda\backslash Q}^{+_Q,J''_a}\,. \tag{4.23}$$

GHS–inequalities imply that $\langle\sigma(t)\,;\,\sigma(t'')\rangle_{\Lambda\backslash Q}^{+_Q,J''_a}$ is decreasing in a; putting $a=0$ we get

$$\langle\sigma(t)\,;\,\sigma(t'')\rangle_{\Lambda\backslash Q}^{+_Q,J''_a}\leqslant\langle\sigma(t)\,;\,\sigma(t'')\rangle_{\Lambda\backslash Q}^{J''_0}=\langle\sigma(t)\,\sigma(t'')\rangle_{\Lambda\backslash Q}^{J_s}\,, \tag{4.24}$$

because the last expectation value is with respect to the Gibbs measure on $\mathbb{L}\setminus\Sigma$ with free boundary condition and consequently by symmetry

$$\langle\sigma(t)\rangle_{\Lambda\backslash Q}^{J_s}=0\,. \tag{4.25}$$

GKS–inequalities imply now

$$\langle\sigma(t)\sigma(t'')\rangle_{\Lambda\backslash Q}^{J_s}\leqslant\langle\sigma(t)\sigma(t'')\rangle_{\Lambda\backslash Q}^{J}\,. \tag{4.26}$$

The conclusion follows from equations (4.18), (4.19), (4.22), (4.24) and (4.26).

6. The first statement follows by monotonicity. To prove the second statement, observe that

$$q_{\Lambda\backslash\Sigma}(\underline{\gamma};J)=q_{\Lambda\backslash\Sigma}(\underline{\gamma};\beta)\geqslant q(\underline{\gamma};\beta)\,, \tag{4.27}$$

where the inequality is a consequence of the first part ($h=1$). Therefore the conclusion follows from point 5. with $Q=\Sigma\cap\Lambda$, using GKS–inequalities and Lemmas 3.1.1 and 3.1.2 to write

$$\langle\sigma(t)\sigma(t'')\rangle_{\Lambda\backslash\Sigma}^{\beta}\leqslant\langle\sigma(t)\sigma(t'')\rangle^{\beta}\leqslant\exp\{-\tau(t-t'';\beta^*)\}\,. \tag{4.28}$$

7. We can suppose without loss of generality that $\mathrm{i}(t_2, \mathcal{E}(\lambda_2)) = 1$. We construct a new graph $\mathcal{G}_{t_2}(\lambda_2)$ as in Lemma 4.1.2. This graph satisfies $Z(\Lambda|\lambda_1 \cup \lambda_2) = Z(\mathcal{G}_{t_2}(\lambda_2)|\lambda_1)$. Since $Z(\mathcal{G}_{t_2}(\lambda_2)) \geqslant Z(\mathcal{G}(\lambda_2))$, we can write

$$\begin{aligned} q_\Lambda(\lambda; J) &= w^*(\lambda_1)w^*(\lambda_2)\frac{Z(\Lambda|\lambda_1 \cup \lambda_2)}{Z(\Lambda)} \\ &= w^*(\lambda_1)\frac{Z(\Lambda|\lambda_1 \cup \lambda_2)}{Z(\mathcal{G}(\lambda_2))} w^*(\lambda_2)\frac{Z(\Lambda|\lambda_2)}{Z(\Lambda)} \\ &\geqslant w^*(\lambda_1)\frac{Z(\mathcal{G}_{t_2}(\lambda_2)|\lambda_1)}{Z(\mathcal{G}_{t_2}(\lambda_2))} w^*(\lambda_2)\frac{Z(\Lambda|\lambda_2)}{Z(\Lambda)} \\ &\geqslant q_\Lambda(\lambda_1; J) q_\Lambda(\lambda_2; J)\,, \end{aligned} \tag{4.29}$$

where the last inequality follows by monotonicity.

□

4.2.3 Some upper bounds

In this section we prove essential upper bounds on some constrained sums over contours. These results are crucial to study the polygonal lines which are introduced in Chapters 6 and 7.

Lemma 4.2.5. *Let $\beta < \beta_c$.*

1. *If $J(e) = \beta$ for all edges e, then for any pair $t_1,\ t_2 \in \Lambda$,*

$$\sum_{\substack{\lambda: \\ \partial\lambda = \{t_1, t_2\}}} q_\Lambda(\lambda) \leqslant \langle \sigma(t_1)\sigma(t_2)\rangle^\beta \leqslant \exp\{-\tau(t_2 - t_1; \beta^*)\}\,.$$

2. *If $\Lambda \subset \mathbb{L}$ and $J(e)$ is defined by* (4.5)*, then for any pair $t_1,\ t_2 \in \Sigma \cap \Lambda$,*

$$\sum_{\substack{\lambda: \\ \partial\lambda = \{t_1, t_2\}}} q_\Lambda(\lambda; J) \leqslant \langle \sigma(t_1)\sigma(t_2)\rangle_{\mathbb{L}}^J \leqslant \exp\{-\tau_{\mathrm{bd}}(t_2 - t_1; \beta^*, h^*)\}\,.$$

3. *Let $Q \subset \mathcal{E}$. If $J(e) > \beta \implies e \in Q$, then for any pair $t_1,\ t_2 \in \Lambda$,*

$$\sum_{\substack{\lambda \subset \mathcal{E}\setminus Q: \\ \partial\lambda = \{t_1, t_2\}}} q_\Lambda(\lambda; J) \leqslant \langle \sigma(t_1)\sigma(t_2)\rangle_\Lambda^\beta \leqslant \exp\{-\tau(t_2 - t_1; \beta^*)\}\,.$$

4. *If $\Lambda \subset \mathbb{L}$ and $J(e)$ is defined by* (4.5)*, then for any pair $t_1,\ t_2 \in \Sigma \cap \Lambda$,*

$$\sum_{\substack{\lambda \subset \mathcal{E}\setminus\mathcal{E}(\Sigma): \\ \partial\lambda = \{t_1, t_2\}}} q_\Lambda(\lambda; J) \leqslant \langle \sigma(t_1)\sigma(t_2)\rangle_\Lambda^\beta \leqslant \exp\{-\tau(t_2 - t_1; \beta^*)\}\,.$$

Proof. 1. This is a consequence of (2.14), GKS inequalities and Lemmas 3.1.1 and 3.1.2.

2. This follows from (2.14), GKS inequalities and Lemmas 3.2.2 and 3.2.3.

3. Let $J'(e) = \min(J(e), \beta)$. Lemma 4.2.4, point 1., implies

$$\begin{aligned}\sum_{\substack{\lambda\subset\mathcal{E}\setminus Q:\\ \partial\lambda=\{t_1,t_2\}}} q_\Lambda(\lambda;J) &\leqslant \sum_{\substack{\lambda\subset\mathcal{E}\setminus Q:\\ \partial\lambda=\{t_1,t_2\}}} q_\Lambda(\lambda;J')\\ &\leqslant \sum_{\substack{\lambda\subset\mathcal{E}:\\ \partial\lambda=\{t_1,t_2\}}} q_\Lambda(\lambda;J')\\ &= \langle\sigma(t_1)\sigma(t_2)\rangle_\Lambda^{J'}\,. \end{aligned}\tag{4.30}$$

   The conclusion follows from GKS inequalities and Lemmas 3.1.1 and 3.1.2.

4. If $h \geqslant 1$, this is a particular case of point 3., otherwise, it follows from point 2. and point 2. of Proposition 3.2.1.

□

Point 4. (and more generally point 3.) of the preceding lemma is of particular importance in the study of boundary effects, see Chapters 6 and 7 for examples of applications.

The following lemma is certainly the core of this type of non-perturbative approaches to phase separation. Such a result was first obtained by Pfister in his study of large deviations [Pf1]. Not only this result is non-perturbative but it hugely simplifies the study of typical configurations of large contours in constrained ensembles, see Chapters 6 and 7.

Definition.

(D103) *Let λ be an open contour such that $\partial\lambda = \{t, t'\}$ and let $t_1, \ldots, t_n \in \mathbb{Z}^2$, $t_i \notin \{t, t'\}\forall i$, and $t_i \neq t_j$ if $i \neq j$. The* **decomposition of λ with cutting points** *$t_1, \ldots, t_n$ is the collection $\lambda_0 \ldots, \lambda_n$ of open contours obtained in the following way:*
- *Consider λ as a unit-speed parameterized curve.*
- *Let $s_i \doteqdot \max\{s : \lambda(s) = t_i\}$, $i = 1, \ldots, n$. We set $s_0 \doteqdot 0$, $s_{n+1} \doteqdot |\lambda|$.*
- *$\lambda_i = \{\lambda(s) : s_i \leqslant s \leqslant s_{i+1}\}$, $i = 0, \ldots, n$.*

(D104) *Let λ be a closed contour and let $t_1, \ldots, t_n \in \mathbb{Z}^2$, $t_i \neq t_j$ if $i \neq j$. The* **decomposition of λ with cutting points** *$t_1, \ldots, t_n$ is the collection $\lambda_1 \ldots, \lambda_n$ of open contours obtained in the following way:*
- *Consider λ as a unit-speed parameterized curve such that $\lambda(0) = \lambda(|\lambda|) = t_1$.*
- *Let $s_i \doteqdot \max\{s : \lambda(s) = t_i\}$, $i = 2, \ldots, n$. We set $s_1 \doteqdot 0$ and $s_{n+1} \doteqdot |\lambda|$.*
- *$\lambda_i \doteqdot \{\lambda(s) : s_i \leqslant s \leqslant s_{i+1}\}$, $i = 1, \ldots, n$.*

Lemma 4.2.6. *Suppose $J(e) \geqslant 0$, for all edges e. Let $t_0, \ldots, t_{n+1} \in \mathbb{Z}^2$, $t_i \neq t_j$ if $i \neq j$, and let $A_0, A_1, \ldots, A_n \subset \mathbb{Z}^2$. Let $\mathcal{Q} \equiv \mathcal{Q}_{t_0,t_{n+1}}(t_1, \ldots, t_n, A_0, \ldots, A_n)$ be the set of all open contours λ such that $\partial\lambda = \{t_0, t_{n+1}\}$ and the decomposition $\lambda_0, \ldots, \lambda_n$ of λ with cutting points $t_1, \ldots, t_n$ satisfies $\lambda_i \subset A_i$, $i = 1, \ldots, n$. Then*

$$\sum_{\lambda\in\mathcal{Q}} q_\Lambda(\lambda;J) \leqslant \prod_i \langle\sigma(t_{i-1})\sigma(t_i)\rangle_{A_i}^J\,.$$

A similar result holds for closed contours and in the case of a family of closed or open contours.

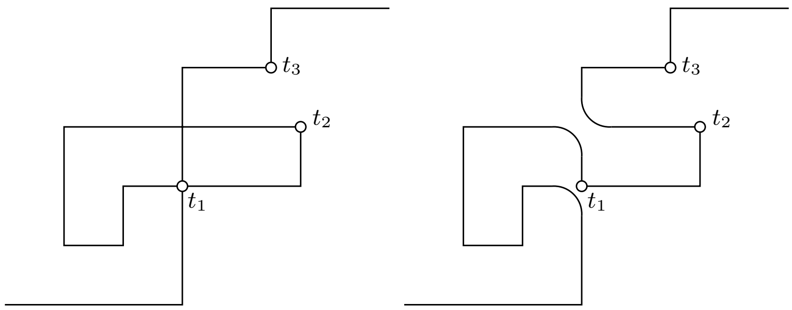

FIGURE 4.4. An open contour and its decomposition with cutting points t_1, t_2 and t_3.

Proof. Let us write $\lambda_0^c \doteqdot \lambda_1 \cup \cdots \cup \lambda_n$ so that $\lambda = \lambda_0 \cup \lambda_0^c$. We also introduce

$$\mathcal{Q}_0^c \doteqdot \{\lambda' \in \mathcal{Q}_{t_1, t_{n+1}}(t_2, \dots, t_n, A_1, \dots, A_n) \,:\, \mathrm{i}(t_1, \mathcal{E}(\lambda')) = 1\}\,. \tag{4.31}$$

Then we can write

$$\sum_{\lambda \in \mathcal{Q}} q_\Lambda(\lambda; J) = \sum_{\lambda_0^c \in \mathcal{Q}_0^c} w^*(\lambda_0^c) \sum_{\substack{\lambda_0 \subset A_0 : \\ \lambda_0 \cup \lambda_0^c \in \mathcal{Q}}} w^*(\lambda_0) \frac{Z(\Lambda | \lambda_0 \cup \lambda_0^c)}{Z(\Lambda)}\,. \tag{4.32}$$

Constructing the graph $\mathcal{G}_{t_1}(\lambda_0^c)$ as in Lemma 4.1.2, we have $Z(\Lambda|\lambda_0 \cup \lambda_0^c) = Z(\mathcal{G}_{t_1}(\lambda_0^c)|\lambda_0)$. Therefore

$$\begin{aligned} \sum_{\substack{\lambda_0 \subset A_0 : \\ \lambda_0 \cup \lambda_0^c \in \mathcal{Q}}} w^*(\lambda_0) \frac{Z(\Lambda | \lambda_0 \cup \lambda_0^c)}{Z(\Lambda)} &= \frac{Z(\mathcal{G}_{t_1}(\lambda_0^c))}{Z(\Lambda)} \sum_{\substack{\lambda_0 \subset A_0 : \\ \lambda_0 \cup \lambda_0^c \in \mathcal{Q}}} w^*(\lambda_0) \frac{Z(\mathcal{G}_{t_1}(\lambda_0^c)|\lambda_0)}{Z(\mathcal{G}_{t_1}(\lambda_0^c))} \\ &\leqslant \frac{Z(\mathcal{G}_{t_1}(\lambda_0^c))}{Z(\Lambda)} \sum_{\substack{\lambda_0 \subset A_0 : \\ \lambda_0 \cup \lambda_0^c \in \mathcal{Q}}} w^*(\lambda_0) \frac{Z(\widehat{\mathcal{G}}_0(\lambda_0^c)|\lambda_0)}{Z(\widehat{\mathcal{G}}_0(\lambda_0^c))} \\ &\leqslant \frac{Z(\mathcal{G}_{t_1}(\lambda_0^c))}{Z(\Lambda)} \langle \sigma(t_0) \sigma(t_1) \rangle_{A_0}\,, \end{aligned} \tag{4.33}$$

where $\widehat{\mathcal{G}}_0(\lambda_0^c)$ is the graph defined by the set of edges

$$\mathcal{E}(A_0) \cap \mathcal{E}(\mathcal{G}_{t_1}(\lambda_0^c))\,. \tag{4.34}$$

We would like to iterate this procedure. However, there are two differences between the remaining sum and the original one. First, λ_0^c must be such that $\mathrm{i}(t_1, \mathcal{E}(\lambda_0^c)) = 1$; second, the partition function which appear in the remaining sum is $Z(\mathcal{G}_{t_1}(\lambda_0^c))$ and not $Z(\mathcal{G}(\lambda_0^c))$. We prove now that these two differences exactly cancel each other.
Let us write, similarly as before, $\lambda_1^c \doteqdot \lambda_2 \cup \cdots \cup \lambda_n$ and

$$\mathcal{Q}_1^c \doteqdot \{\lambda' \in \mathcal{Q}_{t_2, t_{n+1}}(t_3, \dots, t_n, A_2, \dots, A_n) \,:\, \mathrm{i}(t_2, \mathcal{E}(\lambda')) = 1\}\,. \tag{4.35}$$

Again we can write

$$\sum_{\lambda_0^c\in\mathcal{Q}_0^c} w^*(\lambda_1\cup\lambda_1^c)\frac{Z(\mathcal{G}_{t_1}(\lambda_1\cup\lambda_1^c))}{Z(\Lambda)}$$

$$= \sum_{\lambda_1^c\in\mathcal{Q}_1^c} w^*(\lambda_1^c)\frac{Z(\mathcal{G}_{t_2}(\lambda_1^c))}{Z(\Lambda)} \sum_{\substack{\lambda_1\subset A_1:\\ \lambda_1\cup\lambda_1^c\in\mathcal{Q}_0^c}} w^*(\lambda_1)\frac{Z(\mathcal{G}_{t_1}(\lambda_1\cup\lambda_1^c))}{Z(\mathcal{G}_{t_2}(\lambda_1^c))}$$

$$\leqslant \sum_{\lambda_1^c\in\mathcal{Q}_1^c} w^*(\lambda_1^c)\frac{Z(\mathcal{G}_{t_2}(\lambda_1^c))}{Z(\Lambda)} \sum_{\substack{\lambda_1\subset A_1:\\ \lambda_1\cup\lambda_1^c\in\mathcal{Q}_0^c}} w^*(\lambda_1)\frac{Z(\widehat{\mathcal{G}}_1(\lambda_1\cup\lambda_1^c))}{Z(\widehat{\mathcal{G}}_1(\lambda_1^c))}\,, \quad (4.36)$$

where $\widehat{\mathcal{G}}_1(\cdot)$ is defined similarly as $\widehat{\mathcal{G}}_0(\cdot)$. Now observing that

$$Z(\widehat{\mathcal{G}}_1(\lambda\cup\lambda_1^c)) = \sum_{\substack{\underline{\gamma}\,:\,\partial\underline{\gamma}=\varnothing\\ \underline{\gamma}\subset\mathcal{E}(\widehat{\mathcal{G}}_1(\lambda\cup\lambda_1^c))\\ \underline{\gamma}\text{ comp. with }\lambda\cup\lambda_1^c}} w^*(\underline{\gamma})$$

$$= \sum_{\substack{\underline{\gamma}\,:\,\partial\underline{\gamma}=\varnothing\\ \underline{\gamma}\subset\mathcal{E}(\mathcal{G}_1(\lambda\cup\lambda_1^c))\\ \underline{\gamma}\text{ comp. with }\lambda\cup\lambda_1^c}} w^*(\underline{\gamma}) + \sum_{\substack{\underline{\gamma}\,:\,\partial\underline{\gamma}=\varnothing\,,\,\overline{e}\in\underline{\gamma}\\ \underline{\gamma}\subset\mathcal{E}(\widehat{\mathcal{G}}_1(\lambda\cup\lambda_1^c))\\ \underline{\gamma}\text{ comp. with }\lambda\cup\lambda_1^c}} w^*(\underline{\gamma})\,, \quad (4.37)$$

where $\mathcal{G}_1(\lambda\cup\lambda_1^c)$ is the graph defined by the set of edges $\mathcal{E}(\mathcal{G}(\lambda\cup\lambda_1^c))\cap\mathcal{E}(A_1)$ and $\overline{e}$ is the edge (if it exists) of $\mathcal{E}(\widehat{\mathcal{G}}_1(\lambda_1\cup\lambda_1^c))\setminus\mathcal{E}(\mathcal{G}_1(\lambda\cup\lambda_1^c))$, we see that it is possible to remove the constraint on the index at t_1. Indeed, it is enough to glue together λ_1 and the closed contour of $\underline{\gamma}$ which contains $\overline{e}$ (in the second sum) to obtain all possible open contour with index 3 at t_1, while all those of index 1 are obtained with the first sum. Therefore we have

$$\sum_{\substack{\lambda_1\subset A_1:\\ \lambda_1\cup\lambda_1^c\in\mathcal{Q}_0^c}} w^*(\lambda_1)\frac{Z(\widehat{\mathcal{G}}_1(\lambda_1\cup\lambda_1^c))}{Z(\widehat{\mathcal{G}}_1(\lambda_1^c))} = \sum_{\substack{\lambda_1\subset\mathcal{E}(\widehat{\mathcal{G}}_1(\lambda_1^c))\text{-comp.}:\\ \partial\lambda_1=\{t_1,t_2\}}} w^*(\lambda_1)\frac{Z(\widehat{\mathcal{G}}_1(\lambda_1^c)|\lambda_1)}{Z(\widehat{\mathcal{G}}_1(\lambda_1^c))}$$

$$\leqslant \langle\sigma(t_1)\sigma(t_2)\rangle_{A_1}\,. \quad (4.38)$$

This procedure can obviously be iterated to obtain the desired result. □

As a direct application of the above lemma we can prove the Lieb-Simon inequality [Si, L, R],

Lemma 4.2.7. *Suppose $J(e)\geqslant 0$, for all edges e. Let t, $t'\in\mathbb{Z}^2$. Let $A\subset\mathbb{Z}^2$ such that $t\in A$ and $t'\notin A$. Then*

$$\langle\sigma(t)\sigma(t')\rangle_\Lambda^J \leqslant \sum_{u\in\partial A}\langle\sigma(t)\sigma(u)\rangle_A^J\langle\sigma(u)\sigma(t')\rangle_\Lambda^J\,.$$

Proof. Clearly each open contour λ with endpoints t and t' must contain some site of ∂A. Let u be the first such site (according to a parameterization of the λ with $\lambda(0)=t$). Then

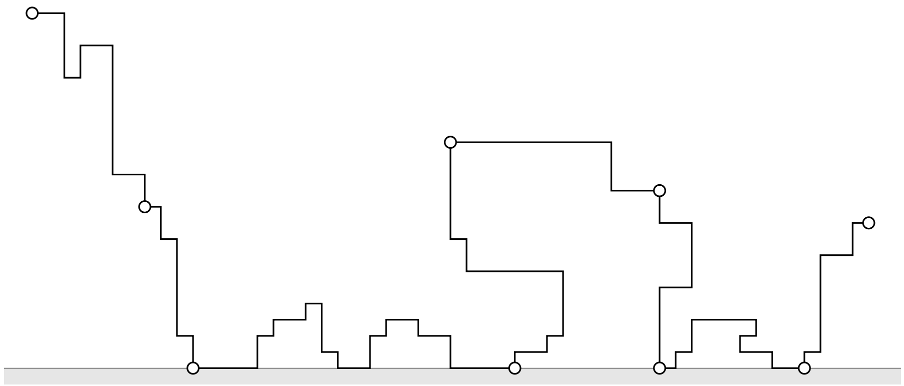

FIGURE 4.5. A contour and some sites to which Lemma 4.2.8 applies.

by Lemma 4.2.6 (using the same notations)

$$\begin{aligned}\langle\sigma(t)\sigma(t')\rangle_\Lambda^J &= \sum_{\substack{\lambda:\\ \partial\lambda=\{t,t'\}}} q_\Lambda(\lambda;J)\\ &\leqslant \sum_{u\in\partial A}\sum_{\lambda\in\mathcal{Q}_{t,\,t'}(u,A,\Lambda)} q_\Lambda(\lambda;J)\\ &\leqslant \sum_{u\in\partial A}\langle\sigma(t)\sigma(u)\rangle_A^J\langle\sigma(u)\sigma(t')\rangle_\Lambda^J\,. \end{aligned} \tag{4.39}$$

□

A more important consequence of Lemma 4.2.6 for our purposes is the following lemma.

Lemma 4.2.8. *Suppose $\beta < \beta_c$ and $J(e)$ is given by* (4.5)*. Let $t_0,\ldots,t_n$ be distinct sites of $\Lambda\subset\mathbb{L}$ and let $\mathcal{A}$ be the set of all open contours λ such that*

- *λ is Λ-compatible;*
- *$\partial\lambda = \{t_0,t_n\}$;*
- *$t_i\in\lambda$, $i=0,\ldots n$;*
- *if $t_i(2)>0$ or $t_{i+1}(2)>0$, then $\lambda(s)(2)>0$ for all $s_i<s<s_{i+1}$, where $s_i=\max\{s' : \lambda(s')=t_i\}$.*

Then

$$\sum_{\lambda\in\mathcal{A}} q_\Lambda(\lambda)\leqslant \prod_{\substack{i:\\ \max(t_i(2),t_{i+1}(2))>0}} \mathrm{e}^{-\tau(t_{i+1}-t_i;\beta^*)} \prod_{\substack{i:\\ \max(t_i(2),t_{i+1}(2))=0}} \mathrm{e}^{-\tau_{\mathrm{bd}}(t_{i+1}-t_i;\beta^*,h^*)}\,.$$

Similar results hold for closed contours and family of contours.

Proof. This is a direct application of Lemmas 4.2.6 and 4.2.5. □

We state now the last result of this section which is similar to the above ones.

Lemma 4.2.9. *Suppose $J(e) \geqslant 0$, for all edges e. Let t_1, t_2 and t_3 be three distinct sites of $\Lambda \subset \mathbb{Z}^2$. Let λ_2 be a Λ-compatible open contour with $\partial\lambda_2 = \{t_2, t_3\}$. Writing $\mathcal{G}_1 \doteqdot \{\lambda_1 : \partial\lambda_1 = \{t_1, t_2\},\ \lambda_1 \cup \lambda_2$ is a Λ-comp. contour$\}$, the following inequality holds,*

$$\sum_{\lambda_1 \in \mathcal{Q}_1} q_\Lambda(\lambda_1 \cup \lambda_2; J) \leqslant 2 q_\Lambda(\lambda_2; J) \sum_{\lambda_1 : \partial\lambda_1 = \{t_1, t_2\}} q_\Lambda(\lambda_1; J)\,.$$

Proof. The beginning of the proof is similar to that of Lemma 4.2.6. We obtain

$$\sum_{\lambda_1 \in \mathcal{Q}_1} q_\Lambda(\lambda_1 \cup \lambda_2; J) \leqslant w^*(\lambda_2) \frac{Z(\mathcal{G}_{t_2}(\lambda_2))}{Z(\Lambda)} \sum_{\lambda_1 : \partial\lambda_1 = \{t_1, t_2\}} q_\Lambda(\lambda_1; J)\,. \tag{4.40}$$

Writing

$$w^*(\lambda_2) \frac{Z(\mathcal{G}_{t_2}(\lambda_2))}{Z(\Lambda)} = w^*(\lambda_2) \frac{Z(\mathcal{G}(\lambda_2))}{Z(\Lambda)} \frac{Z(\mathcal{G}_{t_2}(\lambda_2))}{Z(\mathcal{G}(\lambda_2))}\,, \tag{4.41}$$

and using (2.10) to bound the second quotient by 2, we obtain

$$w^*(\lambda_2) \frac{Z(\mathcal{G}_{t_2}(\lambda_2))}{Z(\Lambda)} \leqslant 2 w^*(\lambda_2) \frac{Z(\Lambda | \lambda_2)}{Z(\Lambda)} = 2 q_\Lambda(\lambda_2)\,. \tag{4.42}$$

□

We have seen in Lemma 4.2.4 that the limit $\lim_{\Lambda \nearrow \mathbb{Z}^2} q_\Lambda(\lambda)$ is well-defined. We may wonder if the relation

$$\langle \sigma(t)\sigma(t') \rangle^\beta = \sum_{\substack{\lambda: \\ \partial\lambda = \{t, t'\}}} q(\lambda) \tag{4.43}$$

still holds. That this is so is proved in the next Lemma.

Lemma 4.2.10. *Suppose that there exists $\overline{\beta}$ with $\beta_c > \overline{\beta} \geqslant J(e) \geqslant 0$ for all edges e. Then*

$$\langle \sigma(t)\sigma(t') \rangle^J = \sum_{\substack{\lambda: \\ \partial\lambda = \{t, t'\}}} q(\lambda)\,.$$

Proof. Let Λ be a finite subset of $\mathbb{Z}^2$. We first have, by Lemma 4.2.4,

$$\langle \sigma(t)\sigma(t') \rangle^J_\Lambda = \sum_{\substack{\lambda: \\ \partial\lambda = \{t, t'\}}} q_\Lambda(\lambda; J) \geqslant \sum_{\substack{\lambda: \\ \partial\lambda = \{t, t'\}}} q(\lambda)\,. \tag{4.44}$$

Taking the limit $\Lambda \nearrow \mathbb{Z}^2$, we obtain the required lower bound. Let us prove the corresponding upper bound. Let $\Lambda' \supset \Lambda$ be two finite subsets of $\mathbb{Z}^2$. Then, by Lemma 4.2.6 and the existence of the massgap, we have

$$\begin{aligned} \sum_{\substack{\lambda: \\ \partial\lambda = \{t, t'\}}} q_\Lambda(\lambda) &= \sum_{\substack{\lambda:\, \partial\lambda = \{t, t'\} \\ \lambda \subset \Lambda}} (q_\Lambda(\lambda) - q_{\Lambda'}(\lambda)) + \sum_{\lambda:\, \partial\lambda = \{t, t'\}} q_{\Lambda'}(\lambda) - \sum_{\substack{\lambda:\, \partial\lambda = \{t, t'\} \\ \lambda \not\subset \Lambda}} q_{\Lambda'}(\lambda) \\ &\geqslant \langle \sigma(t)\sigma(t') \rangle_{\Lambda'} - |\partial\Lambda| \max_{u \in \partial\Lambda} \mathrm{e}^{-(\tau(t-u; \overline{\beta}^*) + \tau(t'-u; \overline{\beta}^*))} \\ &\geqslant (1 - \varepsilon) \langle \sigma(t)\sigma(t') \rangle_{\Lambda'}\,, \end{aligned} \tag{4.45}$$

for any $\varepsilon > 0$ if the distance between t, t' and $\partial\Lambda$ is large enough. The upper bound follows by first taking the limit $\Lambda' \nearrow \mathbb{Z}^2$, and then the limit $\Lambda \nearrow \mathbb{Z}^2$. □

4.3 An estimate on the size of typical contours

The aim of this section is to obtain an estimate on the maximal size of typical contours using results of the preceding section. We first prove such a result for the high-temperature representation and then show how duality can be used to obtain similar informations on the low-temperature representation.
We first need a way to measure the size of a contour. This is done using its diameter,

Definition.
(D105) *The* **diameter of a contour** γ *is defined by*

$$d(\gamma) \doteqdot \max_{x,y\in\gamma} \|y-x\|_1 \,.$$

Lemma 4.3.1. *Let $\beta < \beta_c$, $h \geqslant 0$ and $J(e)$ be given by* (4.5)*. Let $\Lambda \subset \mathbb{L}$. There exists a constant $\alpha = \alpha(J) > 0$ and a constant $C(\alpha)$ such that for all $l \geqslant C(\alpha)$*

$$P_\Lambda[\{\exists\gamma\,,\, d(\gamma) \geqslant l\}; J] \leqslant |\Lambda| \mathcal{O}(l^2) \exp(-\alpha l) \,.$$

Moreover $C(\alpha) = \mathcal{O}(\frac{1}{\alpha}\log\frac{1}{\alpha})$ for small α.

Proof. We consider the (closed) contours as unit-speed parameterized curve with a counterclockwise orientation (see the end of Chapter 1). To each closed contour γ with diameter not smaller than l, we associate a sequence of sites as follows:

1. t_0' is the origin of γ. If $t_0'(2) = 0$ then s_0 is the last time such that $\gamma(s_0)(2) = 0$; we set $t_0 \doteqdot \gamma(s_0)$. Otherwise $t_0 \doteqdot t_0'$.
2. Let s_1 be the first time that γ leaves the square of center t_0 and side $l/2$; we set $t_1 \doteqdot \gamma(s_1)$.
3. Let s_2 be the first time greater than s_1 such that γ leaves the square of center t_1 and side $l/2$; we set $t_2 = \gamma(s_2)$.
4. The procedure is iterated until it stops.

We have thus obtained for each such closed contour a well-defined ordered sequence of points $(t_0', t_0, t_1, \ldots, t_n)$.

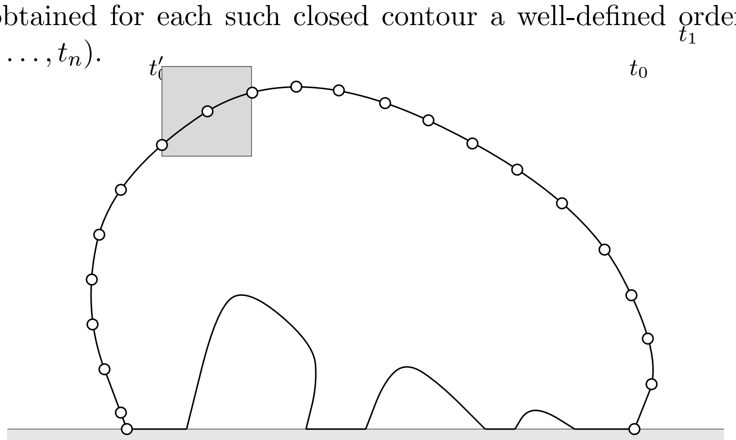

FIGURE 4.6. The coarse–graining procedure.

Since $\beta < \beta_c$, Proposition 3.1.1 implies that $\tau(x;\beta^*) > 0$; we define α as the largest positive constant such that $\tau(x;\beta^*) \geqslant 2\alpha\|x\|_1$, $\forall x \in \mathbb{R}^2$. Clearly

$$P_\Lambda[\{\exists\gamma\,,\, d(\gamma) \geqslant l\}; J] \leqslant \sum_{t\in\Lambda} \sum_{\substack{\gamma\,:\,d(\gamma)\,\geqslant\, l \\ t_0'(\gamma)=t}} q_\Lambda(\gamma; J)\,, \tag{4.46}$$

since $P_\Lambda[\{\exists\gamma\}; J] = q_\Lambda(\gamma; J)$ (see (2.15)). Suppose that the points $t_0', t_0, \ldots, t_n$ are fixed. Then Lemma 4.2.8 implies

$$\begin{aligned} \sum_{\substack{\gamma\,:\,\partial\gamma=\varnothing\,,\,d(\gamma)\,\geqslant\, l \\ t_0',\ldots,t_{n+1}}} q_\Lambda(\gamma) \quad &\leqslant \quad \exp\{-\tau_{\mathrm{bd}}(t_0' - t_0; \beta^*, h^*) - \sum_{i=0}^{n} \tau(t_{i+1} - t_i; \beta^*)\} \\ &\leqslant \quad \exp\{-\tau_{\mathrm{bd}}(t_0' - t_0; \beta^*, h^*)\} \exp\{-\tfrac{1}{2}\alpha n l\}\,, \end{aligned} \tag{4.47}$$

where $t_{n+1} \equiv t_0'$. Therefore

$$\sum_{\substack{\gamma\,:\,d(\gamma)\,\geqslant\, l \\ t_0'(\gamma)=t}} q_\Lambda(\gamma; J) \leqslant \sum_{t_0} \exp\{-\tau_{\mathrm{bd}}(t_0' - t_0; \beta^*, h^*)\} \sum_{n\,\geqslant\, 2} (2[l+2])^n \exp\{-\tfrac{1}{2}\alpha n l\}\,. \tag{4.48}$$

Proposition 3.2.1 implies that $\tau_{\mathrm{bd}}(x; \beta^*, h^*) > 0$, therefore it is possible to choose $C(\alpha)$ large enough so that, for $l \geqslant C(\alpha)$,

$$\sum_{\substack{\gamma\,:\,d(\gamma)\,\geqslant\, l \\ t_0'(\gamma)=t}} q_\Lambda(\gamma; J) \leqslant \mathcal{O}(l^2) \exp\{-\alpha l\}\,. \tag{4.49}$$

□

A result of this kind has already be obtained in [CCS] by different techniques (they used the random–cluster representation of the Ising model; see Chapter 10 for the definition of this representation in the case of the Ashkin–Teller model (which contains the Ising model as a particular case)).

Remark. Notice that the above lemma implies that the probability that there exists contours with diameter larger than $C' \log|\Lambda|$, C' a sufficiently large constant, goes to zero when $|\Lambda|$ goes to ∞.

We want now to obtain a similar result for the contours of the low-temperature representation. This is achieved by using the duality between the two representations.

Lemma 4.3.2. *Let $J(e)$ be given by* (3.30) *with $\beta > \beta_c$ and $h \geqslant 0$. Let Λ be a simply connected subset of $\mathbb{L}$. Then there exists a constant $\alpha' = \alpha'(J) > 0$ and a constant $C'(\alpha')$ such that for all $l \geqslant C'(\alpha')$*

$$P_\Lambda^+[\{\exists\gamma\,,\, d(\gamma) \geqslant l\}; J] \leqslant |\Lambda| \mathcal{O}(l^2) \exp(-\alpha' l)\,,$$

if $h > 0$, and

$$P_\Lambda^+[\{\exists\gamma\,,\, d(\gamma) \geqslant l\}; J] \leqslant |\Lambda|^2 \mathcal{O}(l^2) \exp(-\alpha' l)\,,$$

if $h = 0$. Moreover $C'(\alpha') = \mathcal{O}(\frac{1}{\alpha'} \log \frac{1}{\alpha'})$ for small α'.

Proof. The proof of the first statement is an immediate consequence of Lemmas 4.3.1 (with parameters β^* and h^*) and 2.3.1.
If $h = 0$ then $h^* = \infty$ and therefore Lemma 4.3.1 does not apply. Indeed in that case $\tau_{\text{bd}}(x;\beta) = 0$ and the sum over t_0 is no more bounded uniformly in Λ. However a trivial bound for this sum is $|\Lambda|$ which yields the second statement. □

The previous lemma can be extended to the case $-h_w(\beta) < h < 0$ (see Lemma 4.4.8 for a closely related result).

Similarly, it is not difficult to prove that the magnetic susceptibility of the 2D Ising model with +-b.c. is finite for all $\beta > \beta_c$.

Definition.
(D106) *The* **susceptibility** *of the Ising model is defined by*

$$\chi(\beta) \doteqdot \sum_{t\in\mathbb{Z}^2} (\langle\sigma(0)\sigma(t)\rangle^{+,\beta} - m^*(\beta)^2) \,.$$

Lemma 4.3.3. *Let $J(e) = \beta > \beta_c$, for all edges $e \in \mathcal{E}$. Then $\chi(\beta)$ is finite.*

Proof. Let Λ be some finite square box and let i and j be two points in Λ. We have

$$\begin{aligned}\langle\sigma(t)\sigma(t')\rangle_\Lambda^{+,\beta} &- \langle\sigma(0)\rangle_\Lambda^{+,\beta}\langle\sigma(t)\rangle_\Lambda^{+,\beta} = \\ &4\big(P_\Lambda^{+,\beta}[\{\omega(t)=-1,\omega(t')=-1\}] - P_\Lambda^{+,\beta}[\{\omega(t)=-1\}]P_\Lambda^{+,\beta}[\{\omega(t')=-1\}]\big)\,. \qquad (4.50)\end{aligned}$$

Let's introduce the event $\mathcal{Z} \doteqdot \{\omega \,:\, \exists\gamma \in \underline{\gamma}(\omega)\,,\, t \in \text{int}\gamma,\, t' \in \text{int}\gamma\}$. We can write

$$\begin{aligned}P_\Lambda^{+,\beta}&[\{\omega(t)=-1,\omega(t')=-1\}] \\ &\leqslant P_\Lambda^{+,\beta}[\{\omega(t)=-1,\omega(t')=-1\}\cap\mathcal{Z}^c] + P_\Lambda^{+,\beta}[\mathcal{Z}] \\ &\leqslant \sum_{\mathfrak{c}\in\mathfrak{C}} P_\Lambda^{+,\beta}[\{\omega(t)=-1,\omega(t')=-1\}|\mathfrak{c}]P_\Lambda^{+,\beta}[\mathfrak{c}] + P_\Lambda^{+,\beta}[\mathcal{Z}] \\ &\leqslant \sum_{\mathfrak{c}\in\mathfrak{C}} P_{\Lambda'(\mathfrak{c})}^{+,\beta}[\{\omega(t)=-1\}]P_{\Lambda''(\mathfrak{c})}^{+,\beta}[\{\omega(t')=-1\}]P_\Lambda^{+,\beta}[\mathfrak{c}] + P_\Lambda^{+,\beta}[\mathcal{Z}]\,,\end{aligned} \qquad (4.51)$$

where $\mathfrak{C}$ is the set of all $*$-*connected chains of plus*[5] $\mathfrak{c}$ separating Λ into two components $\Lambda'(\mathfrak{c})$ and $\Lambda''(\mathfrak{c})$ with $t \in \Lambda'(\mathfrak{c})$ and $t' \in \Lambda''(\mathfrak{c})$; the probability of a chain $\mathfrak{c}$ is the probability of the set of configurations ω such that $\mathfrak{c}$ is the first chain of $\mathfrak{C}$ in ω, according to some fixed order.
Since $\{\omega(t) = -1\}$ is a decreasing event, FKG inequalities imply that

$$P_{\Lambda'(\mathfrak{c})}^{+,\beta}[\{\omega(t)=-1\}]P_{\Lambda''(\mathfrak{c})}^{+,\beta}[\{\omega(t')=-1\}] \leqslant P_\Lambda^{+,\beta}[\{\omega(t)=-1\}]P_\Lambda^{+,\beta}[\{\omega(t')=-1\}]\,, \qquad (4.52)$$

and therefore,

$$\langle\sigma(t)\sigma(t')\rangle_\Lambda^{+,\beta} - \langle\sigma(0)\rangle_\Lambda^{+,\beta}\langle\sigma(t)\rangle_\Lambda^{+,\beta} \leqslant 4\,P_\Lambda^{+,\beta}[\mathcal{Z}]\,. \qquad (4.53)$$

[5] A $*$-connected chain of plus is an ordered set of distinct sites $(t_1,\ldots,t_n)$ such that $\|t_{i+1}-t_i\|_2 \leqslant \sqrt{2}$ and $\omega(t_i) = 1$, $\forall i$.

This last probability can be easily estimated using a construction similar (but simpler) to that of the previous lemma. Let u and v be the two points of the dual lattice closest to the intersection of the contour encircling t and t' and the straight line through these two points; if there are more than two such points, we take the two points maximizing $\|v-u\|_1$. Notice that $\|v-u\|_1 \geqslant \|t'-t\|_1$. We then have

$$P_\Lambda^{+,\beta}[\mathcal{Z}] \leqslant \sum_{u,v} P_\Lambda^{+,\beta}[\{\exists\gamma,\, u\in\gamma, v\in\gamma\} \leqslant \sum_{u,v} \exp[-2\tau(v-u)] \leqslant \exp[-\nu\|t'-t\|_1]\,, \quad (4.54)$$

where $\nu > 0$ is some constant. This finally implies that

$$\sum_{t\in\Lambda}(\langle\sigma(0)\sigma(t)\rangle_\Lambda^{+,\beta} - \langle\sigma(0)\rangle_\Lambda^{+,\beta}\langle\sigma(t)\rangle_\Lambda^{+,\beta}) \leqslant 4\sum_{t\in\Lambda} \mathrm{e}^{-\nu\|t'-t\|_1} \leqslant 4\sum_{t\in\mathbb{Z}^2} \mathrm{e}^{-\nu\|t'-t\|_1} < \infty\,, \quad (4.55)$$

uniformly in Λ, which concludes the proof. □

4.4 The random-line representation

In this section we define the random-line representation of the 2-point function and state some of its properties. There are two subsections dealing respectively with the bulk and boundary 2-point function.

4.4.1 The bulk 2-point function

The random-line representation

Let $\beta < \beta_c$. We set $J(e) = \beta$ for all edges $e \in \mathcal{E}$. We consider the set

$$\mathfrak{L} \doteqdot \{\lambda \,:\, \lambda = \varnothing \text{ or } \lambda \text{ is an open contour with } \partial\lambda = \{0,t\}\,,\, 0 \neq t \in \mathbb{Z}^2\}\,. \quad (4.56)$$

There is a natural measure on this set defined by

$$\mathfrak{M}[\{\lambda\}] \doteqdot \begin{cases} q(\lambda;\beta)\,, & \text{if } \lambda \neq \varnothing\,, \\ 0 & \text{otherwise,} \end{cases} \quad (4.57)$$

where $q(\lambda;\beta)$ is the quantity introduced in Lemma 4.2.4, point 3.. $\mathfrak{M}$ is not normalized, however it has a finite mass given by

$$\begin{aligned} \chi &\doteqdot \sum_{\lambda\in\mathfrak{L}} q(\lambda) \\ &= 1 + \sum_{0\neq t\in\mathbb{Z}^2} \sum_{\lambda\,:\,\partial\lambda=\{0,t\}} q(\lambda) \\ &= \sum_{t\in\mathbb{Z}^2} \langle\sigma(0)\sigma(t)\rangle\,. \end{aligned} \quad (4.58)$$

The quantity defined in (4.58) coincides with the one defined in (D106), p. 85, when $\beta < \beta_c$. Indeed in the present case $m^*(\beta) = 0$ and $\langle\sigma(0)\sigma(t)\rangle^{+,\beta} = \langle\sigma(0)\sigma(t)\rangle^\beta$. That the susceptibility is a finite quantity when $\beta < \beta_c$ follows for example from Lemma 4.2.5. It is therefore possible to define a probability measure on $\mathfrak{L}$, however it is more convenient to work with $\mathfrak{M}$ since the results obtained have a nicer formulation with this measure; moreover, the most important statements of this section are given for the measure $\mathfrak{M}$ conditioned on some events.

It is convenient to use the following notation: $\{0 \to t\} \doteqdot \{\lambda \in \mathfrak{L} \,:\, \partial\lambda = \{0, t\}\}$. The bulk 2-point function has the following representation

$$\langle \sigma(0)\sigma(t)\rangle = \mathfrak{M}[\{0 \to t\}]\,. \tag{4.59}$$

The contours in $\mathfrak{L}$ can be interpreted as self-avoiding paths on $\mathcal{E}$. Is it possible to extend some of the techniques and results for self-avoiding random walks (SAW) to this case? This is an interesting question to which we do not answer, however we state in the following subsection, some simple facts inspired by some part of the work [CC] on SAW. The next subsection is devoted to the study of the typical contours in $\mathfrak{L}$ with respect to the measure $\mathfrak{M}(\cdot|\{0 \to t\})$; these results are strongly improved in Section 4.6 at the cost of using the Sharp Triangle Inequality in an essential way.

Cylindrical contours

Definition.

(D107) *An open contour $\lambda \in \mathfrak{L}$ with $\partial\lambda = \{0, t\}$ is called* **cylindrical** *if $\lambda = \lambda' \cup e$ with e an horizontal edge and*

$$\lambda' \subset \{s \in \mathbb{R}^2 \,:\, 0 \leqslant s(1) \leqslant t(1) - 1\}\,.$$

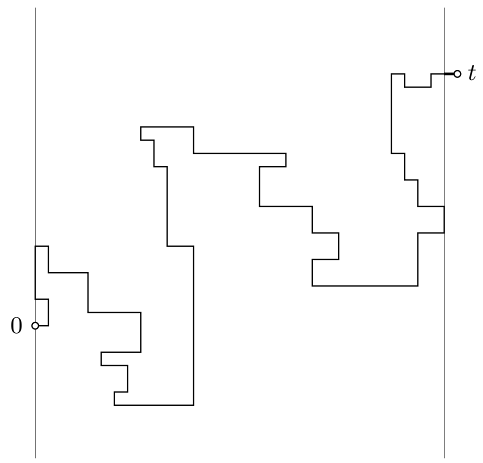

FIGURE 4.7. A cylindrical contour.

We use the notation: $\{0 \xrightarrow{c} t\} \doteqdot \{0 \to t\} \cap \{\lambda \in \mathfrak{L} \,:\, \lambda \text{ cylindrical}\}$. For $a \in \mathbb{N}$, we define

$$\chi(a, 0) \doteqdot \sum_{\substack{t\in\mathbb{Z}^2\,: \\ t(1)=a}} \langle\sigma(0)\sigma(t)\rangle = \sum_{\substack{t\in\mathbb{Z}^2\,: \\ t(1)=a}} \mathfrak{M}[\{0 \to t\}]\,, \tag{4.60}$$

and

$$\chi_c(a,0) \doteqdot \sum_{\substack{t\in\mathbb{Z}^2:\\ t(1)=a}} \mathfrak{M}[\{0 \xrightarrow{c} t\}]\,. \tag{4.61}$$

Lemma 4.4.1. *Let $\beta < \beta_c$ and $J(e) = \beta$ for all edges e; set $\tau^* \doteqdot \tau((1,0);\beta^*)$ and*

$$\chi^* \doteqdot \sum_{\substack{t:\\ t(1)=0}} \langle \sigma(0)\sigma(t)\rangle\,.$$

Then for any $a \in \mathbb{N}$

$$\chi_c(a,0) \leqslant \chi(a,0) \leqslant {\chi^*}^2 \chi_c(a,0)\,;$$

$$\lim_{a\to\infty} -\tfrac{1}{a}\log\chi(a,0) = \lim_{a\to\infty} -\tfrac{1}{a}\log\chi_c(a,0) = \tau^*\,;$$

$$\chi_c(a,0) \leqslant \exp\{-\tau^* a\} \quad , \quad \chi(a,0) \geqslant \exp\{-\tau^* a\}\,.$$

Similar results hold for the quantity

$$\chi(a,a) \doteqdot \sum_{\substack{t\in\mathbb{Z}^2:\\ t(1)+t(2)=a}} \langle \sigma(0)\sigma(t)\rangle\,.$$

Proof. The first statement follows from Lemma 4.2.6 and GKS inequalities. We prove the second statement. Clearly,

$$\lim_{a\to\infty} -\tfrac{1}{a}\log\chi(a,0) \leqslant \lim_{a\to\infty} -\tfrac{1}{a}\log\langle\sigma(0)\sigma((a,0))\rangle \leqslant \tau^*\,. \tag{4.62}$$

Consider the two lines

$$l_1 \doteqdot \{s\in\mathbb{Z}^2 :\ s(2) = 2a - s(1)\} \quad , \quad l_2 \doteqdot \{s\in\mathbb{Z}^2 :\ s(2) = -2a + s(1)\}\,. \tag{4.63}$$

By Lemma 4.2.1 we have

$$\langle\sigma(0)\sigma(t')\rangle \leqslant \langle\sigma(0)\sigma((a,0))\rangle \tag{4.64}$$

if $t' = (a,b)$, $b\in\mathbb{Z}$, or $t' \in l_1$ and $t'(1) \leqslant a$, or $t' \in l_2$ and $t'(1) \leqslant a$. If λ is a cylindrical contour with $\partial\lambda = \{0,t\}$, $t(1) = a$ and $|t(2)| \geqslant a$, then it must intersect l_1 or l_2. Applying Lemma 4.2.6 and (4.64)

$$\chi_c(a,0) = \sum_{\substack{t:\\ t(1)=a}} \mathfrak{M}[\{0 \xrightarrow{c} t\}] \leqslant 2a(1+\chi^*)\langle\sigma(0)\sigma((a,0))\rangle\,. \tag{4.65}$$

The last two statements follow from

$$\chi_c(a_1+a_2,0) \geqslant \chi_c(a_1,0)\chi_c(a_2,0) \tag{4.66}$$

and

$$\chi(a_1+a_2,0) \leqslant \chi(a_1,0)\chi(a_2,0)\,. \tag{4.67}$$

□

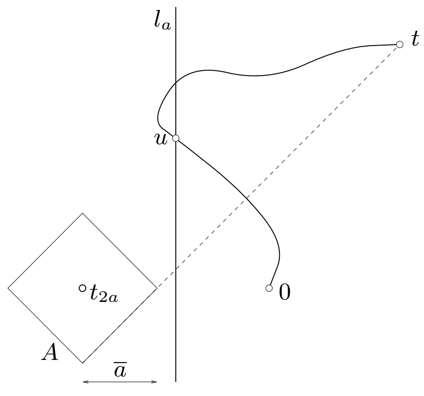

FIGURE 4.8. The reflection trick of Lemma 4.4.2.

Box proposition

We suppose, without loss of generality, that t is such that $t(1) \geqslant t(2) \geqslant 0$. Let

$$\mathcal{B}_t \doteqdot \{u \in \mathbb{Z}^2 \,:\, 0 \leqslant u(1) \leqslant t(1)\,,\ \frac{t(2)-t(1)}{2} \leqslant u(2) \leqslant \frac{t(2)+t(1)}{2}\}\,. \tag{4.68}$$

The aim of this subsection is to compare $\mathfrak{M}[\{0 \to t\}] = \langle\sigma(0)\sigma(t)\rangle$ and

$$\mathfrak{M}[\{0 \to t\} \cap \{\lambda \in \mathfrak{L} \,:\, \lambda \subset \mathcal{B}_t\}] = \sum_{\substack{\lambda\in\mathfrak{L}:\\ \lambda\subset\mathcal{B}_t}} q(\lambda)\,. \tag{4.69}$$

This study is based on a trick, analogous to the reflection principle for random walk, which has been introduced in [Pf1]. We first consider a simpler problem. Let $a \in \mathbb{N}$ and define

$$l_a \doteqdot \{s \in \mathbb{Z}^2 \,:\, s(1) = -a\}\,. \tag{4.70}$$

We want to compare $\mathfrak{M}[\{0 \to t\}]$ with $\mathfrak{M}[\{0 \to t\} \cap \{\lambda \in \mathfrak{L} \,:\, \lambda \cap l_a = \varnothing\}]$. This is done in the next lemma.

Lemma 4.4.2. *Let $\beta < \beta_c$ and $J(e) = \beta$ for all edges $e \in \mathcal{E}$. Let $t \in \mathbb{Z}^2$ such that $t(1) \geqslant 0$ and let $\mathfrak{E}_a(t)$ be the set*

$$\mathfrak{E}_a(t) \doteqdot \{0 \to t\} \cap \{\lambda \in \mathfrak{L} \,:\, \lambda \cap l_a \neq \varnothing\}\,.$$

Then

- $\mathfrak{M}[\mathfrak{E}_a(t)] \leqslant 2|t(2)|\,\mathfrak{M}[\{0 \to t + (2a,0)\}] + \chi(a,0)\,\mathfrak{M}[0 \to t]$.
- *Suppose that $\overline{a} \doteqdot \min\{2a + t(1) - |t(2)|, 2a\}$ is strictly positive. Then*

$$\mathfrak{M}[\mathfrak{E}_a(t)|\{0 \to t\}] \leqslant 8\chi(\overline{a},\overline{a})|t(2)| + \chi(a,0)\,.$$

Proof. We assume, without loss of generality, that $t(2) \geqslant 0$.

We prove the first statement. λ is considered as unit speed parameterized curve with initial point 0. Let s be the first time $\lambda(s) \in l_a$; we set $u \doteqdot \lambda(s)$. Then Lemma 4.2.6 implies

$$\mathfrak{M}[\mathfrak{E}_a(t)] \leqslant \sum_{u\in l_a} \sum_{\substack{\lambda\in\mathfrak{E}_a(t):\\ \lambda\ni u}} q(\lambda) \leqslant \sum_{u\in l_a} \langle\sigma(0)\sigma(u)\rangle\langle\sigma(u)\sigma(t)\rangle\,. \tag{4.71}$$

There are two cases. If $u(2) \leqslant 0$ or $u(2) \geqslant 2\,t(2)$ then by monotonicity and symmetry $\langle\sigma(u)\sigma(t)\rangle \leqslant \langle\sigma(0)\sigma(t)\rangle = \mathfrak{M}[\{0 \to t\}]$. Consequently,

$$\sum_{\substack{u\in l_a:\\ u(2)\notin[0,2\,t(2)]}} \langle\sigma(0)\sigma(u)\rangle\langle\sigma(u)\sigma(t)\rangle \leqslant \chi(a,0)\,\mathfrak{M}[\{0\to t\}]\,. \tag{4.72}$$

Suppose now that $u(2) \in [0, 2\,t(2)]$. Then, using the notation $t_{2a} \doteqdot (-2a, 0)$, we obtain by symmetry, GKS inequalities and translation invariance,

$$\begin{aligned}\langle\sigma(0)\sigma(u)\rangle\langle\sigma(u)\sigma(t)\rangle &= \langle\sigma(t_{2a})\sigma(u)\rangle\langle\sigma(u)\sigma(t)\rangle\\ &\leqslant \langle\sigma(t_{2a})\sigma(t)\rangle\\ &= \langle\sigma(0)\sigma(t-t_{2a})\rangle\,.\end{aligned} \tag{4.73}$$

This concludes the proof of the first statement.
We prove now the second statement. We need to compare $\mathfrak{M}[\{0 \to t + (2a, 0)\}]$ and $\mathfrak{M}[\{0 \to t\}]$. This is done with the use of Simon's inequality (Lemma 4.2.7). We first have to construct a box separating the sites t_{2a} and t; this is the role of the set

$$A \doteqdot \{t' \in \mathbb{Z}^2 \,:\, \|t' - t_{2a}\|_1 \leqslant \overline{a}\}\,. \tag{4.74}$$

Clearly A satisfies the hypotheses of Lemma 4.2.7. Therefore we can write

$$\mathfrak{M}[\{0 \to t + (2a,0)\}] = \langle\sigma(t_{2a})\sigma(t)\rangle \leqslant \sum_{t'\in\partial A} \langle\sigma(t_{2a})\sigma(t')\rangle\langle\sigma(t')\sigma(t)\rangle\,. \tag{4.75}$$

Moreover, the hypothesis on $\overline{a}$ ensures that

$$A \subset \{v \in \mathbb{Z}^2 \,:\, v(1) \leqslant 0\} \cap \{v \in \mathbb{Z}^2 \,:\, v(2) \geqslant v(1) + t(2) - t(1)\}\,. \tag{4.76}$$

Hence we can apply monotonicity properties of the 2-point function to obtain

$$\langle\sigma(t')\sigma(t)\rangle \leqslant \langle\sigma(0)\sigma(t)\rangle\,, \tag{4.77}$$

for every $t' \in \partial A$. Consequently,

$$\mathfrak{M}[\{0 \to t + (2a,0)\}] \leqslant 2\chi(\overline{a},\overline{a})\,\langle\sigma(0)\sigma(t)\rangle\,. \tag{4.78}$$

The conclusion follows from the first statement. □

We have consider a vertical lines l_a. However it is clear that the corresponding statement for horizontal lines can be obtained by symmetry.
We can now consider the case of the square box. We first need a technical result which is inspired by a similar statement proved in [I1].

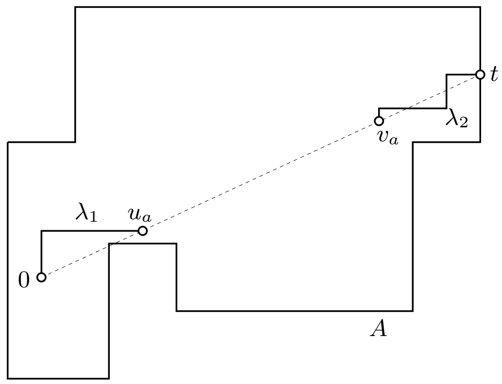

FIGURE 4.9. The setting of Lemma 4.4.3.

Definition.

(D108) *A set $A \subset \mathbb{Z}^2$ is* **connected** *if*

$$\bigcup_{t \in A} p^*(t)$$

is a connected subset of $\mathbb{R}^2$.

Lemma 4.4.3. *Let $\beta < \beta_c$ and $J(e) = \beta$ for all edges $e \in \mathcal{E}$. Suppose $t \in \mathbb{Z}^2$ is such that $0 \leqslant t(2) \leqslant t(1)$. Let $a \in \mathbb{N}$ with $2a < t(1)$. Let $u_a \in \mathbb{Z}^2$ and $v_a \in \mathbb{Z}^2$ such that*

- *u_a is the point on the vertical line $\{t' \in \mathbb{Z}^2 : t'(1) = a\}$ with $u_a(2)$ minimal and $u_a(2) \geqslant a\,(t(2)/t(1))$.*
- *v_a is the point on the vertical line $\{t' \in \mathbb{Z}^2 : t'(1) = t(1) - a\}$ with $v_a(2)$ maximal and $v_a(2) \leqslant t(2) - a\,(t(2)/t(1))$.*

Let $A \subset \mathbb{Z}^2$ be a connected set containing 0, t, u_a and v_a, and such that there exist two open contours λ_1 and λ_2 with

- *$\partial\lambda_1 = \{0, u_a\}$, $\partial\lambda_2 = \{v_a, t\}$;*
- *$|\lambda_1| = \|u_a\|_1$, $|\lambda_2| = \|t - v_a\|_1$;*
- *$\lambda_1 \subset A$, $\lambda_2 \subset A$.*

Then

$$\sum_{\substack{\lambda : \partial\lambda = \{0,t\} \\ \lambda \subset A}} q(\lambda) \geqslant \exp\{-\mathcal{O}(a)\} \sum_{\substack{\lambda : \partial\lambda = \{u_a, v_a\} \\ \lambda \subset A}} q(\lambda)\,.$$

Proof. Let λ_1 and λ_2 be as described in the statement of the lemma.
Let $\lambda' \subset A$ be an open contour with $\partial\lambda' = \{u_a, v_a\}$. We assume that u_a is the initial point. Let $s_1 \in [0, |\lambda'|]$ be the integer time defined by the condition that $t_1 \doteqdot \lambda'(s_1) \in \lambda_1$ so that $t_1(1)$ is minimal; similarly let $s_2 \in [0, |\lambda'|]$ be the integer time defined by the condition that $t_2 \doteqdot \lambda'(s_2) \in \lambda_2$ so that $t_2(1)$ is maximal. This gives a partition of λ' into three open contours; we sum over the first and last ones by using Lemma 4.2.9. We obtain

$$\sum_{\substack{\lambda' : \partial\lambda = \{u_a, v_a\} \\ \lambda' \subset A}} q(\lambda') \leqslant \sum_{t_1, t_2} \sum_{\substack{\lambda : \partial\lambda = \{t_1, t_2\} \\ \lambda \subset A}} 4\, q(\lambda) \langle \sigma(u_a)\sigma(t_1)\rangle \langle \sigma(t_2)\sigma(v_a)\rangle\,. \tag{4.79}$$

Let λ be an open contour of the last sum of (4.79). We extend λ to an open contour $\overline{\lambda} \subset A$ with $\partial\overline{\lambda} = \{0, t\}$: $\overline{\lambda}$ is the union of λ_1', λ, λ_2', with λ_1' the part of λ_1 from 0 to t_1 and λ_2' the part of λ_2 from t_2 to t. By Lemma 4.2.4 point 7. we have

$$q(\overline{\lambda}) \geqslant q(\lambda_1')q(\lambda)q(\lambda_2') \,. \tag{4.80}$$

Using Lemma 4.2.4 point 1. we obtain

$$q(\lambda_j') \geqslant \exp\{-\mathcal{O}(|\lambda_j'|)\} \,, \ \ j = 1, 2 \,. \tag{4.81}$$

Indeed it is enough to put all coupling constants $J(e)$, $e \notin \Delta(\lambda_j')$, to ∞ and make an explicit computation. Thus, since $|\lambda_j'| = \mathcal{O}(a)$,

$$4\, q(\lambda) \leqslant q(\overline{\lambda}) \exp\{\mathcal{O}(a)\} \tag{4.82}$$

Putting these estimates together we obtain the desired result. □

Lemma 4.4.3 allows us to estimate (4.69), in which 0 and t are on the boundary of $\mathcal{B}_t$, by a similar event involving u_a and v_a which are "deep" inside the box. We first show how Lemma 4.4.2 can be used to study this last event.

Lemma 4.4.4. *Let $\beta < \beta_c$ and $J(e) = \beta$ for all edges $e \in \mathcal{E}$. Suppose $t \in \mathbb{Z}^2$ is such that $0 \leqslant t(2) \leqslant t(1)$. Let $a \in \mathbb{N}$ with $2a < t(1)$. Let $\mathcal{B}_t$ be the box (4.68) and u_a, v_a defined as in Lemma 4.4.3 with $A = \mathcal{B}_t$. Then*

$$\sum_{\substack{\lambda\,:\,\partial\lambda=\{u_a,v_a\}\\ \lambda\cap\partial\mathcal{B}_t=\varnothing}} q(\lambda) \geqslant \big[1 - \mathcal{O}(|t(1)|)\exp\{-\mathcal{O}(a)\}\big] \sum_{\lambda\,:\,\partial\lambda=\{u_a,v_a\}} q(\lambda) \,.$$

Proof. It is sufficient to observe that the box $\mathcal{B}_t$ and the points u_a and v_a have been chosen in such a way as to ensure that Lemma 4.4.2 can be applied for each side of the box with $\overline{a} = 2a$. Using this lemma yields

$$\sum_{\substack{\lambda\,:\,\partial\lambda=\{u_a,v_a\}\\ \lambda\cap\partial\mathcal{B}_t\neq\varnothing}} q(\lambda) \leqslant \mathcal{O}(|t(1)|)\exp\{-\mathcal{O}(a)\}\langle\sigma(u_a)\sigma(v_a)\rangle \,, \tag{4.83}$$

where we used $\chi(2a, 0) \leqslant \exp\{-\mathcal{O}(a)\}$ and $\chi(2a, 2a) \leqslant \exp\{-\mathcal{O}(a)\}$ by Lemma 4.4.1. □

Lemma 4.4.4 states a concentration property of the measure $\mathfrak{M}[\,\cdot\,|\{0 \to t\}]$. Indeed let $t_L \doteqdot t \cdot L$; the lemma states that there exists a sequence of boxes $\mathcal{B}_{t_L}'$ such that

$$\mathfrak{M}[\{\lambda \subset \mathcal{B}_{t_L}'\}|\{0 \to t_L\}] \geqslant 1 - L^{-\mathcal{O}(K)} \,. \tag{4.84}$$

(Just construct the boxes $\mathcal{B}_{t_L}'$ such that 0 and t_L play the roles of u_a and v_a in the lemma, with $a = K \log L$, K large enough). This is certainly not an optimal estimate since we expect the typical transverse fluctuations of the contour to be of order $\mathcal{O}(L^{1/2})$, while in this lemma fluctuations of order $\mathcal{O}(L)$ are allowed; see Section 4.6 for (much) better estimates.

Proposition 4.4.1. *Let $\beta < \beta_c$ and $J(e) = \beta$ for all edges $e \in \mathcal{E}$. Suppose $t \in \mathbb{Z}^2$ is such that $0 \leqslant t(2) \leqslant t(1)$. Let $a \in \mathbb{N}$ with $2a < t(1)$. If $\mathcal{B}_t$ is the box (4.68), then*

- $\mathfrak{M}[\{\lambda \in \mathfrak{L} : \lambda \text{ inside } \mathcal{B}_t\} | \{0 \to t\}] \geqslant \big(1 - \mathcal{O}(|t_1|) \exp\{-\mathcal{O}(a)\}\big) \exp\{-\mathcal{O}(a)\}$.
- $\sum_{\substack{\lambda \in \mathfrak{L}: \\ \lambda \text{ inside } \mathcal{B}_t}} q(\lambda) \geqslant \langle \sigma(0)\sigma(t)\rangle \big[1 - \mathcal{O}(|t(1)| \exp\{-\mathcal{O}(a)\})\big] \exp\{-\mathcal{O}(a)\}$,

where λ inside $\mathcal{B}_t$ means $\lambda \subset \mathcal{B}_t$ and $\lambda \cap \partial\mathcal{B}_t = \{0, t\}$.

Proof. From Lemma 4.4.3 we can write

$$\mathfrak{M}[\{0 \to t\} \cap \{\lambda \in \mathfrak{L} : \lambda \text{ inside } \mathcal{B}_t\}] = \sum_{\substack{\lambda : \partial\lambda = \{0,t\} \\ \lambda \text{ inside } \mathcal{B}_t}} q(\lambda) \geqslant \exp\{-\mathcal{O}(a)\} \sum_{\substack{\lambda : \partial\lambda = \{u_a, v_a\} \\ \lambda \text{ inside } \mathcal{B}_t}} q(\lambda)\,, \tag{4.85}$$

The first statement follows easily from Lemma 4.4.4, equation (4.85) and the observation that $\langle \sigma(0)\sigma(t)\rangle \leqslant \langle \sigma(u_a)\sigma(v_a)\rangle$ by monotonicity.
The second statement is just a reformulation of the first one. □

In Section 4.6 we obtain much more precise results on typical contours contributing to the 2-point function. Notice however that, here, we have made no use of the exact solution (neither the Ornstein-Zernicke property (see Section 4.5) nor the Sharp Triangle inequality); unfortunately, the proof of the corresponding statement for the boundary 2-point function requires the use of a lower bound on the 2-point function. However, in some cases, it is possible to avoid completely the use of such information. See Chapter 7 for a consequence of this remark.

4.4.2 The boundary 2-point function

The random-line representation

There is a similar random-line representation for the boundary 2-point function. Let $\beta < \beta_c$ and let $J(e)$ be the coupling constants defined in (4.5). We introduce the following set

$$\mathfrak{L}_{\mathbb{L}} \doteqdot \{\lambda \subset \mathbb{L} : \lambda = \varnothing \text{ or } \partial\lambda = \{0, t\}\,,\, 0 \neq t \in \mathbb{L}\}\,. \tag{4.86}$$

A natural measure on this set is given by

$$\mathfrak{M}_{\mathbb{L}}[\{\lambda\}] \doteqdot \begin{cases} q_{\mathbb{L}}(\lambda; \beta, h)\,, & \text{if } \lambda \neq \varnothing\,, \\ 0 & \text{otherwise,} \end{cases} \tag{4.87}$$

where $q_{\mathbb{L}}(\cdot; \beta, h)$ is defined in Lemma 4.2.4. This measure is not normalized, however it has a finite mass given by

$$\chi_{\mathbb{L}} \doteqdot \sum_{\lambda \in \mathfrak{L}_{\mathbb{L}}} q_{\mathbb{L}}(\lambda) = \sum_{t \in \mathbb{L}} \langle \sigma(0)\sigma(t)\rangle\,, \tag{4.88}$$

which is bounded above by χ by GKS inequalities. The boundary 2-point function has the following representation ($t \in \mathbb{L}$)

$$\langle \sigma(0)\sigma(t)\rangle_{\mathbb{L}} = \mathfrak{M}_{\mathbb{L}}[\{0 \to t\}]\,. \tag{4.89}$$

Lemma 4.2.4 gives an interesting inequality relating $\mathfrak{M}[\{0 \to t\}]$ and $\mathfrak{M}_{\mathbb{L}}[\{0 \to t\}]$ when $h = 1$.

Lemma 4.4.5. *Let $\beta < \beta_c$ and $J(e) = \beta$ for all edges $e \in \mathcal{E}$. Then, for all $t \in \mathbb{L}$,*

$$\mathfrak{M}_{\mathbb{L}}[\{0 \to t\}] \geqslant \mathfrak{M}[\{\lambda \subset \mathbb{L}\} \cap \{0 \to t\}]$$

and

$$\langle \sigma(0)\sigma(t)\rangle_{\mathbb{L}} \geqslant \mathfrak{M}[\{\lambda \subset \mathbb{L}\} | \{0 \to t\}] \, \langle \sigma(0)\sigma(t)\rangle \,.$$

Proof. By Lemma 4.2.4,

$$\begin{aligned} \mathfrak{M}_{\mathbb{L}}[\{0 \to t\}] &= \sum_{\substack{\lambda \subset \mathbb{L} \\ \partial\lambda = \{0,t\}}} q_{\mathbb{L}}(\lambda) \\ &\geqslant \sum_{\substack{\lambda \subset \mathbb{L} \\ \partial\lambda = \{0,t\}}} q(\lambda) \\ &= \mathfrak{M}[\{\lambda \subset \mathbb{L}\} \cap \{0 \to t\}] \,. \end{aligned} \tag{4.90}$$

Therefore

$$\langle \sigma(0)\sigma(t)\rangle_{\mathbb{L}} \geqslant \mathfrak{M}[\{\lambda \subset \mathbb{L}\} \cap \{0 \to t\}] = \mathfrak{M}[\{\lambda \subset \mathbb{L}\} | \{0 \to t\}] \, \langle \sigma(0)\sigma(t)\rangle \,. \tag{4.91}$$

□

Box proposition

The following results are similar to those proved for the bulk 2-point function. Again the estimates are not optimal and are improved in Section 4.6.
Let $t \in \mathbb{L}$ with $t(1) > 0$. We construct a square box

$$\mathcal{B}_t \doteqdot \{u \in \mathbb{Z}^2 \,:\, 0 \leqslant u(1) \leqslant t(1) \,,\, 0 \leqslant u(2) \leqslant t(1)\} \,. \tag{4.92}$$

The aim of this subsection is to compare $\mathfrak{M}_{\mathbb{L}}[\{0 \to t\}] = \langle \sigma(0)\sigma(t)\rangle_{\mathbb{L}}$ and

$$\mathfrak{M}[\{0 \to t\} \cap \{\lambda \in \mathfrak{L}_{\mathbb{L}} \,:\, \lambda \subset \mathcal{B}_t\}] = \sum_{\substack{\lambda \in \mathfrak{L}_{\mathbb{L}} : \\ \lambda \subset \mathcal{B}_t}} q_{\mathbb{L}}(\lambda) \,. \tag{4.93}$$

We first state a result similar to Lemma 4.4.3.

Lemma 4.4.6. *Let $\beta < \beta_c$ and $J(e)$ given by (4.5). Suppose $t \in \Sigma$ is such that $t(1) > 0$. Let $a \in \mathbb{N}$ with $2a < t(1)$. Let $u_a = (a, 0)$ and $v_a = (t(1) - a, 0)$. Let $A \subset \mathbb{L}$ be a connected set containing 0, t, u_a, v_a and the sets $\{t' \in \Sigma \,:\, 0 \leqslant t'(1) \leqslant a\}$ and $\{t' \in \Sigma \,:\, t(1) - a \leqslant t'(1) \leqslant t(1)\}$. Then*

$$\sum_{\substack{\lambda \,:\, \partial\lambda = \{0,t\} \\ \lambda \subset A}} q_{\mathbb{L}}(\lambda) \geqslant \exp\{-\mathcal{O}(a)\} \sum_{\substack{\lambda \,:\, \partial\lambda = \{u_a,v_a\} \\ \lambda \subset A}} q_{\mathbb{L}}(\lambda) \,.$$

Proof. The Lemma is proved in the same way as Lemma 4.4.3. □

We prove now the analogue of Lemma 4.4.4

Lemma 4.4.7. *Let $\beta < \beta_c$ and $J(e)$ given by (4.5). Suppose $t \in \Sigma$ is such that $t(1) > 0$. Let $a \in \mathbb{N}$ with $2a < t(1)$. Let $\mathcal{B}_t$ be the box (4.92) and u_a, v_a defined as in Lemma 4.4.6 with $A = \mathcal{B}_t$. Then*

$$\sum_{\substack{\lambda : \partial\lambda = \{u_a, v_a\} \\ \lambda \text{ inside } \mathcal{B}_t}} q_{\mathbb{L}}(\lambda) \geqslant \left[1 - \mathcal{O}(|t(1)|)^{5/2} \exp\{-2a\tau_{\mathrm{bd}}(\beta^*, h^*)\}\right] \sum_{\lambda : \partial\lambda = \{u_a, v_a\}} q_{\mathbb{L}}(\lambda)\,,$$

where λ inside $\mathcal{B}_t$ means $\lambda \subset \mathcal{B}_t$ and $\lambda \cap \{t' \in \mathcal{B}_t \,:\, t'(1) = 0$ or $t'(1) = t(1)$ or $t'(2) = t(2)\} = \{0, t\}$.

Proof. Consider λ such that $\partial\lambda = \{u_a, v_a\}$ with initial point u_a. Assume that λ touches the boundary of the box $\mathcal{B}_t$ at t_*. Let $\lambda(s_*) := t_*$. There are two cases.

1. $t_*(2) = t(1)$. Then there is a last time s_1 such that $s_1 < s_*$ with $\lambda(s_1) \in \Sigma$ and a first time $s_2 > s_*$ such that $\lambda(s_2) \in \Sigma$. Let $\tau^* := \tau((1,0);\beta^*)$. Using Lemma 4.2.5 and symmetry and monotonicity properties of the surface tension we get

$$\sum_{\substack{\lambda : \partial\lambda = \{u_a, v_a\} \\ t_* \in \lambda}} q_{\mathbb{L}}(\lambda) \leqslant \mathcal{O}(\exp\{-2t(1)\tau^*\})\,. \tag{4.94}$$

We write the right–hand side of (4.94) as

$$\mathcal{O}(\exp\{-2t(1)\tau^*\}) = \frac{\mathcal{O}(\exp\{-2t(1)\tau^*\})}{\langle\sigma(u_a)\sigma(v_a)\rangle_{\mathbb{L}}} \langle\sigma(u_a)\sigma(v_a)\rangle_{\mathbb{L}}\,. \tag{4.95}$$

The lower bound on the boundary two–point function of Section 4.5, Proposition 3.2.1 and $t(1)\tau^* = \tau(t;\beta^*)$, with $t = (t(1), 0)$, imply that

$$\frac{\mathcal{O}(\exp\{-2t(1)\tau^*\})}{\langle\sigma(u_a)\sigma(v_a)\rangle_{\mathbb{L}}} \leqslant \mathcal{O}\Big(|t(1)|^{3/2} \exp\{-\tau(t;\beta^*)\}\Big)\,. \tag{4.96}$$

Replacing $\langle\sigma(u_a)\sigma(v_a)\rangle_{\mathbb{L}}$ by $\langle\sigma(0)\sigma(t)\rangle_{\mathbb{L}}$ and summing over t^*, we get

$$\sum_{\substack{\lambda : \partial\lambda = \{u_a, v_a\} \\ t_* \in \lambda}} q_{\mathbb{L}}(\lambda) \leqslant \langle\sigma(0)\sigma(t)\rangle_{\mathbb{L}} \mathcal{O}\Big(|t(1)|^{5/2} \exp\{-\tau(t;\beta^*)\}\Big)\,. \tag{4.97}$$

2. $t_*(1) = 0$ or $t_*(1) = t(1)$. From Lemma 4.2.6 and GKS inequalities we obtain in a similar manner

$$\begin{aligned}
\sum_{\substack{\lambda : \partial\lambda = \{u_a, v_a\} \\ t_* \in \lambda\,,\, t_*(1) = 0}} q_{\mathbb{L}}(\lambda) &\leqslant \langle\sigma(u_a)\sigma(t_*)\rangle_{\mathbb{L}} \langle\sigma(v_a)\sigma(t_*)\rangle_{\mathbb{L}} \\
&= \langle\sigma(-u_a)\sigma(t_*)\rangle_{\mathbb{L}} \langle\sigma(v_a)\sigma(t_*)\rangle_{\mathbb{L}} \\
&\leqslant \langle\sigma(-u_a)\sigma(v_a)\rangle_{\mathbb{L}} \\
&\leqslant \langle\sigma(u_a)\sigma(v_a)\rangle_{\mathbb{L}} \mathcal{O}\Big(|t(1)|^{3/2} \exp\{-2\tau_{\mathrm{bd}}(u_a;\beta^*, h^*)\}\Big)\,.
\end{aligned} \tag{4.98}$$

Then we sum over t^* and use $\tau_{\mathrm{bd}}(u_a;\beta^*, h^*) = a\tau_{\mathrm{bd}}((1,0);\beta^*, h^*)$. □

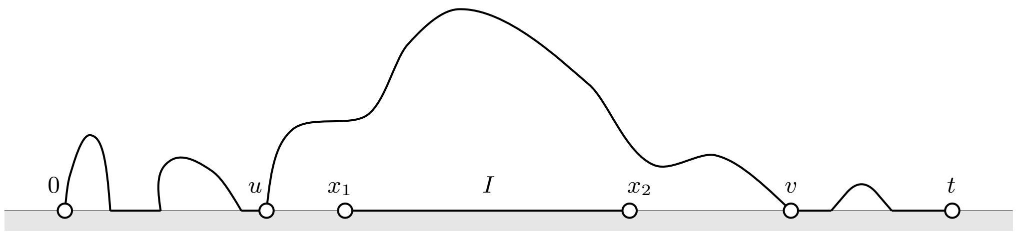

FIGURE 4.10. The construction in the proof of Lemma 4.4.8.

Lemma 4.4.7 states a concentration property of the measure $\mathfrak{M}_{\mathbb{L}}[\,\cdot\,|\{0 \to t\}]$. The same comments as in the case of the bulk 2-point function apply. Notice that Lemma 4.4.8 below shows that when $\tau_{\rm bd}(\beta^*, h^*) < \tau((1,0);\beta^*)$ (i.e. in case of partial wetting) the contour sticks to the wall. This observation can be used to improve hugely the above estimate in such a case. Since a similar (better) estimate is done in Section 4.6, we do not do it here.

We can now state the main result of this subsection, which is similar to Proposition 4.4.1.

Proposition 4.4.2. *Let $\beta < \beta_c$ and $J(e)$ be given by* (3.30)*. Suppose $t \in \mathbb{L}$ is such that $t(1) > 0$. Let $a \in \mathbb{N}$ with $2a < t(1)$. If $\mathcal{B}_t$ is the box of* (4.92)*, then*

- $\mathfrak{M}_{\mathbb{L}}[\{\lambda \in \mathfrak{L}_{\mathbb{L}} : \lambda \subset \mathcal{B}_t\}|\{0 \to t\}] \geqslant \big[1 - \mathcal{O}(|t_1|^{5/2}) \exp\{-2a\tau_{\rm bd}(\beta^*, h^*\}\big] \exp\{-\mathcal{O}(a)\}.$
- $\sum_{\substack{\lambda \in \mathfrak{L}_{\mathbb{L}}:\\ \lambda \subset \mathcal{B}_t}} q_{\mathbb{L}}(\lambda) \geqslant \langle \sigma(0)\sigma(t)\rangle_{\mathbb{L}} \big[1 - \mathcal{O}(|t(1)|^{5/2}) \exp\{-2a\tau_{\rm bd}(\beta^*, h^*\}\big] \exp\{-\mathcal{O}(a)\}.$

Proof. This is done as in the proof of Proposition 4.4.1. □

Again, using the Sharp Triangle Inequality, it is possible to obtain essentially optimal estimates, see Section 4.6.

Before concluding this subsection, we prove a last lemma showing that, when the dual model is in the partial wetting regime $h^* < h_{\rm w}(\beta^*)$, the open contours with both endpoints on Σ really sticks to the wall, in the sense that they return very often to Σ.

Lemma 4.4.8. *Let $\beta < \beta_c$ and $J(e)$ given by* (4.5)*. Suppose $h^* < h_{\rm w}(\beta^*)$. Let $t \in \Sigma$ and $I = \{u \in \Sigma : x_1 \leqslant u(1) \leqslant x_2\}$ be a finite non-empty subset of Σ with $0 < x_1 < x_2 < t(1)$. Then there exist $\varepsilon > 0$, n_ε and C_1 such that for all x_1, x_2 with $|x_2 - x_1| \geqslant n_\varepsilon$*

$$\mathfrak{M}_{\mathbb{L}}[\{\lambda \cap I = \varnothing\}|\{0 \to t\}] \leqslant C_1 \exp\{-\varepsilon|x_2 - x_1|\}\,.$$

Proof. We have

$$\mathfrak{M}_{\mathbb{L}}[\{0 \to t\}] = \langle \sigma(0)\sigma(t)\rangle_{\mathbb{L}}\,. \tag{4.99}$$

Let λ be a random line such that $\partial\lambda = \{0, t\}$ and $\lambda \cap I = \varnothing$. Let s_1 be the last time that λ touches Σ at the left hand side of I, and let s_2 be the first time that λ touches Σ at the right-hand side of I. We set $u \doteqdot \lambda(s_1)$ and $v \doteqdot \lambda(s_2)$. We necessarily have $u(1) < x_1 < x_2 < v(1)$. From Lemmas 4.2.5 and 4.2.6 we get

$$\mathfrak{M}_{\mathbb{L}}[\,\{\lambda \cap I = \varnothing\} \cap \{0 \to t\}\,] \leqslant \sum_{u,v} \exp\{-\tau(v-u;\beta^*)\}\langle \sigma(0)\sigma(u)\rangle_{\mathbb{L}} \langle \sigma(v)\sigma(t)\rangle_{\mathbb{L}}\,. \tag{4.100}$$

By GKS inequalities

$$\langle\sigma(0)\sigma(t)\rangle_{\mathbb{L}} \geqslant \langle\sigma(0)\sigma(u)\rangle_{\mathbb{L}}\langle\sigma(u)\sigma(v)\rangle_{\mathbb{L}}\langle\sigma(v)\sigma(t)\rangle_{\mathbb{L}}\,, \tag{4.101}$$

so that

$$\mathfrak{M}_{\mathbb{L}}[\,\{\lambda\cap I=\varnothing\}\,|\,\{0\to t\}\,] \leqslant \frac{\sum_{u,v}\exp\{-\tau(v-u;\beta^*)\}}{\langle\sigma(u)\sigma(v)\rangle_{\mathbb{L}}}\,. \tag{4.102}$$

We know that

$$\lim_{n\to\infty} -\frac{1}{n}\log\langle\sigma(0)\sigma(nt_1)\rangle_{\mathbb{L}} = {\tau_{\mathrm{bd}}}^*\,, \tag{4.103}$$

where

$${\tau_{\mathrm{bd}}}^* = \tau_{\mathrm{bd}}(t_1;\beta^*,h^*)\quad,\quad t_1=(1,0)\,. \tag{4.104}$$

Let $0<2\varepsilon<\tau^*-{\tau_{\mathrm{bd}}}^*$; $\tau^*=\tau(t_1;\beta^*)$. We can find n_ε so that for all $n\geqslant n_\varepsilon$,

$$\lim_{n} -\frac{1}{n}\log\langle\sigma(0)\sigma(nt_1)\rangle_{\mathbb{L}} \leqslant {\tau_{\mathrm{bd}}}^*+\varepsilon\,, \tag{4.105}$$

so that

$$\langle\sigma(0)\sigma(nt_1)\rangle_{\mathbb{L}} \geqslant \exp\{-n({\tau_{\mathrm{bd}}}^*+\varepsilon)\}\,. \tag{4.106}$$

From this inequality and $\tau(u-v;\beta^*)=|u-v|\cdot\tau^*$

$$\mathfrak{M}_{\mathbb{L}}[\{\lambda\cap I=\varnothing\}\,|\,\{0\to t\}] \leqslant \sum_{u,v}\exp\{-\varepsilon|u-v|\}\,. \tag{4.107}$$

Using $u(1)<x_1<x_2<v(1)$ the lemma follows. □

Remark. Using Proposition 4.5.2 point 1., we can improve Lemma 4.4.8. There exists a constant C such that for any interval $I=[x_1,x_2]$ we have

$$\mathfrak{M}_{\mathbb{L}}[\{\lambda\cap I=\varnothing\}|\{0\to t\}] \leqslant C\exp\{-(\tau^*-{\tau_{\mathrm{bd}}}^*)\cdot|x_2-x_1|\}\,. \tag{4.108}$$

4.5 Lower bounds on the 2-point functions

In Section 4.2.3, we give several upper bounds on 2-point functions. We are interested now in lower bounds on these quantities. These estimates play an important role in the next chapters. Unfortunately most of the results exposed in this section are known to hold non-perturbatively only by explicit computation, and no general non-perturbative argument is available[6]. As in the other sections, we consider first the case of the bulk 2-point function and then of the boundary 2-point function.

[6]See however recent results of Alexander about the same kind of problems for Bernoulli percolation [Al1, Al2].

4.5.1 The bulk 2-point function

Let $\beta < \beta_c$ and $J(e) = \beta$ for all edges e. What we need in later chapters is a result of the following kind: For all $0 \neq t \in \mathbb{Z}^2$,

$$\langle \sigma(0)\sigma(t)\rangle^{\beta} \geqslant \frac{C}{\|t\|_2^k} \exp\{-\tau(t;\beta^*)\} \tag{4.109}$$

for some positive constants C and k (independent of t).
Such a bound, with the correct $k = 1/2$, can be obtained[7] for small β (large β^*) using perturbative techniques, see [P, DKS1, MZ]; see also [G1, BLP1] for earlier works on this problem, in particular see [BF] where the connection with the Central Limit Theorem for random lines is made explicit.
The exact result has also been obtained by explicit calculations, see for example [MW], Chapter XI and XII. These are the results we use in Chapters 6 and 7[8]. Precisely, we have

Proposition 4.5.1. *(Ornstein-Zernicke behaviour of the bulk 2-point function) Let $\beta < \beta_c$ and $J(e) = \beta$ for all edges e. Then there exists a constant C_{OZ} such that, for all $t \neq 0$,*

$$\langle \sigma(0)\sigma(t)\rangle^{\beta} \geqslant \frac{C_{\mathrm{OZ}}}{\sqrt{\|t\|_2}} \exp\{-\tau(t;\beta^*)\}\,.$$

To understand heuristically why the prefactor should be of that form, consider the case when t is such that $t(2) = 0$. Then we can write

$$\begin{aligned}\langle \sigma(0)\sigma(t)\rangle &= \chi(t(1),0)\,\mathfrak{M}[\{t'(2)=0\}|\bigcup_{t'\,:\,t'(1)=t(1)}\{0\to t'\}]\\ &\geqslant \exp\{-\tau(t)\}\,\mathfrak{M}[\{t'(2)=0\}|\bigcup_{t'\,:\,t'(1)=t(1)}\{0\to t'\}]\,,\end{aligned} \tag{4.110}$$

where we used Lemma 4.4.1. The last factor in the last term, which can be interpreted as a conditional probability, is of order $\mathcal{O}(|t(1)|^{-1/2})$, by the Central Limit Theorem, as can be proved perturbatively (see [G1, BLP1, BF]); the general case is treated in [DKS1].

Even if we don't know how to treat the general case non-perturbatively (without explicit calculations), for some special t it is possible to prove such lower bounds using the non-perturbative techniques developed in the previous sections. More precisely the two-point functions $\langle\sigma(0)\sigma(t)\rangle$ with t satisfying any one of the four conditions: $t(1) = 0$, $t(2) = 0$, $t(1) = t(2)$, $t(1) = -t(2)$, can be handled quite easily. This is what we show now.

Lemma 4.5.1. *Let $\beta < \beta_c$ and $J(e) = \beta$ for all edges e. Then, there exists a constant C such that, for all $t \neq 0$ such that $t(1) = 0$ or $t(2) = 0$,*

$$\langle \sigma(0)\sigma(t)\rangle^{\beta} \geqslant \frac{1}{C\sqrt{\|t\|_2\,\log\|t\|_2}} \exp\{-\tau(t;\beta^*)\}\,.$$

[7] In fact these works prove more than just a bound: They show that the 2-point function decays exactly at this rate, up to higher order corrections.

[8] However, some results of Chapter 7 can be obtained without using the exact solution, see the comments there.

Proof. We consider the case $t(2) = 0$. Let $\tau^* \doteqdot \tau((1,0);\beta^*)$. We recall that

$$\chi(a,0) = \sum_{t' : t'(1)=a} \langle\sigma(0)\sigma(t')\rangle \,. \tag{4.111}$$

Lemma 4.4.1 implies that $\chi(a,0) \geqslant \exp\{-\tau^* a\}$. Therefore we can write

$$\begin{aligned}\sum_{\substack{t' : t'(1)=t(1)\\ |t'(2)|<C\sqrt{\|t\|_2\log\|t\|_2}}} &\langle\sigma(0)\sigma(t')\rangle \\ &\geqslant \Bigg[1 - \sum_{\substack{t' : t'(1)=t(1)\\ |t'(2)|\geqslant C\sqrt{\|t\|_2\log\|t\|_2}}} \langle\sigma(0)\sigma(t')\rangle \exp\{\tau^* \|t\|_2\}\Bigg] \exp\{-\tau^* \|t\|_2\} \,. \end{aligned}\tag{4.112}$$

But we have

$$\sum_{\substack{t' : t'(1)=t(1)\\ |t'(2)|<C\sqrt{\|t\|_2\log\|t\|_2}}} \langle\sigma(0)\sigma(t')\rangle \leqslant 2C\sqrt{\|t\|_2}\log\|t\|_2\langle\sigma(0)\sigma(t)\rangle \tag{4.113}$$

by monotonicity, and, using Proposition 3.1.1 point 5.,

$$\begin{aligned}\sum_{\substack{t' : t'(1)=t(1)\\ |t'(2)|\geqslant C\sqrt{\|t\|_2\log\|t\|_2}}} \langle\sigma(0)\sigma(t')\rangle \exp\{\tau^* \|t\|_2\} &\leqslant \sum_{\substack{t' : t'(1)=t(1)\\ |t'(2)|\geqslant C\sqrt{\|t\|_2\log\|t\|_2}}} \exp\{-\tau(t';\beta^*) + \tau^* \|t\|_2\} \\ &\leqslant \sum_{\substack{t' : t'(1)=t(1)\\ |t'(2)|\geqslant C\sqrt{\|t\|_2\log\|t\|_2}}} \exp\{-\tau^*(\|t'\|_2 - \|t\|_2)\} \\ &\leqslant \|t\|_2^{-\mathcal{O}(C)} \,. \end{aligned}\tag{4.114}$$

where the function $\mathcal{O}(C)$ is uniform in t. The conclusion follows easily. □

We cannot use the same technique in the diagonal case, since the surface tension is not minimal in that direction (it is in fact maximal). However, using monotonicity of the 2-point function, we can still prove a similar result (slightly weaker).

Lemma 4.5.2. *Let $\beta < \beta_c$ and $J(e) = \beta$ for all edges e. Then, there exists a constant C such that, for all $t \neq 0$ such that $t(1) = t(2)$,*

$$\langle\sigma(0)\sigma(t)\rangle \geqslant \frac{C}{\|t\|_2} \exp\{-\tau(t;\beta^*)\} \,.$$

Proof. Consider the set B

$$B \doteqdot \{t' \in \mathbb{Z}^2 : \|t'\|_1 \leqslant \|t\|_1\} \,, \tag{4.115}$$

We introduce $t_n \doteqdot n \cdot t$ for any integer $n \geqslant 2$. By Lemma 4.2.7, we have

$$\langle\sigma(0)\sigma(t_n)\rangle \leqslant \sum_{t' \in \partial B} \langle\sigma(0)\sigma(t')\rangle\langle\sigma(t')\sigma(t_n)\rangle \,. \tag{4.116}$$

Using the symmetry properties of the two–point function and monotonicity, we have for any $t' \in \partial B$

$$\langle \sigma(0)\sigma(t')\rangle \leqslant \langle \sigma(0)\sigma(t)\rangle \quad \text{and} \quad \langle \sigma(t')\sigma(t_n)\rangle \leqslant \langle \sigma(t)\sigma(t_n)\rangle \,. \tag{4.117}$$

Therefore

$$\langle \sigma(0)\sigma(t_n)\rangle \leqslant 8\|t\|_1 \langle \sigma(0)\sigma(t)\rangle\langle \sigma(t)\sigma(t_n)\rangle \,. \tag{4.118}$$

By iteration we get

$$\langle \sigma(0)\sigma(t_n)\rangle \leqslant [8\|t\|_2]^n [\langle \sigma(0)\sigma(t)\rangle]^n \,. \tag{4.119}$$

The result follows by taking the logarithm, dividing by n and taking the limit $n \to \infty$. □

4.5.2 The boundary 2-point function

Let $\beta < \beta_c$ and suppose $J(e)$ is given by (4.5).
As in the previous section, we want to obtain lower bounds for the boundary 2-point function showing that the corrections to the exponential decay are polynomial. In this case, however, the situation is more subtle, since the behaviour of the boundary 2-point function depends on the value of h. More precisely there is a transition between two behaviours corresponding to the wetting transition of the dual model:

$$h^* \geqslant h_{\mathrm{w}}(\beta^*) \;:\quad \langle \sigma(0)\sigma(t)\rangle \sim \frac{\exp\{-\tau(t;\beta^*)\}}{\|t\|_2^{3/2}}, \tag{4.120}$$

$$h^* < h_{\mathrm{w}}(\beta^*) \;:\quad \langle \sigma(0)\sigma(t)\rangle \sim \exp\{-\tau_{\mathrm{bd}}(t;\beta^*,h^*)\} \,. \tag{4.121}$$

It is possible to prove non-perturbatively, and without using explicit computation, a lower bound with pure exponential decay in situation (4.121). However in the case (4.120), we have to refer to the exact solution [MW, Pa2]. Nevertheless we can prove that it is enough to obtain a lower bound in the case $h = 1$.

Proposition 4.5.2. *(Ornstein-Zernicke behaviour of the boundary 2-point function) Let $\beta < \beta_c$ and $J(e)$ given by (4.5). Then,*

1. *Let $h^* < h_{\mathrm{w}}(\beta^*)$. Then there exists a constant $C = C(\beta, h)$ such that, for all $t \in \Sigma$,*

$$\langle \sigma(0)\sigma(t)\rangle_{\mathbb{L}}^{\beta,h} \geqslant C\, \exp\{-\tau_{\mathrm{bd}}(t;\beta^*,h^*)\} \,.$$

2. *For all $h \geqslant 0$ and all $t \in \Sigma$,*

$$\langle \sigma(0)\sigma(t)\rangle_{\mathbb{L}}^{\beta,h} \geqslant (\tanh\beta)^2\, \langle \sigma(0)\sigma(t)\rangle_{\mathbb{L}}^{\beta,1} \,.$$

3. *There exists a constant C such that, for all $t \in \Sigma$,*

$$\langle \sigma(0)\sigma(t)\rangle_{\mathbb{L}}^{\beta,1} \geqslant \frac{C}{\|t\|_2^{3/2}} \langle \exp\{-\tau(t;\beta^*)\}\rangle \,.$$

Proof. 1. This is a consequence of Lemma 4.4.8. The proof is similar to that of Lemma 4.5.2. Let $t = (t(1), 0)$ with $t(1) > 0$. We set $t_k \doteqdot kt$ for $k = 1, 2, \ldots, n$. Let $a \in \mathbb{N}$, $t(1) > a$, and I be the interval

$$I \doteqdot \{t \in \mathbb{L} \,:\, t_1(1) - a \leqslant t(1) \leqslant t_1(1)\,,\, t(2) = 0\} \,. \tag{4.122}$$

We have

$$\begin{aligned}\langle\sigma(0)\sigma(t_n)\rangle_{\mathbb{L}} &= \mathfrak{M}_{\mathbb{L}}[\{0\to t_n\}] \\ &= \mathfrak{M}_{\mathbb{L}}[E_I^c|\{0\to t_n\}]\,\mathfrak{M}_{\mathbb{L}}[\{0\to t_n\}] + \mathfrak{M}_{\mathbb{L}}[E_I\cap\{0\to t_n\}]\,,\end{aligned} \tag{4.123}$$

where E_I is the event $\{\lambda\cap I\neq\varnothing\}$ and E_I^c the complementary event. We choose a so that

$$\mathfrak{M}_{\mathbb{L}}[E_I^c|\{0\to t_n\}]\leqslant 1/2\,, \tag{4.124}$$

which is possible according to Lemma 4.4.8 if $t(1)$ is large enough. Thus we have

$$\begin{aligned}\langle\sigma(0)\sigma(t_n)\rangle_{\mathbb{L}} &\leqslant 2\,\mathfrak{M}_{\mathbb{L}}[E_I\cap\{0\to t_n\}] \\ &\leqslant 2\sum_{u\in I}\sum_{\substack{\lambda:\partial\lambda=\{0,t_n\}\\ u\in\lambda}} q_{\mathbb{L}}(\lambda) \\ &\leqslant 2a\,\langle\sigma(0)\sigma(t_1-(a,0))\rangle_{\mathbb{L}}\,\langle\sigma(t_1)\sigma(t_n)\rangle_{\mathbb{L}}\,.\end{aligned} \tag{4.125}$$

We have used Lemma 4.2.5 and the monotonicity property of the boundary two-point function. By GKS inequalities and translation–invariance

$$\begin{aligned}\frac{\langle\sigma(0)\sigma(t_1-(a,0))\rangle_{\mathbb{L}}}{\langle\sigma(0)\sigma(t_1)\rangle_{\mathbb{L}}} &\leqslant \frac{\langle\sigma(0)\sigma(t_1-(a,0))\rangle_{\mathbb{L}}}{\langle\sigma(0)\sigma(t_1-(a,0))\rangle_{\mathbb{L}}\langle\sigma(t_1-(a,0))\sigma(t_1)\rangle_{\mathbb{L}}} \\ &= \frac{1}{\langle\sigma(0)\sigma((a,0))\rangle_{\mathbb{L}}}\,.\end{aligned} \tag{4.126}$$

If we set

$$C_*\doteqdot\frac{\langle\sigma(0)\sigma((a,0))\rangle_{\mathbb{L}}}{2a}\,, \tag{4.127}$$

then

$$\langle\sigma(0)\sigma(t_n)\rangle_{\mathbb{L}}\leqslant C_*^{-1}\langle\sigma(0)\sigma(t_1)\rangle_{\mathbb{L}}\langle\sigma(t_1)\sigma(t_n)\rangle_{\mathbb{L}}\,. \tag{4.128}$$

We can iterate this result,

$$\langle\sigma(0)\sigma(t_n)\rangle_{\mathbb{L}}\leqslant C_*^{-n}\Big(\langle\sigma(0)\sigma(t_1)\rangle_{\mathbb{L}}\Big)^n\,. \tag{4.129}$$

Therefore, if $t(1)$ is large enough, then

$$-\tau_{\mathrm{bd}}(t;\beta^*,h^*)=\lim_{n\to\infty}\frac{1}{n}\log\langle\sigma(0)\sigma(t_n)\rangle_{\mathbb{L}}\leqslant\,-\log C_*+\log\langle\sigma(0)\sigma(t)\rangle_{\mathbb{L}}\,. \tag{4.130}$$

2. By GKS inequalities, the 2-point function decreases if we set $J(e)=0$ for all edges e which are adjacent to a site of Σ, except the two edges $\langle(0,0),(0,1)\rangle$ and $\langle t,t+(0,1)\rangle$. Summing explicitly over $\sigma(0)$ and $\sigma(t)$ yields two factors $\tanh\beta$.
3. Follows from the exact solution, see [MW, Pa2].

□

As before it is possible to understand the prefactor in the third statement of Proposition 4.5.2. By Lemma 4.4.5

$$\langle\sigma(0)\sigma(t)\rangle_{\mathbb{L}}^{\beta,1}\geqslant\mathfrak{M}[\{\lambda\subset\mathbb{L}\}|\{0\to t\}]\,\langle\sigma(0)\sigma(t)\rangle^{\beta}\,. \tag{4.131}$$

The term $\mathfrak{M}[\{\lambda\subset\mathbb{L}\}|\{0\to t\}]$, which can be interpreted as a conditional probability, should be of order $\mathcal{O}(|t(1)|^{-1})$, by analogy with simple random walks. Together with the factor $\mathcal{O}(|t(1)|^{-1/2})$ of $\langle\sigma(0)\sigma(t)\rangle$ we get the $\mathcal{O}(|t(1)|^{-3/2})$ factor.

4.6 Concentration properties

Lemmas 4.4.4 and 4.4.7 provide some characterization on the typical contours contributing to the bulk and boundary 2-point functions. These results, however, are far from optimal. The aim of this section is to improve them. To achieve this, it is necessary to use some information coming from the exact solution, namely the Sharp Triangle Inequality, see Proposition 3.1.1, and lower bounds on the 2-point functions, see Section 4.5. At this cost, it is possible to obtain essentially optimal results. Our philosophy is to avoid any computation at the level of typical fluctuations of the contours[9]. The main reason is that a study of the fluctuations is very difficult to do non-perturbatively (in fact, it already requires a lot of work at a perturbative level, see for example [Hi2, DKS1, DH2]), and can be obtained through exact computation only in some special situations. Our techniques however are much more versatile. Usually, in the study of large deviations, the strict convexity of the rate–function is sufficient to obtain concentration results. We show in this section that the Sharp Triangle Inequality satisfied by the surface tension plays the same role.

For applications in the next chapters, we state the results in a more general way than was done in Section 4.4. Namely we need results for finite volume 2-point functions. Nevertheless we also give the corresponding results for the infinite volume case, since the results become much simpler and possibly more illuminating.

The analogue of the square box of Sections 4.4.1 and 4.4.2 is the following elliptical set. Let $x,\, y \in \mathbb{Z}^2$, and $\rho > 0$. We set

$$\mathcal{S}(x,y,\rho) \doteqdot \{t \in \mathbb{Z}^2 \,:\, \|t-x\|_2 + \|t-y\|_2 \leqslant \|y-x\|_2 + \rho\}\,, \tag{4.132}$$

We also need a different box which is useful when there is partial wetting. Let $x \in \Sigma$, $y \in \Sigma$, with $x(1) < y(1)$, and $\rho > 0$; we set

$$\mathcal{S}'(x,y,\rho) \doteqdot \{t \in \mathbb{Z}^2 \,:\, x(1)-\rho \leqslant t(1) \leqslant y(1)+\rho\,,\, 0 \leqslant t(2) \leqslant \rho\}\,. \tag{4.133}$$

We can now state the finite volume results.

Proposition 4.6.1. *Let $\beta < \beta_c$, $h \geqslant 0$ and $J(e)$ given by* (4.5)*. Let $\Lambda \subset \mathbb{L}$, $x,\, y \in \Lambda$, $x \neq y$, and $\rho > 0$. Let $\mathcal{S}_1 \doteqdot \mathcal{S}(x,y,\rho)$. Then*

1. *Suppose $h^* \geqslant h_{\mathrm{w}}(\beta^*)$. Then*

$$\sum_{\substack{\lambda\,:\,\partial\lambda=\{x,y\}\\ \lambda\not\subset\mathcal{S}_1}} q_\Lambda(\lambda;\beta,h) \leqslant \mathcal{O}(|\partial\,\mathcal{S}_1|^5)\exp\{-\kappa\rho\}\exp\{-\tau(y-x;\beta^*)\}\,.$$

2.

$$\sum_{\substack{\lambda\,:\,\partial\lambda=\{x,y\}\\ \lambda\not\subset\mathcal{S}_1\\ \lambda\cap\Sigma=\varnothing}} q_\Lambda(\lambda;\beta,h) \leqslant |\partial\,\mathcal{S}_1|\exp\{-\kappa\rho\}\exp\{-\tau(y-x;\beta^*)\}\,.$$

[9] Notice however that the two exact results we use implicitly contain information on fluctuations.

3. Suppose $x \notin \Sigma$ and $y \notin \Sigma$. Then

$$\sum_{\substack{\lambda : \partial\lambda = \{x,y\} \\ \lambda \not\subset \mathcal{S}_1}} q_\Lambda(\lambda;\beta,h) \leqslant |\partial \mathcal{S}_1| \exp\{-\kappa\rho\} \exp\{-\tau(y-x;\beta^*)\} \\ + \sum_{z_1,z_2 \in \Sigma\cap\Lambda} \exp\{-(\tau(z_1-x;\beta^*) + \tau_{\mathrm{bd}}(z_2-z_1;\beta^*,h^*) + \tau(y-z_2;\beta^*))\}\,.$$

4. Suppose $x \notin \Sigma$ and $y \in \Sigma$. Then

$$\sum_{\substack{\lambda : \partial\lambda = \{x,y\} \\ \lambda \not\subset \mathcal{S}_1}} q_\Lambda(\lambda;\beta,h) \leqslant |\partial \mathcal{S}_1| \exp\{-\kappa\rho\} \exp\{-\tau(y-x;\beta^*)\} \\ + \sum_{z \in \Sigma\cap\Lambda} \exp\{-(\tau(z-x;\beta^*) + \tau_{\mathrm{bd}}(y-z;\beta^*,h^*))\}\,.$$

5. Suppose $h^* < h_{\mathrm{w}}(\beta^*)$, $x \in \Sigma$ and $y \in \Sigma$. Let $\mathcal{S}_2 \doteqdot \mathcal{S}'(x,y,\rho)$. Then there exists a constant $\overline{C}(\beta) > 0$ such that

$$\sum_{\substack{\lambda : \partial\lambda = \{x,y\} \\ \lambda \not\subset \mathcal{S}_2}} q_\Lambda(\lambda;\beta,h) \leqslant \\ |\partial \mathcal{S}_2| \big[\exp\{-2\rho\tau_{\mathrm{bd}}(\beta^*,h^*)\} + |\rho| \exp\{-\overline{C}\rho\}\big] \exp\{-\tau_{\mathrm{bd}}(y-x;\beta^*,h^*)\}\,.$$

$\kappa(\beta^*)$ is the constant appearing in the definition of the Sharp Triangle inequality.

Proof. 1. Let $\lambda \subset \Lambda$. Let $s \mapsto \lambda(s)$ be a parameterization of the open contour λ from x to y. We introduce a set of points as in figure 4.11. Let s' the first time such that $\lambda(s') \notin \mathcal{S}_1$, $s_1 \leqslant s'$ the first time (if any) such that $\lambda(s_1) \in \Sigma$, $s_1 < s_2 \leqslant s'$ the last time before s' that $\lambda(s_2) \in \Sigma$, $s_3 > s'$ the first time after s' that $\lambda(s_3) \in \Sigma$ and $s_4 > s_3$ the last time such that $\lambda(s_4) \in \Sigma$ and (it is possible that some of these points don't exist, coincide or coincide with x or y).

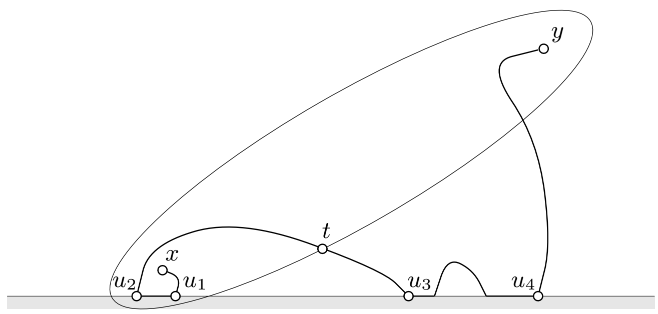

FIGURE 4.11. The construction in the proof of Proposition 4.6.1, point 1.; $t = \lambda(s')$ and $u_i = \lambda(s_i)$, $i = 1, \ldots, 4$.

By Lemma 4.2.8, the fact that $\tau_{\mathrm{bd}}(\beta^*, h^*) = \tau((1,0);\beta^*)$, the Sharp Triangle Inequality and the definition of $\mathcal{S}_1$, we can write

$$\begin{aligned}
&\sum_{t\in\partial\mathcal{S}_1}\ \sum_{u_i\in\Sigma,\, i=1,\dots,4}\ \sum_{\substack{\lambda\,:\,\partial\lambda=\{x,y\}\\ \lambda(s')=t,\,\lambda(s_i)=u_i, i=1,\dots,4}} q_\Lambda(\lambda;\beta,h)\\
&\quad\leqslant \sum_{t\in\partial\mathcal{S}_1}\ \sum_{u_i\in\Sigma,\, i=1,\dots,4} \exp\{-(\tau(u_1-x;\beta^*)+\tau_{\mathrm{bd}}(u_2-u_1;\beta^*,h^*)+\tau(t-u_2;\beta^*)+\\
&\qquad\qquad \tau(u_3-t;\beta^*)+\tau_{\mathrm{bd}}(u_4-u_3;\beta^*,h^*)+\tau(y-u_4;\beta^*))\}\\
&\quad= \sum_{t\in\partial\mathcal{S}_1}\ \sum_{u_i\in\Sigma,\, i=1,\dots,4} \exp\{-(\tau(u_1-x;\beta^*)+\tau(u_2-u_1;\beta^*)+\tau(t-u_2;\beta^*)+\\
&\qquad\qquad \tau(u_3-t;\beta^*)+\tau(u_4-u_3;\beta^*)+\tau(y-u_4;\beta^*))\}\\
&\quad\leqslant \mathcal{O}(|\partial\mathcal{S}_1|^4)\sum_{t\in\partial\mathcal{S}_1}\exp\{-(\tau(t-x;\beta^*)+\tau(y-t;\beta^*))\}\\
&\quad\leqslant \mathcal{O}(|\partial\mathcal{S}_1|^4)\sum_{t\in\partial\mathcal{S}_1}\exp\{-(\tau(y-x;\beta^*)+\kappa(\beta^*)\rho)\}\\
&\quad\leqslant \mathcal{O}(|\partial\mathcal{S}_1|^5)\exp\{-\kappa(\beta^*)\rho\}\exp\{-\tau(y-x;\beta^*)\}\,. \qquad (4.134)
\end{aligned}$$

2. By Lemma 4.2.8,

$$\sum_{\substack{\lambda\,:\,\partial\lambda=\{x,y\}\\ \lambda\not\subset\mathcal{S}_1,\,\lambda\cap\Sigma=\varnothing}} q_\Lambda(\lambda;\beta,h)\leqslant\sum_{t\in\partial\mathcal{S}_1}\exp\{-(\tau(t-x;\beta^*)+\tau(y-t;\beta^*))\}\,, \qquad (4.135)$$

and we conclude as in the first point.

3. Let $\lambda\subset\Lambda$. Let $s\mapsto\lambda(s)$ be a parameterization of the open contour λ from x to y. Let s_1 be the first time (if any) such that $\lambda(s_1)\in\Sigma$ and s_2 the last time such that $\lambda(s_2)\in\Sigma$.

$$\begin{aligned}
\sum_{\substack{\lambda:\partial\lambda=\{x,y\}\\ \lambda\not\subset\mathcal{S}_1}} q_\Lambda(\lambda;\beta,h)&\leqslant\sum_{t\in\partial\mathcal{S}_1}\ \sum_{\substack{\lambda\,:\,\partial\lambda=\{x,y\},\,\lambda\ni t\\ \lambda\cap\Sigma=\varnothing}} q_\Lambda(\lambda;\beta,h)\\
&\quad+\sum_{z_1,z_2\in\Sigma\cap\Lambda}\ \sum_{\substack{\lambda\,:\,\partial\lambda=\{x,y\}\\ \lambda(s_1)=z_1,\,\lambda(s_2)=z_2}} q_\Lambda(\lambda;\beta,h)\,. \qquad (4.136)
\end{aligned}$$

Using Lemma 4.2.8 we get

$$\sum_{t\in\partial\mathcal{S}_1}\ \sum_{\substack{\lambda\,:\,\partial\lambda=\{x,y\},\lambda\ni t\\ \lambda\cap\Sigma=\varnothing}} q_\Lambda(\lambda;\beta,h)\leqslant\sum_{t\in\partial\mathcal{S}_1}\exp\{-(\tau(t-x;\beta^*)+\tau(y-t;\beta^*))\}\,, \quad (4.137)$$

which is estimated as in point 1., and

$$\begin{aligned}
\sum_{z_1,z_2\in\Sigma\cap\Lambda}\ \sum_{\substack{\lambda\,:\,\partial\lambda=\{x,y\}\\ \lambda(s_1)=z_1,\lambda(s_2)=z_2}} q_\Lambda(\lambda;\beta,h)&\leqslant\sum_{z_1,z_2\in\Sigma\cap\Lambda}\exp\{-\tau(z_1-x;\beta^*)\}\\
&\quad\cdot\exp\{-\tau_{\mathrm{bd}}(z_2-z_1;\beta^*,h^*)\}\exp\{-\tau(y-z_2;\beta^*)\}\,. \quad (4.138)
\end{aligned}$$

4. This is proved similarly as the preceding point.
5. Let us write $\tau^* \doteqdot \tau((1,0);\beta^*)$ and $\tau_{\mathrm{bd}}{}^* \doteqdot \tau_{\mathrm{bd}}(\beta^*,h^*)$. We use the fact that $\tau_{\mathrm{bd}}{}^* < \tau^*$. Let $\partial^+\mathcal{S}_2 \doteqdot \{t \in \mathcal{S}_2 \,:\, t(2)=\rho$ or $t(1)=[x(1)-\rho]$ or $t(1)=[y(1)+\rho]\}$; we can write

$$\sum_{\substack{\lambda:x\to y\\ \lambda\not\subset\mathcal{S}_2}} q_\Lambda(\lambda;\beta,h) \leqslant \sum_{t\in\partial^+\mathcal{S}_2}\sum_{\substack{\lambda:x\to y\\ \lambda\ni t}} q_\Lambda(\lambda;\beta,h)$$
$$\leqslant \sum_{\substack{t\in\partial^+\mathcal{S}_2\\ t(2)<\rho_2}}\sum_{\substack{\lambda:x\to y\\ \lambda\ni t}} q_\Lambda(\lambda;\beta,h) + \sum_{\substack{t\in\partial^+\mathcal{S}_2\\ t(2)=\rho_2}}\sum_{\substack{\lambda:x\to y\\ \lambda\ni t}} q_\Lambda(\lambda;\beta,h)\,. \tag{4.139}$$

We treat these sums separately. By symmetry and GKS inequalities

$$\begin{aligned}\sum_{\substack{t\in\partial^+\mathcal{S}_2\\ t(2)<\rho}}\sum_{\substack{\lambda:x\to y\\ \lambda\ni t}} q_\Lambda(\lambda;\beta,h) &\leqslant 2\sum_{\substack{t\in\partial^+\mathcal{S}_2\\ t(1)=x(1)-\rho}} \langle\sigma(x)\sigma(t)\rangle_{\mathbb{L}}\langle\sigma(t)\sigma(y)\rangle_{\mathbb{L}}\\
&= 2\sum_{\substack{t\in\partial^+\mathcal{S}_2\\ t(1)=x(1)-\rho}} \langle\sigma(\overline{x})\sigma(t)\rangle_{\mathbb{L}}\langle\sigma(t)\sigma(y)\rangle_{\mathbb{L}}\\
&\leqslant 2\sum_{\substack{t\in\partial^+\mathcal{S}_2\\ t(1)=x(1)-\rho}} \langle\sigma(\overline{x})\sigma(y)\rangle_{\mathbb{L}}\\
&\leqslant 2\sum_{\substack{t\in\partial^+\mathcal{S}_2\\ t(1)=x(1)-\rho}} \exp\{-2\rho\tau_{\mathrm{bd}}{}^*\}\exp\{-\tau_{\mathrm{bd}}{}^*\|y-x\|_2\}\,,\end{aligned} \tag{4.140}$$

where $\overline{x}$ is the image of x under a reflection of axis $\{u \,:\, u(1)=t(1)\}$.
Let $\lambda \subset \Lambda$, $t \in \lambda$ and $t(2)=\rho$. Let $s \mapsto \lambda(s)$ be a parameterization of the open contour λ from x to y. We set $t=\lambda(s^*)$; we denote by s_1 the last time before s^* such that $\lambda(s_1)\in\Sigma$; we denote by s_2 the first time after s^* such that $\lambda(s_2)\in\Sigma$. We get ($u=\lambda(s_1)$, $v=\lambda(s_2)$)

$$\begin{aligned}\sum_{\substack{t\in\partial^+\mathcal{S}_2\\ t(2)=\rho}}\sum_{\substack{\lambda:x\to y\\ \lambda\ni t}} q_\Lambda(\lambda;\beta,h) &\leqslant \sum_{\substack{t\in\partial^+\mathcal{S}_2\\ t(2)=\rho}}\sum_{u,v}\langle\sigma(x)\sigma(u)\rangle_{\mathbb{L}}\langle\sigma(u)\sigma(t)\rangle\langle\sigma(t)\sigma(v)\rangle\langle\sigma(v)\sigma(y)\rangle_{\mathbb{L}}\\
&\leqslant \sum_{\substack{t\in\partial^+\mathcal{S}_2\\ t(2)=\rho}}\sum_{u,v}\exp\{-(\tau_{\mathrm{bd}}(u-x;\beta^*,h^*)+\tau_{\mathrm{bd}}(y-v;\beta^*,h^*))\}\\
&\quad\cdot\exp\{-\tau(t-u;\beta^*)-\tau(v-t;\beta^*)\}\\
&\leqslant \sum_{\substack{t\in\partial^+\mathcal{S}_2\\ t(2)=\rho}}\sum_{u,v}\exp\{-(\tau_{\mathrm{bd}}(u-x;\beta^*,h^*)+\tau_{\mathrm{bd}}(y-v;\beta^*,h^*))\}\\
&\quad\cdot\exp\{-\tau(u-v;\beta^*)\}\exp\{-\kappa(\|u-t\|_2+\|t-v\|_2-\|u-v\|_2)\}\\
&\leqslant \sum_{\substack{t\in\partial^+\mathcal{S}_2\\ t(2)=\rho}}\sum_{u,v}\exp\{-\tau_{\mathrm{bd}}(x-y;\beta^*,h^*)-(\tau^*-\tau_{\mathrm{bd}}{}^*)\|u-v\|_2\}\\
&\quad\cdot\exp\{-\kappa(\|u-t\|_2+\|t-v\|_2-\|u-v\|_2)\}\,,\end{aligned} \tag{4.141}$$

The conclusion follows from the observation that the summation is over the base of the triangle uvt, and that the term $\exp\{-(\tau^* - \tau_{\mathrm{bd}}{}^*)\|u-v\|_2\}$ allows to control the triangles with a large base, while the term $\exp\{-\kappa(\|u-t\|_2 + \|t-v\|_2 - \|u-v\|_2)\}$ can be used to control the terms in which the base is far from the point t.

□

Proposition 4.6.1 has a simpler counterpart when $\Lambda = \mathbb{Z}^2$ or $\Lambda = \mathbb{L}$ which states concentration properties of the measures $\mathfrak{M}[\,\cdot\,|\{0 \to t\}]$ and $\mathfrak{M}_{\mathbb{L}}[\,\cdot\,|\{0 \to t\}]$.

Proposition 4.6.2. *Let $0 \neq t \in \mathbb{Z}^2$ and $K > 0$. Let $\mathcal{S}(K) \doteqdot \mathcal{S}(0, t, K\log\|t\|_2)$ and $\mathcal{S}'(K) \doteqdot \mathcal{S}'(0, t, K\log\|t\|_2)$. Then*

1. *There exists a constant $C_1 > 0$, independent of t, such that*

$$\mathfrak{M}[\{\lambda \not\subset \mathcal{S}(K)\}|\{0 \to t\}] \leqslant C_1\|t\|_2^{-K\kappa(\beta^*)+\frac{3}{2}}\,,$$

2. *Suppose $t \in \Sigma$ and $h^* \geqslant h_{\mathrm{w}}(\beta^*)$. Then there exists a constant $C_2 > 0$, independent of t, such that*

$$\mathfrak{M}_{\mathbb{L}}[\{\lambda \not\subset \mathcal{S}(K)\}|\{0 \to t\}] \leqslant C_2\|t\|_2^{-K\kappa(\beta^*)+\frac{9}{2}}\,,$$

3. *Suppose $t \in \Sigma$ and $h^* < h_{\mathrm{w}}(\beta^*)$. Then*

$$\mathfrak{M}_{\mathbb{L}}[\{\lambda \not\subset \mathcal{S}'(K)\}|\{0 \to t\}] \leqslant \|t\|_2^{\mathcal{O}(K)}\,,$$

where $\kappa(\beta^*)$ is the constant appearing in the definition of the Sharp Triangle inequality.

Proof. The proof is similar (and slightly simpler) to the proof of Proposition 4.6.1. The only additional work is to use the lower bounds on the 2-point functions to replace the quantities involving surface tension by 2-point functions. For example in point 1., use

$$\exp\{-\tau(t;\beta^*)\} \leqslant \mathcal{O}(\sqrt{\|t\|_2})\langle\sigma(0)\sigma(t)\rangle^{\beta}\,. \tag{4.142}$$

□

Proposition 4.6.2 gives an very accurate description of the set of typical contours contributing to the 2-point function. Indeed, it is known that the scale of the typical fluctuations of the contours of point 1. or 2. are $\mathcal{O}(\sqrt{\|t\|_2})$ lattice sites. On the other hand Lemma 4.4.8 showed that the contours of point 3. are pinned to the wall and that the excursions away from the wall behave as a one-dimensional gas of excitations with exponentially decaying weights; a logarithmic bound is therefore natural. Of course to prove optimality would require the analysis of the fluctuations of these contours, which we cannot do with these techniques.

Chapter 5

The phase of small contours

In this chapter, we consider the low temperature representation of the Ising model. We want to prove some estimates which play a very important role in Chapter 7. One of the main ideas in that chapter, which was introduced in [DKS1] is to introduce a distinction among the contours, treating separately the small and large ones. In Section 5.1 we obtain some results about the low temperature representation of the Ising model constrained to have only small contours. Section 5.2 is dedicated to the study of the large deviations of the magnetization under the constraint on the size of the contours[1]; this is a fundamental topic for Chapter 7.
Let us first give precise definitions. Let $t \in \mathbb{Z}^2$ and $\delta > 0$; we set

$$\mathcal{D}(t,\delta) \doteqdot \{t' \in \mathbb{Z}^2 \,:\, \|t'-t\|_\infty \leqslant \delta/2\}\,. \tag{5.1}$$

We can now define small and large contours.

Definition.
(D109) *A contour γ is s-**small** if there exists $t \in \mathbb{Z}^2$ such that $\overline{\mathrm{int}}\gamma \subset \mathcal{D}(t,s)$.*
(D110) *A contour γ is s-**large** if it is not s-small.*

We sometimes write small instead of s-small and large instead of s-large, when there is no ambiguity. Let $\Lambda \subset \mathbb{Z}^2$. We are interested in the probability of events computed with the conditioned measure

$$P_\Lambda^{+,s}[\,\cdot\,] \doteqdot P_\Lambda^{+}[\,\cdot\,|\{\text{All contours are } s\text{-small}\}]\,. \tag{5.2}$$

Expectation value is denoted $\langle\,\cdot\,\rangle_\Lambda^{+,s}$ or $P_\Lambda^{+,s}[\,\cdot\,]$. This defines the *phase of small contours.* We denote the characteristic function of the event that all contours in Λ are small by I_Λ^s,

$$I_\Lambda^s(\omega) \doteqdot \begin{cases} 1 & \text{if } \omega \text{ satisfies the } \Lambda^+\text{-boundary condition and each contour of } \omega \text{ is } s\text{-small,} \\ 0 & \text{otherwise.} \end{cases} \tag{5.3}$$

Lemma 5.0.1. *The function $\omega \mapsto I_\Lambda^s(\omega)$ is increasing. Moreover, if Λ_1 and Λ_2 are two disjoint components of Λ, then*

$$I_\Lambda^s(\omega) = I_{\Lambda_1}^s(\omega)\, I_{\Lambda_2}^s(\omega)\,.$$

[1]The results of this chapter are part of [PV3].

Proof. The first statement follows from the two following elementary observations:

- If the external contours of any configuration are s-small then all contours of that configuration are s-small.
- If ω is such that $I^s_\Lambda(\omega) = 1$ then changing the value of any spin from -1 to $+1$ cannot make the external contours larger.

The second statement is obvious. □

5.1 Some basic estimates

The main property of the phase of small contours is that events which happen "far from one another" satisfy a strong decoupling property. Indeed since the contours are s-small, as soon as the distance between the support of these events becomes large compared to s there is "screening" by a chain of + spins. The following lemma makes this idea more precise.

Definition.

(D111) *Let $A \subset \mathbb{Z}^2$. The s-**neighbourhood of** A is the set*

$$\mathcal{N}^s(A) \doteqdot \bigcup_{t\,:\,\mathcal{D}(t,s)\cap A\neq\varnothing} \mathcal{D}(t,s)\,.$$

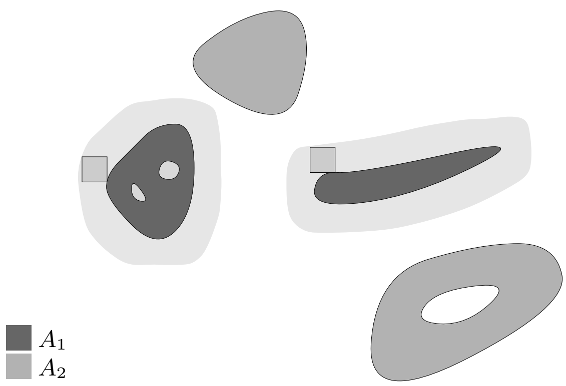

FIGURE 5.1. Two sets A_1 and A_2 as in Lemma 5.1.1. The shaded region indicates the s-neighborhood of A_1 and the two squares are translates of $\mathcal{D}(0,s)$.

Lemma 5.1.1. *Let $J(e) \geqslant 0$ for all edges. Let $\Lambda \subset \mathbb{Z}^2$, $A_1 \subset \Lambda$ and $A_2 \subset \Lambda$ such that*

$$\mathcal{N}^s(A_1) \cap A_2 = \varnothing\,.$$

Let f be a A_1-local function, and g be a A_2-local function. Then, for any $s \in \mathbb{N}$,

1. $|\langle f\, g\rangle^{+,s}_\Lambda - \langle f\rangle^{+,s}_\Lambda \langle g\rangle^{+,s}_\Lambda| \leqslant \max_{A_1\subset\Lambda'\subset\mathcal{N}^s(A_1)} |\langle f\rangle^{+,s}_\Lambda - \langle f\rangle^{+,s}_{\Lambda'}|\, \langle |g|\rangle^{+,s}_\Lambda\,.$
2. *If, furthermore, f is increasing and g is positive, then*

$$\langle f\, g\rangle^{+,s}_\Lambda \geqslant \langle f\rangle^{+}_{\mathcal{N}^s(A_1)} \langle g\rangle^{+,s}_\Lambda\,.$$

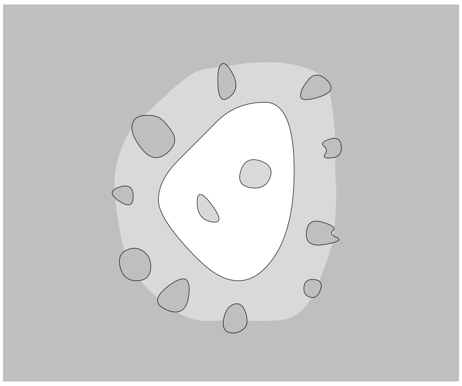

FIGURE 5.2. The white set is A_1; the light–shaded area represents the s-neighborhood of A_1. The dark–shaded region is the random set $\Lambda(\omega)$.

3. If, furthermore, f is decreasing and g is positive, then

$$\langle f\, g\rangle_\Lambda^{+,s} \leqslant \langle f\rangle^+_{\mathcal{N}^s(A_1)}\, \langle g\rangle_\Lambda^{+,s}\,.$$

4. If, furthermore, f and g are both increasing, then

$$\langle f\, g\rangle_\Lambda^{+,s} \geqslant \langle f\rangle^+_{\mathcal{N}^s(A_1)}\, \langle g\rangle^+_{\Lambda\setminus A_1}\,.$$

Proof. 1. Let ω such that $I^s_\Lambda(\omega) = 1$ (the other configurations do not contribute to $\langle f\, g\rangle_\Lambda^{+,s}$). Let γ be an external contour of ω. Clearly, by the definition of s-neighborhood,

$$\mathcal{N}^s(A_1) \not\subset \mathrm{int}\gamma\,. \tag{5.4}$$

The basic observation is that, if

$$\mathrm{int}\gamma \cap (\Lambda \setminus \mathcal{N}^s(A_1)) \neq \varnothing\,, \tag{5.5}$$

then

$$\overline{\mathrm{int}}\gamma \cap A_1 = \varnothing\,. \tag{5.6}$$

Let $\gamma_1(\omega), \ldots, \gamma_2(\omega)$ be all external contours of ω such that

$$\mathrm{int}\gamma_i(\omega) \cap (\Lambda \setminus \mathcal{N}^s(A_1)) \neq \varnothing\,, \quad i = 0, \ldots, n\,; \tag{5.7}$$

we define the random set

$$\Lambda(\omega) \doteqdot (\Lambda \setminus \mathcal{N}^s(A_1)) \cup \bigcup_{i=1,\ldots,n} \overline{\mathrm{int}}\gamma_i \tag{5.8}$$

By Lemma 5.0.1 and the Markov property, we can write

$$\begin{aligned}\langle f\, g\rangle_\Lambda^{+,s} &= \sum_{\Lambda''\subset\Lambda} \langle f\, g|\{\Lambda(\cdot) = \Lambda''\}\rangle_\Lambda^{+,s}\, P_\Lambda^{+,s}[\{\Lambda(\cdot) = \Lambda''\}] \\ &= \sum_{\Lambda''\subset\Lambda} \langle f\rangle^{+,s}_{\Lambda\setminus\Lambda''}\, \langle g|\{\Lambda(\cdot) = \Lambda''\}\rangle_\Lambda^{+,s}\, P_\Lambda^{+,s}[\{\Lambda(\cdot) = \Lambda''\}]\,.\end{aligned} \tag{5.9}$$

If $P_\Lambda^{+,s}[\{\Lambda(\cdot) = \Lambda''\}] \neq 0$ then $A_1 \subset \Lambda \setminus \Lambda'' \subset \mathcal{N}^s(A_1)$. Hence the conclusion follows from

$$\langle f\, g\rangle_\Lambda^{+,s} - \langle f\rangle_\Lambda^{+,s} \langle g\rangle_\Lambda^{+,s} = \sum_{\Lambda''\subset\Lambda} \big(\langle f\rangle_{\Lambda\setminus\Lambda''}^{+,s} - \langle f\rangle_\Lambda^{+,s}\big) \langle g|\{\Lambda(\cdot) = \Lambda''\}\rangle_\Lambda^{+,s} P_\Lambda^{+,s}[\{\Lambda(\cdot) = \Lambda''\}]\,. \quad (5.10)$$

2. We begin as in point 1.. Therefore we have

$$\langle f\, g|\{\Lambda(\cdot) = \Lambda''\}\rangle_\Lambda^{+,s} = \langle f\rangle_{\Lambda\setminus\Lambda''}^{+,s} \langle g|\{\Lambda(\cdot) = \Lambda''\}\rangle_\Lambda^{+,s}\,. \quad (5.11)$$

   Since f is increasing,

$$\langle f\rangle_{\Lambda\setminus\Lambda''}^{+,s} = \frac{\langle f I_{\Lambda\setminus\Lambda''}^s\rangle_{\Lambda\setminus\Lambda''}^+}{\langle I_{\Lambda\setminus\Lambda''}^s\rangle_{\Lambda\setminus\Lambda''}^+} \geqslant \langle f\rangle_{\Lambda\setminus\Lambda''}^+ \geqslant \langle f\rangle_{\mathcal{N}^s(A_1)}^+\,. \quad (5.12)$$

   The conclusion follows easily from (5.11) and (5.12).

3. The proof is done as in point 2..
4. It is similar to the proof of point 2.. Just use

$$\langle g|\{\Lambda(\cdot) = \Lambda''\}\rangle_\Lambda^{+,s} = \langle g\rangle_{\Lambda''}^{+,s} \geqslant \langle g\rangle_{\Lambda''}^+ \geqslant \langle g\rangle_{\Lambda\setminus A_1}^+\,. \quad (5.13)$$

□

Remark. In the proof of last lemma, the only properties of the measure which were used were the Markov property and FKG inequalities; therefore, the conclusion remains true if we replace the measure μ_Λ^+ by another measure sharing the same properties, as the measure of the Ising model in magnetic field, for example.

We are mainly interested in the expectation value of the magnetization in the constrained phase. It is particularly important to know how it compares to the expectation value in the unconstrained phase. This is the content of the next lemma.

Lemma 5.1.2. *Let $J(e)$ be defined by (3.30), with $\beta > \beta_c$ and $h \geqslant 0$. Let $s \in \mathbb{N}$ and $t \in \mathbb{L}$ such that $t(2) > \frac{3}{2}s$. Let $\Lambda \subset \mathbb{L}$ such that $\mathcal{N}^s(\{t\}) \subset \Lambda$. Then there exists a constant $\eta = \eta(\beta)$ such that*

$$\langle \sigma(t)\rangle^{+,\beta} \leqslant \langle \sigma(t)\rangle_\Lambda^{+,s,J} \leqslant \langle \sigma(t)\rangle^{+,\beta} + \mathcal{O}(s^4)\exp\{-\eta\, s\}\,.$$

Proof. The first inequality follows from Lemma 5.1.1, point 2., with $g \equiv 1$, and FKG inequalities

$$\langle \sigma(t)\rangle_\Lambda^{+,s,J} \geqslant \langle \sigma(t)\rangle_{\mathcal{N}^s(\{t\})}^{+,J} = \langle \sigma(t)\rangle_{\mathcal{N}^s(\{t\})}^{+,\beta} \geqslant \langle \sigma(t)\rangle^{+,\beta}\,. \quad (5.14)$$

By (5.11) with $A_1 = \mathcal{D}(t, s)$, we have

$$\langle \sigma(t)\rangle_\Lambda^{+,s,J} = \sum_{\Lambda''\subset\Lambda} \langle \sigma(t)\rangle_{\Lambda\setminus\Lambda''}^{+,J} P^{+,s,J}[\{\Lambda(\cdot) = \Lambda''\}]\,. \quad (5.15)$$

Since only terms with $\mathcal{D}(t,s) \subset \Lambda \setminus \Lambda'' \subset \mathcal{N}^s(\mathcal{D}(t,s))$ give a non-zero contribution, FKG inequalities imply

$$\begin{aligned} \langle \sigma(t) \rangle_\Lambda^{+,s,J} &= \sum_{\Lambda'' \subset \Lambda} \frac{\langle \sigma(t) I^s_{\Lambda\setminus\Lambda''} \rangle^{+,J}_{\Lambda\setminus\Lambda''}}{\langle I^s_{\Lambda\setminus\Lambda''} \rangle^{+,J}_{\Lambda\setminus\Lambda''}} P_\Lambda^{+,s,J}[\{\Lambda(\cdot) = \Lambda''\}] \\ &\leqslant \sum_{\Lambda'' \subset \Lambda} \frac{\langle \sigma(t) I^s_{\mathcal{D}(t,s)} \rangle^{+,J}_{\mathcal{D}(t,s)}}{\langle I^s_{\mathcal{D}(t,s)} \rangle^{+,J}_{\mathcal{D}(t,s)}} \frac{\langle I^s_{\mathcal{D}(t,s)} \rangle^{+,J}_{\mathcal{D}(t,s)}}{\langle I^s_{\Lambda\setminus\Lambda''} \rangle^{+,J}_{\Lambda\setminus\Lambda''}} P_\Lambda^{+,s,J}[\{\Lambda(\cdot) = \Lambda''\}] \,. \end{aligned} \tag{5.16}$$

Observing that all contours in $\mathcal{D}(t,s)$ are s-small, we have

$$\frac{\langle \sigma(t) I^s_{\mathcal{D}(t,s)} \rangle^{+,J}_{\mathcal{D}(t,s)}}{\langle I^s_{\mathcal{D}(t,s)} \rangle^{+,J}_{\mathcal{D}(t,s)}} = \langle \sigma(t) \rangle^{+,s,J}_{\mathcal{D}(t,s)} = \langle \sigma(t) \rangle^{+,\beta}_{\mathcal{D}(t,s)} \,, \tag{5.17}$$

and

$$\langle I^s_{\mathcal{D}(t,s)} \rangle^{+,J}_{\mathcal{D}(t,s)} = 1 \,. \tag{5.18}$$

Moreover, by FKG inequalities and Lemma 4.3.2

$$\langle I^s_{\Lambda\setminus\Lambda''} \rangle^{+,J}_{\Lambda\setminus\Lambda''} \geqslant \langle I^s_{\mathcal{N}^s(\mathcal{D}(t,s))} \rangle^{+,J}_{\mathcal{N}^s(\mathcal{D}(t,s))} \geqslant 1 - \mathcal{O}(s^4) \exp\{-\alpha(\beta^*) s\} \,, \tag{5.19}$$

since a contour with diameter bounded by s is obviously s-small ($\alpha(\beta^*)$ is the quantity defined in Lemma 4.3.2).
Therefore we have

$$\langle \sigma(t) \rangle_\Lambda^{+,s,J} \leqslant \langle \sigma(t) \rangle^{+,\beta}_{\mathcal{D}(t,s)} + \mathcal{O}(s^4) \exp\{-\alpha(\beta^*) s\} \,. \tag{5.20}$$

Lemma A.4.1 then gives

$$|\langle \sigma(t) \rangle^{+,\beta}_{\mathcal{D}(t,s)} - \langle \sigma(t) \rangle^{+,\beta}| \leqslant \mathcal{O}(s) \exp\{-\tfrac{1}{2} s\, \overline{a}(\beta)\} \,, \tag{5.21}$$

from which the conclusion follows by defining $\eta(\beta)$ so that

$$\max\{\exp\{-s\,\alpha(\beta^*)\}, \exp\{-\tfrac{1}{2} s\, \overline{a}(\beta)\}\} \leqslant \exp\{-\eta(\beta)\, s\} \,. \tag{5.22}$$

□

Lemmas 5.1.1 and 5.1.2 allow us to obtain estimates on the decay of the variance in the phase of small contours,

Lemma 5.1.3. *Let* $J(e)$ *be defined by* (3.30)*, with* $\beta > \beta_c$ *and* $h \geqslant 0$*. Let* $s \in \mathbb{N}$ *and* $t \in \mathbb{L}$ *such that* $t(2) > \frac{3}{2} s$*. Let* $\Lambda \subset \mathbb{L}$ *such that* $\mathcal{N}^s(\{t\}) \subset \Lambda$ *and let* $t' \in \Lambda$ *such that* $\min\{|t'(i) - t(i)|\,, i = 1,2\} > s$*. Then*

$$|\langle \sigma(t)\sigma(t') \rangle_\Lambda^{+,s,J} - \langle \sigma(t) \rangle_\Lambda^{+,s,J} \langle \sigma(t') \rangle_\Lambda^{+,s,J}| \leqslant \mathcal{O}(s^4) \exp\{-\eta\, s\} \langle \sigma(t') \rangle_\Lambda^{+,s} \,,$$

where $\eta(\beta)$ *is the constant of Lemma 5.1.2.*

Proof. This follows from Lemma 5.1.1, point 1., and Lemma 5.1.2. □

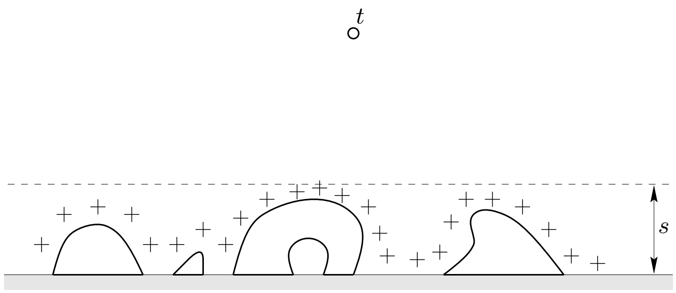

FIGURE 5.3. The idea of the proof of Lemma 5.1.4. The "+" represent the chain of + spins screening the boundary, when the contours touching Σ are fixed (and s-small).

Thanks to Lemma 4.3.2, it is possible to use the phase of small contours to obtain results about the unconstrained phase. This is illustrated in the following useful lemma.

Lemma 5.1.4. *Let $J(e)$ be defined by* (3.30)*, with $\beta > \beta_c$ and $h \geqslant 0$. Let $s \in \mathbb{N}$. Let $\Lambda \subset \mathbb{L}$ and $t \in \Lambda$ such that $t(2) > 2s$. Then*

$$\langle \sigma(t) \rangle^{+,\beta} - P^{+,\beta,h}[\{\exists \gamma \text{ not } s\text{-small}\}] \leqslant \langle \sigma(t) \rangle_\Lambda^{+,\beta,h} \leqslant \langle \sigma(t) \rangle^{+,\beta} + P^{+,\beta,h}[\{\exists \gamma \text{ not } s\text{-small}\}] + \mathcal{O}(s) \exp\{-\overline{a}(\beta)\, s\}\,,$$

where $\overline{a}(\beta)$ is the constant of Lemma A.4.1. There is a analogous statement in the case of $-$-boundary condition.

Proof. Let $\mathcal{U}$ be the event: All contours in ω which have a non-empty intersection with Σ are s-small, and $\mathcal{U}^c$ the complementary event. Then we have

$$\langle \sigma(t) \rangle_\Lambda^{+,\beta,h} = \langle \sigma(t) | \mathcal{U} \rangle_\Lambda^{+,\beta,h} P_\Lambda^{+,\beta,h}[\mathcal{U}] + \langle \sigma(t) | \mathcal{U}^c \rangle_\Lambda^{+,\beta,h} P_\Lambda^{+,\beta,h}[\mathcal{U}^c]\,. \tag{5.23}$$

Therefore, we can write

$$|\langle \sigma(t) \rangle_\Lambda^{+,\beta,h} - \langle \sigma(t) | \mathcal{U} \rangle_\Lambda^{+,\beta,h} P_\Lambda^{+,\beta,h}[\mathcal{U}]| = |\langle \sigma(t) | \mathcal{U}^c \rangle_\Lambda^{+,\beta,h}| P_\Lambda^{+,\beta,h}[\mathcal{U}^c] \leqslant P_\Lambda^{+,\beta,h}[\{\exists \gamma \text{ not } s\text{-small}\}]\,. \tag{5.24}$$

Since t cannot be in the interior of one the contours intersecting Σ when $\mathcal{U}$ holds, FKG inequalities yield

$$\langle \sigma(t) | \mathcal{U} \rangle_\Lambda^{+,\beta,h} \geqslant \langle \sigma(t) \rangle_\Lambda^{+,\beta} \geqslant \langle \sigma(t) \rangle^{+,\beta}\,. \tag{5.25}$$

This gives the lower bound. The upper bound follows by invoking Lemma A.4.1 to show that

$$\langle \sigma(t) | \mathcal{U} \rangle_\Lambda^{+,\beta,h} \leqslant \langle \sigma(t) \rangle^{+,\beta} + \mathcal{O}(s) \exp\{-\overline{a}(\beta)\, s\}\,. \tag{5.26}$$

□

Using Lemma 4.3.2 it is possible to obtain estimates on $P_\Lambda^{+,\beta,h}[\{\exists \gamma \text{ not } s\text{-small}\}]$ when Λ is simply connected, thus giving precise informations on the effect of the magnetic field on the spins which are not in the vicinity of the wall.

5.2 Large deviations in the phase of small contours

The aim of this section is to obtain an estimate on the probability of large deviations of the magnetization in the phase of small contours. Due to the decoupling properties of this phase, the idea is to introduce suitable block-spins to transform this event into a large deviation event for these independent random variables. This idea was first used in [Pi] and [I2], then it was simplified in [ScSh1]. We follow an improved, simpler version which appeared in [PV3]. We first state an elementary large deviation result for independent random variables.

Lemma 5.2.1. *Let $Y_1, \ldots, Y_n$ be independent random variables whose expectation values satisfy $\mathbb{E}[\exp(aY_i)] < \infty$ for all $a \in \mathbb{R}$ and all $i = 1, \ldots, n$. Let*

$$m_n \doteqdot \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}[Y_i]\,,$$

$$f_n(a) \doteqdot \frac{1}{n}\log \mathbb{E}[\exp(a\sum_{i=1}^{n} Y_i)]\,,$$

and

$$\mathrm{var}_n^{\#} \doteqdot \sup_{a\in\mathbb{R}} \frac{\mathrm{d}^2}{\mathrm{d}a^2} f_n(a)\,.$$

Then, for any $x > 0$,

$$\mathrm{Prob}[\{|\sum_{i=1}^{n}(Y_i - \mathbb{E}[Y_i])| \geqslant nx\}] \leqslant 2\exp\{-n\frac{x^2}{2\mathrm{var}_n^{\#}}\}\,.$$

Proof. The proof is inspired by a similar result of [Pf1]. We first consider the event

$$\{\sum_{i=1}^{n}(Y_i - \mathbb{E}[Y_i]) \geqslant nx\}\,. \tag{5.27}$$

Then, for any $a \geqslant 0$, we have by Bernstein inequality,

$$\begin{aligned} \mathrm{Prob}[\{\sum_{i=1}^{n}(Y_i - \mathbb{E}[Y_i]) \geqslant nx\}] &\leqslant \frac{\mathbb{E}[\exp\{a\sum_{i=1}^{n}(Y_i - \mathbb{E}[Y_i])\}]}{\exp\{anx\}} \\ &\leqslant \exp\{-nxa - nm_n a + nf_n(a)\} \\ &\leqslant \exp\{-nxa + \tfrac{1}{2}na^2\mathrm{var}_n^{\#}\} \\ &\leqslant \exp\{-n\frac{x^2}{2\mathrm{var}_n^{\#}}\}\,. \end{aligned} \tag{5.28}$$

The proof for the event $\{\sum_{i=1}^{n}(Y_i - \mathbb{E}[Y_i]) \leqslant -nx\}$ is similar. □

The proof of the above lemma is not restricted to independent random variables; indeed it is possible to prove for example

Lemma 5.2.2. *Let $J(e) \geqslant 0$ for all edges and let*

$$\mathrm{var}_\Lambda^+ \doteqdot \frac{1}{|\Lambda|} \sum_{t,t \in \Lambda} \left(\langle \sigma(t)\sigma(t') \rangle_\Lambda^{+,J} - \langle \sigma(t) \rangle_\Lambda^{+,J} \langle \sigma(t') \rangle_\Lambda^{+,J} \right).$$

Then, for any $x \geqslant 0$,

$$P_\Lambda^{+,J}[\{\sum_{t \in \Lambda} (\sigma(t) - \langle \sigma(t) \rangle_\Lambda^{+,J}) \geqslant x|\Lambda|\}] \leqslant \exp\{-|\Lambda| \frac{x^2}{2\mathrm{var}_\Lambda^+}\},$$

and for any $x \leqslant 0$,

$$P_\Lambda^{-,J}[\{\sum_{t \in \Lambda} (\sigma(t) - \langle \sigma(t) \rangle_\Lambda^{-,J}) \leqslant x|\Lambda|\}] \leqslant \exp\{-|\Lambda| \frac{x^2}{2\mathrm{var}_\Lambda^-}\},$$

where $\mathrm{var}^- \doteqdot \frac{1}{|\Lambda|} \sum_{t,t \in \Lambda} \left(\langle \sigma(t)\sigma(t') \rangle_\Lambda^{-,J} - \langle \sigma(t) \rangle_\Lambda^{-,J} \langle \sigma(t') \rangle_\Lambda^{-} \right) = \mathrm{var}^+$*, by symmetry.*

Proof. The proof is identical to the one of Lemma 5.2.1. Let

$$f_\Lambda(a) \doteqdot \frac{1}{|\Lambda|} \log \langle \exp[a \sum_{t \in \Lambda} \sigma(t)] \rangle_\Lambda^{+,J}, \tag{5.29}$$

and observe that GHS inequalities yield

$$\sup_{a \geqslant 0} \frac{\mathrm{d}^2}{\mathrm{d}a^2} f_\Lambda(a) \leqslant \mathrm{var}_\Lambda^+. \tag{5.30}$$

The second statement is proved in the same way. □

Definition.

(D112) *Let $\Lambda \subset \mathbb{Z}^2$, and $\underline{\gamma}$ a Λ^+-compatible family of contours. Then*

$$\Lambda_\#(\underline{\gamma}) \doteqdot \Lambda \setminus (\bigcup_{\gamma \in \underline{\gamma}} \overline{\mathrm{int}}\gamma \cap \overline{\mathrm{ext}}\gamma).$$

Proposition 5.2.1. *1. Let $J(e) \geqslant 0$ for all edges, Λ be a simply connected subset of $\mathbb{Z}^2$ and $s \in \mathbb{N}$. Then, for any $x \geqslant 0$,*

$$P_\Lambda^{+,s,J}\left[\{\sum_{t \in \Lambda} (\sigma(t) - \langle \sigma(t) \rangle_\Lambda^{+,J}) \geqslant x|\Lambda|\}\right] \leqslant \left(1 - P_\Lambda^{+,J}[\{\exists \gamma \text{ not } s\text{-small}\}]\right)^{-1} \exp\{-|\Lambda| \frac{x^2}{2\mathrm{var}_\Lambda^+}\},$$

where var_Λ^+ is defined in Lemma 5.2.2.

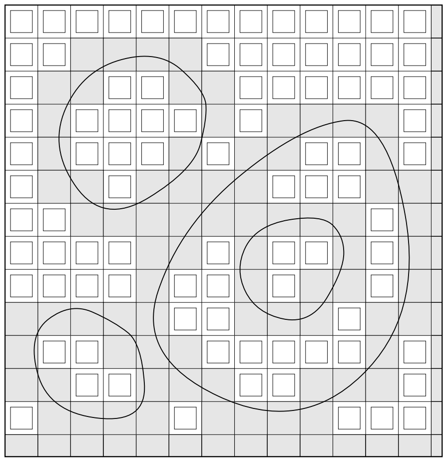

FIGURE 5.4. The partition in the proof of Proposition 5.2.1. The "bad" boxes are shaded.

2. *Let $J(e)$ be defined by (3.30), with $\beta > \beta_c$ and $h \geqslant 0$. Let $0 < x < 1$ and $s > 0$. Let $C_1 > 0$, $C_2 > 0$, $C_3 > 0$. Let $\Lambda \subset \mathbb{L}$ with $|\Lambda| = C_1 L^2$ and $|\partial\Lambda| = C_2 L$. Let $\underline{\Gamma}$ be a Λ^+-compatible family of s-large contours such that $|\underline{\Gamma}| < C_3 L$. Let us write $P^\star[\,\cdot\,] \doteqdot P^{\omega_{\underline{\Gamma}},s,J}_{\Lambda_\#(\underline{\Gamma})}[\,\cdot\,]$, where $\omega_{\underline{\Gamma}}$ is the only configuration satisfying $+$-b.c. in Λ which has $\underline{\Gamma}$ as its set of contours; we also write $\langle\,\cdot\,\rangle^\star$ for the corresponding expectation value. Then, there exists a sufficiently large constant $K = K(C_1, C_2, C_3)$ such that, if $Lx^2 > K^2 s$ and $Lx > Ks$,*

$$P^\star\Big[\{|\sum_{t\in\Lambda}(\sigma(t) - \langle\sigma(t)\rangle^\star)| \geqslant |\Lambda| x\}\Big] \leqslant \exp\{-\mathcal{O}(|\Lambda| x^4/s^2)\}\,.$$

3. *Let $J(e)$ be defined by (3.30), with $\beta > \beta_c$ and $h \geqslant 0$. Let $c \in \mathbb{R}$ and $s = L^\delta$, $\delta > 0$, such that $2c + \delta < 1$. Let $C_1 > 0$, $C_2 > 0$, $C_3 > 0$, $C_4 > 0$. Let $\Lambda \subset \mathbb{L}$ with $|\Lambda| = C_1 L^2$ and $|\partial\Lambda| = C_2 L$. Let $\underline{\Gamma}$ be a Λ^+-compatible family of s-large contours such that $|\underline{\Gamma}| < C_3 L$. Then,*

$$P^\star\Big[\{|\sum_{t\in\Lambda}(\sigma(t) - \langle\sigma(t)\rangle^\star)| \geqslant C_4 |\Lambda| L^{-c}\}\Big] \leqslant \exp\{-\mathcal{O}(L^{2-4c-2\delta})\}\,.$$

Proof. 1. This is a consequence of Lemmas 5.2.2 and 4.3.2.

2. We introduce

$$\Lambda_+ \doteqdot \{t \in \Lambda_\#(\underline{\Gamma}) \,:\, \omega_{\underline{\Gamma}}(t) = 1\} \quad \Lambda_- \doteqdot \{t \in \Lambda_\#(\underline{\Gamma}) \,:\, \omega_{\underline{\Gamma}}(t) = -1\}\,. \tag{5.31}$$

We suppose that $|\Lambda_+| \geqslant |\Lambda_-|$ (the other case is treated in the same way). By hypotheses, there exist constants C_+, C_-, C'_+, C'_-, μ and ν such that

$$\begin{aligned} |\Lambda_+| &= C_+ L^2\,, && \tfrac{1}{2}C_1 \leqslant C_+ \leqslant C_1\,; && \\ |\Lambda_-| &= C_- L^\mu\,, && 0 \leqslant C_- \leqslant \tfrac{1}{2}C_1\,, && 0 \leqslant \mu \leqslant 2\,; \\ |\partial\Lambda_+| &= C'_+ L\,, && C'_+ \leqslant C_2 + C_3\,; && \\ |\partial\Lambda_-| &= C'_- L^\nu\,, && C'_- \leqslant C_3\,, && \tfrac{1}{2}\mu \leqslant \nu \leqslant 1\,. \end{aligned} \tag{5.32}$$

The functions $\mathcal{O}(\cdot)$ and $o(\cdot)$ below depend on the value of the constants C_i, $i = 1, \dots, 4$.
The idea of the proof is similar to that of Lemma 5.1.1. Let $R \doteqdot K^{1/2}s/x$. We consider a grid whose cells are translate of $\mathcal{D}(0, R)$. The cells of the grid are denoted $\mathcal{C}_1, \dots, \mathcal{C}_M$. Clearly $M = \mathcal{O}(L^2/R^2)$. We first estimate the number N of cells entirely contained in either Λ_+ or Λ_-. To make this computation, we construct a set of square boxes $\mathcal{B}_1, \dots, \mathcal{B}_K$, which are translates of $\mathcal{D}(0, s)$, such that $\underline{\Gamma} \subset \bigcup_{i=1,\dots,K} \mathcal{B}_i$. We consider the contours of $\underline{\Gamma}$ as unit-speed parameterized curves. We construct the set of boxes with the following procedure:

(a) W set $t_0^1 = \Gamma_1(0)$ and $\mathcal{B}_1 = \mathcal{D}(t_0^1, s)$;

(b) Let s_1 be the first time such that $\Gamma_1(s_1) \notin \mathcal{D}(t_0^1, s)$; we set $t_1^1 \doteqdot \Gamma_1(s_1)$ and $\mathcal{B}_2 = \mathcal{D}(t_1^1, s)$;

(c) Let s_2 be the first time after s_1 such that $\Gamma_1(s_2) \notin \mathcal{D}(t_1^1, s)$; we set $t_2^1 \doteqdot \Gamma_1(s_2)$ and $\mathcal{B}_3 = \mathcal{D}(t_0^2, s)$;

(d) This procedure is iterated until it stops; then we do the same thing to the next contour Γ_i.

Since $K^{1/2}s/x > s$, if K is large enough, the cells of the grid are larger than the boxes we just defined. Therefore each such box can intersect at most 4 cells of the grid. Since the number of boxes is $\mathcal{O}(L/s)$, the number of cells entirely contained inside Λ is

$$N = M - \mathcal{O}(L/s) = \mathcal{O}(L^2/R^2)\,, \tag{5.33}$$

where we use the hypothesis that $Lx^2 > K^2 s$. Observe that the total volume of the boxes not entirely inside Λ_+ or entirely inside Λ_- is at most $\mathcal{O}(Ls) \ll |\Lambda|x$, if K is large enough.
In the center of each cells $\mathcal{C}_i$ we put another smaller square box $\mathcal{C}_i'$ which is a translate of $\mathcal{D}(0, R-2s)$. We have the following estimate on the total volume of the corridors between the boxes $\mathcal{C}_i'$:

$$|\bigcup_i \mathcal{C}_i \setminus \mathcal{C}_i'| \leqslant \mathcal{O}(sR)\mathcal{O}(L^2/R^2) = \mathcal{O}(sL^2/R)\,. \tag{5.34}$$

Notice that $sL^2/R = L^2x/K^{1/2} \ll L^2x$.
We set

$$Y_i \doteqdot \frac{1}{|\mathcal{C}_i|}\sum_{t\in\mathcal{C}_i'} \sigma(t)\,. \tag{5.35}$$

We can then write

$$P^\star[\{|\sum_{t\in\Lambda}(\sigma(t) - \langle\sigma(t)\rangle^\star)| \geqslant |\Lambda|x\}] \leqslant$$

$$P^\star[\{|\sum_{j=1}^N Y_j - \frac{1}{|\mathcal{C}_1|}\sum_{t\in\Lambda}\langle\sigma(t)\rangle^\star| \geqslant Nx/2\}]\,, \tag{5.36}$$

if K is large enough. We define a random set $\Lambda(\omega)$. Let $\gamma_1(\omega), \dots, \gamma_n(\omega)$ be all external contours of ω such that $\mathrm{int}\gamma_k$ has a non-empty intersection with at least two different cells. We set

$$\Lambda(\omega) \doteqdot \bigcup_{i=1}^n \overline{\mathrm{int}}\gamma_i\,. \tag{5.37}$$

By construction,

$$\Lambda(\omega) \cap \mathcal{C}'_j = \varnothing\,, \quad j = 1, \ldots, n\,, \tag{5.38}$$

for all ω such that all contours are s-small. We then have

$$P^\star[\{|\sum_{j=1}^{N} Y_j - \frac{1}{|\mathcal{C}_1|}\sum_{t\in\Lambda}\langle\sigma(t)\rangle^\star| \geqslant Nx/2\}] =$$

$$\sum_{\Lambda'\subset\Lambda} P^\star[\{|\sum_{j=1}^{N} Y_j - \frac{1}{|\mathcal{C}_1|}\sum_{t\in\Lambda}\langle\sigma(t)\rangle^\star| \geqslant Nx/2\}|\{\Lambda(\cdot) = \Lambda'\}]\, P^\star[\{\Lambda(\cdot) = \Lambda'\}]\,. \tag{5.39}$$

Let Λ' be such that $P^\star[\{\Lambda(\cdot) = \Lambda'\}] \neq 0$. Using Lemma 5.1.2, we obtain

$$|\frac{1}{|\mathcal{C}_1|}\sum_{t\in\Lambda}\langle\sigma(t)\rangle^\star - \sum_{j=1}^{N}\langle Y_j|\Lambda(\cdot) = \Lambda'\rangle^\star| \leqslant N\left(\mathcal{O}(s/R) + \mathcal{O}(s/L) + \mathcal{O}(L^{\nu-2}/s)\right). \tag{5.40}$$

But clearly

$$\mathcal{O}(s/L) < \mathcal{O}(s/R) = \mathcal{O}(x/K^{1/2}) \ll x \tag{5.41}$$

and

$$L^{\nu-2}s < s/L = xR/(K^{1/2}L) \ll x\,. \tag{5.42}$$

Therefore, if K is large enough, we have

$$P^\star[\{|\sum_{t\in\Lambda}(\sigma(t) - \langle\sigma(t)\rangle^\star)| \geqslant |\Lambda|x\}] \leqslant$$

$$\sum_{\Lambda'\subset\Lambda} P^\star[\{|\sum_{j=1}^{N}\big(Y_j - \langle Y_j|\Lambda(\cdot) = \Lambda'\rangle^\star\big)| \geqslant Nx/3\}|\{\Lambda(\cdot) = \Lambda'\}]\, P^\star[\{\Lambda(\cdot) = \Lambda'\}]\,. \tag{5.43}$$

The variables Y_i, $i = 1, \ldots, n$ are independent with respect to the probability measure $P^\star[\cdot|\{\Lambda(\cdot) = \Lambda'\}]$. We can therefore use Lemma 5.2.1 to estimate this probability and we finally obtain

$$P^\star[\{|\sum_{t\in\Lambda}(\sigma(t) - \langle\sigma(t)\rangle^\star)| \geqslant |\Lambda|x\}] \leqslant 2\exp\{-Nx^2/18\}$$
$$= \exp\{-\mathcal{O}(|\Lambda|x^4/s)\}\,, \tag{5.44}$$

where we use the fact that the variances of the random variables Y_i are bounded by one, since the random variables are bounded by one.

3. This is a particular case of 2.. Clearly, the hypotheses imply that $Lx^2 = C_4^2 L^{1-2c} \gg L^\delta K^2 = sK^2$ and $Lx = C_4 L^{1-c} \gg KL^\delta$, for any constant K as soon as L is large enough.

□

Remark.

1. In Proposition 5.2.1, point 2., we used a trivial bound on the variances of the variables Y_j. Since there is a constraint on the size of the contours, the variance goes to zero when $|\Lambda| \to \infty$; a better estimate would improve the result.
2. To use Lemma 5.2.2 and Proposition 5.2.1, it is necessary to control the quantities $\langle\sigma(t)\rangle_\Lambda^+$, var_Λ^+ and $\langle\sigma(t)\rangle_\Lambda^{+,s}$. We make some comments on this subject.
   (a) If $J(e) = \beta$, for all edges, and $\beta > \beta_c$, then FKG inequalities give

$$\langle\sigma(t)\rangle_\Lambda^+ \geqslant m^*(\beta) \quad \text{and} \quad \langle\sigma(t)\rangle_\Lambda^- \leqslant -m^*(\beta) \tag{5.45}$$

   and we can use Lemma A.4.1 to estimate

$$|\langle\sigma(t)\rangle_\Lambda^+ - m^*(\beta)| \quad \text{and} \quad |\langle\sigma(t)\rangle_\Lambda^- + m^*(\beta)|\,. \tag{5.46}$$

   Moreover, GHS inequalities yield

$$\begin{aligned}\mathrm{var}_\Lambda^- = \mathrm{var}_\Lambda^+ &= \frac{1}{|\Lambda|}\sum_{t,t'\in\Lambda}\big(\langle\sigma(t)\sigma(t')\rangle_\Lambda^+ - \langle\sigma(t)\rangle_\Lambda^+\langle\sigma(t')\rangle_\Lambda^+\big)\\ &\leqslant \frac{1}{|\Lambda|}\sum_{t,t'\in\Lambda}\big(\langle\sigma(t)\sigma(t')\rangle^+ - \langle\sigma(t)\rangle^+\langle\sigma(t')\rangle^+\big)\\ &\leqslant \sum_{t\in\mathbb{Z}^2}\big(\langle\sigma(0)\sigma(t)\rangle^{+,\beta} - m^*(\beta)^2\big)\\ &= \chi(\beta)\,.\end{aligned} \tag{5.47}$$

   (b) Let $\Lambda \subset \mathbb{L}$. If $J(e)$ is given by (3.30) with $\beta > \beta_c$ and $h \geqslant 1$, then GHS inequalities imply

$$\mathrm{var}_\Lambda^{-,\beta,h} = \mathrm{var}_\Lambda^{+,\beta,h} \leqslant \mathrm{var}_\Lambda^{+,\beta,1} \leqslant \chi(\beta)\,. \tag{5.48}$$

   Moreover, for all $t \in \mathbb{L}$, $t(2) \geqslant 1$, we have by FKG inequalities

$$\langle\sigma(t)\rangle_{\Lambda\backslash\Sigma}^{-,\beta,1} \leqslant \langle\sigma(t)\rangle_\Lambda^{-,\beta,h} \leqslant \langle\sigma(t)\rangle_\Lambda^{-,\beta,1} \tag{5.49}$$

   so that we can use Lemma A.4.1 to compare $\langle\sigma(t)\rangle_\Lambda^{-,\beta,h}$ with $-m^*(\beta)$.
   (c) Let Λ be simply connected. If $J(e)$ is given by (3.30) with $\beta > \beta_c$ and $0 \leqslant h \leqslant 1$, we use Lemmas 5.1.4 and 4.3.2. To get an upper bound on $\mathrm{var}_\Lambda^{+,\beta,h}$, we use GKS inequalities,

$$\langle\sigma(t)\sigma(t')\rangle_\Lambda^{+,\beta,h} \leqslant \langle\sigma(t)\sigma(t')\rangle_\Lambda^{+,\beta,1}\,, \tag{5.50}$$

   and use Lemmas 5.1.4 and A.4.1 to estimate

$$\langle\sigma(t)\rangle_\Lambda^{+,\beta,h}\langle\sigma(t')\rangle_\Lambda^{+,\beta,h} - \langle\sigma(t)\rangle_\Lambda^{+,\beta,1}\langle\sigma(t)\rangle_\Lambda^{+,\beta,1}\,. \tag{5.51}$$

There is one case in which it is possible to improve the above proposition; it is interesting because it is exactly the kind of event which was considered in [I2] and [ScSh1]. In this case the techniques used above give the exact exponent for the exponential decay of the probability as can be seen by using the techniques of Lemma 7.6.3 to construct a lower bound for this probability.

Proposition 5.2.2. *Let $J(e) = \beta$ for all edges e, with $\beta > \beta_c$. Let $s = L^\delta$, $\delta > 0$. Let $C_1 > 0$ and $C_2 > 0$. Let $\Lambda \subset \mathbb{L}$ with $|\Lambda| = C_1 L^2$ and $|\partial\Lambda| = C_2 L$. Then, for any $\varepsilon > 0$,*

$$P^{+,s,\beta}[\{\frac{1}{|\Lambda|}\sum_{t\in\Lambda}\sigma(t) \leqslant m^*(\beta) - \varepsilon\}] \leqslant \exp\{-\mathcal{O}(L^{2-\delta})\}\,.$$

Proof. The proof is very similar to that of Proposition 5.2.1, so that we only explain what is different in this case.
Let $C(\varepsilon)$ be a sufficiently large constant. We partition Λ into cells $\mathcal{C}_i$, which are translate of the box $\mathcal{D}(0,[C(\varepsilon)+2]s)$. The total number of boxes is therefore $N = \mathcal{O}(L^{2-2\delta})$. At the center of each of these boxes we put a translate of $\mathcal{D}(0, C(\varepsilon)s)$ which we denote by $\mathcal{C}'_i$. We have

$$|\bigcup_i \mathcal{C}_i \setminus \mathcal{C}'_i| = \mathcal{O}(L^2/C(\varepsilon)) \ll \varepsilon L^2\,, \tag{5.52}$$

if $C(\varepsilon)$ is large enough. Therefore, we have

$$\{\frac{1}{|\Lambda|}\sum_{t\in\Lambda}\sigma(t) \leqslant m^*(\beta) - \varepsilon|\Lambda|\} \subset \{\frac{1}{N|\mathcal{C}'_i|}\sum_i\sum_{t\in\mathcal{C}'_i}\sigma(t) \leqslant m^*(\beta) - \tfrac{1}{2}\varepsilon|\Lambda|\}\,, \tag{5.53}$$

if $C(\varepsilon)$ is large enough. Let $\mu \doteqdot \varepsilon/(4(1+m^*(\beta)))$. This new event can be realized only if at least μN boxes have a magnetization at most $m^*(\beta) - \varepsilon/4$. Indeed, suppose this is false, then, denoting by $\mathcal{M}$ the set of cells having a magnetization at most $m^*(\beta) - \varepsilon/4$,

$$\begin{aligned}\frac{1}{N|\mathcal{C}'_i|}\sum_i\sum_{t\in\mathcal{B}'_i}\sigma(t)(\omega) &= \frac{1}{N|\mathcal{C}'_i|}\Big(\sum_{\substack{i:\\ \mathcal{C}'_i\notin\mathcal{M}}}\sum_{t\in\mathcal{B}'_i}\sigma(t)(\omega) + \sum_{\substack{i:\\ \mathcal{C}'_i\in\mathcal{M}}}\sum_{t\in\mathcal{B}'_i}\sigma(t)(\omega)\Big)\\ &\geqslant (m^*(\beta)-\varepsilon/4)(1-\mu) - \mu\\ &= m^*(\beta) - \varepsilon/2 + \frac{\varepsilon^2}{16(1+m^*(\beta))}\\ &> m^*(\beta) - \varepsilon/2\,.\end{aligned} \tag{5.54}$$

We therefore have to compute the probability that there is a volume order large deviation in μN cells $\mathcal{C}'_i$. We decouple the boxes $\mathcal{C}'_i$ as is done in the proof of Proposition 5.2.1. Conditioned on the random set, these events are independent. We would like to use the results of [S2][2] to estimate their probability. To do this we first remove the constraint on the size of contours (notice that these events are decreasing) and then use monotonicity in the volume (use FKG inequalities) to obtain expectation value in $\mathcal{C}_i$. Using Schonmann's upper bound we conclude that the probability of such a deviation in one box is at most $\exp\{-\mathcal{O}(L^\delta)\}$. Therefore

$$P^{+,s,\beta}[\{\frac{1}{|\Lambda|}\sum_{t\in\Lambda}\sigma(t) \leqslant m^*(\beta) - \varepsilon|\Lambda|\}] \leqslant \exp\{-\mathcal{O}(L^\delta)\}^{\mathcal{O}(L^{2-2\delta})} = \exp\{-\mathcal{O}(L^{2-\delta})\}\,. \tag{5.55}$$

□

[2]His results imply that there exist two positive constants c_1 and c_2 such that the probability p of a volume order large deviation in Λ satisfies $\exp(-c_1|\partial\Lambda|) \leqslant p \leqslant \exp(-c_2|\partial\Lambda|)$.

The preceding propositions show that the probability of (sufficiently) large deviations in the phase of small contours goes to zero exponentially fast with some power of L. However, the conditions of validity of these results may be too strong in some cases. In particular, when proving the lower bounds in Chapter 7, it is sufficient to know that the probability goes to zero when L goes to infinity, but it is useful to have a result which holds also for smaller deviations. This is the aim of the next elementary proposition.

Proposition 5.2.3. *Let $J(e)$ be defined by* (3.30)*, with $\beta > \beta_c$ and $h \geqslant 0$. Let $\Lambda \subset \mathbb{L}$, $s \in \mathbb{N}$ and $\mathcal{B}_s \doteqdot \{t \in \Lambda : t(2) \leqslant 2s\}$. Let $\underline{\Gamma}$ be a Λ^+-compatible family of s-large contours and define $P^\star[\,\cdot\,] \doteqdot P^{\omega_{\underline{\Gamma}},s,J}_{\Lambda_\#(\underline{\Gamma})}[\,\cdot\,]$, where $\omega_{\underline{\Gamma}}$ is the only configuration satisfying $+$-b.c. in Λ which has $\underline{\Gamma}$ as its set of contours; $\langle\,\cdot\,\rangle^\star$ denotes the corresponding expectation value. Then,*

$$P^\star[\{\sum_{t\in\Lambda}(\sigma(t) - \langle\sigma(t)\rangle^\star_\Lambda) > x\} \leqslant \frac{\exp\{-\mathcal{O}(s)\} + \mathcal{O}(|\mathcal{B}_s|^2/|\Lambda|^2) + \mathcal{O}(s^2/|\Lambda|)}{x^2}\,.$$

Proof. Let

$$Y \doteqdot \frac{1}{|\Lambda|}\sum_{t\in\Lambda}(\sigma(t) - \langle\sigma(t)\rangle^\star_\Lambda)\,. \tag{5.56}$$

Chebyshev's inequality gives

$$P^\star[\{|Y| > x\} \leqslant \frac{\langle Y^2\rangle^\star_\Lambda}{x^2}\,, \tag{5.57}$$

which shows that we only have to estimate the variance of the random variable Y, which is easily done using Lemma 5.1.3. Indeed, we have

$$Y^2 = \frac{1}{|\Lambda|^2}\sum_{t,t'\in\Lambda_L}\big(\sigma(t)\sigma(t') + \langle\sigma(t)\rangle^\star_\Lambda\langle\sigma(t')\rangle^\star_\Lambda - 2\sigma(t)\langle\sigma(t')\rangle^\star_\Lambda\big)\,, \tag{5.58}$$

which implies, by Markov property and Lemma 5.1.3,

$$\begin{aligned}\langle Y^2\rangle^\star_\Lambda &= \sum_{t,t'\in\Lambda_L}\big(\langle\sigma(t)\sigma(t')\rangle^\star_\Lambda - \langle\sigma(t)\rangle^\star_\Lambda\langle\sigma(t')\rangle^\star_\Lambda\big)\\ &\leqslant \frac{1}{|\Lambda|^2}\Big[\sum_{(t,t')\in\mathcal{K}}\big(\langle\sigma(t)\sigma(t')\rangle^\star_\Lambda - \langle\sigma(t)\rangle^\star_\Lambda\langle\sigma(t')\rangle^\star_\Lambda\big) + \mathcal{O}(|\mathcal{B}_s|^2) + \mathcal{O}(|\Lambda|s^2)\Big]\\ &\leqslant \exp\{-\mathcal{O}(s)\} + \mathcal{O}((\frac{|\mathcal{B}_s|}{|\Lambda|})^2) + \mathcal{O}(\frac{s^2}{|\Lambda|})\,,\end{aligned} \tag{5.59}$$

where we have denoted by $\mathcal{K}$ the set of pairs of points which belong to a same component of Λ_L and which satisfy the hypotheses of Lemma 5.1.3. □

Chapter 6

Interface Pinning

In this chapter we give a first application of the tools introduced in the preceding chapters to a problem of phase separation. There are several ways to induce phase separation: We can impose a canonical constraint on the system, i.e. specify the total number of spins having a given value, or we can choose appropriate boundary conditions to ensure the presence of more than one phase. Chapter 7 and Part II are devoted to the first situation in the case of the Ising and Ashkin-Teller models. We want to study now the second situation, which is simpler since it avoids several lengthy technicalities which result from the canonical constraint. The case we consider is probably the simplest one which contains interesting physics, however the techniques we use can be applied to much more complicated situations.

Consider an Ising model in some square box Λ, with boundary conditions inducing the plus phase on the upper part of the box, and the minus phase on the lower part. Suppose the spins on the bottom line are subject to a magnetic field h. The question we want to answer is the following one: Looking at the box at a macroscopic scale (this will be made precise), what are the typical configurations; in particular, what is the behaviour of the interface between the two phases? We show that the techniques of the previous chapters are very convenient to study such problems; moreover they provide very precise answers to such questions. The same problem has been studied in the case of the SOS model in [Pa1] and similar questions were investigated using exact computations in the 2D Ising model in [PU].

We first state the problem and introduce some notations in Section 6.1. In Section 6.2 we describe (and solve) the corresponding thermodynamical variational problem. Section 6.4 is dedicated to the proof of the main results; it relies on two technical results proved in Section 6.3. The phase transition is discussed in Section 6.5. In the last section, we show that these low-temperature results have an interesting high-temperature counterpart, which gives an explicit (and elementary) example of a situation where there is no equivalence between the "short" and "long" correlation lengths (see Section 3.1.3)[1].

[1]These results can be found in [PV4]

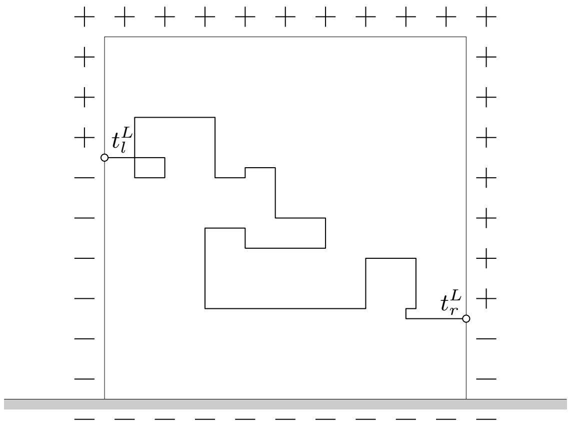

FIGURE 6.1. The box Λ_L with ab-boundary condition. The shaded band represents the wall, and the two points are t_l^L and t_r^L. The open contour joining these two points is the phase separation line.

6.1 Definitions, notations

Let $Q \subset \mathbb{R}^2$ be the square box

$$Q \doteqdot \{x \in \mathbb{R}^2 \,:\, |x(1)| \leqslant 1/2\,,\, 0 \leqslant x(2) \leqslant 1\}\,, \tag{6.1}$$

and $\Lambda_L \subset \mathbb{Z}^2$ (L an even integer)

$$\Lambda_L \doteqdot \{t \in \mathbb{Z}^2 \,:\, |t(1)| \leqslant L/2\,,\, 0 \leqslant t(2) \leqslant L\}\,. \tag{6.2}$$

Let the coupling constants $J(e)$ be given by (3.30) with $\beta > \beta_c$ and $h > 0$ [2]. Let $0 < a < 1$ and $0 < b < 1$ [3]. Let $\overline{\omega}^{ab} \in \Omega$ be the configuration defined by

$$\overline{\omega}^{ab}(t) \doteqdot \begin{cases} -1\,, & \text{if } t(2) \leqslant a\,L,\ t(1) < 0, \\ -1\,, & \text{if } t(2) \leqslant b\,L,\ t(1) \geqslant 0, \\ +1\,, & \text{otherwise.} \end{cases} \tag{6.3}$$

Definition.
(D113) $\overline{\omega}^{ab}$ *defines the* ab**-boundary condition***.*

We write $\mu^{ab}_{\Lambda_L}(\cdot)$ for the Gibbs measure in Λ_L with ab-boundary condition and $P^{ab}_{\Lambda_L}[\cdot]$, $\langle\cdot\rangle^{ab}_{\Lambda_L}$ or $\langle\cdot\rangle^{ab,\beta,h}_{\Lambda_L}$ for the corresponding expectation values.
Let ω satisfying the Λ^{ab}_L-b.c. (i.e. the ab-b.c. in Λ_L). It is not difficult to see that, similarly to what is done in Section 2.1, we have the following characterization of $\underline{\gamma}(\omega)$.

Lemma 6.1.1. *Let ω a configuration satisfying the Λ^{ab}_L-b.c.. Then $\underline{\gamma}(\omega) = (\lambda(\omega), \underline{\widehat{\gamma}}(\omega))$, where $\underline{\widehat{\gamma}}$ are closed contours and λ is an open contour such that $\partial\lambda = \{t^L_l, t^L_r\}$, with t^L_l and $t^L_r \in \mathbb{Z}^{2*}$ independent of ω.*

[2]The case $h \leqslant 0$ can be treated similarly.
[3]We only consider the case $0 < a < 1$ and $0 < b < 1$. This is done to simplify the exposition, however the proofs given in this chapter can easily be modified to study the remaining cases.

Definition.

(D114) *The open contour* $\lambda \in \underline{\gamma}(\omega)$ *is called the* **phase separation line**.

(D115) *A family of contours* $(\lambda', \underline{\gamma}')$ *is* Λ_L^{ab}*-compatible if it is* Λ_L^**-compatible and there exists a configuration* ω *satisfying the* Λ^{ab}*-b.c. such that* $\lambda(\omega) = \lambda'$ *and* $\widehat{\underline{\gamma}}(\omega) = \underline{\gamma}'$.

The following quantities are the analogues of those of Section 2.1,

Definition.

(D116) $$Z^{ab}(\Lambda_L|\lambda) \doteqdot \sum_{\substack{\underline{\gamma}:\partial\underline{\gamma}=\varnothing \\ \underline{\gamma}\cup\lambda\ \Lambda^*\text{-comp.}}} w(\underline{\lambda})$$

(D117) $$Z^{ab}(\Lambda_L) \doteqdot \sum_{\substack{\lambda:\Lambda^*\text{-comp.} \\ \partial\lambda=\{t_l^L,t_r^L\}}} w(\lambda) Z^{ab}(\Lambda_L|\lambda).$$

(D118) $$q^{ab}_{\Lambda_L}(\lambda) \doteqdot w(\lambda)\frac{Z^{ab}(\Lambda_L|\lambda)}{Z^-(\Lambda_L)}.$$

Remark. $q^{ab}_{\Lambda_L}(\lambda)$ is *not* the probability that λ is a contour of the configuration since we do not divide by $Z^{ab}(\Lambda_L)$ but by $Z^-(\Lambda_L)$. However, this is a natural quantity to work with, as can be seen below.

Before proceeding, it may be important to discuss some more conceptual issues. We want to advocate the importance of discerning the concepts of *phase separation line* and *interface*. The first of these objects is defined on the scale of the lattice and can undergo wild fluctuations on that scale. On the contrary, the interface is defined only at a macroscopic (or possibly mesoscopic) scale and is a *deterministic* object. To be more precise, let us consider a specific case. Let $\Lambda'_L \doteqdot \{t \in \mathbb{Z}^2 : \|t\|_\infty \leqslant L\}$. The boundary conditions are given by the configuration $\overline{\omega}^\pm$ such that $\overline{\omega}^\pm(t) = \operatorname{sign} t(2)$ [4,5]. In this case, it is well known (see [G1, BF, BLP1, AR, AU, Hi2, DKS1] for example) that the fluctuations of the phase separation line are of the order $\mathcal{O}(L^{1/2})$ lattice spacings. As a consequence of these giant fluctuations, when L becomes large, the expectation value of any local function will behave as if it was with equal probabilities either in the + or the − phase. Indeed, it can be shown (see [MM]) that the limiting Gibbs state is given by $\frac{1}{2}\mu^+ + \frac{1}{2}\mu^-$ (we have already seen in Chapter 1 that all limiting Gibbs states of the 2D Ising model can be written has a convex combination of these two extremal Gibbs measures). To summarize, the theory of Gibbs states tells us that, in the thermodynamic limit, the system is translation invariant. That's nice, but is it really satisfactory from a physical point of view? Of course it depends on which properties one finds most important, but let us just make a simple computation of order of magnitude. Let us suppose $|\Lambda_L| = 10^{20}$ (that is, we have a macroscopic number of spins). Then the results of Section 6.4 show that, with probability essentially one, the phase separation line is entirely contained inside an elliptical set of

[4] $\operatorname{sign} x \doteqdot \begin{cases} 1, & \text{if } x(2) \geqslant 0, \\ -1, & \text{otherwise.} \end{cases}$

[5] Notice that this is exactly the $\frac{1}{2}\frac{1}{2}$-b.c. in Λ_L, up to a global translation.

width of order $\mathcal{O}(10^5)$ lattice sites[6]. That means that if the box Q has sides of, say, 10cm then the phase separation line will not go outside a set of width 10^{-4}cm. Above this set the system is in the + phase, below it is the − phase. Can we decently say that the system is *translation invariant*? We believe that, when we are interested in phase separation, the relevant limit is not the thermodynamic limit, but rather some kind of continuum limit (see also Chapter 7 for an explicit continuum limit in a more complicated case).

6.2 The variational problem

Before trying to prove anything, we have first to understand what is the expected behaviour of the corresponding thermodynamical system[7].

The thermodynamical problem corresponding to the problem defined in the preceding section is the following. In the box Q, there are two coexisting phases separated by an interface $\mathcal{C}$ whose endpoints A and B are fixed: $A = (-\frac{1}{2}, a)$, $B = (\frac{1}{2}, b)$. The interface $\mathcal{C}$ is a simple rectifiable curve. The free energy associated to the bottom side of Q, which we call w_Q, is modified. The principles of Thermodynamics tell us that the equilibrium shape of the interface is characterized by its free energy functional which is given by[8]

$$\mathfrak{F}(\mathcal{C}) \doteqdot \int_0^{|\mathcal{C}|} \tau(\dot{u}(s), \dot{v}(s))\, \mathrm{d}s + |\mathcal{C} \cap w_Q| \left(\tau_{\mathrm{bd}} - \tau((1,0))\right), \tag{6.4}$$

where $(u(s), v(s))$ is a unit-speed parameterization of $\mathcal{C}$, $|\mathcal{C} \cap w_Q|$ is the length of the part of the interface which is in contact with the wall w_Q, and $\tau(x) = \tau(x;\beta)$ and $\tau_{\mathrm{bd}} = \tau_{\mathrm{bd}}(\beta, h)$ are respectively the surface tension and wall free energy of the 2D Ising model[9]. Thermodynamics states that the equilibrium shape of the interface is the solution to the following

Variational problem: *Find the minimum of the functional $\mathfrak{F}$ among all simple rectifiable open curve in Q with extremities A and B.*

We denote by $\mathcal{D}$ the straight line from A to B, and by $\mathcal{W}$ the curve composed of the following three straight line segments: From A to a point w_1 on the wall, then along the wall from w_1 to w_2, and finally from w_2 to B. The points w_1 and w_2 are such that the angles between the first segment and the wall and between the last segment and the wall are equal, chosen in the interval $[0, \pi/2]$, and solutions of [10,11]

$$\cos\vartheta_{\mathrm{Y}}\, \tau(\vartheta_{\mathrm{Y}}) - \sin\vartheta_{\mathrm{Y}}\, \tau'(\vartheta_{\mathrm{Y}}) = \tau_{\mathrm{bd}} \tag{6.5}$$

which is the Herring-Young equation (see Section 7.2)[12].

[6]This deterministic set is what we call the interface (notice that it is really what an experimentalist would call the interface if he was looking at this system in his labs).

[7]Notice it is not always necessary to be able to solve the variational problem (this can be very difficult!) to prove that the continuum limit is given by its solution; in Chapter 7, we use no information on the variational problem (for example, we need neither existence nor unicity of the solution). However we need some informations on this problem if we want, as is the case in the current chapter, essentially optimal estimates.

[8]We use the fact that $\tau((n_1, n_2)) = \tau((-n_2, n_1))$.

[9]Notice that we have used the positive homogeneity and the symmetry of the surface tension: $\tau(x(1), x(2)) = \tau(-x(2), x(1))$.

[10]Let the unit sphere in $\mathbb{R}^2$ be parameterized by an angle ϑ, $\boldsymbol{n}(\vartheta) \doteqdot (\cos\vartheta, \sin\vartheta)$; we set $\tau(\vartheta) \doteqdot \tau(\boldsymbol{n}(\vartheta))$.

[11]$\tau'(\vartheta) \doteqdot \frac{\mathrm{d}}{\mathrm{d}\vartheta}\tau(\vartheta)$.

[12]The existence of ϑ_{Y} is a consequence of the Winterbottom construction, see Appendix B.

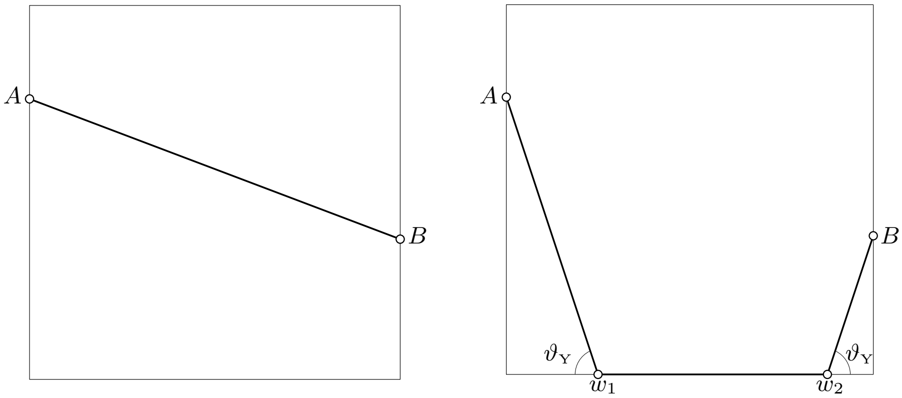

FIGURE 6.2. a) The straight line $\mathcal{D}$. b) The broken line $\mathcal{W}$.

Proposition 6.2.1. *Let ϑ_{Y} be the solution of the Herring-Young equation* (6.5). *Then*

1. *If $\tan\vartheta_{\mathrm{Y}} \leqslant a+b$, then the minimum of the variational problem is given by the curve $\mathcal{D}$.*
2. *If $\pi/2 > \vartheta_{\mathrm{Y}} > \arctan(a+b)$, then the minimum of the variational problem is given by $\mathcal{D}$ if $\mathfrak{F}(\mathcal{D}) < \mathfrak{F}(\mathcal{W})$, by $\mathcal{W}$ if $\mathfrak{F}(\mathcal{D}) > \mathfrak{F}(\mathcal{W})$, and by both $\mathcal{D}$ and $\mathcal{W}$ if $\mathfrak{F}(\mathcal{D}) = \mathfrak{F}(\mathcal{W})$.*

Proof. The proof is an easy consequence of the two following lemmas. □

Lemma 6.2.1 states that the minimum is a polygonal line.

Lemma 6.2.1. *Let $\mathcal{C}$ be some simple rectifiable parameterized curve with initial point A and final point B. Then*

1. *If $\mathcal{C} \cap w_Q = \varnothing$, then*
$$\mathfrak{F}(\mathcal{C}) \geqslant \mathfrak{F}(\mathcal{D}),$$
with equality if and only if $\mathcal{C} = \mathcal{D}$.
2. *If $\mathcal{C} \cap w_Q \neq \varnothing$, let t_1 and t_2 be the first and last times $\mathcal{C}$ touches the wall. Let $\widehat{\mathcal{C}}$ be the curve given by three segments from A to $\mathcal{C}(t_1)$, from $\mathcal{C}(t_1)$ to $\mathcal{C}(t_2)$ and from $\mathcal{C}(t_2)$ to B. Then*
$$\mathfrak{F}(\mathcal{C}) \geqslant \mathfrak{F}(\widehat{\mathcal{C}}).$$
Equality holds if and only if $\mathcal{C} = \widehat{\mathcal{C}}$.

Proof. Since τ is convex and homogeneous, we have in the first case by Jensen's inequality

$$\mathfrak{F}(\mathcal{C}) = |\mathcal{C}|\frac{1}{|\mathcal{C}|}\int_0^{|\mathcal{C}|}\tau(\dot{u}(s),\dot{v}(s))\,\mathrm{d}s \geqslant |\mathcal{C}|\tau(\frac{1}{|\mathcal{C}|}\int_0^{|\mathcal{C}|}\dot{u}(s)\,\mathrm{d}s, \frac{1}{|\mathcal{C}|}\int_0^{|\mathcal{C}|}\dot{v}(s)\,\mathrm{d}s) = \mathfrak{F}(\mathcal{D}). \tag{6.6}$$

The inequality is strict if $\mathcal{C} \neq \mathcal{D}$ as is easily seen using the Sharp Triangle Inequality (see Proposition (3.1.1)).

In the second case we apply Jensen's inequality to the part of $\mathcal{C}$ between A and $\mathcal{C}(t_1)$ and between $\mathcal{C}(t_2)$ and B to compare with the corresponding straight segments of $\widehat{\mathcal{C}}$. Combining Jensen's inequality and the fact that $\tau_{\mathrm{bd}} \leqslant \tau((1,0))$ (see Proposition 3.2.1), we can also compare the part of $\mathcal{C}$ between $\mathcal{C}(t_1)$ and $\mathcal{C}(t_2)$ with the corresponding straight segment of $\widehat{\mathcal{C}}$. □

Lemma 6.2.2. *Let $\widehat{\mathcal{C}}$ be a polygonal line from A to $\widehat{w}_1 \in w_Q$, then from $\widehat{w}_1$ to $\widehat{w}_2 \in w_Q$, and finally from $\widehat{w}_2$ to B. Let ϑ_{Y} be the solution of the Herring-Young equation* (6.5)*. Then*

1. *If $\pi/2 > \vartheta_{\mathrm{Y}} > \arctan(a+b)$, then*
$$\mathfrak{F}(\widehat{\mathcal{C}}) \geqslant \mathfrak{F}(\mathcal{W}),$$
with equality if and only if $\widehat{\mathcal{C}} = \mathcal{W}$.
2. *If $\arctan(a+b) \geqslant \vartheta_{\mathrm{Y}}$, then*
$$\mathfrak{F}(\widehat{\mathcal{C}}) > \mathfrak{F}(\mathcal{D}).$$

Proof. Let $\vartheta_1 \in (0, \pi/2)$ be the angle of the straight segment of $\widehat{\mathcal{C}}$ from A to $\hat{w}_1$, with the wall w_Q, and $\vartheta_2 \in (0, \pi/2)$ be the angle of the straight segment of $\widehat{\mathcal{C}}$ from $\hat{w}_2$ to B, with the wall w_Q. We have

$$\begin{aligned}\mathfrak{F}(\widehat{\mathcal{C}}) &= \tau(\vartheta_1)\frac{a}{\sin\vartheta_1} + \tau_{\mathrm{bd}}(1 - \frac{a}{\tan\vartheta_1} - \frac{b}{\tan\vartheta_2}) + \tau(\vartheta_2)\frac{b}{\sin\vartheta_2} \\ &= g(\vartheta_1, a) + g(\vartheta_2, b),\end{aligned}$$

where we have introduced

$$g(\vartheta, x) \doteqdot \tau(\vartheta)\frac{x}{\sin\vartheta} + \tau_{\mathrm{bd}}(1/2 - \frac{x}{\tan\vartheta}). \tag{6.7}$$

Let ϑ_{Y} be defined as the solution of the Herring-Young equation (6.5), so that

$$\frac{\partial}{\partial\vartheta} g(\vartheta_{\mathrm{Y}}, x) = \frac{x}{\sin^2\vartheta_{\mathrm{Y}}}(\sin\vartheta_{\mathrm{Y}}\,\tau'(\vartheta_{\mathrm{Y}}) - \cos\vartheta_{\mathrm{Y}}\,\tau(\vartheta_{\mathrm{Y}}) + \tau_{\mathrm{bd}}) = 0. \tag{6.8}$$

The second derivative of $g(\vartheta, x)$ is

$$\frac{\partial^2}{\partial\vartheta^2} g(\vartheta, x) = \frac{x(\tau(\vartheta) + \tau''(\vartheta))}{\sin\vartheta} - \frac{2}{\tan\vartheta}\frac{\partial}{\partial\vartheta} g(\vartheta, x). \tag{6.9}$$

Therefore, for $\vartheta \in (0, \pi/2)$, we have

$$\frac{\partial}{\partial\vartheta} g(\vartheta, x) = x\int_{\vartheta_{\mathrm{Y}}}^{\vartheta} \exp\{-\int_{\gamma}^{\vartheta} \frac{2}{\tan\alpha}\,\mathrm{d}\alpha\}\,\frac{\tau(\gamma) + \tau''(\gamma)}{\sin\gamma}\,\mathrm{d}\gamma. \tag{6.10}$$

Since τ has positive stiffness, i.e. $\tau(\vartheta) + \tau''(\vartheta) > 0$, (6.10) implies that ϑ_{Y} is an absolute minimum of $g(\vartheta, x)$ over the interval $(0, \pi/2)$, and that g is strictly monotonous over the intervals $(\vartheta_{\mathrm{Y}}, \pi/2)$ and $(0, \vartheta_{\mathrm{Y}})$.

A necessary and sufficient condition, that we can construct a simple polygonal line $\widehat{\mathcal{C}}$ as above, is

$$\frac{a}{\tan\vartheta_1} + \frac{b}{\tan\vartheta_2} \leqslant 1. \tag{6.11}$$

In particular $\vartheta_1 \in [\vartheta_a, \pi/2]$ where $\vartheta_a \doteqdot \arctan a$, and $\vartheta_2 \in [\vartheta_b, \pi/2]$ where $\vartheta_b \doteqdot \arctan b$. Similarly $\mathcal{W}$ is a simple curve in Q if and only if

$$\vartheta_{\mathrm{Y}} \in [\arctan a + b, \pi/2). \tag{6.12}$$

From the preceding results we have

$$\mathfrak{F}(\widehat{\mathcal{C}}) \geqslant g(\vartheta_1^*, a) + g(\vartheta_2^*, b)\,, \tag{6.13}$$

with

$$\vartheta_1^* = \begin{cases} \vartheta_{\mathrm{Y}}\,, & \text{if } \vartheta_{\mathrm{Y}} \in [\vartheta_a, \pi/2] \\ \vartheta_a\,, & \text{otherwise,} \end{cases}$$
$$\vartheta_2^* = \begin{cases} \vartheta_{\mathrm{Y}}\,, & \text{if } \vartheta_{\mathrm{Y}} \in [\vartheta_b, \pi/2]\,, \\ \vartheta_b\,, & \text{otherwise.} \end{cases} \tag{6.14}$$

If (6.12) holds, then (6.13) implies $\mathfrak{F}(\widehat{\mathcal{C}}) \geqslant \mathfrak{F}(\mathcal{W})$. If (6.12) does not hold, then the two segments from A to the wall, and from B to the wall intersect at some point P. Let $\widehat{\mathcal{W}}$ be the simple polygonal line going from A to P, then from P to B. We have (this follows from Lemma 6.2.1 and $\tau((1,0)) \geqslant \tau_{\mathrm{bd}}$)

$$g(\vartheta_{\mathrm{Y}}, a) + g(\vartheta_{\mathrm{Y}}, b) \geqslant \mathfrak{F}(\widehat{\mathcal{W}})\,. \tag{6.15}$$

Applying again Lemma 6.2.1 we get

$$\mathfrak{F}(\widehat{\mathcal{W}}) > \mathfrak{F}(\mathcal{D})\,. \tag{6.16}$$

□

6.3 Technical results

To show that the prescriptions of Thermodynamics yield the correct result for the Ising system is our aim now. Moreover, we want to obtain estimates on the finite volume corrections.

We study the probability of having a given phase separation line, and show that the corresponding probability measure satisfy a concentration property on a small set of phase separation lines which describes the interface. The proof requires two technical results: We first prove a lower bound which plays an essential role in the following (it is some kind of Ornstein-Zernicke property in the dual model), then we obtain a basic estimate on the probability of a given phase separation line in terms of the surface tension of a coarse-grained version of it.

6.3.1 A lower bound

Proposition 6.3.1. *Let $J(e)$ be given by* (3.30) *with $\beta > \beta_c$ and $h > 0$. Let $0 < a < 1$, $0 < b < 1$ and $\mathfrak{F}^* = \mathfrak{F}^*(\beta, h)$ be the minimum of the functional $\mathfrak{F}$. Then there exist constants $C > 0$ and $L_0 = L_0(\beta, h)$ such that, for all $L \geqslant L_0$,*

$$Z^{ab}(\Lambda_L) \geqslant \frac{\exp\{-\mathfrak{F}^*\, L\}}{L^C}\, Z^-(\Lambda_L)\,.$$

Remark. Since, by Lemma 2.3.1,

$$\begin{aligned} Z^{ab}(\Lambda_L;\beta,h) &= Z^-(\Lambda_L;\beta,h) \sum_{\lambda\,:\,\partial\lambda=\{t_l^L,t_r^L\}} q_L^{ab}(\lambda;\beta,h) \\ &= Z^*(\Lambda_L^*;\beta^*,h^*) \sum_{\lambda\,:\,\partial\lambda=\{t_l^L,t_r^L\}} q_{\Lambda_L^*}(\lambda;\beta^*,h^*)\,, \end{aligned} \tag{6.17}$$

the statement of the proposition can be written

$$\langle \sigma(t_l^L)\sigma(t_r^L)\rangle_{\Lambda_L^*}^{\beta^*,h^*} \geqslant \frac{\exp\{-\mathfrak{F}^*\,L\}}{L^C}\,. \tag{6.18}$$

Therefore the statement of the lemma can be seen as an Ornstein-Zernicke property of the dual 2-point function. Notice that this implies that, when $\mathfrak{F}^* \neq \mathfrak{F}(\mathcal{D})$, the decay-rate of this 2-point function is *not* given by the massgap in the direction $(t_r^L - t_l^L)/\|t_r^L - t_l^L\|_2$. This is discussed in more details in Section 6.6.

Proof. Let $\mathcal{C}^*$ be the simple rectifiable curve in Q which realizes the minimum of the variational problem (or one of the minima in case of degeneracy). Let $K_1 > 0$ be a sufficiently large constant to be chosen later.
We first consider the case $\mathcal{C}^* = \mathcal{D}$. Let $u_l^L \in \Lambda_L^*$ and $u_r^L \in \Lambda_L^*$ be such that (see Fig. 6.3.1)

- u_l^L is the point on the vertical line $\{t' \in \mathbb{Z}^{2*} \,:\, t'(1) = t_l^L(1) + [K_1 \log L]\}$ with $u_l^L(2)$ minimal and $u_l^L(2) \geqslant t_l^L(2) + K_1 \log L\,(t_r^L(2) - t_l^L(2))/(t_r^L(1) - t_l^L(1))$.
- u_r^L is the point on the vertical line $\{t' \in \mathbb{Z}^{2*} \,:\, t'(1) = t_r^L(1) - [K_1 \log L]\}$ with $u_r^L(2)$ maximal and $u_r^L(2) \leqslant t_r^L(2) - K_1 \log L\,(t_r^L(2) - t_l^L(2))/(t_r^L(1) - t_l^L(1))$.

Let $C' > 2$. We introduce $\mathcal{B} \doteqdot \{t \in \Lambda_L^* \,:\, t(2) > C' \log L\}$. By (6.17), Lemma 4.2.4 point 6. and Lemma 4.4.3, we have, if L is large enough,

$$\begin{aligned} \frac{Z^{ab}(\Lambda_L;\beta,h)}{Z^-(\Lambda_L;\beta,h)} &\geqslant \sum_{\substack{\lambda\,:\,\partial\lambda=\{t_l^L,t_r^L\} \\ \lambda\subset\mathcal{B}}} q_{\Lambda_L^*}(\lambda;\beta^*,h^*) \\ &\geqslant \tfrac{1}{2} \sum_{\substack{\lambda\,:\,\partial\lambda=\{t_l^L,t_r^L\} \\ \lambda\subset\mathcal{B}}} q(\lambda;\beta^*) \\ &\geqslant \exp\{-\mathcal{O}(K_1\log L)\} \sum_{\substack{\lambda\,:\,\partial\lambda=\{u_l^L,u_r^L\} \\ \lambda\subset\mathcal{B}}} q(\lambda;\beta^*) \\ &\geqslant \exp\{-\mathcal{O}(K_1\log L)\} \Bigg(\sum_{\lambda\,:\,\partial\lambda=\{u_l^L,u_r^L\}} q(\lambda;\beta^*) - \sum_{\substack{\lambda\,:\,\partial\lambda=\{u_l^L,u_r^L\} \\ \lambda\not\subset\mathcal{B}}} q(\lambda;\beta^*) \Bigg)\,. \end{aligned} \tag{6.19}$$

Let $\rho > 0$ such that the set $\mathcal{S}_1 \doteqdot \mathcal{S}(u_l^L, u_r^L, \rho K_1 \log L)$ (see (4.132)) satisfies $\mathcal{S}_1 \subset \mathcal{B}$ (that this is always possible is not difficult to check).
Then, by Proposition 4.6.2 point 1., we have, if K_1 is large enough,

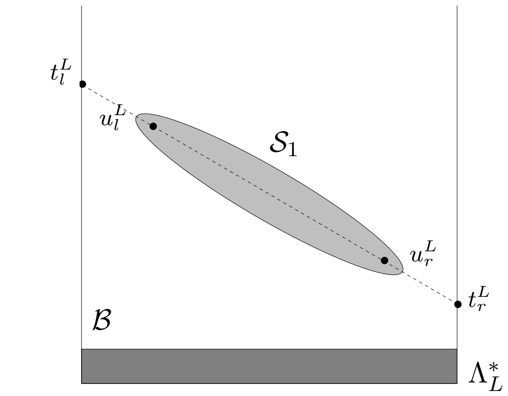

FIGURE 6.3. The ellipse $\mathcal{S}_1$; the box $\mathcal{B}$ is the whole box minus the bottom strip of height $C' \log L$.

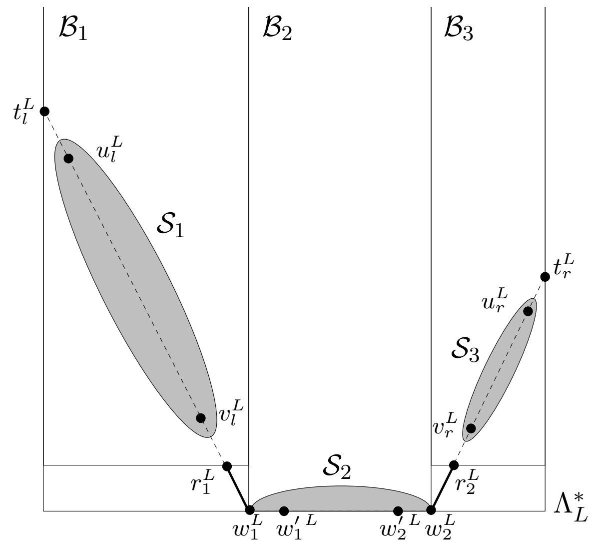

FIGURE 6.4. The three boxes $\mathcal{B}_i$, $i = 1, \ldots, 3$ and their elliptical subsets; the two bold segments represent the two shortest contours $\overline{\lambda}_i$, $i = 1, 2$.

$$\sum_{\substack{\lambda\,:\,\partial\lambda=\{u_l^L,u_r^L\}\\ \lambda\not\subset\mathcal{B}}} q(\lambda;\beta^*) \leqslant \sum_{\substack{\lambda\,:\,\partial\lambda=\{u_l^L,u_r^L\}\\ \lambda\not\subset\mathcal{S}_1}} q(\lambda;\beta^*) \leqslant \exp\{-\mathcal{O}(K_1 \log L)\} \langle \sigma(u_l^L)\sigma(u_r^L)\rangle^{\beta^*}. \tag{6.20}$$

And consequently, by Proposition 4.5.1,

$$\begin{aligned} \frac{Z^{ab;\beta,h}(\Lambda_L)}{Z^-(\Lambda_L;\beta,h)} &\geqslant \exp\{-\mathcal{O}(K_1 \log L)\} \langle \sigma(u_l^L)\sigma(u_r^L)\rangle^{\beta^*} \\ &\geqslant \frac{\exp\{-\tau(t_r^L - t_l^L)\}}{L^C} \end{aligned} \tag{6.21}$$

for some positive constant C.

Let us now consider the case $\mathcal{C}^* = \mathcal{W}$. Since $h > 0$, the angle ϑ_{Y} satisfies $0 < \vartheta_{\mathrm{Y}} < \pi/2$. Denote by w_1^L and w_2^L the two points on Σ^* which are closest to the corners of the polygonal line $\mathcal{W}$ scaled by L.

We define three rectangular boxes (see Fig. 6.4)

$$\begin{aligned} \mathcal{B}_1 &= \{t \in \Lambda_L^* \,:\, t(1) \leqslant w_1^L(1)\,,\, t(2) > [C' \log L]\}\,, \\ \mathcal{B}_2 &= \{t \in \Lambda_L^* \,:\, w_1^L(1) \leqslant t(1) \leqslant w_2^L(1)\}\,, \\ \mathcal{B}_3 &= \{t \in \Lambda_L^* \,:\, w_2^L(1) \leqslant t(1)\,,\, t(2) > [C' \log L]\}\,. \end{aligned} \tag{6.22}$$

Moreover let r_1^L, resp. r_2^L, be the point of Λ_L^* closest to the straight line through t_l^L and w_1^L, resp. t_r^L and w_2^L, and such that $r_1^L(2) = [C' \log L] + 1/2$, resp. $r_2^L(2) = [C' \log L] + 1/2$. Let $\overline{\lambda}_1$, resp. $\overline{\lambda}_2$, be a shortest open contour from r_1^L to w_1^L, resp. from r_2^L to w_2^L. Introducing the set $\mathfrak{A}$ of open contours $\lambda = \lambda_1 \cup \overline{\lambda}_1 \cup \lambda_2 \cup \overline{\lambda}_2 \cup \lambda_3$ with

- $\lambda_i \subset \mathcal{B}_i$, $i = 1, \ldots, 3$;
- $\partial \lambda_1 = \{t_l^L, r_1^L\}$;
- $\partial \lambda_2 = \{w_1^L, w_2^L\}$;
- $\partial \lambda_3 = \{r_2^L, t_r^L\}$;

we can write

$$\begin{aligned} \frac{Z^{ab}(\Lambda_L;\beta,h)}{Z^-(\Lambda_L;\beta,h)} &\geqslant \sum_{\lambda \in \mathfrak{A}} q_{\Lambda_L^*}(\lambda;\beta^*,h^*) \\ &\geqslant \exp\{-\mathcal{O}(\log L)\} \prod_i \sum_{\substack{\lambda_i \subset \mathcal{B}_i : \\ \lambda_i \text{ as above}}} q_{\Lambda_L^*}(\lambda_i;\beta^*,h^*) \\ &\geqslant \exp\{-\mathcal{O}(\log L)\} \sum_{\substack{\lambda_1 \subset \mathcal{B}_1 : \\ \partial\lambda_1 = \{t_l^L, r_1^L\}}} q(\lambda_1;\beta^*) \sum_{\substack{\lambda_2 \subset \mathcal{B}_2 : \\ \partial\lambda_2 = \{w_1^L, w_2^L\}}} q_{\mathbb{L}}(\lambda_2;\beta^*,h^*) \times \\ &\qquad\qquad \times \sum_{\substack{\lambda_3 \subset \mathcal{B}_3 : \\ \partial\lambda_3 = \{t_r^L, r_2^L\}}} q(\lambda_3;\beta^*)\,. \end{aligned} \tag{6.23}$$

Let $u_l^L \in \mathcal{B}_1$ and $v_l^L \in \mathcal{B}_1$ such that

- u_l^L is the point on the vertical line $\{t' \in \mathbb{Z}^{2*} \,:\, t'(1) = t_l^L(1) + [K_1 \log L]\}$ with $u_l^L(2)$ minimal and $u_l^L(2) \geqslant t_l^L(2) + K_1 \log L\,(r_1^L(2) - t_l^L(2))/(r_1^L(1) - t_l^L(1))$.
- u_l^L is the point on the vertical line $\{t' \in \mathbb{Z}^{2*} \,:\, t'(1) = r_1^L(1) - [K_1 \log L]\}$ with $u_l^L(2)$ maximal and $u_l^L(2) \leqslant r_1^L(2) - K_1 \log L\,(r_1^L(2) - t_l^L(2))/(r_1^L(1) - t_l^L(1))$.

Let u_r^L and v_r^L be the corresponding point in the box $\mathcal{B}_3$, constructed using the points r_2^L and t_r^L. Let ${w'_1}^L \doteqdot w_1^L + ([K_1 \log L], 0)$, ${w'_2}^L \doteqdot w_2^L - ([K_1 \log L], 0)$.
We finally introduce three elliptical sets:

$$\begin{aligned} \mathcal{S}_1 &\doteqdot \mathcal{S}(u_l^L, v_l^L, \rho_1 K_1 \log L)\,, \\ \mathcal{S}_2 &\doteqdot \mathcal{S}({w'_1}^L, {w'_2}^L, 2[K_1 \log L])\,, \\ \mathcal{S}_3 &\doteqdot \mathcal{S}(u_r^L, v_r^L, \rho_3 K_1 \log L)\,, \end{aligned} \tag{6.24}$$

where $\rho_1 > 0$ and $\rho_3 > 0$ are chosen such that $\mathcal{S}_1 \subset \mathcal{B}_1$ and $\mathcal{S}_3 \subset \mathcal{B}_3$.
Applying Lemmas 4.4.3 and Proposition 4.6.2, point 1., to the sum over λ_1 and λ_3 as we

did in the first part of the proof, and applying similarly Lemma 4.4.6 and Proposition 4.6.2, point 2. or 3., to the sum over λ_2 yield the desired result,

$$\frac{Z^{ab}(\Lambda_L;\beta,h)}{Z^{-}(\Lambda_L;\beta,h)} \geqslant \frac{\exp\{-L\mathfrak{F}^*\}}{L^C} \tag{6.25}$$

for some positive constant C. □

6.3.2 An upper bound

The lower bound of Proposition 6.3.1 allows us to prove that the surface tension of a (very!) coarse-grained version of the phase separation line cannot be too large compared to the minimum $\mathfrak{F}^*$ of the functional $\mathfrak{F}$.

Let λ be the open contour. We construct a polygonal line approximation $\mathcal{P} \doteqdot \mathcal{P}(\lambda)$ of λ. Let $s \mapsto \lambda(s)$ be a unit-speed parameterization of λ. If $\lambda(s)(2) > -1/2$ for all s, then let $\mathcal{P}$ be the straight line from t_l^L to t_r^L. Otherwise, let s_1 be the first time such that $\lambda(s)(2) = -1/2$ and s_2 the last time such that $\lambda(s)(2) = -1/2$; we write $\hat{w}_i^L = \lambda(s_i)$, $i = 1, 2$. We also introduce $[\mathcal{P}] \doteqdot \{\omega : \mathcal{P}(\lambda(\omega)) = \mathcal{P}\}$.

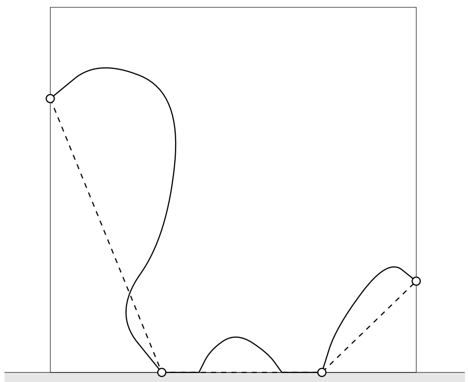

FIGURE 6.5. The coarse-graining of the phase separation line. The dashed line is the polygonal approximation.

By construction, if $s < s_1$ or $s > s_2$ then $\lambda(s)(2) \neq -1/2$. We can therefore apply Lemma 4.2.8 to estimate the probability of the event $[\mathcal{P}]$,

$$P_{\Lambda_L}^{ab,\beta,h}[[\mathcal{P}]] \leqslant \exp\{-\mathfrak{F}(\mathcal{P})\}\,. \tag{6.26}$$

Proposition 6.3.2. *Let $J(e)$ be given by* (3.30) *with $\beta > \beta_c$ and $h > 0$; let $0 < a < 1$, $0 < b < 1$. Then there exists $L_0 = L_0(\beta, h)$ such that, for all $L \geqslant L_0$ and $T > 0$,*

$$P_{\Lambda_L}^{ab,\beta,h}[\{\mathfrak{F}(\mathcal{P}(\lambda)) \geqslant \mathfrak{F}^* L + T\}] \leqslant \exp\{-T + \mathcal{O}(\log L)\}\,.$$

Proof. Let

$$\mathfrak{I}(T) \doteqdot \{\lambda \subset \Lambda_L^* : \partial\lambda = \{t_l^L, t_r^L\}\,,\, \mathfrak{F}(\mathcal{P}(\lambda)) \geqslant \mathfrak{F}^* L + T\}\,. \tag{6.27}$$

Then, from Lemma (2.3.1), Proposition 6.3.1, and equation (6.26),

$$\begin{aligned}
P^{ab,\beta,h}_{\Lambda_L}[\{\mathfrak{F}(\mathcal{P}(\lambda)) \geqslant \mathfrak{F}^* L + T\}] &= \frac{Z^-(\Lambda_L)}{Z^{ab}(\Lambda_L)} \sum_{\lambda\in\mathfrak{I}(T)} q_{\Lambda^*_L}(\lambda;\beta^*,h^*) \\
&\leqslant \exp\{\mathfrak{F}^* L + \mathcal{O}(\log L)\} \sum_{\lambda\in\mathfrak{I}(T)} q_{\Lambda^*_L}(\lambda) \\
&\leqslant \exp\{\mathfrak{F}^* L + \mathcal{O}(\log L)\} \sum_{\substack{\mathcal{P}:\\ \mathfrak{F}(\mathcal{P}) \geqslant \mathfrak{F}^* L+T}} \sum_{\substack{\lambda:\\ \mathcal{P}(\lambda)=\mathcal{P}}} q_{\Lambda^*_L}(\lambda) \\
&\leqslant \exp\{\mathfrak{F}^* L + \mathcal{O}(\log L)\} \sum_{\substack{\mathcal{P}:\\ \mathfrak{F}(\mathcal{P}) \geqslant \mathfrak{F}^* L+T}} \exp\{-\mathfrak{F}(\mathcal{P})\} \\
&\leqslant \exp\{-T + \mathcal{O}(\log L)\}\,, \qquad (6.28)
\end{aligned}$$

since the number of different coarse–grained polygonal lines is bounded by $\mathcal{O}(L^2)$. □

The problem now is to show that we have not lost too much information on the phase separation line during the coarse-graining. More precisely, we need to show that, given a polygonal line $\mathcal{P}$, the typical phase separation lines for which $\mathcal{P}$ is the polygonal approximation are close (in some appropriate sense) to $\mathcal{P}$. This is the last step of the proof and it is done in the next section.

6.4 Concentration properties

In Section 4.6, precise concentration properties of the probability measure of open contours with fixed endpoints are obtained in various situations. We show how these results can be used to prove concentration properties of the probability measure of our phase separation line on a deterministic set which converges to the solution of the variational problem. The results are essentially optimal, see discussion at the end of the section.

To state the results, we introduce the following notations for the probability measure on the set of phase separation lines (with fixed endpoints t^L_l and t^L_r).

$$\mathfrak{P}^{ab}_L[\lambda;\beta,h] \doteqdot \frac{q_{\Lambda^*_L}(\lambda;\beta^*,h^*)}{\langle\sigma(t^L_l)\sigma(t^L_r)\rangle^{\beta^*,h^*}_{\Lambda^*_L}}\,. \qquad (6.29)$$

Let $\mathcal{D}$ and $\mathcal{W}$ be the curves in Q introduced in Section 6.2 (if $\mathcal{W}$ does not fit inside Q, or has self-intersection, then we say it does not exist). We set

$$I^L_i \doteqdot \{x\in\Sigma^* \,:\, \|x-w^L_i\|_1 \leqslant (ML\log L)^{1/2}\}\,,\quad i=1,2\,, \qquad (6.30)$$

and

$$\rho_L \doteqdot M\log L\,. \qquad (6.31)$$

We define two sets of contours. The set $\mathfrak{T}_{\mathcal{D}}$ contains all $\lambda\subset\Lambda^*_L$ such that

a1 $\partial\lambda = \{t^L_l, t^L_r\}$;

a2 $\lambda\subset\mathcal{S}(t^L_l, t^L_r, \rho_L)$.

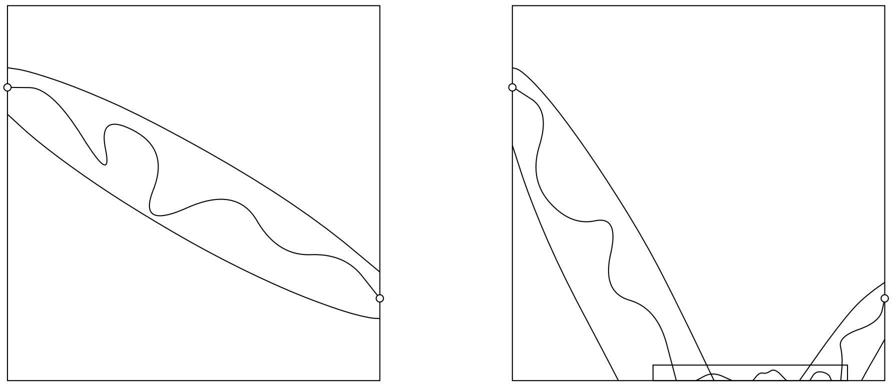

FIGURE 6.6. Left: A contour in $\mathfrak{T}_{\mathcal{D}}$. Right: A contour in $\mathfrak{T}_{\mathcal{W}}$.

If $\mathcal{W}$ exists, the set $\mathfrak{T}_{\mathcal{W}}$ contains all $\lambda \subset \Lambda_L^*$, considered as parameterized curves $s \mapsto \lambda(s)$, such that ($\mathcal{S}$ and $\mathcal{S}'$ are defined in Section 4.6)

b1 $\partial\lambda = \{t_l^L, t_r^L\}$, $\lambda(0) = t_l^L$;

b2 $\exists s_1$ such that $\lambda(s_1) \in I_1^L$ and, for all $s < s_1$, $\lambda(s) > 1/2$.

b3 $\lambda_1 \doteqdot \{\lambda(s) \,:\, s \leqslant s_1\} \subset \mathcal{S}(t_l^L, \lambda(s_1), \rho_L)$;

b4 $\exists s_2$ such that $\lambda(s_2) \in I_2^L$ and, for all $s > s_2$, $\lambda(s) > 1/2$.

b5 $\lambda_3 \doteqdot \{\lambda(s) \,:\, s \geqslant s_2\} \subset \mathcal{S}(\lambda(s_2), t_r^L, \rho_L)$;

b6 $\lambda_2 \doteqdot \{\lambda(s) \,:\, s_1 \leqslant s \leqslant s_2\} \subset \mathcal{S}'(\lambda(s_1), \lambda(s_2), \rho_L)$.

The main result of this chapter is that the probability measure $\mathfrak{P}_L^{ab}[\,\cdot\,;\beta,h]$ satisfies a concentration property on one (or both in case of degeneracy) of these sets as $L \to \infty$.

Theorem 6.4.1. *Let $J(e)$ be given by* (3.30) *with $\beta > \beta_c$ and $h > 0$; let $0 < a < 1$ and $0 < b < 1$. There exist two constants $M > 0$ and $L_0 = L_0(\beta, h, M)$ such that, for all $L \geqslant L_0$,*

1. If the solution of the variational problem in Q is the curve $\mathcal{D}$, then

$$\mathfrak{P}_L^{ab}[\mathfrak{T}_{\mathcal{D}};\beta,h] \geqslant 1 - L^{-\mathcal{O}(M)}\,.$$

2. If the solution of the variational problem in Q is the curve $\mathcal{W}$, then

$$\mathfrak{P}_L^{ab}[\mathfrak{T}_{\mathcal{W}};\beta,h] \geqslant 1 - L^{-\mathcal{O}(M)}\,.$$

3. If the solution of the variational problem in Q is not unique, then

$$\mathfrak{P}_L^{ab}[\mathfrak{T}_{\mathcal{D}} \cup \mathfrak{T}_{\mathcal{W}};\beta,h] \geqslant 1 - L^{-\mathcal{O}(M)}\,.$$

Proof. 1. Suppose that the minimum of the variational problem is given by $\mathcal{D}$, $\mathfrak{F}(\mathcal{D}) = \mathfrak{F}^*$. Let $\mathfrak{F}^{**}$ be the minimum of the functional over all simple curves in Q, with end–points A and B, and which touch the wall w_Q. By hypothesis there exists $\delta > 0$

with $\mathfrak{F}^{**} = \mathfrak{F}^* + \delta$. Consequently, applying Proposition 6.3.2 to phase separation lines touching the wall, we obtain

$$\begin{aligned}\mathfrak{P}_L^{ab}[\{\lambda \cap \mathcal{E}^*(\Sigma^*) \neq \varnothing\}; \beta, h] &\leqslant P_{\Lambda_L}^{ab;\beta,h}[\{\mathfrak{F}(\mathcal{P}(\lambda)) \geqslant (\mathfrak{F}^* + \delta)L\}] \\ &\leqslant \exp\{-\delta\, L + \mathcal{O}(\log L)\}\,. \end{aligned} \tag{6.32}$$

Therefore we can restrict our attention to contours λ such that $\lambda \cap \mathcal{E}^*(\Sigma^*) = \varnothing$. We set $\mathcal{S}_1 \doteqdot \mathcal{S}(t_l^L, t_r^L, \rho_L)$; for L large enough $\mathcal{S}_1 \cap \Sigma^* = \varnothing$ since $a > 0$ and $b > 0$. We apply Proposition 6.3.1. We have

$$\begin{aligned}\mathfrak{P}_L^{ab}[\{\lambda \notin \mathfrak{T}_{\mathcal{D}}\,,\, \lambda \cap \mathcal{E}^*(\Sigma^*) = \varnothing\}; \beta, h] &= \frac{1}{\langle \sigma(t_l^L)\sigma(t_r^L)\rangle_{\Lambda_L^*}^{\beta^*,h^*}} \sum_{\substack{\lambda \notin \mathfrak{T}_{\mathcal{D}} \\ \lambda \cap \mathcal{E}^*(\Sigma^*) = \varnothing}} q_{\Lambda_L^*}(\lambda; \beta^*, h^*) \\ &\leqslant L^C \exp\{\mathfrak{F}^* L\} \sum_{\substack{\lambda \notin \mathfrak{T}_{\mathcal{D}} \\ \lambda \cap \mathcal{E}^*(\Sigma^*) = \varnothing}} q_{\Lambda_L^*}(\lambda; \beta^*, h^*)\,. \end{aligned} \tag{6.33}$$

We apply Proposition 4.6.1, point 2..

$$\sum_{\substack{\lambda \notin \mathfrak{T}_{\mathcal{D}} \\ \lambda \cap \mathcal{E}^*(\Sigma^*) = \varnothing}} q_{\Lambda_L^*}(\lambda; \beta^*, h^*) \leqslant |\partial \mathcal{S}_1| \exp\{-\kappa \rho_L\} \exp\{-\tau(t_r^L - t_l^L; \beta)\}\,. \tag{6.34}$$

This proves the first statement.

2. Suppose that the minimum of the variational problem is given by $\mathcal{W}$, $\mathfrak{F}(\mathcal{W}) = \mathfrak{F}^*$. Then there exists $\delta > 0$ such that $\mathfrak{F}(\mathcal{D}) = \mathfrak{F}^* + \delta$. We estimate $\mathfrak{P}_L^{ab}[\{\lambda \notin \mathfrak{T}_{\mathcal{W}}\}; \beta, h]$ in several steps. Notice that condition b1 is always satisfied.
1. The probability that condition b2 is satisfied, but not b3, can be estimated by Proposition 4.6.1, point 2.; it is smaller than $\mathcal{O}(L^{C+1})/L^{\kappa M}$.
2. The probability that condition b4 is satisfied, but not b5, is estimated in the same way; it is smaller than $\mathcal{O}(L^{C+1})/L^{\kappa M}$.
3. The probability that conditions b2 and b4 are satisfied, but not b6, can be estimated by Proposition 4.6.1, point 5.[13]; it is smaller than $L^{-\mathcal{O}(M)}$.
4. We estimate the probability that condition b2 is not satisfied. The case with condition b5 is similar. If λ does not intersect Σ^*, then this probability is smaller than $\exp\{-\delta L + \mathcal{O}(\log L)\}$, since $\mathfrak{F}(\mathcal{D}) = \mathfrak{F}^* + \delta$.
Suppose that there exist s_1 and s_2, with $\lambda(s_i) \in \Sigma^*$, $\lambda(s) \cap \Sigma^* = \varnothing$ for all $s < s_1$ and $\lambda(s) \cap \Sigma^* = \varnothing$ for all $s > s_2$. Let $p_i^L \doteqdot \lambda(s_i)$, $i = 1, 2$. Under these conditions, b2 is not satisfied if and only if $p_1^L \notin I_1^L$. Let $\mathcal{C}(p_1^L, p_2^L)$ be the polygonal curve from t_l^L to p_1^L, then from p_1^L to p_2^L, and finally from p_2^L to t_r^L. Then the probability of this event is bounded above by

$$\begin{aligned}&\sum_{p_1^L \in \Sigma^* \setminus I_1^L} \sum_{p_2^L \in \Sigma^*} \exp\{-\mathfrak{F}(\mathcal{C}(p_1^L, p_2^L))\} \leqslant \\ &\qquad \mathcal{O}(L^2)\, \min\big\{\exp\{-\mathfrak{F}(\mathcal{C}(p_1^L, p_2^L))\}\,|\, p_1^L \in \Sigma^* \setminus I_1^L\,,\, p_2^L \in \Sigma^*\big\}\,. \end{aligned} \tag{6.35}$$

[13]Notice that if $\mathcal{W}$ is the minimum, then $\tau_{\rm bd} < \tau((1,0))$, by Jensen's inequality.

Suppose that $\mathcal{C}$ denotes the polygonal line giving the minimum; scaled by $1/L$ we get a polygonal line in Q, denoted by $\mathcal{C}^*$, from A to some point P_1^*, then from P_1^* to P_2^*, and finally from P_2^* to B. Let ϑ^* be the angle between the straight line from A to P_1^* with the wall. We have

$$\mathfrak{F}(\mathcal{C}) = L\mathfrak{F}(\mathcal{C}^*) \geqslant L(g(\vartheta^*, a) + g(\vartheta_{\mathrm{Y}}, b)) . \tag{6.36}$$

By hypothesis

$$|\vartheta^* - \vartheta_{\mathrm{Y}}| \geqslant \frac{1}{L^{1/2}} \mathcal{O}((M \log L)^{1/2}) . \tag{6.37}$$

Therefore (use a Taylor expansion of g around ϑ_{Y} and the monotonicity of $g(\vartheta, x)$ on $[0, \vartheta_{\mathrm{Y}}]$, respectively $[\vartheta_{\mathrm{Y}}, \pi/2]$) there exists a positive constant α such that

$$\begin{aligned} \mathfrak{F}(\mathcal{C}^*) &\geqslant g(\vartheta_{\mathrm{Y}}, a) + g(\vartheta_{\mathrm{Y}}, b) + \frac{\alpha M \log L}{L} \\ &= \mathfrak{F}^* + \frac{\alpha M \log L}{L} . \end{aligned} \tag{6.38}$$

We conclude that the probability, that condition b2 is not satisfied, is bounded above by $O(L^{C+2})/L^{\alpha M}$. If M is large enough, the second statement of the theorem follows.

3. The proof of the third statement of the theorem is similar.

□

For a given value of the parameters β and h, it is thus possible to define a deterministic set in which the phase separation is contained with probability going to 1 when L goes to ∞. The sets obtained are essentially optimal. Indeed, we proved that the fluctuations of the parts of the phase separation line which are not pinned to the wall can be neglected at a scale $\mathcal{O}(\sqrt{L \log L})$; on the other hand, we know that the typical fluctuations are of order $\mathcal{O}(\sqrt{L})$. Similarly, we proved that there are typically no excursion out of a band of height $\mathcal{O}(\log L)$ for the part of the phase separation line pinned to the wall, which is characteristic of a gas of excitations with exponentially decaying weights.

We would like to identify this deterministic set with the interface, that is we have to show that it separates + and − phase. This is easily done using Lemma A.4.1. Let M be the constant of Theorem 6.4.1 and $\mathfrak{T}$ be the set of Theorem 6.4.1 corresponding to β and h. Let us denote the two connected components of the set (see (D111), p. 108)

$$\Lambda_L \setminus \big(\mathcal{N}^{K' \log L}(\partial \Lambda \cup \mathfrak{T})\big) \tag{6.39}$$

by Λ_L^+ (for the component which is above $\mathfrak{T}$) and Λ_L^-. We first have

$$\langle \sigma_A \rangle_{\Lambda_L}^{ab,\beta,h} = \sum_{\lambda \in \mathfrak{T}} \langle \sigma_A | \lambda \rangle_{\Lambda_L}^{ab,\beta,h} \mathfrak{P}_L^{ab}[\lambda; \beta, h] + \sum_{\lambda \notin \mathfrak{T}} \langle \sigma_A | \lambda \rangle_{\Lambda_L}^{ab,\beta,h} \mathfrak{P}_L^{ab}[\lambda; \beta, h] . \tag{6.40}$$

But Lemma A.4.1 implies that, for any $\lambda \in \mathfrak{T}$ and for any $A \subset \Lambda_L^+$,

$$|\langle \sigma_A | \lambda \rangle_{\Lambda_L}^{ab,\beta,h} - \langle \sigma_A \rangle^+| \leqslant |A| L^{-\mathcal{O}(M)} . \tag{6.41}$$

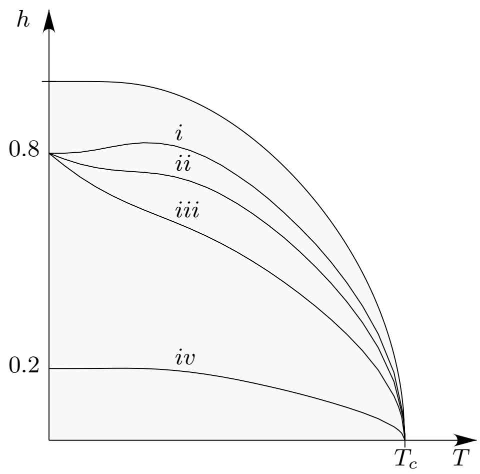

FIGURE 6.7. A sequence of phase transition lines, separating the phase in which the interface is a straight line and the phase in which it is pinned to the wall. The shaded area correspond to the value of (T, h) so that $\tau_{\text{bd}}(\beta, h) < \tau((1,0);\beta)$. The four curves correspond to: i) $a = 0.1$, $b = 0.1$; ii) $a = 0.1$, $b = 0.2$; iii) $a = 0.1$, $b = 0.4$; iv) $a = 0.4$, $b = 0.4$. Observe that the system in case i) exhibits reentrance (see also Fig. 6.8).

Therefore, by Theorem 6.4.1,

$$|\langle\sigma_A\rangle_{\Lambda_L}^{ab,\beta,h} - \langle\sigma_A\rangle^+| \leqslant |A| L^{-\mathcal{O}(M)} . \tag{6.42}$$

This shows that, if B is a finite subset of $\mathbb{Z}^2$ such that $B \subset \Lambda_L^+$ when L is large enough, the expectation value of any B-local function converges to its expectation value in the $+$ phase. Of course we can make the same argument for Λ^-. The interpretation of $\mathfrak{T}$ as an interface is therefore justified.

6.5 Phase transition and reentrance

The results of the preceding section show that, when the parameters a and b are well-chosen, the system under consideration can undergo a phase transition from a phase in which the interface is pinned to the wall on a macroscopic distance to a phase in which it does not touch the wall. It is interesting to consider the corresponding phase diagram. Figure 6.7 shows a set of phase transition lines, depending on the parameters a and b, in the T-h plane ($T = 1/k\beta$ being the temperature). The shaded area corresponds to the set of parameters

$$\{(T, h) \,:\, \tau_{\text{bd}}(\beta, h) < \tau((1,0);\beta)\} . \tag{6.43}$$

In other words, the boundary of that region is the wetting transition line: If we set $a = b = 0$, then for values of the temperature and boundary magnetic field inside this set, the phase separation line is pinned to the wall microscopically (partial wetting), while for values of the parameters outside this set it takes off and fluctuates far from the wall (complete wetting). Notice that in the continuum limit it is not possible to distinguish these two kinds of behaviour: Macroscopically, the interface is always on the wall in this case. Clearly, for $a, b > 0$, the phase transition line must be inside the shaded region.

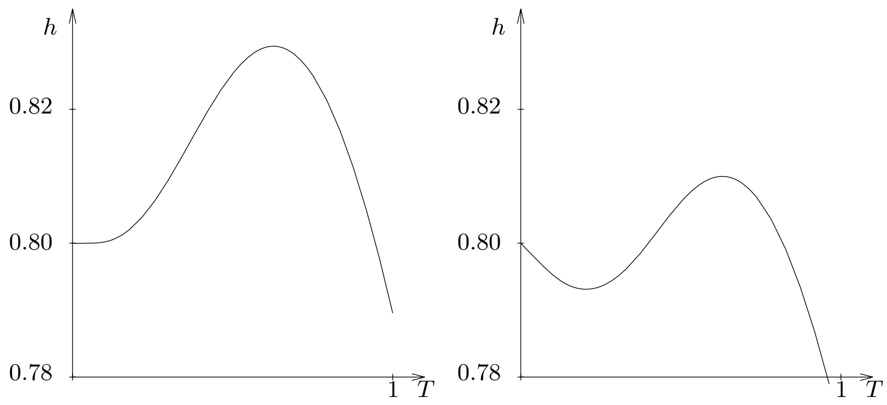

FIGURE 6.8. This figure shows part of the phase coexistence line for $a = 0.1, b = 0.1$ (left), and $a = 0.1, b = 0.12$ (right). For values of the parameters T and h below these curves the interface is pinned, while it is a straight line above these curves. In the second case, the system has even one more transition in temperature for h slightly smaller than 0.8.

"Above" the phase transition line, the system's interface is the straight line, while under the curve it is pinned.

An interesting phenomenon occurs for some values of the parameters (see Fig. 6.7 and 6.8). Suppose $a = b = 0.1$ and h slightly above 0.8. At very low temperature, the interface does not touch the wall; if we increase the temperature, then there is a first transition and the interface becomes tied to the wall; if we increase further the temperature, then a second transition takes place and the interface is again the straight line; finally, at $T = T_c$, the system becomes disordered. In such a case, the system is said to show *reentrance*. In fact for slightly different values of a and b, there can even be one more transition, as shown in Figure 6.8.

Another way to discuss these transitions is by taking the thermodynamic in the half-plane: $\Lambda_L \nearrow \mathbb{L}$, $L \to \infty$. Then depending on whether the interface is pinned to the wall or not, the system will be in the + or − phase. Then reentrance manifests itself in the following way: by increasing the temperature starting from a sufficiently low one, the system is successively in the +, −, +, − and disordered phases.

6.6 Finite size effects for the 2-point function

The purpose of this last section is to show that the results of Propositions 6.3.1 and 6.3.2 have an interesting high-temperature counterpart. Let A and B be two distinct points in Q, and let t_l^L and t_r^L be the two points in Λ_L^* which are closest to $L \cdot A$ and $L \cdot B$. The results of the preceding sections can be written as[14]

$$\left| \frac{1}{\|t_l^L - t_r^L\|_2} \log \langle \sigma(t_l^L)\sigma(t_r^L) \rangle_{\Lambda_L^*}^{ab,\beta^*,h^*} + \frac{\mathfrak{F}^*}{\|B - A\|_2} \right| \leqslant M \frac{\log L}{L}, \tag{6.44}$$

[14]Looking at the proofs of Propositions 6.3.1 and 6.3.2 (and Theorem 6.4.1), observe that the fact that t_l^L and t_r^L are on the boundary is not necessary (except for the interpretation of the open contour as a phase separation line at low temperature).

where M is a sufficiently large constant.
The quantity

$$\overline{\alpha}(\beta^*, h^*) \doteqdot \lim_{L\to\infty} -\frac{1}{\|t_l^L - t_r^L\|_2} \log \langle \sigma(t_l^L)\sigma(t_r^L)\rangle_{\Lambda_L^*}^{ab,\beta^*,h^*} \tag{6.45}$$

is what we called "long" correlation length to distinguish it from the ("short") correlation length (the massgap)

$$\alpha(\beta^*) = -\lim_{L\to\infty} \frac{1}{\|t_r^L - t_l^L\|_2} \log \langle \sigma(t_l^L)\sigma(t_r^L)\rangle^{\beta^*}. \tag{6.46}$$

This last quantity obviously does not depend on h since the Gibbs state is unique at high temperature. On the other hand, the "long" correlation length, $\overline{\alpha}$, does depend on the magnetic field (at least for well-chosen points A and B). Therefore this yields an example where these two quantities are not equivalent[15]. More precisely, for well-chosen points A and B, we can find $\beta^* < \beta_c$ and a finite constant $h_c^*(A, B; \beta^*)$ such that

$$\alpha(\beta^*) = \overline{\alpha}(\beta^*, h^*) \Leftrightarrow h < h^*(A, B; \beta^*). \tag{6.47}$$

The terminology "short" and "long" correlation lengths has been chosen by analogy to the terminology of [SML] where the relation between "short" and "long" long-range order was discussed.

As a consequence of the possible discrepancy between α and $\overline{\alpha}$, we have a finite size effect at high temperature. Indeed, suppose you consider a *finite* box $\Lambda \subset \mathbb{Z}^2$. Then taking two spins in the box and trying to compute an approximation of the massgap by computing the correlation length between these two spins may not yield an approximation of the correct quantity, α, but instead could give an approximation of $\overline{\alpha}$.

Remark. Although the only quantities involved are 2-point functions, and all coupling constants are ferromagnetic, the system exhibits reentrance, which is a non-monotonic behaviour.

[15]Of course, when $h = 1$, $\mathcal{W}$ is never the minimum of the variational problem (by Jensen's inequality) and therefore this does not contradict the result of Lemma 3.1.3.

Chapter 7

Large deviations

This chapter is dedicated to the study of the large (and moderately large) deviations of the magnetization in the 2D Ising model with a boundary magnetic field[1]. The study of large deviations in the Ising model has been the subject of many works recently (see the historical remarks of Section 7.1), but none of the previous works has ever considered the effect of the boundary conditions as more than a nuisance. The techniques of the previous chapters give a very powerful set of tools to study this kind of problem. Another novelty in our approach to the problem of large deviations (and of the continuum limit) is that we do not have to suppose that we know how to solve the corresponding variational problem: We do not need stability results, neither unicity, nor even existence of the solution. This is important because there are many cases (even for the relatively simple problem we are considering) in which the variational problem is very complicated (and unsolved!). However, there is a price to pay (at least with current techniques) if we do not use information on the variational problem, we will still be able to obtain convergence in an appropriate continuum limit to the minimum of the variational problem (similarly to what is done in Chapter 6) but we won't have information on the rate of convergence.

This chapter is organized as follows. In Section 7.1 we make some historical and other general remarks. The variational problem is discussed in Section 7.2. Section 7.4 is concerned with the study of subvolumic large deviations. A similar analysis is done for volume order large deviations in Section 7.5. The idea of the proof of such results is given in Section 7.3 so that the reader can first understand the structure of the proof before reading the technical exposition. Finally, we treat the continuum limit in Section 7.6.

7.1 Introduction

Let us consider an Ising model in a square box of side L and with a bulk magnetic field equal to ρ. It can be shown that the rate function describing the volume order large deviations of the magnetization (for any b.c.) is given by the following function (see e.g. [El])

$$I(m|\beta,\rho) = \beta\big[\sup_t (mt - p(t,\beta)) + p(\rho,\beta) - m\rho\big]\,, \tag{7.1}$$

[1] All these results, except the moderately large deviations, are contained in [PV3]

where $p(t,\beta)$ is the (bulk) free energy of the system, which is a strictly convex function. The function $I(m|\beta,\rho)$ is convex and non-negative. We then have

$$\liminf_L \frac{1}{|\Lambda_L|} \log P^{+,\beta,\rho}_{\Lambda_L}[\{\frac{1}{|\Lambda_L|}\sum_{t\in\Lambda}\omega(t)\in\mathcal{O}\}] \geqslant -\inf_{m\in\mathcal{O}} I(m|\beta,\rho)\,, \quad \mathcal{O} \text{ open},$$
$$\limsup_L \frac{1}{|\Lambda_L|} \log P^{+,\beta,\rho}_{\Lambda_L}[\{\frac{1}{|\Lambda_L|}\sum_{t\in\Lambda}\omega(t)\in\mathcal{C}\}] \leqslant -\inf_{m\in\mathcal{C}} I(m|\beta,\rho)\,, \quad \mathcal{C} \text{ closed}. \tag{7.2}$$

When $\beta > \beta_c$ and $\rho = 0$, there are several Gibbs states and the free energy is not differentiable in ρ; its left and right derivatives give the spontaneous magnetization of the system, $\pm m^*(\beta)$. To this non-differentiability corresponds by convex duality a non-trivial affine part of its Legendre transform $\sup_t(mt - p(t,\beta))$, which is constant and minimal on sets of the form $\{(\beta,m) : \beta > \beta_c, -m^*(\beta) \leqslant m \leqslant m^*(\beta)\}$. When $\beta < \beta_c$, or $\rho \neq 0$, the rate function is positive, equal to zero at some unique value $\overline{m}(\beta,\rho)$ and therefore the above equations show that the probability of a large deviation of the magnetization goes to zero exponentially with the size of the system. When $\beta > \beta_c$ and $\rho = 0$, however, the rate function is zero on the interval $[-m^*(\beta), m^*(\beta)]$, and the above equations do not provide any useful information for deviations inside this interval. In such a case, the leading term is no more of order $\mathcal{O}(|\Lambda_L|)$, and we have to study the corrections. The leading order in this case appears to be $\mathcal{O}(\sqrt{|\Lambda_L|})$. The reason for this change of behaviour is the possibility for the system to use the coexistence of phases to make large deviations at minimum cost. Indeed, by separating the two phases, say, by putting a droplet of minus phase inside the plus phase, the increase of free energy only results from the interface between the two phases, since the bulk terms are equal; but the cost (in free energy) of an interface of length $\mathcal{O}(L)$ around a droplet of volume $\mathcal{O}(L^2)$ is only $\mathcal{O}(L)$, and the probability that such a droplet appears in the system behaves like $\exp\{-\mathcal{O}(L)\}$. We see that the large deviations in the phase coexistence region are deeply related to phase separation issues; it is the aim of this chapter to study these problems.

The first work on this problem goes back as far as 1967, when Minlos and Sinai published two papers of major importance [MS1, MS2], which were going to have a lot of influence on later Rigorous Statistical Physics. They studied the large deviations and phase separation in the Ising model at low temperature and any dimension $D \geqslant 2$. They proved that phase separation indeed occurs, and that the droplet of minus phase is "close" to a D-dimensional cube (their notion of closeness was sufficiently weak to allow for the rounding of the corners of the actual Wulff shape at strictly positive temperature). For twenty years, no real progress had been made on this subject (this is not exactly true, since there were some generalizations of the work of Minlos and Sinai: Kuroda, for example, extended part of their results to systems with more than two phases [Ku1, Ku2]). The next important step has been made by Schonmann in 1987, and (using different techniques) by Föllmer and Ort in 1988 [FO, Or]. In these works lower and upper bounds were obtained *non-perturbatively* for the 2D Ising model. However, these bounds were not optimal, and the exact rate (at leading order) of the large deviations was still unknown (or at least unproved). The next breakthrough was done by Dobrushin, Kotecký and Shlosman [DKS1, DKS2], who announced, in the late 80s, exact large deviations bounds. Moreover they were able to obtain a precise description of the typical configurations, showing, among other results, that the droplet indeed has the Wulff shape. Their results are valid for the 2D Ising model at low temperature with periodic boundary conditions. This work was continued in [DS1, DS2], where very precise description of moderate deviations is done for the same model. After the announcement of the results of [DKS1], Pfister proved similar results for +-boundary conditions [Pf1]. More importantly, he proved several crucial estimates non-perturbatively; in particular, he obtained sharp upper bounds on the probability of large contours (the first version of Lemma 4.2.6). His basic tools were correlation inequalities and duality, but for some part of the analysis, he still had to use cluster expansion techniques. Consequently, his results also are valid only at low temperature. His techniques, however, are less sensitive to the boundary conditions, and works for example in the case of periodic boundary conditions. Notice that the results *depend* on the choice of boundary conditions (which should not be surprising, since we are looking at surface effects). In 1994–95, using the non-perturbative estimates of Pfister and several important new ideas about how to treat the large deviations in the phase of small contours, Ioffe was able to obtain *non-perturbatively* the exact large deviations rate [I1, I2]. In none of these works, however, the boundary effects were treated as more than an annoying complication (this is the reason why only periodic boundary condition was treated in [DKS1], and why m was supposed to be not too different from m^* in [Pf1] and [I1, I2]); the effects of the boundary was discussed on a qualitative level in [Sh1]; rigorous studies in the case of free and +-b.c. have been done in [CGMS] and [ScSh2]. The first serious analysis of the effect of the boundary was done in [PV3] (a large part of this thesis comes from this paper). We were able not only to study what happens when the droplet becomes too large to fit in the box without being deformed, but also to analyze the effect of a real boundary magnetic field on the large deviations. A suitable continuum limit was studied and provided a precise description of the typical configurations. In particular, these results gave an accurate description of the wetting phenomenon in the canonical ensemble [PV1][2]. Ioffe (with Schonmann) has recently announced results for deviations of any order; in particular the transition between exponential and Gaussian behaviour for deviations of order $\mathcal{O}(L^{4/3})$ is proved non-perturbatively. A first step in the direction of obtaining the next-to-leading order behaviour has recently been made in [DH1, DH2], where it is proved that the phase separation line of an Ising model (with boundary condition corresponding to what is called d-b.c. in this thesis) with a volume constraint, converges to some random process.

Other works on large deviations in other systems are [ACC], in which exact non-perturbative large deviations estimates are obtained for the 2D Bernoulli percolation model; [DePi, Pi], where non-perturbative bounds on the large deviations are obtained for the percolation, Ising and Potts models in $\mathrm{D} \geqslant 3$. Recently, there has been a lot of interest in the large deviations and phase separation of the 2D Ising model with Kac potential, [BMP, BBP].

These results on large deviations for the $2D$ Ising model have been used to derive interesting informations on the behaviour of the model when a vanishingly small magnetic field is applied, providing a precise description of metastability [ScSh1, ScSh2, GS]. They also give rise to useful tools in the study of the dynamics of the model, see for example [CGMS, ScSh3].

[2]Such a problem has previously been studied in the simpler case of the SOS model, see [CDR, MR].

7.2 The variational problem

In this section, we discuss a thermodynamical variational problem which plays a fundamental role in the rest of this chapter. We use several elementary results from convex analysis and refer to Appendix B for a brief introduction to these notions and for the notations and definitions. Contrarily to the variational problem of Chapter 6, the present one is quite complicated; there are even some situations in which the solution is not known. For this reason, we do not give proofs of the main assertions in this section, but refer to the existing literature.

7.2.1 The Generalized Isoperimetric Inequality

Let $\mathcal{W}$ be a compact convex body whose support function we denote by $\tau_{\mathcal{W}}$. Let ∂V be a closed rectifiable curve, which is the boundary of an open set V. Let $(u(s), v(s))$ be a unit-speed parameterization of ∂V (counterclockwise). We introduce a functional on such curves defined by

$$\mathfrak{F}_{\mathcal{W}}(\partial V) \doteqdot \int_0^{|\partial V|} \tau((-\dot{v}(s), \dot{u}(s))\,\mathrm{d}s\,. \tag{7.3}$$

First observe that this functional is positive. Indeed, consider a translate $\mathcal{W}' \doteqdot \mathcal{W} + a$ of $\mathcal{W}$ which contains the origin as an interior point; the support function of this translate is strictly positive at $x \neq 0$. Moreover, since $\tau_{\mathcal{W}'}(x) = \tau_{\mathcal{W}}(x) + \langle a, x\rangle$ and

$$\int_{\partial V} \langle a, (-\dot{v}(s), \dot{u}(s))\rangle\,\mathrm{d}s = 0\,, \tag{7.4}$$

the functional is independent of a.

The main result of this subsection is the following fundamental inequality, which generalizes the usual isoperimetric inequality.

Theorem 7.2.1 (Generalized Isoperimetric Inequality)**.** *Let $\mathcal{W}$ be a convex body in $\mathbb{R}^2$ and V be an open set in $\mathbb{R}^2$ whose boundary is a rectifiable curve ∂V. The Lebesgue measure of $\mathcal{W}$ and V are denoted by $|\mathcal{W}|$ and $|V|$. Then*

$$\mathfrak{F}_{\mathcal{W}}(\partial V) \geqslant 2|\mathcal{W}|^{1/2}|V|^{1/2}\,. \tag{7.5}$$

Equality holds if and only if V equals, up to translation and dilation, the set $\mathcal{W}$.

This theorem is proved[3] in [D, Tay, Fo] for example. The basic idea of the proof (see [D]) is to express the functional $\tau_{\mathcal{W}}$ as

$$\tau_{\mathcal{W}}(\partial V) = \lim_{\varepsilon\to 0} \frac{|V + \varepsilon\mathcal{W}| - |V|}{\varepsilon}\,, \tag{7.6}$$

where $V + \varepsilon\mathcal{W} \doteqdot \bigcup_{x\in V}(x + \varepsilon\mathcal{W})$, $\varepsilon\mathcal{W} \doteqdot \{\varepsilon x \,:\, x \in \mathcal{W}\}$. The result then follows by applying the Brunn-Minkowski inequality to $V + \varepsilon\mathcal{W}$,

$$|V + \varepsilon\mathcal{W}|^{1/2} \geqslant |V|^{1/2} + |\varepsilon\mathcal{W}|^{1/2}\,. \tag{7.7}$$

The above result can even be completed by strong stability results, the so-called *Bonnesen's inequalities* (see [DKS1] for example), which we do not state since we don't need them in the following.

[3] Under even less restrictive assumptions in some of these references.

7.2.2 Unconstrained variational problem

We first consider the following variational problem [Wu]:

Wulff Variational Problem : Find the minimum of the functional $\mathfrak{F}_{\mathcal{W}}$ among all rectifiable closed curves, which are the boundary of an open set of Lebesgue measure $|V|$.

The solution to this problem is an immediate consequence of the Generalized Isoperimetric Inequality. Indeed it states that the minimum is taken on any translate of the convex body $\mathcal{W}$ after scaling it so that it has volume $|V|$. The value of the functional at the minimum is then equal to $2|V|$.

From a physical point of view, we are first given a positively homogeneous, convex function τ, the surface tension. The minimum of the variational problem is then given by (a suitably rescaled version) of the convex body whose support function is τ, which is called in such a context the *Wulff shape*. It can be obtained through the *Wulff construction*, see Appendix B.

We consider now a more complicated variational problem. Let us denote by $\mathcal{H}$ the half–plane,

$$\mathcal{H} \doteqdot \{x \in \mathbb{R}^2 \,:\, x(2) \geqslant 0\}\,, \tag{7.8}$$

and

$$w \doteqdot \{x \in \mathbb{R}^2 \,:\, x(2) = 0\}\,. \tag{7.9}$$

Let ∂V be a closed rectifiable curve in $\mathcal{H}$ which is the boundary of an open set V. Let us introduce a new functional on such curves. Let $\tau_{\text{bd}} \in \mathbb{R}$ be such that $|\tau_{\text{bd}}| \leqslant \tau((0,-1))$; we set

$$\mathfrak{W}(\partial V) \doteqdot \mathfrak{F}_{\mathcal{W}}(\partial V) + (\tau((0,-1)) - \tau_{\text{bd}})|\partial V \cap w| \tag{7.10}$$

where $|\partial V \cap w|$ is the Lebesgue measure of $\partial V \cap w$.

Physically, w plays the role of a wall, and τ_{bd} is the corresponding wall free energy. The last term in (7.10) gives the modification to the functional in the bulk, $\mathfrak{F}_{\mathcal{W}}$, when the curve touches the wall. The new variational problem is

Winterbottom Variational Problem : Find the minimum of the functional $\mathfrak{W}$ among all rectifiable closed curves inside $\mathcal{H}$, which are the boundary of an open set of Lebesgue measure $|V|$.

We consider three cases.

Case 1. $\tau_{\text{bd}} = \tau((0,-1))$.

In this case, the last term in (7.10) is equal to zero, and we have the same variational problem as before. Therefore, the solution is given by any Wulff shape of Lebesgue measure $|V|$ entirely contained inside $\mathcal{H}$.

Case 2. $|\tau_{\text{bd}}| < \tau((0,-1))$.

In this case the minimum is given by a (suitably rescaled version of [4]) the *Winterbottom shape* which is the convex body (see Fig. 7.1)

$$\mathcal{W}^w \doteqdot \mathcal{W} \cap \{x \in \mathbb{R}^2 \,:\, \langle x, (0,-1)\rangle \leqslant \tau_{\text{bd}}\}\,. \tag{7.11}$$

[4] Observe that the problem is still scale–invariant.

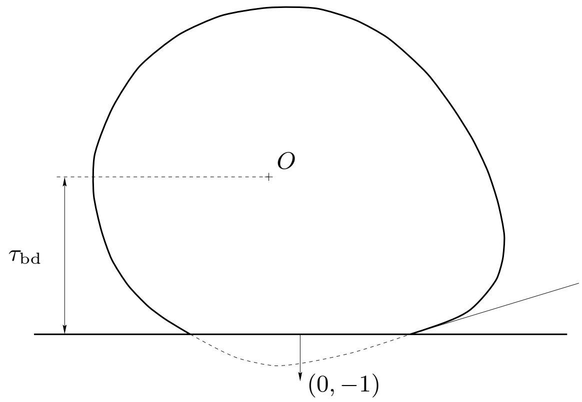

FIGURE 7.1. The Winterbottom shape.

This can be easily understood, using the technique of [KP]. Observe that a *monotonicity principle* holds. Suppose it is possible to find a convex body $\mathcal{C}$ such that

$$\mathfrak{W}(\partial V) \geqslant \mathfrak{F}_{\mathcal{C}}(\partial V)\,, \tag{7.12}$$

and

$$\mathfrak{W}(\partial \mathcal{C}) = \mathfrak{F}_{\mathcal{C}}(\partial \mathcal{C})\,. \tag{7.13}$$

Then applying the Generalized Isoperimetric Inequality to $\mathcal{C}$ yields

$$\mathfrak{W}(\partial V) \geqslant \mathfrak{F}_{\mathcal{C}}(\partial V) \geqslant 2|\mathcal{C}|^{1/2}|V|^{1/2}\,, \tag{7.14}$$

with equality if and only if V is equal to $\mathcal{C}$ (up to translation and dilation). If the Winterbottom shape satisfy these two conditions, then it is necessarily the (unique) minimum of the variational problem. That it satisfies (7.13) is immediate. Let us show that it also satisfy (7.12). Let us introduce

$$\tilde{\tau}(\widehat{x}) = \begin{cases} \tau(\widehat{x}) & \widehat{x} \neq (0,-1)\,, \\ \tau_{\mathrm{bd}} & \widehat{x} = (0,-1)\,. \end{cases} \tag{7.15}$$

Notice that the Legendre transform of $\tilde{\tau}$, $\tilde{\tau}^*$, is the indicator function of $\mathcal{W}^w$ (by the Wulff construction, see Appendix B). Therefore, we must have

$$\tilde{\tau}(\widehat{x}) \geqslant (\tilde{\tau}^*)^*(\widehat{x}) = \tau_{\mathcal{W}^w}(\widehat{x})\,, \tag{7.16}$$

from which (7.12) follows.

Remark. This simple idea is quite powerful. It also allows to prove stability results for such problems from the corresponding stability results in the Wulff case, see [KP].

Case 3. $\tau_{\mathrm{bd}} = -\tau((0,-1))$.
In this case, the problem is degenerate. Indeed in this case the minimum value of $\mathfrak{W}$ is equal to zero and can be reached (for example) by a sequence of rectangles $\mathcal{R}_l$ whose base is on w and has a length l.

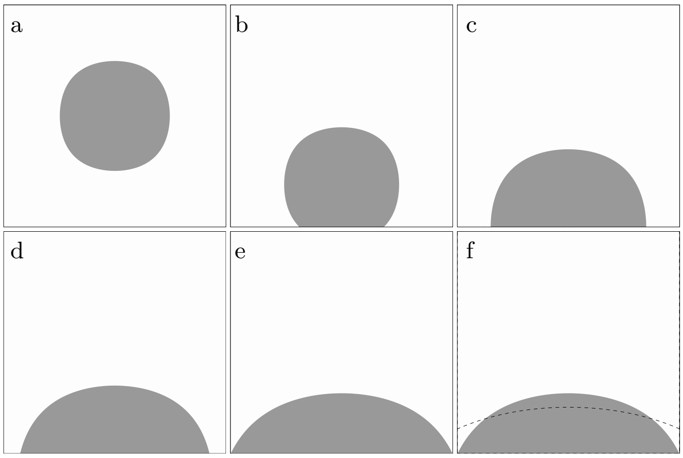

FIGURE 7.2. A sequence of equilibrium shapes when τ and τ_{bd} are the surface tension and wall free energy of the Ising model at parameters β and h. a: $h > h_{\text{w}}(\beta)$; b: $0 < h < h_{\text{w}}(\beta)$; c: $h = 0$; d, e, f: A sequence of droplets for decreasing values of $-h_{\text{w}}(\beta) < h < 0$. The droplet spreads until it begins to touch the vertical sides of the box (e). Further reduction of the magnetic field does not modify the shape of the droplet, but makes it unstable in the sense that the removal of the vertical walls would result in a spreading of the droplet. f: The dashed line shows only a part of the droplet which would be obtained by removing the walls when $0 > h > -h_{\text{w}}(\beta)$. For $h \leqslant -h_{\text{w}}(\beta)$, the droplet remains unchanged (like in e and f) but is completely unstable, in the sense that however large the box is made, the droplet always wet the whole bottom wall.

7.2.3 Constrained variational problem

We now state the most complicated of these variational problems. Let r_1 and r_2 be two positive integers and set

$$Q \doteqdot \{x \in \mathbb{R}^2 \,:\, |x(1)| \leqslant r_1 \,,\, 0 \leqslant x(2) \leqslant 2r_2\}\,. \tag{7.17}$$

The constrained variational problem is given by

The constrained Variational Problem : Find the minimum of the functional $\mathfrak{W}$ among all rectifiable closed curves inside Q, which are the boundary of an open set of Lebesgue measure $|V| \leqslant |Q|$.

This problem is much more complicated in general. If $|V|$ is sufficiently small so that a translate of the minimum of the Winterbottom Variational Problem fits inside the box, then of course it is the solution. In the other cases, the solution is a squeezed Wulff or Winterbottom shape and there are a variety of cases to consider. Most cases have been solved, using the monotonicity principle, in [KP], but there are some cases in which no proof has been given. We don't want to discuss these problems here, since we do not use information about this variational problem in the following. However, we just make some comments: The degenerate case which happened in the Winterbottom Variational Problem when $\tau_{\text{bd}} = -\tau((0,1))$ cannot take place in this new setting. Indeed, take a volume $|V|$ small enough so that a translate of the Winterbottom shape fits inside Q for some $\tau_{\text{bd}} < \tau((0,1))$. If we decrease the value of the wall free energy, then the droplet spreads along the bottom wall. This happens until the wall free energy reaches a critical

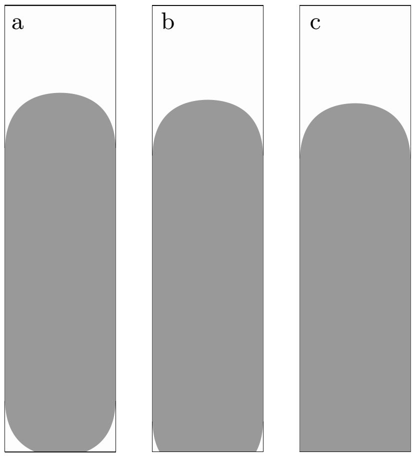

FIGURE 7.3. Sequence of big droplets in a tube for decreasing value of the magnetic field (2D Ising model). a,b: The upper part of the droplet has the Wulff shape, while the lower part has the winterbottom shape; they are joined by a rectangle such that the total volume is conserved; c: The droplet completely wets the lower wall, the droplet is build up from a half Wulff shape and a rectangle.

value $\tau_{\mathrm{bd}}^{\mathrm{c}} > -\tau((1,0))$, after that, the droplet does not change anymore (see Fig. 7.2 and 7.3 for an illustration in the case of the 2D Ising model).

There is one thing that we can prove about the solution of this variational problem, which is true in all cases and which helps us in the following sections: The minimum (if it exists) is necessarily taken on the boundary of a *convex* subset of Q. Let D be a non-convex open subset of Q and $\widehat{D}$ be its convex envelope. First notice that $\partial\widehat{D}\cap w \supset \partial D\cap w$. Let a and b be two points in ∂D such that the part of ∂D between a and b does not belong to $\partial\widehat{D}$. By Jensen's inequality, we decrease the value of the functional $\mathfrak{F}$ if we replace this part of ∂D by a straight line segment (it is the same argument as in Section 6.2). If, by doing this, we increase the length of the part of the droplet which touches the wall, then the value of the functional $\mathfrak{W}$ can only become smaller (since $\tau_{\mathrm{bd}} < \tau((0,-1))$. Therefore $\mathfrak{W}(\partial D) \geqslant \mathfrak{W}(\partial\widehat{D})$. Moreover $\mathrm{vol}\,\widehat{D} \geqslant \mathrm{vol}\,D$ and therefore, scaling [5] $\widehat{D}$ so that it has the correct volume still lowers the value of $\mathfrak{W}$ (notice that this functional is always positive on closed curves since $\tau_{\mathrm{bd}} \geqslant -\tau((0,-1))$).

7.2.4 Some remarks in the case of the 2D Ising model

In the rest of this chapter, we consider the case of the 2D Ising model. The functional is $\mathfrak{W} = \mathfrak{W}(\beta, h)$, with τ and τ_{bd} given respectively by the surface tension and wall free energy of the 2D Ising model at inverse temperature $\beta > \beta_c$ and boundary magnetic field h. For this model, the Wulff shape has neither corners, nor facets for any $\beta_c < \beta < \infty$. Using the symmetries of τ, we can show that the angles of contact between the Winterbottom

[5] Observe that it would be more complicated if both the top and bottom wall had a modified wall free energy; indeed, shrinking the droplet would prevent it from touching both walls and may therefore increase the value of the corresponding functional. This should not be surprising, since in such a case we may have a non–convex droplet of minus phase.

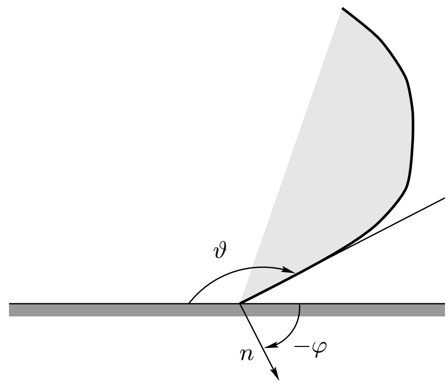

FIGURE 7.4. Angles ϑ and φ

shape and the wall are given by the Herring-Young equation[6],

$$\cos\vartheta\,\tau(\vartheta) - \sin\vartheta\,\tau'(\vartheta) = \tau_{\mathrm{bd}}\,. \tag{7.18}$$

This is a consequence of Lemma B.1.6. Indeed, let n be the unit normal to the interface defining the angle ϑ (see Fig. 7.4); from this lemma we have[7]

$$\mathrm{grad}\tau(\boldsymbol{n})(2) = -\tau_{\mathrm{bd}}\,. \tag{7.19}$$

Let us show that this equation is equivalent to the Herring-Young equation. In polar coordinates, the surface tension is

$$\tau(r,\varphi) \doteqdot r\,\tau(\varphi)\,, \tag{7.20}$$

so that, using $x(2) = r\sin(\varphi)$, $\varphi + \vartheta = \pi/2$ we obtain $(\tau(1,\varphi) \equiv \tau(\varphi))$

$$\begin{aligned}\mathrm{grad}\tau(n)(2) &= \sin(\varphi)\tau(\varphi) + \cos(\varphi)\tau'(\varphi)\\ &= \cos(\vartheta)\tau(\vartheta) - \sin(\vartheta)\tau'(\vartheta)\\ &= \tau_{\mathrm{bd}}\,.\end{aligned} \tag{7.21}$$

7.3 Principle of the proof of large deviations

Before studying large and moderately large deviations in the Ising model, we think it may be useful to give a short sketch of the main steps of the proof which is composed of several technical estimates. We consider the case of volume order large deviations, the idea of the proof being identical in the case of moderately large deviations.
The problem is to obtain an estimate of the probability of the event

$$\mathcal{A}(m,c) \doteqdot \{\omega\,:\,|\sum_{t\in\Lambda_L}\omega(t) - m|\Lambda_L||\leqslant|\Lambda_L|\,L^{-c}\}\,, \tag{7.22}$$

[6]As before, $\tau(\vartheta) \doteqdot \tau(\boldsymbol{n}(\vartheta))$, $\boldsymbol{n}(\vartheta) \doteqdot (\cos\vartheta,\sin\vartheta)$, and $\tau'(\vartheta) \doteqdot \frac{\mathrm{d}}{\mathrm{d}\vartheta}\tau(\vartheta)$.
[7]Since τ is differentiable, the subdifferential at $\boldsymbol{n}$ is unique and is equal to $\mathrm{grad}\tau(\boldsymbol{n})$.

which is exact at leading order. The box Λ_L is given by

$$\Lambda_L \doteqdot \{t \in \mathbb{Z}^2 \,:\, -r_1 L \leqslant t(1) < r_1 L\,,\, 0 \leqslant t(2) < 2r_2 L\}\,. \tag{7.23}$$

The proof is done in two main steps: First, we prove a lower bound on this probability, second we prove an upper bound which coincides with the lower one at leading order.

7.3.1 The lower bound

The idea behind the proof of the lower bound is quite simple. Let us first suppose that we know the minimum of the variational problem is given by some curve $\mathcal{C} \subset Q$. We then scale the curve $\mathcal{C}$ by a factor L so that the curve obtained, $L\mathcal{C}$, is in Λ_L^* (Λ_L^* playing the same role for $L\mathcal{C}$ as Q for $\mathcal{C}$). We then construct a polygonal approximation of the curve $\mathcal{C}$ with vertices in Λ_L^*. On each side of the polygonal curve we construct a box (either a square box or an elliptical set). We then sum over all contours going through all these vertices and included in the boxes. Conditioned on the presence of such a contour, the probability of the event $\mathcal{A}(m, c)$ is close to 1 (since the expectation value of the magnetization in this case is close to m). It is therefore enough to estimate the probability of such contours, which can be done using the tools of Chapter 4 (the polygonal approximation and the boxes are constructed in such a way as to ensure that the hypotheses of the relevant propositions of that chapter are satisfied).

If the minimum of the variational problem is not known (and possibly does not even exist), then a slight modification of the proof is required. Instead of considering the curve $\mathcal{C}$ which realizes the minimum, we consider any convex body in Q with the right volume and do the same proof as before with its boundary, taking care of obtaining only estimates which are uniform in the convex body. Optimization over such curves provides the lower bound.

7.3.2 The upper bound

The proof is slightly more involved, but the idea is still quite simple. The main idea is to introduce some intermediate scale $L^{\delta'}$ and to work only on that scale. More precisely, we consider the family of $L^{\delta'}$-large contours in the configuration and make a coarse-grained description of them. The polygonal lines thus obtained have a volume close to the volume of the original contours, and are constructed in such a way as to make possible the application of Lemma 4.2.8. Using this lemma, we show that the probability of a given family of polygonal lines (defined as the probability of the set of configurations whose large contours' coarse-graining are these polygonal lines) decays exponentially with their surface tension.

The next step is to find a class of polygonal lines which are typical in $\mathcal{A}(m, c)$. To do this, we show that the probability that the volume of the minus phase delimited by the polygonal lines is "very" different from $|\Lambda_L|(m^* - m)/2m^*$, knowing that $\mathcal{A}(m, c)$ occurs, is vanishingly small.

We thus obtain that the probability of $\mathcal{A}(m, c)$ is smaller than the probability of all families of polygonal lines delimiting a volume of minus phase close to $|\Lambda_L|(m^* - m)/2m^*$. The conclusion follows from the fact that the surface tension of any such family of polygonal lines must be larger or equal than the surface tension of the solution of the variational problem, and that the number of such families can be controlled.

7.4 Moderately large deviations

Let $J(e)$ be given by (3.30) with $\beta > \beta_c$ and $h \geqslant 0$. Let

$$\Lambda_L \doteqdot \{t \in \mathbb{Z}^2 \,:\, -r_1 L \leqslant t(1) < r_1 L\,,\, 0 \leqslant t(2) < 2r_2 L\}\,. \tag{7.24}$$

In this and the following sections, we would like to be able to work with a negative magnetic field $h < 0$; however several of the tools developed in the previous sections are valid only in the case $h \geqslant 0$. The simple solution to this problem is to realize that $\mu_{\Lambda_L}^{+,\beta,h} = \mu_{\Lambda_L}^{\pm,\beta,-h}$, where $\mu_{\Lambda_L}^{\pm,\beta,h}$ is the Gibbs measure with $\pm$-boundary condition, which is defined by the configuration $\omega^{\pm}$[8],

$$\omega^{\pm}(t) \doteqdot \operatorname{sign} t(2)\,. \tag{7.25}$$

Therefore it is possible to replace negative magnetic fields by their absolute value, provided we change the boundary condition accordingly. The main difference between these two of boundary conditions is that the contours of any configuration satisfying the Λ_L^+-b.c. are closed, whereas the contours of a configuration satisfying the $\Lambda_L^{\pm}$-b.c. always contain one (unique) open contour; see Section 7.4.2 for more details. When we speak, in the following, of Gibbs measure with negative magnetic field $h < 0$ we always mean the Gibbs measure $\mu_{\Lambda_L}^{\pm,\beta,|h|}$.

This section is devoted to the determination of the exact leading term in the probability of having a moderately large deviation of the magnetization. More precisely, let $C_1 > 0$ and $c > \nu > 0$; we want to compute the asymptotic behaviour of the expectation value of the following event

$$\mathcal{A}(C_1,\nu,c) \doteqdot \{\omega \,:\, |\sum_{t\in\Lambda_L} \omega(t) - m^*(\beta) + C_1|\Lambda_L|L^{-\nu}| \leqslant |\Lambda_L|L^{-c}\}\,, \tag{7.26}$$

with the Gibbs measures $\mu_{\Lambda_L}^{+}$ and $\mu_{\Lambda_L}^{\pm}$.

This case is technically simpler than the volume order large deviations one. Indeed, in the present case, if L is sufficiently large, then there is always enough room in the box Λ_L to put a translate of the Wulff (or Winterbottom depending on the value of h) shape of volume $(C_1/2m^*)|\Lambda_L|L^{-\nu}$ (at least when the system is not in the regime of complete wetting, i.e. $h > -h_{\mathrm{w}}(\beta)$). Therefore we can work with a single curve (the solution of the variational problem) which simplifies considerably the analysis; see Section 7.5 for the treatment of the more complicated case when the volume constraint plays an important role.

The analysis contain two parts, similar to what has been done in Chapter 6. We first prove lower bounds in Sections 7.4.1 ($h \geqslant 0$) and 7.4.2 ($h < 0$), then we prove upper bounds in Sections 7.4.3 ($h \geqslant 0$) and 7.4.4 ($h < 0$). In Section 7.4.5 we discuss heuristically the behaviour of the typical fluctuations of the magnetization when there is complete wetting.

[8] This corresponds to the d-boundary condition, where d is the horizontal line going through $(0,0)$.

7.4.1 Lower bound: Positive magnetic field

Theorem 7.4.1. *Suppose* $J(e)$ *is given by* (3.30) *with* $\beta > \beta_c$ *and* $h \geqslant 0$. *Let* $C_1 > 0$, $1 > c > \nu > 0$ *be given. Let* $\mathfrak{W}^*(\beta, h, C_1, \nu)$ *be the minimum of the functional* $\mathfrak{W}(\cdot; \beta, h)$ *(without constraints) among curves enclosing a volume* $|Q|(C_1/2m^*)$. *Then there exists* $L_0 = L_0(\beta, h, C_1, \nu, c)$ *such that, for all* $L \geqslant L_0$,

$$P^{+,\beta,h}_{\Lambda_L}[\mathcal{A}(C_1, \nu, c)] \geqslant \exp\{-L^{1-\nu/2}\, \mathfrak{W}^*(\beta, h, C_1, \nu) - \mathcal{O}(L^{1/2-\nu/4} \log L)\}\,.$$

Proof. We first consider the case $h > 0$.
Let $\mathcal{C} = \mathcal{C}(\beta, h)$ be the curve which realizes the minimum of the unconstrained variational problem, corresponding to the parameters β and h, and enclosing a volume $|Q|(C_1/2m^*)L^{-\nu}$. We suppose that L is large enough to ensure that there exists a translate of $\mathcal{C}$ such that

- If $h \geqslant h_{\mathrm{w}}$, then it is entirely contained inside Q and is at a finite distance from ∂Q.
- If $h < h_{\mathrm{w}}$, then it is contained in Q, its bottom affine part is on w_Q and it is at a finite distance from the other sides of Q.

We still denote by $\mathcal{C}$ this translate. The idea of the proof is the following: We first construct a polygonal approximation of $\mathcal{C}$, then we sum over all contours going through each vertex of a rescaled version of this polygonal approximation (respecting the order). The probability of such contours can then be estimated using the results of Chapter 4. The proof is divided in five steps.

Step 1. Definition of a polygonal approximation of $\mathcal{C}$.

Let $L \in \mathbb{N}$; we set

$$\delta_L \doteqdot L^{-\rho}\,, \qquad \rho = 1/2 + \nu/4\,. \tag{7.27}$$

We define a polygonal approximation $\mathcal{P}_L$ of $\mathcal{C}$.

1. If $h < h_{\mathrm{w}}$, we approximate the part of $\mathcal{C}$ which intersects the wall, starting from the left extremity, by a polygonal curve with vertices on $\mathcal{C}$ and sides of length δ_L, except possibly for the last one which may be shorter.
2. We approximate the remaining part $\mathcal{C} \setminus w_Q$ by a polygonal curve with vertices on $\mathcal{C}$ and sides of length δ_L, except possibly the last one which may be shorter.

Since $\tau(\cdot)$ is convex, Jensen's inequality implies ($\boldsymbol{n}(s)$ is the unit normal at s)

$$\int_{\mathcal{C}} \tau(\boldsymbol{n}(s))\, \mathrm{d}s \geqslant \int_{\mathcal{P}_L} \tau(\boldsymbol{n}(s))\, \mathrm{d}s\,. \tag{7.28}$$

Therefore,

$$\mathfrak{W}(\mathcal{C}) \geqslant \mathfrak{W}(\mathcal{P}_L)\,. \tag{7.29}$$

Moreover, we have

$$\mathrm{vol}\, \mathcal{P}_L = \mathrm{vol}\, \mathcal{C} - \mathcal{O}(L^{-2\rho}) = \mathrm{vol}\, \mathcal{C} - \mathcal{O}(L^{-1-\nu/2})\,. \tag{7.30}$$

Indeed, consider two successive vertices of $\mathcal{P}_L$. Let $\rho_{\min}$ be the minimal radius of curvature of the Wulff shape (which is strictly positive by positive stiffness and of order $L^{-\nu/2}$ since the volume of $\mathcal{C}$ is $\mathcal{O}(L^{-\nu})$); observe that $\rho_{\min} \gg \delta_L$. Consider the circle of radius $\rho_{\min}$ which goes through these two vertices and which has its largest part inside $\mathcal{P}_L$ (see Fig. 7.5).

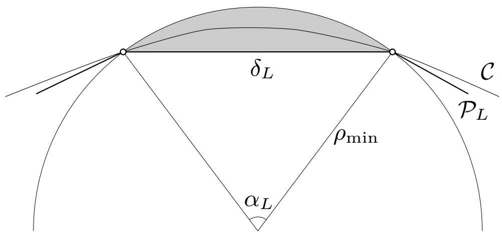

FIGURE 7.5. Estimation of the difference in volume between the curve $\mathcal{C}$ and its polygonal approximation $\mathcal{P}_L$ (This is only a sketch, in reality $\rho_{\min} \gg \delta_L$).

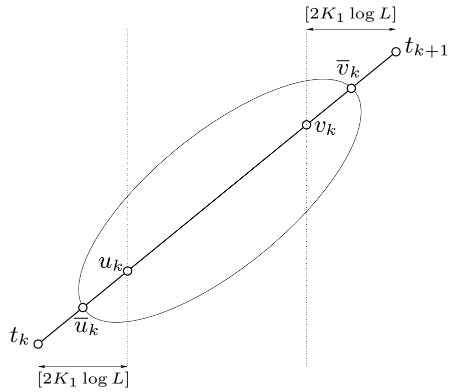

FIGURE 7.6. The points u_k, $\overline{u}_k$, v_k and $\overline{v}_k$ associated to a pair of successive vertices t_k, t_{k+1} not both on the wall. The elliptical set $\mathcal{S}(u_k, v_k, [2K_1 \log L])$ is also indicated.

Then, since $\mathcal{C}$ is convex, has a volume of order $\mathcal{O}(L^{-\nu})$ and from the choice of $\rho_{\min}$, the part of $\mathcal{C}$ between these two vertices is contained inside the set whose border is the segment of straight line connecting the two vertices and the exterior arc of the circle. A direct estimate of the volume of this set yields the result, using the fact that there is $\mathcal{O}(L^{\rho-\nu/2})$ such pairs of vertices (if α_L is as in Fig. 7.5, then the volume of the shaded area in the figure is of order $\mathcal{O}(\rho_{\min}^2 \alpha_L^3)$).

Step 2. Scaling and definition of a set of closed contours $\mathcal{G}_L$

Let $L\mathcal{P}_L$ be the polygon obtained by scaling $\mathcal{P}_L$ by a factor L, translating it by $(0, -1/2)$ and modifying slightly the position of its vertices so that they are in Λ_L^*.
Let $K_1 > 0$ be some sufficiently large constant. We denote by t_k, $k = 1, \ldots, N_L$ the vertices of $L\mathcal{P}_L$, numbering them counterclockwise; we also set $t_{N+1} \equiv t_1$. To each pair of successive vertices t_k and t_{k+1} such that $\max\{t_k(2), t_{k+1}(2)\} > -1/2$ and the corresponding side of $\mathcal{P}_L$ has length δ_L, we associate four other points of Λ_L^*, u_k, $\overline{u}_k$, v_k and $\overline{v}_k$ defined by (see Fig. 7.6)[9]:

- u_k is the point on the vertical line $\{t' \in \mathbb{Z}^{2*} : t'(1) = t_k(1) + [2K_1 \log L]\}$ with $u_k(2)$ minimal and $u_k(2) \geqslant t_k(2) + K_1 \log L\,(t_{k+1}(2) - t_k(2))/(t_{k+1}(1) - t_k(1))$.
- v_k is the point on the vertical line $\{t' \in \mathbb{Z}^{2*} : t'(1) = t_{k+1}(1) - [2K_1 \log L]\}$ with $v_k(2)$ maximal and $v_k(2) \leqslant t_{k+1}(2) - K_1 \log L\,(t_{k+1}(2) - t_k(2))/(t_{k+1}(1) - t_k(1))$.
- $\overline{u}_k \doteqdot ([\frac{1}{2}(t_k(1) + u_k(1))], [\frac{1}{2}(t_k(2) + u_k(2))])$.

[9] We give the construction in the case $|t_{k+1}(1) - t_k(1)| \geqslant |t_{k+1}(2) - t_k(2)|$. The construction for the other case is done symmetrically.

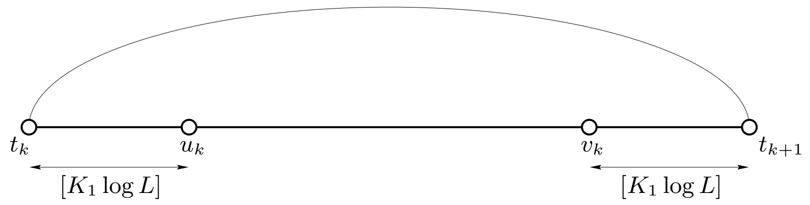

FIGURE 7.7. The points u_k, and v_k associated to a pair of successive vertices t_k, t_{k+1} both on the wall. The (semi-)elliptical set $\mathcal{S}'(u_k, v_k, 2[K_1 \log L])$ is also indicated.

- $\overline{v}_k \doteqdot ([\frac{1}{2}(t_{k+1}(1) + v_k(1))], [\frac{1}{2}(t_{k+1}(2) + v_k(2))])$.

To each such side of $L\mathcal{P}_L$ we associate a set $\mathcal{S}_k \doteqdot \mathcal{S}(u_k, v_k, [2K_1 \log L])$ so that $\overline{u}_k$ and $\overline{v}_k$ belong to the boundary of $\mathcal{S}_k$[10].
To each pair of successive vertices t_k and t_{k+1} such that $\max\{t_k(2), t_{k+1}(2)\} = -1/2$ and the corresponding side of $\mathcal{P}_L$ has length δ_L, we associate two other points of Λ_L^*, $u_k \doteqdot t_k + ([K_1 \log L], 0)$ and $v_k \doteqdot t_{k+1} - ([K_1 \log L], 0)$ (see Fig. 7.7). To each such side of $L\mathcal{P}_L$, we associate a set $\mathcal{S}_k \doteqdot \mathcal{S}'(u_k, v_k, 2[K_1 \log L])$ so that t_k and t_{k+1} belong to the boundary of $\mathcal{S}_k$.
Notice that by construction $\mathcal{S}_k \cap \mathcal{S}_{k'} = \varnothing$, if $k \neq k'$ (and L is large enough).

Let $C' > 0$ be some constant. We define a set of closed contours: $\mathfrak{G}_L$ is the set of all closed contours $\Gamma = \Gamma_{\text{bd}} \cup \Gamma_{\text{bulk}}$ such that

1. Let t_K be the last vertex belonging to Σ^*. Γ_{bd} goes through all sites t_k, $k < K$ belonging to Σ^* successively. The part of Γ_{bd} between t_k and t_{k+1}, $k < K - 1$ is contained in $\mathcal{E}^*(\mathcal{S}_k)$. The part of Γ_{bd} between t_{K-1} and t_K is any fixed contour of minimal length between these two sites.
2. Γ_{bulk} goes through the following sites successively: $t_k, \overline{u}_k, \overline{v}_k, \overline{u}_{k+1}, \overline{v}_{k+1}, \ldots \overline{v}_{N_L - 1}, t_1$. The part of Γ_{bulk} between $\overline{u}_k$ and $\overline{v}_k$ is contained inside $\mathcal{E}^*(\mathcal{S}_k)$. The parts of Γ_{bulk} between $\overline{v}_k$ and $\overline{u}_{k+1}$ or between t_K and $\overline{u}_K$ or between $\overline{v}_{N_L-1}$ and t_1 is any shortest contour between the two sites.

The total length of the fixed parts of Γ is of order

$$\mathcal{O}(L^{\rho - \nu/2} \log L) + \mathcal{O}(L^{1-\rho}) = \mathcal{O}(L^{1/2 - \nu/4} \log L) \,. \tag{7.31}$$

We can now use the tools of Chapter 4 and 5 to estimate the probability of $\mathcal{A}(C_1, \nu, c)$.

Step 3. Estimation of $P_{\Lambda_L}^{+,\beta,h}[\mathcal{A}(C_1, \nu, c) | \{\Gamma; \gamma \neq \Gamma \, s\text{-small}\}]$.

Let $s = L^\delta$, $1 - c > \delta > 0$. Let $\Gamma \in \mathfrak{G}_L$. We estimate

$$\begin{aligned} &P_{\Lambda_L}^{+,\beta,h}[\mathcal{A}(C_1, \nu, c) | \{\Gamma; \gamma \neq \Gamma \, s\text{-small}\}] \\ &= 1 - P_{\Lambda_L}^{+,\beta,h}[\{|\sum_{t \in \Lambda_L} \omega(t) - m^*(\beta) + C_1 |\Lambda_L| \, L^{-\nu}| > |\Lambda_L| \, L^{-c}\} | \{\Gamma; \gamma \neq \Gamma \, s\text{-small}\}] \,. \end{aligned} \tag{7.32}$$

We want to use Proposition 5.2.3. We first have to estimate

$$\langle \sum_{t \in \Lambda_L} \omega(t) | \{\Gamma; \gamma \neq \Gamma \, s\text{-small}\} \rangle \,. \tag{7.33}$$

[10]It may happen that $\overline{u}_k$ and $\overline{v}_k$ do not belong to $\partial \mathcal{S}_k$; in such a case we move these points so that they satisfy this condition.

This is not difficult using Lemma 5.1.2. Just observe that, by definition of the elliptical sets,

$$||\overline{\text{int}}\Gamma| - \text{vol}\, L\mathcal{P}_L| \leqslant \mathcal{O}(L^{3/2-\rho/2-\nu}(\log L)^{1/2}) \ll |\Lambda| L^{-c}\,, \tag{7.34}$$

and

$$|\text{vol}\, L\mathcal{C} - \text{vol}\, L\mathcal{P}_L| \leqslant \mathcal{O}(L^{2-2\rho}) \ll |\Lambda| L^{-c}\,, \tag{7.35}$$

by the hypothesis on c. Consequently,

$$\Big|\langle \sum_{t\in\Lambda_L} \omega(t) | \{\Gamma; \gamma \neq \Gamma\; s\text{-small}\}\rangle - m^*(\beta) + C_1 |\Lambda_L| L^{-\nu}\Big| \ll |\Lambda_L| L^{-c}\,. \tag{7.36}$$

Using Proposition 5.2.3, we find

$$P^{+,\beta,h}_{\Lambda_L}[\{|\sum_{t\in\Lambda_L} \omega(t) - m^*(\beta) + C_1|\Lambda_L|\, L^{-\nu}| > |\Lambda_L|\, L^{-c}\} | \{\Gamma; \gamma \neq \Gamma\; s\text{-small}\}]$$
$$\leqslant \mathcal{O}(L^{-2(1-c-\delta)})\,, \tag{7.37}$$

which implies that

$$P^{+,\beta,h}_{\Lambda_L}[\mathcal{A}(C_1,\nu,c) | \{\Gamma; \gamma \neq \Gamma\; s\text{-small}\}] \geqslant 1 - \mathcal{O}(L^{-2(1-c-\delta)})\,. \tag{7.38}$$

Step 4. Estimation of $P^{+,\beta,h}_{\Lambda_L}[\{\Gamma; \gamma \neq \Gamma\; s\text{-small}\}]$.

Define

$$\Lambda_L(\text{ext}\Gamma) \doteqdot \Lambda_L \setminus \overline{\text{int}}\Gamma\,, \tag{7.39}$$

and

$$\Lambda_L(\text{int}\Gamma) \doteqdot \Lambda_L \setminus \overline{\text{ext}}\Gamma\,. \tag{7.40}$$

We have

$$P^{+,\beta,h}_{\Lambda_L}[\{\Gamma; \gamma \neq \Gamma\; s\text{-small}\}] = w^*(\Gamma) \frac{Z^{+,s}(\Lambda_L(\text{ext}\Gamma)) Z^{+,s}(\Lambda_L(\text{int}\Gamma))}{Z^+(\Lambda_L)}\,, \tag{7.41}$$

where $Z^{+,s}(\Lambda')$ is defined as $Z^+(\Lambda')$ in (D69), p. 46, but by summing only over s–small contours. Although $\Lambda_L(\text{ext}\Gamma)$ is not simply connected, any $\Lambda_L(\text{ext}\Gamma)^*$–compatible family of s-small closed contours is $\Lambda_L(\text{ext}\Gamma)^+$-compatible, and consequently we also have $Z^{+,s}(\Lambda_L(\text{ext}\Gamma)) = Z^s(\Lambda_L(\text{ext}\Gamma)^*)$. Therefore, dividing and multiplying by $Z(\Lambda^*_L|\Gamma)$ and using Lemma 2.3.1, we obtain

$$P^{+,\beta,h}_{\Lambda_L}[\{\Gamma; \gamma \neq \Gamma\; s\text{-small}\}] = q_{\Lambda^*_L}(\Gamma; \beta^*, h^*)\, \langle\{\gamma\; s\text{-small}\}\rangle_{\Lambda_L(\text{ext}\Gamma)^*}\, \langle\{\gamma\; s\text{-small}\}\rangle_{\Lambda_L(\text{int}\Gamma)^*}\,. \tag{7.42}$$

Lemma 4.3.1 implies (if diameter $d(\gamma) \leqslant s$, then γ is s-small)

$$\langle\{\gamma\; s\text{-small}\}\rangle_{\Lambda_L(\text{ext}\Gamma)^*} \geqslant 1 - \mathcal{O}(L^{2+2\delta} \exp\{-\alpha L^\delta\})\,. \tag{7.43}$$

A similar estimate holds for $\langle\{\gamma\ s\text{-small}\}\rangle_{\Lambda_L(\mathrm{int}\Gamma)^*}$.

Before doing the last step it is useful to summarize all the above estimates. We have

$$\begin{aligned}
P^{+,\beta,h}_{\Lambda_L}&[\mathcal{A}(C_1,\nu,c)]\\
&\geqslant \sum_{\Gamma\in\mathfrak{G}_L} P^{+,\beta,h}_{\Lambda_L}[\mathcal{A}(C_1,\nu,c)|\{\Gamma;\gamma\neq\Gamma s\text{-small}\}]\,P^{+,\beta,h}_{\Lambda_L}[\{\Gamma;\gamma\neq\Gamma\ s\text{-small}\}]\\
&\geqslant \big(1-\mathcal{O}(L^{-2(1-c-\delta)})\big)\sum_{\Gamma\in\mathfrak{G}_L} P^{+,\beta,h}_{\Lambda_L}[\{\Gamma;\gamma\neq\Gamma s\text{-small}\}]\\
&\geqslant \big(1-\mathcal{O}(L^{-2(1-c-\delta)})\big)\big(1-\mathcal{O}(L^{2+2\delta}\exp\{-\alpha L^{\delta}\})\big)^2\sum_{\Gamma\in\mathfrak{G}_L} q_{\Lambda^*_L}(\Gamma;\beta^*,h^*)\,.
\end{aligned}\tag{7.44}$$

Step 5. Estimation of $P^{+,\beta,h}_{\Lambda_L}[\mathcal{A}(C_1,\nu,c)]$.

It remains to control the last sum. Lemmas 4.2.4, points 4. and 7., give

$$\begin{aligned}
\sum_{\Gamma\in\mathfrak{G}_L} q_{\Lambda^*_L}(\Gamma;\beta^*,h^*) &\geqslant \sum_{\Gamma=\{\gamma_i\}\in\mathfrak{G}_L}\prod_i q_{\Lambda^*_L}(\gamma_i)\\
&\geqslant \sum_{\Gamma=\{\gamma_i\}\in\mathfrak{G}_L}\prod_i q_{\mathbb{L}}(\gamma_i)\,.
\end{aligned}\tag{7.45}$$

We use Lemma 4.2.4, point 5., to replace $q_{\mathbb{L}}(\gamma_k)$ by $q(\gamma_k)$ whenever $\partial\gamma_k=\{\overline{u}_k,\overline{v}_k\}$ (notice that the distance of the the set $\mathcal{S}_k$ from the wall is $\mathcal{O}(K_1\log L)$). From the definition of $\mathfrak{G}_L$, the sums over the γ_i which are not fixed are independent, so that we can use Lemmas 4.4.3, 4.4.6 and Proposition 4.6.2, point 1. and 3., to estimate them (see the proof of Proposition 6.3.1 for similar estimates). Using Propositions 4.5.1 and 4.5.2, we can find a constant K_2 such that

$$\sum_{\substack{\gamma_i\,:\,\partial\gamma_i=\{t_i,t_{i+1}\}\\ \gamma_i\subset\mathcal{S}_i}} q_{\mathbb{L}}(\gamma_i;\beta^*,h^*)\geqslant\big(1-\mathcal{O}(L^{-K_1})\big)\exp\{-\tau_{\mathrm{bd}}(t_{i+1}-t_i;\beta,h)\}\,,\tag{7.46}$$

and

$$\sum_{\substack{\gamma_i\,:\,\partial\gamma_i=\{\overline{u}_i,\overline{v}_i\}\\ \gamma_i\subset\mathcal{S}_i}} q(\gamma_i;\beta^*)\geqslant\big(1-\mathcal{O}(L^{-K_1})\big)\frac{\exp\{-\tau(t_{i+1}-t_i;\beta)\}}{|t_{i+1}-t_i|^{K_2}}\,.\tag{7.47}$$

The total length of the fixed γ_i does not exceed $O(L^{1/2-\nu/4}\log L)$ and the total number of sides is $\mathcal{O}(L^{1/2-\nu/4})$, so that the above estimates imply the theorem.

We now consider the case $h=0$. The proof is done as before, with two simple modifications: First, we use Lemma 4.3.2 (in which the case $\tau_{\mathrm{bd}}=0$ is considered); second, we do not introduce the boxes along the wall. Then the result follows easily. □

Remark. 1. We emphasize the fact that the next-to-leading order term in the lower bound is *not* $\mathcal{O}(L^{1/2-\nu/4}\log L)$. Indeed, we authorize fluctuations of order L^{2-c} around $m^*(\beta)-C_1|\Lambda_L|L^{-\nu}$, and consequently the system will realize the maximal magnetization

possible in the interval (that is, the smallest droplet). Therefore, the leading term, which we took centered in the interval, is somewhat misleading for it implicitly contains the largest error term.
2. When $\nu > 2/3$, the estimate which is given in Theorem 7.4.1 is not optimal. This is due to the fact that the system will not create a large droplet of $-$ phase in such a case.

7.4.2 Lower bound: Negative magnetic field

In this section, we work with the measure $\mu^{\pm,\beta,|h|}_{\Lambda_L}$. Let us make some comments on the modifications it implies. The $\pm$-b.c. is a particular case of the d-b.c., where d is the horizontal line going through $(0,0)$. We use the terminology of Section 2.1.2 but we write $Z^{\pm}(\Lambda_L|\lambda)$ instead of $Z^d(\Lambda_L|\lambda)$.
If Λ_L is simply connected, Lemma 2.3.1 implies the following important identity

$$\frac{Z^{\pm}(\Lambda_L;J)}{Z^{+}(\Lambda_L;J)} = \sum_{\Gamma^*} w(\Gamma^*)\frac{Z^{\pm}(\Lambda_L|\Gamma^*;J)}{Z^{+}(\Lambda_L;J)} = \langle\sigma(t_1^*)\sigma(t_2^*)\rangle^{J^*}_{\Lambda_L^*}\,. \tag{7.48}$$

This quantity can be controlled by Propositions 3.2.1 and 4.5.2.

Theorem 7.4.2. *1. Suppose that $J(e)$ is given by (3.30) with $\beta > \beta_c$ and $-h_w < h < 0$. Let $C_1 > 0$, $1 > c > \nu > 0$ be given. Let $\mathfrak{W}^*(\beta,h,C_1,\nu)$ be the minimum of the functional $\mathfrak{W}(\cdot;\beta,h)$ (without constraints) among curves enclosing a volume $|Q|(C_1/2m^*)$. Then there exists $L_0 = L_0(\beta,h,C_1,\nu,c)$ such that, for all $L \geqslant L_0$,*

$$P^{\pm,\beta,|h|}_{\Lambda_L}[\mathcal{A}(C_1,\nu,c)] \geqslant \exp\{-L^{1-\nu/2}\,\mathfrak{W}^*(\beta,h,C_1,\nu) - \mathcal{O}(L^{1/2-\nu/4}\log L)\}\,.$$

2. Suppose that $J(e)$ is given by (3.30) with $\beta > \beta_c$ and $h \leqslant -h_w$. Let $C_1 > 0$, $1/4 > \nu > 0$ and $1/2 > c > \nu$ be given. Let $\mathfrak{R}^*_L(\beta,h,C_1,\nu)$ be the minimum[11] of the functional $\mathfrak{W}(\cdot;\beta,h)$ among curves contained in Q enclosing a volume $|Q|(C_1/2m^*)L^{-\nu}$. Then
a) The constant $\mathfrak{W}^*(\beta,h,C_1,\nu)$ defined by

$$\mathfrak{W}^*(\beta,h,C_1,\nu) \doteqdot \lim_{L\to\infty} L^{2\nu}\,\mathfrak{R}^*_L(\beta,h,C_1,\nu)$$

is finite and strictly positive.
b) There exists $L_0 = L_0(\beta,h,C_1,\nu,c)$ such that, for all $L \geqslant L_0$,

$$P^{\pm,\beta,|h|}_{\Lambda_L}[\mathcal{A}(C_1,\nu,c)] \geqslant \exp\{-L^{1-2\nu}\,\mathfrak{W}^*(\beta,h,C_1,\nu) - \mathcal{O}(L^{1/2}\log L) - \mathcal{O}(L^{1-3\nu})\}\,.$$

Remark. Notice that the complete wetting transition has a remarkable manifestation on the behaviour of moderately large deviations: It modifies the scale of these large deviations! In particular, observe that the exponent $1-2\nu$ becomes zero when $\nu = 1/2$ (our results do not cover this case, since we need $\nu < 1/4$, however we expect that the theorem holds true for all $\nu < 1/2$). This implies that the change of behaviour between large and normal deviations (which happens at $L^{4/3}$, i.e. $\nu = 2/3$, when $h > -h_w$), must happen sooner than usually. This can be understood, at least heuristically, see Section 7.4.5.

---

[11] The curve realizing this minimum is the Winterbottom shape with w_Q as basis and volume $|Q|(C_1/2m^*)L^{-\nu}$.

Proof. 1. The proof is almost identical to the case of positive magnetic field, the differences coming from the fact that we are working with the Gibbs measure $\mu_{\Lambda_L}^{\pm,\beta,|h|}$. We only point out the modifications.

Let $\mathcal{C} = \mathcal{C}(\beta, h)$ be the curve which realizes the minimum of the unconstrained variational problem, corresponding to the parameters β and $h < 0$, and enclosing a volume $|Q|(C_1/2m^*)L^{-\nu}$. We suppose that L is large enough to ensure that $\mathcal{C}$ is contained in Q with its bottom affine part on w_Q and is at a finite distance from the other sides of Q. We do not make a polygonal approximation of $\mathcal{C}$ but of the open curve $\overline{\mathcal{C}} = \mathcal{C} \triangle w_Q$[12]. The construction is the same as before, and the rest of the proof is unchanged except that we use

$$w(\Gamma)\frac{Z^{\pm}(\Lambda_L|\Gamma;\beta,h)}{Z^{\pm}(\Lambda_L;\beta,h)} = \frac{1}{\langle\sigma(t_1^*)\sigma(t_2^*)\rangle_{\Lambda_L^*}^{J^*}} q_{\Lambda_L}(\Gamma;\beta^*,h^*)\,. \tag{7.49}$$

The conclusion follows by observing that

$$\begin{aligned} P_{\Lambda_L}^{\pm,\beta,|h|}&[\mathcal{A}(C_1,\nu,c)] \\ &\geqslant \frac{1}{\langle\sigma(t_1^*)\sigma(t_2^*)\rangle_{\Lambda_L^*}^{J^*}} \exp\{-L^{1-\nu/2}\,\mathfrak{W}(\overline{\mathcal{C}};\beta,|h|) - \mathcal{O}(L^{1/2-\nu/4}\log L)\} \\ &\geqslant \exp\{\tau_{\text{bd}}(t_2^* - t_1^*)\}\exp\{-L^{1-\nu/2}\,\mathfrak{W}(\overline{\mathcal{C}};\beta,|h|) - \mathcal{O}(L^{1/2-\nu/4}\log L)\} \\ &= \exp\{-L^{1-\nu/2}\,\mathfrak{W}(\mathcal{C};\beta,h) - \mathcal{O}(L^{1/2-\nu/4}\log L)\} \\ &= \exp\{-L^{1-\nu/2}\,\mathfrak{W}^*(\beta,h) - \mathcal{O}(L^{1/2-\nu/4}\log L)\}\,. \end{aligned} \tag{7.50}$$

2. The existence of $\mathfrak{R}^*$ follows from straightforward computations (we use smoothness of τ, and the positive stiffness to prove strict positivity.). In fact, we obtain a little more, namely

$$\mathfrak{R}^* = \mathfrak{W}^*\,L^{-2\nu} + \mathcal{O}(L^{-3\nu})\,. \tag{7.51}$$

The second statement is proved as the corresponding statement of point 1., so that we only sketch the proof.

We first set

$$\delta_L \doteqdot L^{-\rho}\,, \qquad \rho = 1/2\,. \tag{7.52}$$

Let $\mathcal{C}$ be the Winterbottom shape of basis w_Q and volume $|Q|(C_1/2m^*)L^{-\nu}$ (this is the solution of the constrained variational problem, as is easy to see). We have

$$\operatorname{vol}\mathcal{P}_L = \operatorname{vol}\mathcal{C} - \mathcal{O}(L^{-2\rho-\nu}) = \operatorname{vol}\mathcal{C} - \mathcal{O}(L^{-1-\nu})\,. \tag{7.53}$$

As in point 1., we consider another curve, which is given by $\overline{\mathcal{C}} \doteqdot \mathcal{C}\backslash w_Q$. We construct the polygonal approximation of $\overline{\mathcal{C}}$ as before. The two main differences between this case and the case of point 1. is that: 1) The length of the polygon is $\mathcal{O}(L)$, and 2) we cannot put elliptical sets near the extremities of $\overline{\mathcal{C}}$ otherwise they would not be contained inside the box (the droplet, now, is *extremely* flat!). So the error term is slightly larger than before: The total length of the fixed contours in this case is

[12] $\triangle$ denotes the operation of symmetric difference: $A \triangle B \doteqdot (A \setminus B) \cup (B \setminus A)$.

(observe that the angle between $\overline{\mathcal{C}}$ and the wall at each extremities is at least of order $L^{-\nu}$)

$$\mathcal{O}(L^{1-\rho}) + \mathcal{O}(L^{\rho}\log L) = \mathcal{O}(L^{1/2}\log L)\,. \tag{7.54}$$

The proof then proceeds as before. Since, at the end, we want that the error term be negligeable with respect to the first order term, ν must satisfy $1-2\nu > 1/2$, which requires $\nu < 1/4$.

□

7.4.3 Upper bound: Positive magnetic field

By Theorem 7.4.1, if $0<\nu<c<1$ and $C_1>0$, then

$$P^{+,\beta,h}_{\Lambda_L}[\mathcal{A}(C_1,\nu,c)] \geqslant \exp\{-L^{1-\nu/2}\,\mathfrak{W}^*(\beta,h,C_1,\nu) - \mathcal{O}(L^{1/2-\nu/4}\log L)\}\,, \tag{7.55}$$

for L large enough, where

$$\mathfrak{W}^*(\beta,h,C_1,\nu) \doteqdot \inf\{\mathfrak{W}(\mathcal{C})\,:\,\mathcal{C}\subset\mathbb{R}^2\,,\,\mathrm{vol}\,\mathcal{C} = |Q|(C_1/2m^*)\}\,, \tag{7.56}$$

We show that this lower bound is optimal to leading order when ν is small enough. To do this we analyze the conditioned measure in terms of large contours. The basic idea is to make a coarse–grained description of the large contours.

The basic estimates come from Lemma 4.2.8 and Proposition 5.2.1. As pointed out in the first remark following its proof, the bound proved in this proposition is not sharp. This is the reason why we have to take $c<1/4$ (and therefore also $\nu<1/4$)[13].

Since the magnetic field is positive, Lemma 3.2.1 implies $\tau_{\mathrm{bd}}\geqslant 0$. Hence

$$\mathfrak{W}(\mathcal{C}) \geqslant \int \tau(\dot u^+(t),\dot v^+(t))\,\mathrm{d}t\,, \tag{7.57}$$

where $(u^+(t),v^+(t))$ is a parameterization of the curve $\mathcal{C}_+ \doteqdot \mathcal{C}\setminus w_Q$.
Let $r_1,r_2\in\mathbb{N}$ and $\Lambda_L=\Lambda_L(r_1,r_2)$ be the box of equation (7.24). We treat the case of the Λ_L^+–boundary condition. The constants c, C_1 and ν are fixed; we set $c=1/4-\delta$, $\delta>0$. The cut–off between small and large contours is chosen as

$$s \doteqdot [L^{\delta'}]\,. \tag{7.58}$$

with $2\delta>\delta'>0$.

In each configuration ω with Λ_L^+–boundary condition we denote the large contours by $\Gamma_1,\Gamma_2,\ldots$. The origin of Γ_i is the first point of Γ_i. The parameterization of Γ_i, $s\mapsto\Gamma_i(s)$, is chosen so that it is counterclockwise and $\Gamma_i(0)$ is the origin of Γ_i. The coarse-grained description of Γ_i consists in defining a sequence of points $S_i=(t_{i0},t_{i1},\ldots,t_{in_i})$. The procedure is similar to the one used in the proof of Lemma 4.3.1, but here we must treat the points of Γ_i on the line Σ^* with special care. Indeed, we would like to be able to apply Lemma 4.2.8, so that we have to cut the contour in such a way as to be sure that it does

[13] In fact it is possible to slightly enlarge the domain of ν if we take c and ν close enough, for example $c=2/7-\delta$ and $\nu=2/7-2\delta$. However this is still much smaller than the real domain of validity (which should go up to $\nu=2/3$) so we do not consider this case.

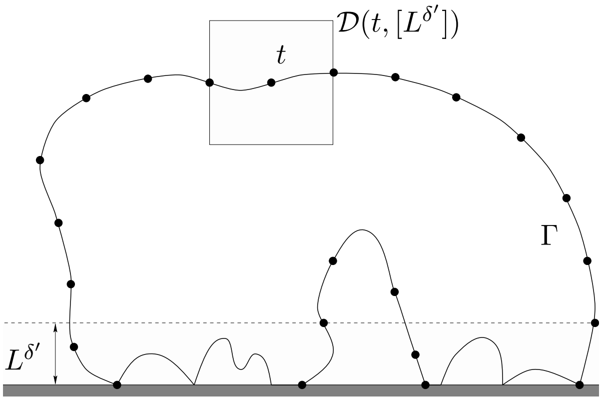

FIGURE 7.8. Coarse-graining of a large contour Γ touching the lower wall; the dots represent the sequence of points $S_i = \{t_{i0}, \dots, t_{in_i}\}$.

not touch Σ^* between successive vertices t_{ij}, t_{ij+1} with at least one of the two points not on the boundary. We cannot do this as simply as in Section 6.3.2 because we need to have a good control on the volume of the contours having a given polygonal approximation. The following procedure does the trick.

1. We set $t_{i0} \doteqdot \Gamma_i(0)$.
2. If $t_{i0}(2) = -1/2$ then go to 6. Otherwise go to 3.
3. Let s_1 be the first integer time such that Γ_i is outside the square $\mathcal{D}(t_{i0}, [L^{\delta'}])$ (see (5.1)). We set $t_{i1} \doteqdot \Gamma_i(s_1)$.
4. Let s_2 be the first integer time greater than s_1 such that Γ_i is outside the square $\mathcal{D}(t_{i1}, [L^{\delta'}])$. We set $t_{i2} \doteqdot \Gamma_i(s_2)$.
5. The procedure is iterated until it stops.
6. If $t_{i0}(2) = -1/2$, then there exists $s \in \mathbb{N}$ such that $\Gamma_i(s)(2) = -1/2$. We set $t_{i1} \doteqdot \Gamma_i(s_1)$ with s_1 the largest integer time such that
$$\Gamma_i(s_1)(2) = -1/2 \text{ and } \Gamma_i(s)(2) \leqslant [L^{\delta'}] \quad \forall s \in [0, s_1]\,. \tag{7.59}$$
7. If, for all $s > s_1$, $\Gamma_i(s)(2) \neq -1/2$, then apply the procedure 3. to 5. to the part of Γ_i, $\{\Gamma_i(s) \,:\, s \geqslant s_1\}$. Otherwise go to 8.
8. Let s_2 be the first integer time greater than s_1, such that $\Gamma_i(s_2)(2) > [L^{\delta'}]$. We set $t_{i2} \doteqdot \Gamma_i(s_2)$. Let s^* be the first integer time greater than s_2 such that $\Gamma_i(s^*)(2) = -1/2$. Apply the procedure 3. to 5. to the part of Γ_i, $\{\Gamma_i(s) \,:\, s_2 \leqslant s \leqslant s^*\}$. Then apply the procedure starting at 2. to the part of Γ_i, $\{\Gamma_i(s) \,:\, s \geqslant s^*\}$.

Let $S \doteqdot (t_1, \dots, t_n)$ be an ordered sequence of points and $\mathcal{P}(S)$ be the corresponding closed polygonal line with vertices $(t_1, \dots\dots, t_n)$. To each Γ_i we associate a closed polygonal line $\mathcal{P}(\Gamma_i)$[14]:
$$\mathcal{P}(\Gamma_i) \doteqdot \mathcal{P}(S_i)\,, \tag{7.60}$$
where $S_i = (t_{i0}, t_{i1}, \dots, t_{in_i})$ is the ordered sequence of points defined by the above procedure. Given a polygonal line, we need to be sure that the contours to which it is the

[14]Notice that, contrarily to what is done in [DKS1, Pf1, I2], to a given contour corresponds a unique polygonal line.

polygonal approximation are close enough to it (since we want to be able to control the volume of these contours knowing only the polygonal line). With this in mind, we introduce the following set,

$$B(S_i) \doteqdot \{t \in \Lambda_L \,:\, t(2) \leqslant [L^{\delta'}]\} \bigcup_{t_{ij}\in S_i} \Big(\mathcal{D}(t_{ij}, [L^{\delta'}]) \cap \Lambda_L\Big)\,. \tag{7.61}$$

We then have: $\mathcal{P}(\Gamma) = \mathcal{P}(S_i) \implies \Gamma \subset B(S_i)$.
We would like now to prove that a typical family of polygonal lines cannot have too large a surface tension,

$$\mathfrak{W}(S_1,\ldots,S_k) \doteqdot \sum_{j=1}^{k} \mathfrak{W}(\mathcal{P}(S_j))\,, \tag{7.62}$$

similarly to what is done in Chapter 6.
First observe that, whenever $\Gamma(s_j)(2) = t_j(2) \neq -1/2$ or $\Gamma(s_{j+1})(2) = t_{j+1}(2) \neq -1/2$,

$$\{\Gamma(s) \,:\, s_j < s < s_{j+1}\} \cap \{t \in \mathbb{Z}^{2*} \,:\, t(2) = -1/2\} = \varnothing\,. \tag{7.63}$$

Using this property we can estimate $P^{+,\beta,h}_{\Lambda_L}[\{S_1,\ldots,S_k\}]$ in terms of the functional $\mathfrak{W}$ (see Lemma 4.2.8),

$$P^{+,\beta,h}_{\Lambda_L}[\{S_1,\ldots,S_k\}] \leqslant \exp\{-\mathfrak{W}(S_1,\ldots,S_k)\} \tag{7.64}$$

(we used the fact that $Z^{+,s}(\Lambda_L|\underline{\Gamma}) \leqslant Z^{+}(\Lambda_L|\underline{\Gamma}) \leqslant Z(\Lambda_L^*|\underline{\Gamma})$). Notice that Lemma 4.2.8 still holds in the case $h^* = \infty$, which corresponds to $h = 0$ (since, for finite h^*, replacing τ_{bd} by zero provides an upper bound, and therefore the limit $h^* \to \infty$ is well-defined).

Let $\omega_{\underline{\Gamma}}$ be the unique configuration satisfying the Λ_L^+-boundary condition which has $\underline{\Gamma} \doteqdot (\Gamma_1, \Gamma_2, \ldots, \Gamma_k)$ as the complete set of its contours. We introduce some terminology[15].

Definition.

(D119) *The* **interior of** $\mathcal{P}(S_i)$ *is*

$$\operatorname{Int}\mathcal{P}(S_i) \doteqdot \{t \in \Lambda_L \,:\, \omega_{\Gamma_i}(t) = -1\} \setminus B(S_i)\,,$$

where Γ_i *is any contour such that* $\mathcal{P}(\Gamma_i) = \mathcal{P}(S_i)$.

(D120) *The* **volume of** $\mathcal{P}(S_i)$ *is*

$$\operatorname{Vol}\mathcal{P}(S_i) \doteqdot |\operatorname{Int}\mathcal{P}(S_i)|\,.$$

(D121) *The* **closure of** $\operatorname{Int}\mathcal{P}(S_i)$ *is*

$$\overline{\operatorname{Int}}\,\mathcal{P}(S_i) \doteqdot \operatorname{Int}\mathcal{P}(S_i) \cup B(S_i)\,.$$

(D122) *The* **interior of** $\underline{S} \doteqdot (S_1,\ldots,S_k)$ *is*

$$\operatorname{Int}\underline{S} \doteqdot \{t \in \Lambda_L \,:\, \omega_{\underline{\Gamma}}(t) = -1\} \setminus \bigcup_i B(S_i)\,,$$

where $\underline{\Gamma} \doteqdot (\Gamma_1,\ldots,\Gamma_k)$ *is any set of contours such that* $\mathcal{P}(\Gamma_i) = \mathcal{P}(S_i)$, $i = 1,\ldots,k$.

[15] We emphasize that these definitions are different from the analogous ones in [DKS1, Pf1, I2].

(D123) *The* **phase volume of** $\underline{S}$ *is*

$$\alpha(\underline{S})|\Lambda_L| \doteqdot |\mathrm{Int}\,\underline{S}|\,.$$

(D124) *Let* $\underline{\Gamma}$ *be a* Λ_L^+*-compatible family of contours. The* **phase volume of** $\underline{\Gamma}$ *is*

$$\alpha(\underline{\Gamma})|\Lambda_L| \doteqdot |\{t \in \Lambda_L \,:\, \omega_{\underline{\Gamma}}(t) = -1\}|\,.$$

We can now prove a result similar to Proposition 6.3.2.

Lemma 7.4.1. *We assume that the coupling constants are defined by* (3.30) *with* $\beta > \beta_c$*, and* $h \geqslant 0$*. Then for any* $\eta < \delta'$ *and* $T > 0$

$$P_{\Lambda_L}^{+,\beta,h}\Big[\{\sum_{j\,\geqslant\,1} \mathfrak{W}(\mathcal{P}(S_j)) \geqslant T\}\Big] \leqslant \exp\Big\{-T[1-\mathcal{O}(L^{\eta-\delta'})]\Big\}\,. \tag{7.65}$$

Proof. We follow the proof of the corresponding statement in [Pf1].
We have

$$P_{\Lambda_L}^{+,\beta,h}[\{\sum_{j\,\geqslant\,1} \mathfrak{W}(\mathcal{P}(S_j)) \geqslant T\}] = \sum_{k\,\geqslant\,1}\sum_{S_1,\dots S_k} P_{\Lambda_L}^{+,\beta,h}[\{\mathfrak{W}(S_1,\dots,S_k) \geqslant T\}]\,. \tag{7.66}$$

Let $q(x,k)$ be the number of integer solutions of $1 \leqslant \alpha_1 \leqslant \,\dots\, \leqslant \alpha_k \leqslant x$, $\sum_{i=1}^k \alpha_i = x$, k fixed, and $q(x)$ the number of integer solutions of $1 \leqslant \alpha_1 \leqslant \,\dots\, \leqslant \alpha_k \leqslant x$ and $\sum_{i=1}^k \alpha_i = x$, k arbitrary. For large x,

$$q(x) \sim \frac{1}{4\sqrt{3}x} \exp\Big\{2\pi\sqrt{x/6}\Big\}\,. \tag{7.67}$$

Let us consider k polygonal lines $\mathcal{P}(S_1),\dots,\mathcal{P}(S_k)$, where $S_i = (t_{i0}, t_{i1}, \dots, t_{in_i})$. $L^{\mathcal{O}(N)}$ is a rough estimate of the number of families of k polygonal lines with $n_1 + \cdots + n_k = N$. Therefore the number of families of polygonal lines with $n_1 + \cdots + n_k = N$, k arbitrary, is bounded by

$$\sum_k q(N,k) L^{\mathcal{O}(N)} = \exp\{N\mathcal{O}(\ln L)\}\,. \tag{7.68}$$

Using (7.57) it is possible to obtain a bound on $N \equiv n_1 + \cdots + n_k$ in terms of the total surface tension of the family of polygonal lines

$$\mathfrak{W}(S_1,\dots,S_k) = \sum_{i=1}^k \mathfrak{W}(\mathcal{P}(S_i))\,. \tag{7.69}$$

Suppose that $\mathfrak{W}(S_1,\dots,S_k) = T'$; then

$$N = n_1 + \cdots + n_k \leqslant K_1 T' L^{-\delta'}\,, \tag{7.70}$$

with $K_1 \leqslant 2(\min_{\|\boldsymbol{n}\|_2=1} \tau(\boldsymbol{n}))^{-1}$. Therefore

$$\begin{aligned} P_{\Lambda_L}^{+,\beta,h}[\{S_1,\dots,S_k\}] &\leqslant \exp\{-\mathfrak{W}(S_1,\dots,S_k)\} \\ &= \exp\{-\mathfrak{W}(S_1,\dots,S_k) + NL^{\eta}\}\exp\{-NL^{\eta}\} \\ &\leqslant \exp\Big\{-\mathfrak{W}(S_1,\dots,S_k)(1-\mathcal{O}(L^{-\delta'+\eta}))\Big\}\exp\{-NL^{\eta}\}\,. \end{aligned} \tag{7.71}$$

Therefore

$$P_{\Lambda_L}^{+,\beta,h}[\{\sum_{j\geqslant 1}\mathfrak{W}(\mathcal{P}(S_j))\geqslant T\}] \leqslant \exp\left\{-T[1-\mathcal{O}(L^{\eta-\delta'})]\right\}\sum_{N\geqslant 1}\exp\{N\mathcal{O}(\ln L)-NL^{\eta}\} \leqslant \exp\left\{-T[1-\mathcal{O}(L^{\eta-\delta'})]\right\}. \quad (7.72)$$

□

Our aim is to characterize the set of configurations contributing to the event $\mathcal{A}(C_1,\nu,c)$ by describing the family of polygonal approximations of the large contours of typical such configurations. Lemma 7.4.1 provides a first criterion, namely surface tension should be as low as possible. However, there is a second constraint: If we want realize the event $\mathcal{A}(C_1,\nu,c)$, it is necessary that the phase volume of the family of polygonal approximations is sufficiently close to $(C_1/2m^*)L^{-\nu}$. This is the purpose of the next lemma.

Lemma 7.4.2. *We assume that the coupling constants are defined by* (3.30) *with* $\beta>\beta_c$ *and* $h\geqslant 0$. *Let* $1/4>c>\nu>0$, $c=1/4-\delta$, *and* $C_1>0$. *For any* $\eta>0$,

$$P_{\Lambda_L}^{+,\beta,h}\left[\{|\alpha(\underline{S})-(C_1/2m^*)L^{-\nu}|\geqslant\frac{1+\eta}{2m^*L^c}\}\,\Big|\,\mathcal{A}(C_1,\nu,c)\right]\leqslant\exp\{-\mathcal{O}(L^{1-\nu/2})\},$$

provided L *is large enough.*

Proof. We set

$$E_1(C_1,\nu,c)\doteqdot\left\{\left|\alpha(\underline{S})-(C_1/2m^*)L^{-\nu}\right|\geqslant\frac{1+\eta}{2m^*L^c}\right\}. \quad (7.73)$$

We partition $E_1(C_1,\nu,c)$ into sets indexed by the set of their large contours. Let

$$[\underline{\Gamma}]\doteqdot\{\omega\,:\,\underline{\Gamma}\text{ is the family of large contours of }\omega\}. \quad (7.74)$$

We write

$$P_{\Lambda_L}^{+,\beta,h}[E_1(m;c)\,|\,\mathcal{A}(C_1,\nu,c)]=\sum_{\substack{\underline{\Gamma}:\\ [\underline{\Gamma}]\subset E_1(C_1,\nu,c)}}P_{\Lambda_L}^{+,\beta,h}[\mathcal{A}(C_1,\nu,c)\,|\,[\underline{\Gamma}]]\,\frac{P_{\Lambda_L}^{+,\beta,h}[[\underline{\Gamma}]]}{P_{\Lambda_L}^{+,\beta,h}[\mathcal{A}(C_1,\nu,c)]}. \quad (7.75)$$

Since (7.55) and Lemma 7.4.1 hold we can find a constant K such that

$$P_{\Lambda_L}^{+,\beta,h}[E_1(C_1,\nu,c)\cap\{\sum_i\mathfrak{W}(\mathcal{P}(S_i))\geqslant KL^{1-\nu/2}\}\,|\,\mathcal{A}(C_1,\nu,c)] \leqslant P_{\Lambda_L}^{+,\beta,h}[\{\sum_i\mathfrak{W}(\mathcal{P}(S_i))\geqslant KL^{1-\nu/2}\}\,|\,\mathcal{A}(C_1,\nu,c)]\leqslant\exp\{-\mathcal{O}(L^{1-\nu/2})\}. \quad (7.76)$$

It is therefore sufficient to control in (7.75) the terms with $\underline{\Gamma}$ such that

$$\sum_i\mathfrak{W}(\mathcal{P}(S_i))\leqslant KL^{1-\nu/2}. \quad (7.77)$$

We suppose that this condition is satisfied in the rest of the proof. In particular the total length of the polygonal lines is at most $\mathcal{O}(L^{1-\nu/2})$. Suppose that $\underline{\Gamma} = \{\Gamma_1, \ldots, \Gamma_k\}$ and that $\mathcal{P}(\Gamma_j) = \mathcal{P}(S_j)$, $j = 1, \ldots, k$. In particular, this implies that

$$|\bigcup_i B(S_i)| \leqslant \mathcal{O}(L^{1-\nu/2+\delta'}) \tag{7.78}$$

for any $\underline{\Gamma}$ such that (7.77) holds. The volumes of phase $\alpha(\underline{S})$ and $\alpha(\underline{\Gamma})$ are easily compared, using (7.78) and the observation that $\Gamma_i \subset B_i$,

$$|\alpha(\underline{S}) - \alpha(\underline{\Gamma})||\Lambda_L| \leqslant |\bigcup_i B(S_i)| \leqslant \mathcal{O}(L^{1-\nu/2+\delta'})\,. \tag{7.79}$$

We denote by $\star$ the boundary condition defined by the configuration $\omega_{\underline{\Gamma}}$. Then we can write ($\Lambda_{\#}(\underline{\Gamma})$ is defined in (D112), p. 114)

$$P^{+,\beta,h}_{\Lambda_L}[\mathcal{A}(C_1,\nu,c)\,|\,[\underline{\Gamma}]] = P^{\star,s}_{\Lambda_{\#}(\underline{\Gamma})}[\mathcal{A}(C_1,\nu,c)]\,. \tag{7.80}$$

From Lemma 5.1.2 and (7.78), we have

$$\begin{aligned}\langle \sum_{t\in\Lambda_L} \sigma(t)\rangle^{\star,s}_{\Lambda_{\#}(\underline{\Gamma})} &= m^*|\Lambda_L|(1-2\alpha(\underline{\Gamma})) \pm \mathcal{O}(L^{1-\nu/2+\delta'}) \\ &= m^*|\Lambda_L|(1-2\alpha(\underline{S})) \pm \mathcal{O}(L^{1-\nu/2+\delta'})\,. \end{aligned} \tag{7.81}$$

Since (writing $m \doteqdot m^*(\beta) - C_1|\Lambda_L|L^{-\nu}$)

$$\sum_{t\in\Lambda_L}\sigma(t)(\omega) - m|\Lambda_L| = \Big(\sum_{t\in\Lambda_L}\sigma(t)(\omega) - \langle\sum_{t\in\Lambda_L}\sigma(t)\rangle^{\star,s}_{\Lambda_{\#}(\underline{\Gamma})}\Big) + \Big(\langle\sum_{t\in\Lambda_L}\sigma(t)\rangle^{\star,s}_{\Lambda_{\#}(\underline{\Gamma})} - m|\Lambda_L|\Big)\,, \tag{7.82}$$

we have, for every $\omega \in \mathcal{A}(C_1,\nu,c)$ and L large enough,

$$\begin{aligned}|\sum_{t\in\Lambda_L}\sigma(t)(\omega) - \langle\sum_{t\in\Lambda_L}\sigma(t)\rangle^{\star,s}_{\Lambda_{\#}(\underline{\Gamma})}| &\geqslant |\langle\sum_{t\in\Lambda_L}\sigma(t)\rangle^{\star,s}_{\Lambda_{\#}(\underline{\Gamma})} - m|\Lambda_L|| - |\sum_{t\in\Lambda_L}\sigma(t)(\omega) - m|\Lambda_L|| \\ &\geqslant \frac{1+\eta}{L^c}|\Lambda_L| - \mathcal{O}(L^{1-\nu/2+\delta'}) - \frac{|\Lambda_L|}{L^c} \\ &\geqslant \frac{\eta}{2}\frac{|\Lambda_L|}{L^c}\,. \end{aligned} \tag{7.83}$$

Consequently,

$$P^{+,\beta,h}_{\Lambda_L}[\mathcal{A}(C_1,\nu,c)\,|\,[\underline{\Gamma}]] \leqslant P^{\star,s}_{\Lambda_{\#}(\underline{\Gamma})}\Big[|\sum_{t\in\Lambda_L}\sigma(t)(\omega) - \langle\sum_{t\in\Lambda_L}\sigma(t)\rangle^{\star,s}_{\Lambda_{\#}(\underline{\Gamma})}| \geqslant \frac{\eta}{2}|\Lambda_L|L^{-c}\Big]\,. \tag{7.84}$$

We estimate (7.84) by Proposition 5.2.1,

$$P^{+,\beta,h}_{\Lambda_L}[\mathcal{A}(C_1,\nu,c)\,|\,[\underline{\Gamma}]] \leqslant \exp\{-\mathcal{O}(L^{1+4\delta-2\delta'})\}\,, \tag{7.85}$$

provided L is large enough. the conclusion follows easily (remember that $2\delta > \delta'$). □

Remark. It is possible to improve the previous lemma. Indeed, in (7.75), we control the denominator using only the conditional probability; however, the probability of $[\underline{\Gamma}]$ can be used to cancel the leading order term of the denominator when the phase volume is not too far from $C_1|\Lambda_L|L^{-\nu}$.

It is now possible to state the main result of this section, which gives the exact asymptotics for the probability of moderately large deviations when the boundary magnetic field is positive; moreover, the typical configurations with respect to the conditioned measure are described in terms of the polygonal approximations.

Theorem 7.4.3. *Assume that the coupling constants are defined by* (3.30) *with* $\beta > \beta_c$ *and* $h \geqslant 0$. *Let* $C_1 > 0$, $1/4 > c > \nu > 0$, $c = 1/4 - \delta$ *be given. Let* $\mathfrak{W}^*(\beta, h, C_1, \nu)$ *be the minimum of the functional* $\mathfrak{W}(\cdot;\beta,h)$ *(without constraints) among curves enclosing a volume* $|Q|(C_1/2m^*)$. *Let* $0 < \delta' < 2\delta$ *such that* $\delta' + \frac{1}{2}\delta < \frac{1}{8}$, *and* $\eta = \delta' + (\nu - c)/2$. *We set*

$$\mathcal{A}(C_1,\nu,c) \doteqdot \{\omega \,:\, |\sum_{t\in\Lambda_L}\omega(t) - m|\Lambda_L|| \leqslant |\Lambda_L|\,L^{-c}\}\,;$$

$$E_1(C_1,\nu,c) \doteqdot \left\{\left|\alpha(\underline{S}) - (C_1/2m^*)L^{-\nu}\right| < \frac{1+\eta}{2m^* L^c}\right\};$$

$$E_2(C_1,\nu,c) \doteqdot \left\{\sum_i \mathfrak{W}(S_i;\beta,h) \leqslant L^{1-\nu/2}\,\mathfrak{W}^*(\beta,h,C_1,\nu)\Big[1+\mathcal{O}(L^{\eta-\delta'})\Big]\right\}.$$

Then, for L *large enough,*

$$P^{+,\beta,h}_{\Lambda_L}[E_1(C_1,\nu,c)\cap E_2(C_1,\nu,c)\,|\,A(C_1,\nu,c)\,] \geqslant 1 - \exp\Big\{-\mathcal{O}(L^{1-c/2})\Big\}$$

and

$$|\frac{1}{L^{1-\nu/2}}\log P^{+,\beta,h}_{\Lambda_L}[\mathcal{A}(C_1,\nu,c)] + \mathfrak{W}^*(\beta,h,C_1,\nu)| \leqslant \mathcal{O}(L^{-(c-\nu)/2})\,.$$

Proof. The first affirmation follows from Theorem 7.4.1, Lemma 7.4.1 and Lemma 7.4.2. We prove the second affirmation. For L large enough Theorem 7.4.1 implies that

$$-\frac{1}{L^{1-\nu/2}}\log P^{+,\beta,h}_{\Lambda_L}[\mathcal{A}(C_1,\nu,c)\,] \leqslant \mathfrak{W}^*(\beta,h,C_1,\nu) + \mathcal{O}(L^{-1/2+\nu/4}\log L)\,. \tag{7.86}$$

Let $\widetilde{E}_1(C_1,\nu,c)$ be the complementary event of $E_1(C_1,\nu,c)$. We can write, setting $\mathcal{A} = \mathcal{A}(C_1,\nu,c)$,

$$\begin{aligned}P^{+,\beta,h}_{\Lambda_L}[\mathcal{A}\,] &= P^{+,\beta,h}_{\Lambda_L}[\mathcal{A}\cap E_1] + P^{+,\beta,h}_{\Lambda_L}[\mathcal{A}\cap\widetilde{E}_1]\\ &= P^{+,\beta,h}_{\Lambda_L}[\mathcal{A}\cap E_1] + P^{+,\beta,h}_{\Lambda_L}[\widetilde{E}_1\,|\,\mathcal{A}\,]\,P^{+,\beta,h}_{\Lambda_L}[\mathcal{A}]\,.\end{aligned} \tag{7.87}$$

Therefore,

$$(1 - P^{+,\beta,h}_{\Lambda_L}[\widetilde{E}_1\,|\,\mathcal{A}])\,P^{+,\beta,h}_{\Lambda_L}[\mathcal{A}] \leqslant P^{+,\beta,h}_{\Lambda_L}[E_1]\,. \tag{7.88}$$

Let $\mathfrak{W}_*(m;\beta,h)$ be defined by

$$\mathfrak{W}_*(m;\beta,h) \doteqdot \inf\{\mathfrak{W}(\mathcal{C};\beta,h)\,:\,\mathcal{C}\subset Q\,,\,\mathrm{vol}\,\mathcal{C} = |Q|\frac{C_1L^{-\nu}}{2m^*}\}\,, \tag{7.89}$$

and let $m \doteqdot m^* - C_1 L^{-\nu}$.
The inequality

$$\sum_i \operatorname{Vol} \mathcal{P}(S_i) \geqslant \alpha(\underline{S})|\Lambda_L| \geqslant \Big((C_1/2m^*)L^{-\nu} - \frac{1+\eta}{2m^* L^c}\Big)|\Lambda_L| \tag{7.90}$$

implies that

$$\sum_i \mathfrak{W}(\mathcal{P}(S_i);\beta,h) \geqslant \mathfrak{W}_*(m + \frac{1+\eta}{L^c};\beta,h)L\,. \tag{7.91}$$

Let $V_1 \subset Q$ be a convex body realizing the minimum $\mathfrak{W}_*(m+(1+\eta)/L^c;\beta,h)$ and $V_2 \subset Q$ be a disk of volume $(1+\eta)/2m^* L^c$. We can choose these convex bodies so that their union is a set of volume $|Q|C_1 L^{-\nu}/2m^*$. Thus

$$\mathfrak{W}_*(m + \frac{1+\eta}{L^c};\beta,h) + \mathfrak{W}(\partial V_2;\beta,h) \geqslant \mathfrak{W}_*(m;\beta,h) = \mathfrak{W}^*(\beta,h,C_1,\nu)L^{-\nu/2}\,. \tag{7.92}$$

Therefore

$$\begin{aligned}(1 - P_{\Lambda_L}^{+,\beta,h}[\widetilde{E}_1 \,|\, \mathcal{A}])\, &P_{\Lambda_L}^{+,\beta,h}[\mathcal{A}] \\ &\leqslant P_{\Lambda_L}^{+,\beta,h}\Big[\Big\{\sum_i \mathfrak{W}(\mathcal{P}(S_i)) \geqslant \mathfrak{W}_*(m + \frac{1+\eta}{L^c})L\Big\}\Big] \\ &\leqslant P_{\Lambda_L}^{+,\beta,h}\Big[\Big\{\sum_i \mathfrak{W}(\mathcal{P}(S_i)) \geqslant \mathfrak{W}^*(\beta,h,C_1,\nu)L^{1-\nu/2} - \mathfrak{W}(\partial V_2)L\Big\}\Big]\,.\end{aligned} \tag{7.93}$$

Lemma 7.4.1 implies that for L large enough

$$-\frac{1}{L^{1-\nu/2}} \log P_{\Lambda_L}^{+,\beta,h}[\mathcal{A}(C_1,\nu,c)] \geqslant \mathfrak{W}^*(\beta,h,C_1,\nu) - \mathcal{O}(L^{\eta-\delta'})\,. \tag{7.94}$$

□

7.4.4 Upper bound: Negative magnetic field

Let the coupling constants be given by (3.30) with $\beta > \beta_c$ and $h < 0$. The remarks of subsection 7.4.2 apply.
We show that the lower bounds obtained in Theorem 7.4.2 are optimal to leading order, using the same techniques as in the previous section.

By definition the open contour Γ^* is a large contour. We associate to Γ^* a sequence of points $S^* \doteqdot (t_{*0},\ldots,t_{*N})$ using the same procedure as for the other contours. $\mathcal{P}(S^*)$ is the open polygonal line with vertices S^*. We thus obtain a family,

$$(\mathcal{P}(S_1),\ldots,\mathcal{P}(S_q),\mathcal{P}(S^*),\mathcal{P}(S'_1),\ldots,\mathcal{P}(S'_p)) \tag{7.95}$$

of polygonal lines. We have distinguished between the polygonal lines with no edge belonging to the line $\{t \in \mathbb{R}^2 \,:\, t(2) = -1/2\}$, which are denoted by $(\mathcal{P}(S_1),\ldots,\mathcal{P}(S_q))$, and the other ones denoted by $(\mathcal{P}(S'_1),\ldots,\mathcal{P}(S'_p))$. We will now associate to the set of polygonal lines $(\mathcal{P}(S^*),\mathcal{P}(S'_1),\ldots,\mathcal{P}(S'_p))$ a new set of *closed* polygonal lines $(\mathcal{P}(S_{q+1}),\ldots,\mathcal{P}(S_k))$.
This is done in the following way:

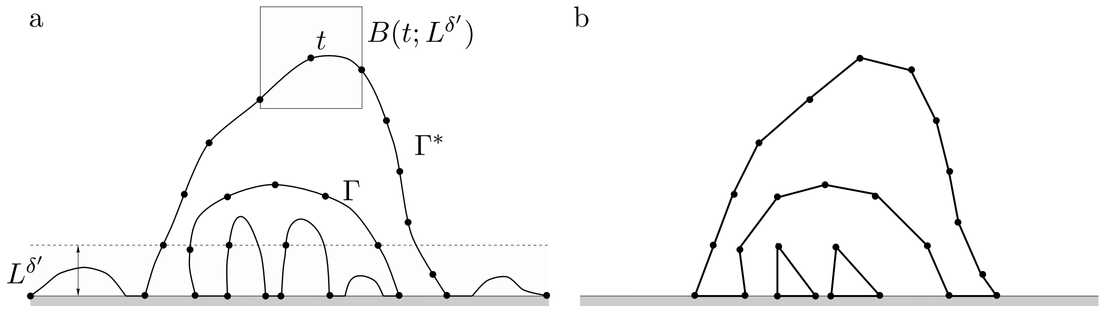

FIGURE 7.9. a) Coarse–graining of a large contour Γ touching the lower wall and of the open contour Γ^*; the dots represent the sequence of points obtained by the coarse–graining procedure described. b) The *three* resulting closed polygonal lines.

1. Consider the family of polygonal lines $(\mathcal{P}(S^*), \mathcal{P}(S'_1), \ldots, \mathcal{P}(S'_p))$; let $\mathcal{E}^*$ be the set of edges formed by all edges of this family, which belong to the line $\{t \in \mathbb{R}^2 : t(2) = -1/2\} \cap \Lambda^*_L$. Remove $\mathcal{E}^*$ from the set of all edges of $(\mathcal{P}(S^*), \mathcal{P}(S'_1), \ldots, \mathcal{P}(S'_p))$.
2. Close the polygonal lines obtained in 1. by adding the set

$$(\Sigma^* \cap \Lambda^*) \setminus \mathcal{E}^* . \tag{7.96}$$

This defines a set of closed polygonal lines denoted $\mathcal{P}(S_{q+1}), \ldots, \mathcal{P}(S_k)$.

Remark. We do not modify the large contours. The relation between the family $(S_1, \ldots, S_k)$ and the large contours of the configuration is that these contours must be compatible with the original family $(S_1, \ldots, S_q, S^*, S'_1, \ldots, S'_p)$.

Notice that the above construction is such that we have the identity

$$\mathfrak{W}(S_1, \ldots, S_k; \beta, h) = \mathfrak{W}(S_1, \ldots, S_q, S^*, S'_1, \ldots, S'_p; \beta, |h|) - \tau_{\mathrm{bd}}(2r_1 L + 1) \tag{7.97}$$

where

$$\mathfrak{W}(S_1, \ldots, S_k; \beta, h) \doteqdot \sum_{i=1}^{k} \mathfrak{W}(\mathcal{P}(S_i); \beta, h) . \tag{7.98}$$

Lemma 7.4.3. *In the setting described above, there exists a constant K_2 such that*

$$P^{\pm,\beta,|h|}_{\Lambda_L}[\{S_1, \ldots, S_k\}] \leqslant K_2 \exp\{-\mathfrak{W}(S_1, \ldots, S_k; \beta, h)\} .$$

if $h > -h_{\mathrm{w}}(\beta)$, and

$$P^{\pm,\beta,|h|}_{\Lambda_L}[\{S_1, \ldots, S_k\}] \leqslant K_2 \, L^{3/2} \exp\{-\mathfrak{W}(S_1, \ldots, S_k; \beta, h)\} .$$

if $h \leqslant -h_{\mathrm{w}}(\beta)$.

Proof. We write $P^{\pm,\beta,|h|}_{\Lambda_L}[\{S_1, \ldots, S_k\}]$ as a quotient

$$P^{\pm,\beta,|h|}_{\Lambda_L}[\{S_1, \ldots, S_k\}] \ =: \ \frac{Z^{\pm}(\Lambda_L | S_1, \ldots, S_k; \beta, |h|)}{Z^{\pm}(\Lambda_L; \beta, |h|)} . \tag{7.99}$$

Dividing and multiplying by $Z^+(\Lambda_L;\beta,|h|)$ we must consider the quotients

$$\frac{Z^{\pm}(\Lambda_L|S_1,\ldots,S_k;\beta,|h|)}{Z^+(\Lambda_L;\beta,|h|)} \quad , \quad \frac{Z^{\pm}(\Lambda_L;\beta,|h|)}{Z^+(\Lambda_L;\beta,|h|)}\,. \tag{7.100}$$

The first quotient is estimated using Lemma 4.2.8 and the above remark,

$$\frac{Z^{\pm}(\Lambda_L|S_1,\ldots,S_k;\beta,|h|)}{Z^+(\Lambda_L;\beta,|h|)} \leqslant \exp\{-\mathfrak{W}(S_1,\ldots,S_q,S^*,S'_1,\ldots,S'_p;\beta,|h|)\} \tag{7.101}$$

The second quotient is estimated as in Section 7.4.2, using Proposition 4.5.2,

$$\frac{Z^{\pm}(\Lambda_L;\beta,|h|)}{Z^+(\Lambda_L;\beta,|h|)} = \langle\sigma(t_1^*)\sigma(t_2^*)\rangle_{\Lambda_L^*}^{\beta^*,|h|^*} \geqslant C\,\exp\{-\tau_{\mathrm{bd}}(t_2^*-t_1^*;\beta^*,|h|^*)\}\,, \tag{7.102}$$

if $h>-h_{\mathrm{w}}(\beta)$, and

$$\frac{Z^{\pm}(\Lambda_L;\beta,|h|)}{Z^+(\Lambda_L;\beta,|h|)} = \langle\sigma(t_1^*)\sigma(t_2^*)\rangle_{\Lambda_L^*}^{\beta^*,|h|^*} \geqslant C\,(2r_1L)^{-3/2}\exp\{-\tau_{\mathrm{bd}}(t_2^*-t_1^*;\beta^*,|h|^*)\}\,, \tag{7.103}$$

if $h\leqslant -h_{\mathrm{w}}(\beta)$.
These inequalities give respectively, using (7.97),

$$P_{\Lambda_L}^{\pm,\beta,|h|}[\{S_1,\ldots,S_k\}] \leqslant C^{-1}\,\exp\{-\mathfrak{W}(S_1,\ldots,S_k;\beta,h)\}\,, \tag{7.104}$$

if $h>-h_{\mathrm{w}}(\beta)$, and

$$P_{\Lambda_L}^{\pm,\beta,|h|}[\{S_1,\ldots,S_k\}] \leqslant C^{-1}(2r_1)^{3/2}\,L^{3/2}\exp\{-\mathfrak{W}(S_1,\ldots,S_k;\beta,h)\}\,. \tag{7.105}$$

if $h\leqslant -h_{\mathrm{w}}(\beta)$. □

Lemma 7.4.4. *Suppose the coupling constants are defined by* (3.30) *with* $\beta>\beta_c$*, and* $h<0$*. Then for any* $\eta<\delta'$ *and* $T>0$

1. *If* $h>-h_{\mathrm{w}}(\beta)$*, then*

$$P_{\Lambda_L}^{\pm,\beta,|h|}[\{\sum_{j\,\geqslant\,1}\mathfrak{W}(\mathcal{P}(S_j);\beta,h)\geqslant T\}] \leqslant \exp\left\{-T[1-\mathcal{O}(L^{\eta-\delta'})]\right\}\,.$$

2. *If* $h\leqslant -h_{\mathrm{w}}(\beta)$*, then*

$$P_{\Lambda_L}^{\pm,\beta,|h|}[\{\sum_{j\,\geqslant\,1}\mathfrak{W}(\mathcal{P}(S_j);\beta,h)\geqslant T\}] \leqslant \exp\left\{-T[1-\mathcal{O}(L^{\eta-\delta'})]+\mathcal{O}(L^{1+\eta-\delta'})\right\}\,.$$

Proof. The proof is essentially the same as that of Lemma 7.4.1. The complication comes from the fact that the functional $\mathfrak{W}(\cdot;\beta,h)$ is not positive when $h<0$. We have

$$P_{\Lambda_L}^{\pm,\beta,|h|}[\{\sum_{j\,\geqslant\,1}\mathfrak{W}(\mathcal{P}(S_j);\beta,h)\geqslant T\}] = \sum_{k\,\geqslant\,1}\sum_{S_1,\ldots S_k}P_{\Lambda_L}^{\pm,\beta,|h|}[\{\mathfrak{W}(S_1,\ldots,S_k;\beta,h)\geqslant T\}]\,. \tag{7.106}$$

We decompose the surface tension in the following way:

$$\mathfrak{W}(S_1,\ldots,S_k;\beta,h)=T'\equiv T'_+-T'_-\,, \tag{7.107}$$

with T'_+, resp. T'_-, the positive, resp. negative, part of the functional $\mathfrak{W}(\cdot;\beta,h)$. The total number N of vertices of the polygonal lines $\mathcal{P}(S_i)$, $i=1,\ldots,k$, is bounded by (see (7.70))

$$N\leqslant T'_+K_1L^{-\delta'}\,. \tag{7.108}$$

Now we have to consider separately the cases $h>-h_{\mathrm{w}}(\beta)$ and $h\leqslant -h_{\mathrm{w}}(\beta)$.

1. Suppose first that $h>-h_{\mathrm{w}}(\beta)$. Since $T'_+\geqslant\frac{\tau((1,0);\beta)}{\tau_{\mathrm{bd}}(\beta,h)}T'_-$, we have

$$N\leqslant\frac{K_1\tau((1,0);\beta)}{\tau((1,0);\beta)-\tau_{\mathrm{bd}}(\beta,h)}L^{-\delta'}\,T'\,. \tag{7.109}$$

   and we can therefore proceed as in Lemma 7.4.1.
2. We consider the case $h\leqslant -h_{\mathrm{w}}(\beta)$. Since $|T'_-|$ is at most $O(L)$,

$$\begin{aligned}P^{\pm,\beta,|h|}_{\Lambda_L}[\{S_1,\ldots,S_k\}]&\leqslant\exp\{-\mathfrak{W}(S_1,\ldots,S_k;\beta,h)\}\,K_2L^{3/2}\\&=\exp\left\{-T'_++T'_-+NL^\eta\right\}K_2L^{3/2}\exp\{-NL^\eta\}\\&\leqslant\exp\left\{-T'_+(1-\mathcal{O}(L^{-\delta'+\eta}))+T'_-\right\}K_2L^{3/2}\exp\{-NL^\eta\}\\&\leqslant\exp\left\{-\mathfrak{W}(S_1,\ldots,S_k;\beta,h)(1-\mathcal{O}(L^{-\delta'+\eta}))+\mathcal{O}(L^{1-\delta'+\eta})\right\}\\&\qquad\times K_2L^{3/2}\exp\{-NL^\eta\}\,.\end{aligned} \tag{7.110}$$

   The end of the proof is the same as that of Lemma 7.4.1.

$$\begin{aligned}&P^{\pm,\beta,|h|}_{\Lambda_L}[\{\sum_{j\geqslant1}\mathfrak{W}(\mathcal{P}(S_j);\beta,h)\geqslant T\}]\\&\leqslant\exp\left\{-T[1-\mathcal{O}(L^{\eta-\delta'})]+\mathcal{O}(L^{1-\delta'+\eta})\right\}\sum_{N\geqslant1}K_2L^{3/2}\exp\{N\mathcal{O}(\ln L)-NL^\eta\}\\&\leqslant\exp\left\{-T[1-\mathcal{O}(L^{\eta-\delta'})]+\mathcal{O}(L^{1-\delta'+\eta})\right\}\,.\end{aligned} \tag{7.111}$$

□

We define $\alpha(S_1,\ldots S_k)\doteqdot\alpha(S_1,\ldots,S_q,S^*,S'_1,\ldots,S'_p)$.

Lemma 7.4.5. *We assume that the coupling constants are defined by* (3.30) *with* $\beta>\beta_c$ *and* $h<0$.

1. *Suppose* $h>-h_{\mathrm{w}}(\beta)$. *Let* $1/4>c>\nu>0$, $c=1/4-\delta$, $C_1>0$ *and* $0<\delta'<2\delta$. *Then, for any* $\eta>0$,

$$P^{\pm,\beta,|h|}_{\Lambda_L}\left[\left\{|\alpha(\underline{S})-(C_1/2m^*)L^{-\nu}|\geqslant\frac{1+\eta}{2m^*L^c}\right\}\,\middle|\,\mathcal{A}(C_1,\nu,c)\right]\leqslant\exp\{-\mathcal{O}(L^{1-\nu/2})\}\,,$$

   provided L *is large enough.*

2. *Suppose $h \leqslant -h_{\mathrm{w}}(\beta)$. Let $1/8 > c > \nu > 0$ and $C_1 > 0$. Suppose that $c + \delta' < 1$ and $4c + 2\delta' - 2\nu < 1$. Then, for any $\eta > 0$,*

$$P_{\Lambda_L}^{\pm,\beta,|h|}\Big[\{|\alpha(\underline{S}) - (C_1/2m^*)L^{-\nu}| \geqslant \frac{1+\eta}{2m^* L^c}\}\,\Big|\,\mathcal{A}(C_1,\nu,c)\Big] \leqslant \exp\{-\mathcal{O}(L^{1-2\nu})\}\,,$$

provided L is large enough.

Proof. 1. The proof is the same as that of Lemma 7.4.2.

2. The only modification comes from the fact that the *a priori* bound on the total length of the polygonal lines is $\mathcal{O}(L)$ (and not $\mathcal{O}(L^{1-\nu/2})$). This is the reason why there are these supplementary conditions on the parameters.

□

From the results of this section, we finally have

Theorem 7.4.4. *Assume that the coupling constants are defined by (3.30) with $\beta > \beta_c$ and $h < 0$.*

1. *Let $C_1 > 0$, $1/4 > c > \nu > 0$, $c = 1/4 - \delta$ be given. Let $\mathfrak{W}^*(\beta,h,C_1,\nu)$ be the minimum of the functional $\mathfrak{W}(\cdot;\beta,h)$ (without constraints) among curves enclosing a volume $|Q|(C_1/2m^*)$. Let $0 < \delta' < 2\delta$ such that $\delta' + \frac{1}{2}\delta < \frac{1}{8}$, and $\eta = \delta' + (\nu - c)/2$. We set*

$$\mathcal{A}(C_1,\nu,c) \doteqdot \{\omega : |\sum_{t\in\Lambda_L}\omega(t) - m|\Lambda_L|| \leqslant |\Lambda_L|\,L^{-c}\}\,;$$

$$E_1(C_1,\nu,c) \doteqdot \left\{\left|\alpha(\underline{S}) - (C_1/2m^*)L^{-\nu}\right| < \frac{1+\eta}{2m^* L^c}\right\};$$

$$E_2(C_1,\nu,c) \doteqdot \left\{\sum_i \mathfrak{W}(S_i;\beta,h) \leqslant L^{1-\nu/2}\,\mathfrak{W}^*(\beta,h,C_1,\nu)\Big[1 + \mathcal{O}(L^{\eta-\delta'})\Big]\right\}.$$

Then, for L large enough,

$$P_{\Lambda_L}^{+,\beta,h}[E_1(C_1,\nu,c) \cap E_2(C_1,\nu,c)\,|\,A(C_1,\nu,c)\,] \geqslant 1 - \exp\Big\{-\mathcal{O}(L^{1-c/2})\Big\}$$

and

$$|\frac{1}{L^{1-\nu/2}}\log P_{\Lambda_L}^{+,\beta,h}[\mathcal{A}(C_1,\nu,c)] + \mathfrak{W}^*(\beta,h,C_1,\nu)| \leqslant \mathcal{O}(L^{-(c-\nu)/2})\,.$$

2. *Suppose $h \leqslant -h_{\mathrm{w}}(\beta)$. Let $C_1 > 0$, $\frac{5}{32} > c > 5\nu > 0$ be given. Let (see Theorem 7.4.2)*

$$\mathfrak{W}^*(\beta,h,C_1,\nu) \doteqdot \lim_{L\to\infty} L^{2\nu}\,\mathfrak{R}_L^*(\beta,h,C_1,\nu)\,.$$

Let $\delta' \doteqdot 5\nu$ and $\eta \doteqdot \delta'/2$. We set

$$\mathcal{A}(C_1,\nu,c) \doteqdot \{\omega : |\sum_{t\in\Lambda_L}\omega(t) - m|\Lambda_L|| \leqslant |\Lambda_L|\,L^{-c}\}\,;$$

$$E_1(C_1,\nu,c) \doteqdot \left\{\left|\alpha(\underline{S}) - (C_1/2m^*)L^{-\nu}\right| < \frac{1+\eta}{2m^* L^c}\right\};$$

$$E_2(C_1,\nu,c) \doteqdot \left\{\sum_i \mathfrak{W}(S_i;\beta,h) \leqslant L^{1-2\nu}\,\mathfrak{W}^*(\beta,h,C_1,\nu)\Big[1 + \mathcal{O}(L^{2\nu+\eta-\delta'})\Big]\right\}.$$

Then, for L large enough,

$$P^{+,\beta,h}_{\Lambda_L}[E_1(C_1,\nu,c)\cap E_2(C_1,\nu,c)\,|\,A(C_1,\nu,c)\,] \geqslant 1-\exp\Big\{-\mathcal{O}(L^{5\nu/2})\Big\}$$

and

$$|\frac{1}{L^{1-2\nu}}\log P^{+,\beta,h}_{\Lambda_L}[\mathcal{A}(C_1,\nu,c)] + \mathfrak{W}^*(\beta,h,C_1,\nu)| \leqslant \mathcal{O}(L^{-\nu/2})\,.$$

Proof. 1. The proof is exactly the same as in the case of positive h.

2. The proof is the same. One just has to check that the various conditions on the parameters are satisfied for the particular choice made in the statement of the theorem.

□

7.4.5 Some heuristic remarks in the case of complete wetting

Let

$$\mathcal{A}(C_1,\nu,c) \doteqdot \{\omega\,:\,|\sum_{t\in\Lambda_L}\omega(t) - m^*(\beta) + C_1|\Lambda_L|L^{-\nu}| \leqslant |\Lambda_L|L^{-c}\}\,, \tag{7.112}$$

From the previous subsections, we know that (at least when ν is small enough)

$$P^{+,\beta,h}_{\Lambda_L}(\mathcal{A}(C_1,\nu,c)) \sim \begin{cases}\exp(\mathcal{O}(L^{1-\nu/2})) & h > -h_{\mathrm{w}}(\beta)\,,\\ \exp(\mathcal{O}(L^{1-2\nu})) & h \leqslant -h_{\mathrm{w}}(\beta)\,.\end{cases} \tag{7.113}$$

We discuss heuristically some questions related to the behaviour in the complete wetting regime. In particular, we want to argue that not only the large deviations are qualitatively affected by the complete wetting, but also the typical ones.

The techniques used in the previous subsections do not allow us to prove (7.113) for all values at which the result holds. What do we expect? In the first case ($h > -h_{\mathrm{w}}(\beta)$) (7.113) should remain valid up to $\nu = 2/3$; after that the system should have a Gaussian behaviour, i.e. $P^{+,\beta,h}_{\Lambda_L}(\mathcal{A}(C_1,\nu,c)) \sim \exp\mathcal{O}(L^{2-2\nu})$. This is proved at low temperature in the case $h=1$ [16] and we cannot see any reason why this should change when $h\neq 1$. The second case ($h \leqslant -h_{\mathrm{w}}(\beta)$) is much more interesting. Clearly, (7.113) cannot remain true when $\nu \geqslant 1/2$; in particular, something must happen before ν reaches $2/3$. We would like to give some arguments to show that what should happen is the following: The scale of the typical fluctuations of the magnetization in the complete wetting regime is not $\mathcal{O}(L)$ anymore, but is in fact much larger, of order $\mathcal{O}(L^{3/2})$. We have no proof of this (this is a quite difficult problem, even perturbatively), so we give some heuristic justifications.

Let us first find what should be the mean value of the magnetization in the complete wetting regime. We write Γ^* the open contour in the $\pm$-b.c. and $\mathrm{vol}\,\Gamma^* \doteqdot |\{t\in\Lambda_L\,:$

[16] A non-perturbative derivation for $h=1$ (and without considering boundary effects) has been announced by Ioffe and Schonmann.

$\omega_{\Gamma^*}(t) = -1\}|$; let $c' > 0$ be some sufficiently large constant. Then

$$\begin{aligned}\langle \sum_{t\in\Lambda_L} \sigma(t)\rangle_{\Lambda_L}^{+,\beta,h} &= \sum_{\Gamma^*} \langle \sum_{t\in\Lambda_L} \sigma(t) \,|\, \Gamma^*\rangle_{\Lambda_L}^{\pm,\beta,|h|} P_{\Lambda_L}^{\pm,\beta,|h|}(\Gamma^*) \\ &= \sum_{\Gamma^*\,:\,|\Gamma^*|<c'L} (m^*(\beta)(|\Lambda_L| - 2\mathrm{vol}\,\Gamma^*) + \mathcal{O}(L))P_{\Lambda_L}^{\pm,\beta,|h|}(\Gamma^*) + \exp(-\mathcal{O}(L)) \\ &= m^*(\beta)|\Lambda_L| - 2\langle \mathrm{vol}\,\Gamma^*\rangle_{\Lambda_L}^{\pm,\beta,|h|} + \mathcal{O}(L)\,. \end{aligned} \tag{7.114}$$

The point now is to realize that $\langle \mathrm{vol}\,\Gamma^*\rangle_{\Lambda_L}^{\pm,\beta,|h|}$ should be of order $\mathcal{O}(L^{3/2})$. Indeed, the phase separation line should cease to be pinned to the wall when $h \leqslant h_{\mathrm{w}}$; the results of [FP3, FP4], show that the excursions must become unbounded (otherwise the Gibbs state would not be unique), however they do not give information on the typical geometry of the phase separation line. One can modelize such a situation with a Bernoulli excursion (i.e. a random walk in a half plane, with both endpoints fixed on the wall.) which should give the right scales. For such a model, it is possible to compute explicitly the distribution of the area under the path (see [Ta]) and it is found that the mean value is of order $\mathcal{O}(L^{3/2})$. Therefore this argument shows that when complete wetting occurs the magnetization is quite far from $m^*(\beta)$ (it is already, spontaneously, in what would be call large deviations regime when $h = 1$); more importantly, the main role is played by the phase separation line. Let us now discuss the fluctuations of magnetization.

There are two main contributions to the fluctuations of magnetization: first, the usual bulk fluctuations and boundary effects which are of order $\mathcal{O}(L)$; second, the fluctuation of the phase separation line which should be of order $\mathcal{O}(L^{3/2})$. This can be again understood by comparing our case to the much simpler case of Bernoulli excursion, for which fluctuations of that order have been proved [Ta].

To make the argument possibly more convincing, we may say that a suitably rescaled version of the phase separation line should converge to the Brownian excursion process (Brownian motion in a half plane, with both endpoints fixed on the wall). In the simpler case of a phase separation line in the middle of the box (i.e. far from the boundary) convergence of a suitably rescaled version of the phase separation line to the Brownian bridge (Brownian motion with both endpoints fixed, but not constrained to lie in a half plane) has been proved at low temperature in [Hi2]. The distribution of the area under the path of a Brownian excursion can be obtained from the corresponding distribution for the Bernoulli excursion, see [Ta].

7.5 Large deviations

In this section we obtain the exact leading order for the asymptotics of volume order large deviations. This problem is more difficult than the corresponding one for moderately large deviations. Indeed, in the present case, it is possible that there is no translate of the Wulff shape of the required volume that fits inside the box Q. The corresponding variational problem, which has been studied in [KP], is more complicated. Even though solutions (and even stability) have been obtained for many situations, there are some cases which are not covered. Since the variational problem can become very involved, it is interesting, in our opinion, to have techniques which allow to deal with the statistical mechanical parts of these questions without having to use information from the variational part. This is the philosophy of this and the following sections.

Let $r_1, r_2 \in \mathbb{N}$ and let $\Lambda_L \subset \mathbb{Z}^2$ be the box

$$\Lambda_L \doteqdot \{t \in \mathbb{Z}^2 \,:\, -r_1 L \leqslant t(1) < r_1 L\,,\, 0 \leqslant t(2) < 2r_2 L\}\,. \tag{7.115}$$

The coupling constants are given by (3.30) with $\beta > \beta_c$. We consider the case of positive and negative boundary magnetic field. In the case $h < 0$, the remarks of Section 7.4.2 apply and we work with the measure $\mu_{\Lambda_L}^{\pm,\beta,|h|}$ instead of the measure $\mu_{\Lambda_L}^{+,\beta,h}$, as explained in that section.

Let $-m^*(\beta) < m < m^*(\beta)$ and $c > 0$. We are interested in the large volume asymptotics of the event

$$\mathcal{A}(m,c) \doteqdot \{\omega \,:\, |\sum_{t\in\Lambda_L} \omega(t) - m|\Lambda_L|| \leqslant |\Lambda_L|\, L^{-c}\}\,. \tag{7.116}$$

This section is composed of two subsections. In subsection 7.5.1, we prove lower bounds for the large deviations. Optimality (at leading order) of these bounds is proved in Subsection 7.5.2.

7.5.1 The lower bounds

We derive lower bounds for the large volume asymptotics of

$$P_{\Lambda_L}^{+,\beta,h}[\mathcal{A}(m,c)] \quad \text{and} \quad P_{\Lambda_L}^{\pm,\beta,h}[\mathcal{A}(m,c)] \tag{7.117}$$

in terms of the Wulff functional. These lower bounds are shown to be optimal in Sections 7.5.2.

As explained in the introductory part of this section, we don't want to use information from the variational problem. The only information we use is that we can restrict our attention to convex curves, which is an elementary consequence of Jensen's inequality, see Section 7.2. The strategy is then to prove lower bounds for the boundary $\mathcal{C}$ of *any* convex body in Q of volume $|Q|\frac{m^*(\beta)-m}{2m^*(\beta)}$. Since we obtain these results uniformly in $\mathcal{C}$, the best lower bound will be given by the infimum of the lower bounds over all these convex bodies, that is it will be given by the infimum of the constrained variational problem.

The proof is similar to the corresponding proof for moderately large deviations. However, in contradistinction with what is done in Sections 7.4.1 and 7.4.2, we do not use the concentration results of Section 4.6 but use instead the less precise box propositions of Section 4.4. There are three reasons for this decision: first, it shows how to apply these weaker concentration properties; second, it is easier to describe; third, if we are not interested in an estimate of the rate of convergence, then it is possible to prove such asymptotics *without using any information from the exact solution*, see the end of the section.

Theorem 7.5.1. *Assume that the coupling constants are defined by* (3.30) *with* $\beta > \beta_c$ *and* $h \in \mathbb{R}$. *Let* $-m^*(\beta) < m < m^*(\beta)$ *and* $c \doteqdot 1/2 - \delta$, $\delta > 0$. *Then there exists* $L_0(\beta, h, m, c, Q)$ *such that, for any simple closed rectifiable curve* $\mathcal{C}$, *which is the boundary* *of a convex body of volume* $|Q|\frac{m^*(\beta)-m}{2m^*(\beta)}$ *in the rectangle* Q*, and for all* $L \geqslant L_0$*,*[17]

$$P^{+,\beta,h}_{\Lambda_L}[\mathcal{A}(m,c)] \geqslant \exp\Big\{-L\cdot\mathfrak{W}(\mathcal{C};\beta,h)-\mathcal{O}(\beta L^{1/2}\log L)\Big\}\,,$$

where $P^{+,\beta,h}_{\Lambda_L}[\,\cdot\,] \doteqdot P^{\pm,\beta,|h|}_{\Lambda_L}[\,\cdot\,]$ *if* $h<0$*.*

Proof. 1) We first prove the case $h>0$.
As in the proof of Theorem 7.4.1, given the boundary $\mathcal{C}$ of a convex body V, we define a polygonal approximation of it. Then, by summing over all large contours passing through the vertices of the polygonal approximation we can estimate the probability of the event $\mathcal{A}(m,c)$ in terms of the functional $\mathfrak{W}$ using Propositions 4.4.1 and 4.4.2. The main difference with the above mentioned proof is that now we are not working with the Wulff shape, but with any convex curve, so that we have to be careful to have only estimates which are uniform in $\mathcal{C}$ (in particular, we don't want L_0 to depend on $\mathcal{C}$!). As before, we divide the proof into five steps.

Step 1. Definition of a polygonal approximation of $\mathcal{C}$.

Consider the convex body V whose boundary is $\partial V=\mathcal{C}$. Let $L\in\mathbb{N}$ and set

$$\delta_L \doteqdot L^{-1/2}\ln L\,. \tag{7.118}$$

Let

$$Q_L \doteqdot \{x\in Q\,:\,\min_{y\notin Q}\|y-x\|_2 \geqslant \delta_L\}\,, \tag{7.119}$$

and set $V_L \doteqdot V\cap Q_L$.

We define a polygonal approximation $\mathcal{P}_L$ of ∂V_L. First define a polygonal approximation $\mathcal{P}^0_L$. Let Δ_L be the square

$$\Delta_L \doteqdot \{x\in\mathbb{R}^2\,:\,\|x\|_1=\frac{\delta_L}{\sqrt{2}}\}\,, \tag{7.120}$$

and denote its four sides, which have a length δ_L, J_1, J_2, J_3 and J_4 (counterclockwise). Since V is convex, $\operatorname{vol}V_L \geqslant \operatorname{vol}V-\mathcal{O}(\delta_L)$ and $V_L\subset Q_L$, there exists L_0 independent of V, such that $\operatorname{int}V$ contains a translate of Δ_L.

1. We choose first four disjoint segments isometric to J_k, $k=1,\dots,4$, with extremities on ∂V_L. If this is not possible, then we choose one corner isometric to $J_k\cup J_{k+1}$ with extremities on ∂V_L, but not necessarily its apex, and two disjoint segments isometric to J_m, J_n, $m,n\neq k,k+1$, or two corners isometric to $J_k\cup J_{k+1}$ and $J_n\cup J_{n+1}$ with extremities on ∂V_L, but not necessarily their apexes[18]. Starting from these four segments we construct a polygonal approximation of $\partial V_L\setminus\partial Q_L$ with a maximal number of segments of length δ_L. We obtained a polygonal approximation of $\partial V_L\setminus\partial Q_L$ with at most 8 segments of length smaller than δ_L. The union of this polygonal approximation and $\partial V_L\cap\partial Q_L$ defines $\mathcal{P}^0_L$.

[17]The proof gives an estimate of the function $\mathcal{O}(\beta L^{1/2}\log L)$, which is at most $75\beta L^{1/2}\log L$.

[18]We don't have to consider other cases, for otherwise the volume of V cannot be large enough (remember that it is convex by hypothesis).

Since $\tau(\cdot)$ is convex, Jensen's inequality implies ($\boldsymbol{n}(s)$ is the unit normal at s)

$$\int_{\partial V_L} \tau(\boldsymbol{n}(s))\, \mathrm{d}s \geqslant \int_{\mathcal{P}_L^0} \tau(\boldsymbol{n}(s))\, \mathrm{d}s\,. \tag{7.121}$$

For each side of $\mathcal{P}_L^0 \setminus \partial Q_L$ of length δ_L we construct a square box by translating and possibly rotating (by an angle $\pi/2$) the box (4.68) so that the two extremities of the side play the role of 0 and t in (4.68). The construction of $\mathcal{P}_L^0$ assures that all these boxes are pairwise disjoint.

2. Let $[t,s] \doteqdot \{x \in \mathcal{P}_L^0 \,:\, x(2) = \delta_L\}$. If $[t,s] \neq \varnothing$ and $\|t-s\|_2 > 0$, then we replace $[t,s]$ by the broken line from t to $(t(1),0)$, then $(t(1),0)$ to $(s(1),0)$ and finally from $(s(1),0)$ to s. Then we subdivide the segment $(t(1),0)$ to $(s(1),0)$ into segments of length $\frac{1}{2}\delta_L$ (except possibly the last one which may be shorter). We do a similar construction with the other parts of $\mathcal{P}_L^0 \cap \partial Q_L$.

The polygonal approximation $\mathcal{P}_L$ of ∂V_L is given by the modification of $\mathcal{P}_L^0$ by 2.; the vertices of $\mathcal{P}_L$ are denoted by t_k. For each segment of length δ_L of $\mathcal{P}_L \cap \partial Q$, we construct a square box by translating the box (4.92) so that the extremities of the side play the role of 0 and t in (4.92). We have ($\tau(x;\beta) \leqslant 2\beta$)

$$\mathfrak{W}(\mathcal{C}) \geqslant \mathfrak{W}(\mathcal{P}_L) - 16\beta\delta_L\,. \tag{7.122}$$

Step 2. Scaling and definition of a set of closed contours $\mathcal{G}_L$.

Let $L\mathcal{P}_L$ be the polygon obtained by scaling $\mathcal{P}_L$ by a factor L, shifting it by $(0,-1/2)$ and modifying slightly the position of its vertices, if necessary, so that all of them are in Λ_L^*.

We define a set of closed contours $\mathcal{G}_L = \{\Gamma\}$.

1. Each $\Gamma \in \mathcal{G}_L$ is closed and passes through all vertices of $L\mathcal{P}_L$ (counterclockwise). We denote by $[Lt_k, Lt_{k+1}]$ the side of $L\mathcal{P}_L$ between two consecutive vertices, say Lt_k and Lt_{k+1}.
2. If there is a box B_k associated with $[Lt_k, Lt_{k+1}]$, then γ_k, the part of Γ between Lt_k and Lt_{k+1}, is such that $\gamma_k \cap \partial B_k = Lt_k, Lt_{k+1}$. Otherwise $\gamma_k = \eta_k$, a fixed contour of minimal length from Lt_k and Lt_{k+1}.

The total length of the fixed part of Γ is smaller than $26L\delta_L$.

We can now conclude as in the proof of Theorem 7.4.1. We only sketch the rest of the proof (notice that the numbering of the steps in this proof is the same as the numbering in the proof of Theorem 7.4.1).

Step 3. Estimation of $P_{\Lambda_L}^{+,\beta,h}[\mathcal{A}(m,c)\,|\,\{\Gamma\,;\,\gamma \neq \Gamma\, s\text{-small}\}]$.

Let $s \doteqdot L^\delta$. We use Proposition 5.2.3 to estimate $P_{\Lambda_L}^{+,\beta,h}[\mathcal{A}(m,c)\,|\,\{\Gamma\,;\,\gamma \neq \Gamma\, s\text{-small}\}]$. Since the total volume of the boxes B_k is smaller than $\mathcal{O}(L^{3/2}\log L)$, uniformly in V[19] (the length of $\mathcal{C} = \partial V$ is bounded by the length of ∂Q, and thus the number of sides of $\mathcal{P}_L$ is uniformly bounded by $\mathcal{O}(L^{1/2}/\log L)$). The difference of the volumes of LV and $L\mathcal{P}_L$

[19] Of course this is in this estimate that the fact of using square boxes instead of ellipses play a role. In particular, this is the reason we have to take $c < 1/2$ instead of $c < 1$.

is bounded by $\mathcal{O}(L^{3/2}\log L)$, uniformly in V[20]. Therefore, we get, uniformly in V,

$$|\langle \sum_{t\in\Lambda_L} \sigma(t) \,|\, \{\Gamma\,;\, \gamma \neq \Gamma\, s\text{-small}\} \rangle^{+,\beta,h}_{\Lambda_L} - m|\Lambda_L|| \leqslant \mathcal{O}(L^{3/2}\log L)\,. \tag{7.123}$$

Since $\mathcal{O}(L^{3/2}\log L) \ll L^{2-c}$ and the hypotheses of that Proposition are clearly satisfied, we obtain

$$P^{+,\beta,h}_{\Lambda_L}[\mathcal{A}(m,c)^c \,|\, \{\Gamma\,;\, \gamma \neq \Gamma\, s\text{-small}\}] \leqslant \mathcal{O}(L^{-2(1-c-\delta)})\,. \tag{7.124}$$

Step 4. Estimation of $P^{+,\beta,h}_{\Lambda_L}[\{\Gamma\,;\, \gamma \neq \Gamma\, s\text{-small}\}]$.

There is no modification in this step.

Step 5. Estimation of $P^{+,\beta,h}_{\Lambda_L}[\mathcal{A}(m,c)]$ in terms of the functional $\mathfrak{W}$.

There is no modification in this step, except that we use Propositions 4.4.1 and 4.4.2, with $a = K'\log L$, K' large enough, instead of Proposition 4.6.2. Notice that in this step are the only places where one uses the exact solution (the Ornstein-Zernicke behaviour of the 2-point function). We use it twice: First, when we estimate the contribution of contours inside a box which lies along the boundary (since the corresponding Proposition 4.4.2 uses it) and, second, when we replace the 2-point functions by their lower bounds in terms of surface tension.

Since the total number of sides is $\mathcal{O}(L^{1/2}/\log L)$ and the total length of the fixed contours is at most $28L^{1/2}\log L$, and since replacing $|\gamma_k|$ by $\exp\{-\mathfrak{W}([t_k,t_{k+1}])\}$ for the fixed contours induces an error at most $\exp\{2\beta|\gamma_k|\}$, the theorem follows.

2) The modifications in the case of zero and negative boundary magnetic field are treated in the same way as in Section 7.4.2. The fact that the curve $\mathcal{C}$ is arbitrary (but convex) only involves minor changes which we explain now. We use the terminology of Section 7.4.2.

We first construct the polygonal approximation $\mathcal{P}_L$ as in the proof of Theorem 7.5.1. Let $I \doteqdot \mathcal{P}_L \cap \{x \in Q \,:\, x(2) = 0\}$. If $I = \varnothing$, then we subdivide the set $\{x \in Q \,:\, x(2) = 0\}$ into segments of length $\frac{1}{2}\delta_L$ and introduce translates of the box (4.92). The open contour Γ^* is constrained to pass though the extremities of these segments and to stay inside these boxes. We can repeat the proof of Theorem 7.5.1 since the construction of Theorem 7.5.1 does not interfere with the open contour in that case.

Suppose now that $I = [a,b]$. We define a new polygonal line $\mathcal{P}'_L$. $\mathcal{P}'_L$ goes from the bottom left corner of Q up to a along $\{x \in Q \,:\, x(2) = 0\}$, then it follows $\mathcal{P}_L \setminus I$ up to b, and finally goes along $\{x \in Q \,:\, x(2) = 0\}$ up to the bottom right corner of Q. The proof is now the same as that of Theorem 7.5.1 with the polygonal line $\mathcal{P}'_L$, up to minor, straightforward modifications.

Dividing and multiplying by $Z^+(\Lambda_L;\beta,|h|)$ and using identity (7.48) we can conclude. □

It is interesting to observe that the following version of the lower bound can be proved without any information from the exact solution.

[20] We don't want to use the fact that positive stiffness holds, this is why we have this quite rough bound.

Theorem 7.5.2. *Assume that the coupling constants are defined by* (3.30) *with* $\beta > \beta_c$ *and* $h \in \mathbb{R}$. *Let* $-m^*(\beta) < m < m^*(\beta)$ *and* $1/2 > c > 0$. *Then, for any* $\varepsilon > 0$, *there exists* $L_0(\varepsilon, \beta, h, m, c, Q)$ *such that, for any simple closed rectifiable curve* $\mathcal{C}$, *which is the boundary of a convex body of volume* $|Q|\frac{m^*(\beta)-m}{2m^*(\beta)}$ *in the rectangle* Q, *and for all* $L \geqslant L_0$,

$$P_{\Lambda_L}^{+,\beta,h}[\mathcal{A}(m,c)] \geqslant \exp\{-L\mathfrak{W}(\mathcal{C};\beta,h) - \varepsilon\mathcal{O}(L)\} ,$$

where $P_{\Lambda_L}^{+,\beta,h}[\,\cdot\,] \doteqdot P_{\Lambda_L}^{\pm,\beta,|h|}[\,\cdot\,]$ *if* $h < 0$.

Since the upper bound does not rely on the exact solution and ε can be taken arbitrarily small provided L is large enough, this gives exact large volume asymptotics for the large deviations of the magnetization without information from the exact solution.

Proof. The only place where the exact solution is used is step 5.. However, since the massgap $\alpha(\boldsymbol{n})$ converges to its limit uniformly in $\boldsymbol{n} \in S^1$, it is enough to observe that, for any $\varepsilon > 0$,

$$\langle\sigma(0)\sigma(t)\rangle \geqslant \exp\{-\tau(t) - \varepsilon\|t\|_2\}\,, \tag{7.125}$$

as soon as $\|t\|_2 = L^{1/2}\log L$ is large enough (uniformly in the direction); moreover if L is large enough this is also true for the boundary 2-point function (with the same ε). Therefore (7.125) can be used instead of the lower bounds from Proposition 4.5.2. In the same way, in the proof of Proposition 4.4.2 (in which lower bounds on the 2-point functions where used), it is easy to see that it is not necessary to use more than (7.125), in our current setting. Indeed, we can choose $a = \varepsilon L\delta_L$ for the boxes along the boundary. Because of these two modifications, the total error is $\varepsilon\mathcal{O}(L)$. □

7.5.2 The upper bounds

Let the coupling constants be given by (3.30) with $\beta > \beta_c$ and $h \in \mathbb{R}$. Let $-m^*(\beta) < m < m^*(\beta)$ and $1/2 > c > 0$. Theorem 7.5.1 states that there exists $L_0(\beta, h, m, c, Q)$ such that, for all $L \geqslant L_0$,

$$P_{\Lambda_L}^{+,\beta,h}[\mathcal{A}(m,c]) \geqslant \exp\{-\mathfrak{W}^*(m;\beta,h)\,L - \mathcal{O}(L^{1/2}\log L)\}\,, \tag{7.126}$$

where

$$\mathfrak{W}^*(m;\beta,h) \doteqdot \inf\{\mathfrak{W}(\mathcal{C};\beta,h)\,:\,\mathcal{C}\subset Q\,,\,\mathrm{vol}\,\mathcal{C} = |Q|\frac{m^*(\beta)-m}{2m^*(\beta)}\}\,. \tag{7.127}$$

The leading term in this lower bound is optimal. The proof being identical to that of the case of moderately large deviations, we do not write it. Notice that, for volume order large deviations, the case of complete wetting $h < -h_{\mathrm{w}}(\beta)$ does not yield worse estimates than the other cases. The reason for this is that the error term coming from the fact that the functional $\mathfrak{W}(\cdot;\beta,h)$ is not positive is still of order $\mathcal{O}(L^{1+\eta-\delta'})$ (see Lemma 7.4.4), but that, in the present case, the leading order is $\mathcal{O}(L)$ (before it was $\mathcal{O}(L^{1-2\nu})$, with $\nu > 0$!) and therefore always dominate. We have the following theorem [21],

[21] We use the same terminology for large and small contours as in the case of moderate deviations, that is we set $s \doteqdot L^{\delta'}$ with $0 < \delta' < 2\delta$, where $\delta = 1/4 - c$.

Theorem 7.5.3. *Assume that the coupling constants are defined by* (3.30) *with* $\beta > \beta_c$ *and* $h \in \mathbb{R}$. *Let* $-m^*(\beta) < m < m^*(\beta)$ *and* $c \doteqdot 1/4 - \delta$, $\delta > 0$. *Let* $\mathfrak{W}^*(m;\beta,h)$ *be defined by* (7.127). *Let* $\eta < \delta' < \delta$ *and*

$$\mathcal{A}(m;c) \doteqdot \{\omega \,:\, |\sum_{t\in\Lambda_L} \omega(t) - m|\Lambda_L|| \leqslant |\Lambda_L|\, L^{-c}\}\,;$$

$$E_1(m;c) \doteqdot \left\{\left|\alpha(\underline{S}) - \frac{m^*-m}{2m^*}\right| < \frac{1+\eta}{2m^* L^c}\right\}\,;$$

$$E_2(m;c) \doteqdot \left\{\sum_i \mathfrak{W}(S_i;\beta,h) \leqslant L\,\mathfrak{W}^*(m;\beta,h)\Big[1+\mathcal{O}(L^{\eta-\delta'})\Big]\right\}\,.$$

Then, for L large enough,

$$P^{+,\beta,h}_{\Lambda_L}[E_1(m;c)\cap E_2(m;c)\,|\,\mathcal{A}(m;c)\,] \geqslant 1 - \exp\Big\{-\mathcal{O}(L^{1+\eta-\delta'})\Big\}$$

and

$$|\frac{1}{L}\log P^{+,\beta,h}_{\Lambda_L}[\mathcal{A}(m;c)] + \mathfrak{W}^*(m;\beta,h)| \leqslant \mathcal{O}(L^{\eta-\delta'})\,.$$

If $h<0$, *we use the identity* $P^{+,\beta,h}_{\Lambda_L}[\cdot] = P^{\pm,\beta,|h|}_{\Lambda_L}[\cdot]$.

7.6 Continuum limit

Theorem 7.5.3 provides a description of the typical configurations, with respect to the measure conditioned by the event that a large deviation occurs, in terms of the polygonal approximations of their family of large contours. We would like to show how these informations can be used to prove that, in a suitable continuum limit, the conditioned measure concentrates on the set of configurations corresponding to the solution (or solutions) of the thermodynamical variational problem.

The proof proceeds in two main steps. First we show that, nearly everywhere in the box, the magnetization is very close to either $m^*(\beta)$ or $-m^*(\beta)$ (Subsection 7.6.1). Then we study the geometry of typical phase separation lines (Subsection 7.6.2). These two estimates can then be put together to prove the desired convergence result (Subsection 7.6.3).

In all this section, we suppose that the coupling constants are given by (3.30) with $\beta > \beta_c$ and $h \in \mathbb{R}$ and that $\Lambda_L = \Lambda_L(r_1, r_2)$ is defined in (7.24). Let $-m^*(\beta) < m < m^*(\beta)$ and $c = 1/4 - \delta > 0$ be given.

Definition.

(D125) *The* **canonical Gibbs state**, $\langle\,\cdot\,|m\rangle^{\beta,h}$, *is defined by*

$$\langle\,\cdot\,|m\rangle^{\beta,h}_L \doteqdot \begin{cases} \langle\,\cdot\,|\mathcal{A}(m;c)\rangle^{+,\beta,h}_{\Lambda_L}\,, & h\geqslant 0\,,\\ \langle\,\cdot\,|\mathcal{A}(m;c)\rangle^{\pm,\beta,|h|}_{\Lambda_L}\,, & h<0\,.\end{cases}$$

We write the corresponding probability measure Prob[·]. Similarly, we write

$$\langle\,\cdot\,\rangle_{\Lambda_L}^{+,\beta,h} \doteqdot \begin{cases} \langle\,\cdot\,\rangle_{\Lambda_L}^{+,\beta,h}, & h \geqslant 0\,, \\ \langle\,\cdot\,\rangle_{\Lambda_L}^{\pm,\beta,|h|}, & h < 0\,. \end{cases} \tag{7.128}$$

As in the preceding section, a contour is small if there exists a translate of the box $\mathcal{D}(0, L^{\delta'})$, $0 < \delta' < 2\delta$ in which it is contained. δ' will be chosen later; it will be small. The polygonal lines are constructed as before, using the same intermediate scale $L^{\delta'}$. We partition these polygonal lines into two classes according to their size. Let $0 < a < 1$ and $\mu > 0$ such that $a + \mu < 1$; a will be chosen later and will be close to 1.

Definition.

(D126) *A polygonal line is* **small** *if there exists a translate of the box* $\mathcal{D}(0, L^{a+\mu})$ *containing* $\overline{\mathrm{Int}}\mathcal{P}(S_i)$*; otherwise the polygonal line is* **large***.*

The analysis is made in the box Λ_L; at the end, we will scale everything by a factor $1/L$ and take the limit of lattice spacing going to zero. We partition Λ_L into cells. Since we will use several such partitions in this section, we introduce the following notation.

Definition.

(D127) *Let* $1 > m > 0$ *and let* $\{\mathcal{C}_i\,,\, i \in \mathbb{Z}\}$ *be a partition of* $\mathbb{Z}^2$ *into disjoint translates of the box* $\mathcal{D}(0, L^m)$*. The* **grid** $\mathcal{L}(m)$ *is the smallest subset of* $\{\mathcal{C}_i\,,\, i \in \mathbb{Z}\}$ *such that*

$$\Lambda_L \subset \bigcup_{\mathcal{C} \in \mathcal{L}(m)} \mathcal{C}\,.$$

The boxes $\mathcal{C}_i \in \mathcal{L}(m)$ *are called* **cells of the grid** $\mathcal{L}(m)$*.*

Let $\omega \in \Omega$ and let $\underline{\mathcal{S}}$ be the family of the polygonal approximations of its large contours. We consider the grid $\mathcal{L}(a)$, and partition its cells into four classes[22]. Let $\eta'' > 0$, η'' will be chosen later and will be small.

[22] The partition depends on ω, since it depends on $\underline{\mathcal{S}}$.

Definition.

(D128) *A cell* $\mathcal{C} \in \mathcal{L}(a)$ *is* **polluted** *if*

$$\Big|C \cap \big(\bigcup_{\mathcal{P}(S) \text{ small}} \overline{\mathrm{Int}}\mathcal{P}(S)\big)\Big| \geqslant L^{2a-\eta''}\,.$$

(D129) *A cell* $\mathcal{C} \in \mathcal{L}(a)$ *is an* **interface-cell** *if it is not polluted and has a non-empty intersection with* $B(S_i)$ *for some large polygonal line* $\mathcal{P}(S_i)$*, where, in this section,*

$$B(S_i) \doteqdot \bigcup_{t_{ij}\in S_i} \big(\mathcal{D}(t_{ij}, L^{\delta'}) \cap \Lambda_L\big)\,.$$

(D130) *A cell* $\mathcal{C} \in \mathcal{L}(a)$ *is a* **phase-cell** *if it is neither polluted, nor an interface-cell and it is entirely contained inside* Λ_L*.*

(D131) *A cell* $\mathcal{C} \in \mathcal{L}(a)$ *is a* **boundary-cell** *if it not polluted, not a interface-cell and not a phase-cell.*

The observation is that, typically, most of these cells are phase-cells, and that with high probability the magnetization inside phase-cells is close to $\pm m^*(\beta)$. We prove now the first of these statements, while the second one is proved in the next subsection.

In all this section, $E_1(m;c)$ and $E_2(m;c)$ are the events defined in Theorem 7.5.3.

Lemma 7.6.1. *Let* $\omega \in E_1(m;c) \cap E_2(m;c)$ *and suppose that* $\delta' < a$ *and* $a + \mu < 1 - \eta''$*. Then, uniformly in* ω*,*

$$\#\{\mathcal{C} \in \mathcal{L}(a) \,:\, \mathcal{C} \text{ is polluted}\} \leqslant \mathcal{O}(L^{1-a+\mu+\eta''})\,,$$
$$\#\{\mathcal{C} \in \mathcal{L}(a) \,:\, \mathcal{C} \text{ is an interface-cell}\} \leqslant \mathcal{O}(L^{1-a})\,,$$
$$\#\{\mathcal{C} \in \mathcal{L}(a) \,:\, \mathcal{C} \text{ is a boundary-cell}\} \leqslant \mathcal{O}(L^{1-a})\,.$$

Proof. We estimate the total volume of the region containing small polygonal lines. We partition the small polygonal lines into families. The first family contains all small polygonal lines $\mathcal{P}(S)$ with $\mathrm{Int}\,\mathcal{P}(S) = \varnothing$. We then partition the remaining polygonal lines into families so that for each family

$$[L^{a+\mu}]^2 \leqslant \Big|\bigcup_{\mathcal{P}(S)} \mathrm{Int}\,\mathcal{P}(S)\Big| \leqslant 10[L^{a+\mu}]^2 \tag{7.129}$$

(except possibly for the last family which may not satisfy the lower bound). The total length of the members of a family satisfying the latter inequalities is at least $K_3L^{a+\mu}$ (isoperimetric inequality). Since the total length of the polygonal lines is at most $K'L$, we have at most $\mathcal{O}(L^{1-a-\mu})$ families. Consequently, the total volume of these small polygonal lines is bounded by $\mathcal{O}(L^{1+a+\mu})$. The volume of $B(\underline{S})$ is bounded by $\mathcal{O}(L^{1+\delta'})$. Hence the total volume of the closure of the interior of these small polygonal lines is at most $\mathcal{O}(L^{1+a+\mu})$.

The number of polluted cells is therefore at most $\mathcal{O}(L^{1+a+\mu})/L^{2a-\eta''} = \mathcal{O}(L^{1-a+\mu+\eta''})$.

To count the number of interface-cells we estimate the number of points we need in order to make a coarse-grained description of large polygonal lines using a reference box $\mathcal{D}(0, L^a)$ according to the method of the previous sections. Since the total length of the polygonal lines is at most $K'L$ and the box $\mathcal{D}(0, L^a)$ is larger than the box $\mathcal{D}(0, L^{\delta'})$, the total number of interface-cells is at most $8K'L^{1-a}$.

The number of boundary cells is bounded by $\mathcal{O}(L^{1-a})$. □

The values of the parameters a, μ and η'' will be chosen later in such a way as to ensure that all these estimates are very small compared to the total number of cells, which is $\mathcal{O}(L^{2-2a})$.

7.6.1 The local magnetization

We want to show that the typical magnetization inside the phase-cells is either close to $m^*(\beta)$, or close to $-m^*(\beta)$. We introduce the notion of empirical magnetization, which measures the magnetization inside a cell.

Definition.

(D132) *The* **empirical magnetization inside the cell** $\mathcal{C}$ *is defined by*

$$m_{\mathcal{C}}(\omega) \doteqdot \frac{1}{|\mathcal{C}|}\sum_{t\in\mathcal{C}} \sigma(t)(\omega)\,.$$

Observe that, by definition, a phase-cell cannot be surrounded by a small polygonal line, otherwise it would be polluted. Let $\varepsilon(L)$ be a positive function such that

$$\lim_{L\to\infty} \varepsilon(L) = 0\,. \tag{7.130}$$

We define the event E_3: In any phase cell $\mathcal{C}$ the empirical magnetization satisfies

$$|m_{\mathcal{C}}(\omega) - m^*(\beta)| \leqslant \varepsilon(L)\,, \tag{7.131}$$

if the phase-cell is outside all external large contours, or inside an even number of large contours. Otherwise,

$$|m_{\mathcal{C}}(\omega) + m^*(\beta)| \leqslant \varepsilon(L)\,. \tag{7.132}$$

Theorem 7.6.1. *Let the coupling constants be given by* (3.30) *with* $\beta > \beta_c$ *and* $h \in \mathbb{R}$. *Let* $-m^*(\beta) < m < m^*(\beta)$ *and* $c = 1/4 - \delta > 0$. *Let* $\langle\,\cdot\,|m\rangle_L^{\beta,h}$ *be the canonical Gibbs state. Let* E_1 *and* E_2 *be the events defined in Theorem 7.5.3. Let* $\eta' > 0$ *be such that* $2a - \delta' - 3\eta' > 1$. *Suppose that*

$$\lim_{L\to\infty} \frac{\max(L^{-\eta'}, L^{-\eta''})}{\varepsilon(L)} = 0\,.$$

Then there exists a positive constant κ *(see* (7.136)*) such that, for* L *large enough,*

$$\mathrm{Prob}[E_3|E_1 \cap E_2] \geqslant 1 - \exp\{-\mathcal{O}(L^\kappa)\}\,,$$

and

$$\mathrm{Prob}[E_3 \cap E_1 \cap E_2] \geqslant 1 - \exp\{-\mathcal{O}(L^\kappa)\}\,.$$

Proof. Let $\mathcal{A} \equiv \mathcal{A}(m;c)$, E_3^c be the complementary event to E_3, and $E_{1,2} \doteqdot E_1 \cap E_2$. By definition

$$\begin{aligned}
\operatorname{Prob}[E_3^c \,|\, E_{1,2}] &= \frac{\langle E_3^c \cap E_{1,2} | m\rangle_L^{\beta,h}}{\langle E_{1,2} | m\rangle_L^{\beta,h}} \\
&= \frac{\langle E_3^c \cap E_{1,2} \cap \mathcal{A}\rangle_{\Lambda_L}^{+,\beta,h}}{\langle E_{1,2} \cap \mathcal{A}\rangle_{\Lambda_L}^{+,\beta,h}} \\
&= \langle \mathcal{A} \,|\, E_3^c \cap E_{1,2}\rangle_{\Lambda_L}^{+,\beta,h} \frac{\langle E_3^c \,|\, E_{1,2}\rangle_{\Lambda_L}^{+,\beta,h}}{\langle \mathcal{A} \,|\, E_{1,2}\rangle_{\Lambda_L}^{+,\beta,h}} \\
&\leqslant \frac{\langle E_3^c \,|\, E_{1,2}\rangle_{\Lambda_L}^{+,\beta,h}}{\langle \mathcal{A} \,|\, E_{1,2}\rangle_{\Lambda_L}^{+,\beta,h}} \,. \qquad (7.133)
\end{aligned}$$

The numerator and denominator are estimated in the two following lemmas. Lemma 7.6.2 yields

$$\langle E_3^c \,|\, E_{1,2}\rangle_{\Lambda_L}^{+,\beta,h} \leqslant \exp\{-\mathcal{O}(L^{2a-2\delta'})\varepsilon(L)^4\}\,, \qquad (7.134)$$

while Lemma 7.6.3 gives

$$\langle \mathcal{A} \,|\, E_{1,2}\rangle_{\Lambda_L}^{+,\beta,h} \geqslant \exp\{-\mathcal{O}(L^{2-c-\delta'})\}\,. \qquad (7.135)$$

Recall that $\varepsilon(L) \gg L^{-\eta'}$ and $c = 1/4 - \delta > 0$. The result follows if we can find a such that $1 > a > 0$, δ' such that $0 < \delta' < 2\delta$ and $0 < \eta'$ so that the hypothesis of the theorem is satisfied and

$$\kappa \doteqdot 2a - \delta' - 4\eta' - 2 + c > 0\,. \qquad (7.136)$$

(7.136) is equivalent to

$$a > 1 - \frac{c}{2} + \frac{\delta'}{2} + 2\eta'\,, \qquad (7.137)$$

which is true for suitable a, δ' and η'. The last affirmation

$$\operatorname{Prob}[E_3 \cap E_1 \cap E_2] \geqslant 1 - \exp\{-\mathcal{O}(L^{\kappa})\} \qquad (7.138)$$

is a consequence of the first statement and Theorem 7.5.3. □

Lemma 7.6.2. *Suppose the hypotheses of Theorem 7.6.1 are satisfied. Then*

1. *If the phase–cell C is outside all external large contours or inside an even number of large contours, then for L large enough*

$$\langle\{|m_{\mathcal{C}}(\omega) - m^*(\beta)| \geqslant \varepsilon(L)\} \,|\, E_{1,2})\rangle_{\Lambda_L}^{+,\beta,h} \leqslant \exp\{-\mathcal{O}(L^{2a-2\delta'})\varepsilon(L)^4\}\,.$$

2. *If the phase–cell C is inside an odd number of large contours, then for L large enough*

$$\langle\{|m_{\mathcal{C}}(\omega) + m^*(\beta)| \geqslant \varepsilon(L)\} \,|\, E_{1,2})\rangle_{\Lambda_L}^{+,\beta,h} \leqslant \exp\{-\mathcal{O}(L^{2a-2\delta'})\varepsilon(L)^4\}\,.$$

Proof. We prove 1. Let $\underline{\Gamma}$ be a family of large contours; $E(\underline{\Gamma})$ is the set of configurations with $\underline{\Gamma}$ as family of large contours. $\underline{\Gamma}$ has a coarse–grained description $\underline{S}$. $E(\underline{S})$ is the set of configurations such that the large contours have the coarse–grained description $\underline{S}$. It is sufficient to prove

$$\langle \{|m_{\mathcal{C}}(\omega) - m^*(\beta)| \geqslant \varepsilon(L)\} \,|\, E(\underline{\Gamma})\rangle_{\Lambda_L}^{+,\beta,h} \leqslant \exp\{-\mathcal{O}(L^{2a-2\delta'})\varepsilon(L)^4\}\,, \tag{7.139}$$

with $\mathcal{O}(L^{2a-2\delta'})$ uniform in $\underline{\Gamma}$ such that $E(\underline{\Gamma}) \subset E(\underline{S}) \subset E_{1,2}$. Since $\mathcal{C}$ is a phase cell, the total volume of large contours inside $\mathcal{C}$ is small. We first get rid of the corresponding regions. Let

$$\mathcal{C}^* \doteqdot \mathcal{C} \cap \Big(\bigcup_{\mathcal{P}(S) \text{ small}} \overline{\mathrm{Int}}\,\mathcal{P}(S) \Big)\,. \tag{7.140}$$

For L large enough (use $\varepsilon(L) \gg L^{-\eta''}$)

$$\langle \{|m_{\mathcal{C}}(\omega) - m^*(\beta)| \geqslant \varepsilon(L)\} \,|\, E(\underline{\Gamma})\,\rangle_{\Lambda_L}^{+,\beta,h} \leqslant \langle \{|m_{\mathcal{C}\setminus\mathcal{C}^*}(\omega) - m^*(\beta)| \geqslant 2\varepsilon(L)/3\} \,|\, E(\underline{\Gamma})\rangle_{\Lambda_L}^{+,\beta,h}\,. \tag{7.141}$$

We have $\mathcal{C} \setminus \mathcal{C}^* \subset \Lambda_{\#}(\underline{\Gamma})$ (see (D112), p. 114) and consequently

$$\langle \{|m_{\mathcal{C}\setminus\mathcal{C}^*}(\omega) - m^*(\beta)| \geqslant 2\varepsilon(L)/3\} \,|\, E(\underline{\Gamma})\rangle_{\Lambda_L}^{+,\beta,h} = \langle \{|m_{\mathcal{C}\setminus\mathcal{C}^*}(\omega) - m^*(\beta)| \geqslant 2\varepsilon(L)/3\}\rangle_{\Lambda_{\#}(\underline{\Gamma})}^{\star,s}\,, \tag{7.142}$$

$\langle\,\cdot\,\rangle_{\Lambda_{\#}(\underline{\Gamma})}^{\star,s}$ being the Gibbs measure in $\Lambda_{\#}(\underline{\Gamma})$ with boundary condition given by $\omega_{\underline{\Gamma}}$ (see section 5.2), conditioned on the fact that there are only small contours. Using Lemmas A.4.1 and 5.1.2 we get

$$|\langle m_{\mathcal{C}\setminus\mathcal{C}^*}(\omega)\rangle_{\Lambda_{\#}(\underline{\Gamma})}^{\star,s} - m^*(\beta)| \leqslant \exp\{-\mathcal{O}(L^{\delta'})\}\,. \tag{7.143}$$

We apply a variant of Proposition 5.2.1, point 2.. It is proved in the same way, except that we only make a partition of $\mathcal{C}$ instead of Λ_L. Fixing the small contours intersecting the boundary of $\mathcal{C}$, we decouple the cell from the rest of Λ_L. The proof then proceeds in the same way. It gives, for L large enough,

$$\langle \{|m_{\mathcal{C}\setminus\mathcal{C}^*}(\omega) - m^*(\beta)| \geqslant \tfrac{2}{3}\varepsilon(L)\}\rangle_{\Lambda_{\#}(\underline{\Gamma})}^{\star,s} \leqslant \langle \{|m_{\mathcal{C}\setminus\mathcal{C}^*}(\omega) - \langle m_{\mathcal{C}\setminus\mathcal{C}^*}(\omega)\rangle_{\Lambda_{\#}(\underline{\Gamma})}^{\star,s}| \geqslant \tfrac{1}{2}\varepsilon(L)\}\rangle_{\Lambda_{\#}(\underline{\Gamma})}^{\star,s} \leqslant \exp\{-\mathcal{O}(L^{2a-2\delta'})\varepsilon(L)^4\}\,. \tag{7.144}$$

□

Lemma 7.6.3. *For L large enough*

$$\langle \mathcal{A}(m;c) \,|\, E_{1,2}\rangle_{\Lambda_L}^{+,\beta,h} \geqslant \exp\{-\mathcal{O}(L^{2-c-\delta'})\}\,.$$

Proof. Let $\underline{\Gamma}$ be given, $E(\underline{\Gamma}) \subset E_{1,2}$. It is sufficient to prove that

$$\langle \mathcal{A}(m;c) \,|\, E(\underline{\Gamma})\,\rangle_{\Lambda_L}^{+,\beta,h} \geqslant \exp\{-\mathcal{O}(L^{2-c-\delta'})\}\,, \tag{7.145}$$

uniformly in $\underline{\Gamma} \subset E(\underline{S}) \subset E_{1,2}$. All contours $\gamma \notin \underline{\Gamma}$ in $\omega \in E(\underline{\Gamma})$ are s–small, $s = L^{\delta'}$. Since $E(\underline{\Gamma}) \subset E_{1,2}$ the phase volume $\alpha(\underline{S})$ satisfies

$$\Big| \alpha(\underline{S}) - \frac{m^*(\beta) - m}{2m^*} \Big| < \frac{1+\eta}{2m^* L^c}\,, \tag{7.146}$$

with η some fixed positive number smaller than δ'. We have $|\Lambda_L \setminus \Lambda_\#(\underline{\Gamma})| \leqslant 2K' L^{1+\delta'}$ (see (D112), p. 114); hence, with the notations of Proposition 5.2.1,

$$\Big| \langle \sum_{t \in \Lambda_L} \sigma(t) \,|\, E(\underline{\Gamma}) \rangle^{+,\beta,h}_{\Lambda_L} - \langle \sum_{t \in \Lambda_\#(\underline{\Gamma})} \sigma(t) \rangle^{\star,s}_{\Lambda_\#(\underline{\Gamma})} \Big| \leqslant \mathcal{O}(L^{1+\delta'})\,. \tag{7.147}$$

On the other hand,

$$\langle \sum_{t \in \Lambda_L} \sigma(t) \,|\, E(\underline{\Gamma}) \rangle^{+,\beta,h}_{\Lambda_L} = m^*(\beta) |\Lambda_L| (1 - 2\alpha(\underline{S})) \pm \mathcal{O}(L^{1+\delta'})\,. \tag{7.148}$$

Therefore

$$\Big| \langle \sum_{t \in \Lambda_\#(\underline{\Gamma})} \sigma(t) \rangle^{\star,s}_{\Lambda_\#(\underline{\Gamma})} - m|\Lambda_L| \Big| \leqslant \frac{1+2\eta}{L^c} |\Lambda_L|\,, \tag{7.149}$$

for L large enough. (7.149) show that the magnetization may still be quite far from m. There are two regimes: Either the expectation value in (7.149) is very close to m or it is not. In the first case, a large fluctuation will be required in order *not* to realize the event $\mathcal{A}(m;c)$; in the second case, we have to construct a set of configuration realizing $\mathcal{A}(m;c)$. Let us consider the first situation. Suppose that

$$\Big| \langle \sum_{t \in \Lambda_\#(\underline{\Gamma})} \sigma(t) \,\rangle^{\star,s}_{\Lambda_\#(\underline{\Gamma})} - m|\Lambda_L| \Big| \leqslant \frac{1-\eta}{L^c} |\Lambda_L|\,, \tag{7.150}$$

then, using Proposition 5.2.1,

$$\begin{aligned}\langle \mathcal{A}(m;c) \,|\, E(\underline{\Gamma}) \rangle^{+,\beta,h}_{\Lambda_L} &\geqslant 1 - P^{\star,s}_{\Lambda_\#(\underline{\Gamma})} [\{ \big| \sum_{t \in \Lambda_\#(\underline{\Gamma})} \sigma(t)(\omega) - \langle \sum_{t \in \Lambda_\#(\underline{\Gamma})} \sigma(t) \rangle^{\star,s}_{\Lambda_\#(\underline{\Gamma})} \big| > \frac{\eta}{2L^c} |\Lambda_L| \}] \\ &> \frac{1}{2}\,,\end{aligned} \tag{7.151}$$

if L is large enough. Suppose now that we are in the second situation,

$$\Big| \langle \sum_{t \in \Lambda_\#(\underline{\Gamma})} \sigma(t) \rangle^{\star,s}_{\Lambda_\#(\underline{\Gamma})} - m|\Lambda_L| \Big| > \frac{1-\eta}{L^c} |\Lambda_L|\,. \tag{7.152}$$

To be specific we consider the case $(0 < \varepsilon \leqslant 3\eta)$

$$\langle \sum_{t \in \Lambda_\#(\underline{\Gamma})} \sigma(t) \rangle^{\star,s}_{\Lambda_\#(\underline{\Gamma})} = m|\Lambda_L| + \frac{1-\eta+\varepsilon}{L^c} |\Lambda_L|\,. \tag{7.153}$$

In this case, the mean magnetization is too large in $\Lambda_\#(\underline{\Gamma})$. The idea is the following. Without the constraint on the size of the contour, the system would create a large droplet of minus phase to lower the magnetization (notice that we are still in the regime of (moderately) large deviations, since $L^{2-c} \gg L^{4/3}$). However, since the constraint prevents this behaviour, it is natural to expect that the system will instead create a lot of droplets of maximal allowed size. Our aim is to describe a set of configurations in which this happens, to choose these configurations so that the event $\mathcal{A}(m;c)$ is satisfied, and finally to compute the probability of such configurations.

Let Λ^+ be the component of $\Lambda_\#(\underline{\Gamma})$ where the $\star$ boundary condition corresponds to $+$ boundary condition. We construct a region $\Delta \subset \Lambda^+$ of suitable volume and we impose zero magnetization inside Δ in order to reduce the total magnetization (that's the region in which we are going to put all these droplets). First let us compute the volume of Δ. It is specified by the condition

$$\langle \sum_{t\in\Lambda_\#(\underline{\Gamma})\setminus\Delta} \sigma(t)\rangle^{\star,s}_{\Lambda_\#(\underline{\Gamma})} = m|\Lambda_L|\,, \tag{7.154}$$

that is,

$$\begin{aligned}\langle \sum_{t\in\Lambda_\#(\underline{\Gamma})} \sigma(t)\rangle^{\star,s}_{\Lambda_\#(\underline{\Gamma})} - \langle\sum_{t\in\Delta}\sigma(t)\rangle^{\star,s}_{\Lambda_\#(\underline{\Gamma})}& \\ = m|\Lambda_L| + \frac{1-\eta+\varepsilon}{L^c}|\Lambda| - |\Delta| m^*(\beta)& \\ = m|\Lambda_L|\,,&\end{aligned} \tag{7.155}$$

which implies that

$$|\Delta| = \frac{1-\eta+\varepsilon}{m^*(\beta)L^c}|\Lambda|\,. \tag{7.156}$$

We now show that we can construct Δ as a union of cubes which are translate of $\mathcal{D}(0, L^{\delta'})$. This is useful in two respects: First, it is easier to put droplets inside boxes having a nice shape; second, in these boxes all contours are small and therefore, if we are able to decouple all of them, we will have to consider the probability of a large deviation in an unconstrained phase inside each of these boxes. We introduce the grid $\mathcal{L}(\delta')$. The number of cells of $\mathcal{L}(\delta')$ which intersects some $B(S_i)$ is bounded by $\mathcal{O}(L^{1-\delta'})$. The total number of cells of $\mathcal{L}(\delta')$ is $\mathcal{O}(L^{2-2\delta'})$ so that it is always possible to find $\mathcal{O}(L^{2-c-2\delta'})$ cells not intersecting any $B(S_i)$, provided L is large enough. Let $0<\delta''<\delta'$. Inside each selected cells $\mathcal{B}_j$ we place in the center a translate $\mathcal{B}'_j$ of the box $\mathcal{D}(0, L^{\delta'} - 2L^{\delta''})$. We define the event $\widetilde{\mathcal{A}}$:

1. all contours which have a non–empty intersection with $\Lambda_\#(\underline{\Gamma}) \setminus \Delta$ or with at least two $\mathcal{B}_j$ are $L^{\delta''}$-small;
2. $$\Big|\sum_{t\in\Lambda_\#(\underline{\Gamma})\setminus\Delta} \sigma(t) - m|\Lambda_L|\Big| \leqslant |\Lambda_L|/2L^c\,; \tag{7.157}$$
3. for each box $\mathcal{B}'_j$ we have $$\Big|\sum_{t\in\mathcal{B}'_j}\sigma(t)\Big| \leqslant |\mathcal{B}'_j|/L^{c\delta'}\,. \tag{7.158}$$

By definition $\widetilde{\mathcal{A}} \subset \mathcal{A}(m;c)$. Therefore

$$\langle \mathcal{A}(m;c) \,|\, E(\underline{\Gamma}) \rangle_{\Lambda_L}^{+,\beta,h} \geqslant \langle \widetilde{\mathcal{A}} \rangle_{\Lambda_{\#}(\underline{\Gamma})}^{\star,s} . \tag{7.159}$$

Let $\widetilde{\mathcal{A}}_{1,2}$ be the event defined by conditions 1. and 2. only. Then

$$\langle \widetilde{\mathcal{A}} \rangle_{\Lambda_{\#}(\underline{\Gamma})}^{\star,s} = \langle \widetilde{\mathcal{A}} \,|\, \widetilde{\mathcal{A}}_{1,2} \rangle_{\Lambda_{\#}(\underline{\Gamma})}^{\star,s} \, \langle \widetilde{\mathcal{A}}_{1,2} \rangle_{\Lambda_{\#}(\underline{\Gamma})}^{\star,s} . \tag{7.160}$$

We want to estimate the term $\langle \widetilde{\mathcal{A}} \,|\, \widetilde{\mathcal{A}}_{1,2} \rangle_{\Lambda_{\#}(\underline{\Gamma})}^{\star,s}$ using Theorem 7.5.1. To do this, we first have to decouple the different cells $\mathcal{B}'_j$. Denote by $\underline{\gamma}(\omega)$ all external contours in ω which have a non–empty intersection with $\Lambda_{\#}(\underline{\Gamma}) \setminus \Delta$ or with at least two $\mathcal{B}_j$, and by $\widetilde{\mathcal{A}}_{1,2}(\underline{\gamma}')$ the set of $\omega \in \widetilde{\mathcal{A}}_{1,2}$ such that $\underline{\gamma}(\omega) = \underline{\gamma}'$. Then

$$\langle \widetilde{\mathcal{A}} \,|\, \widetilde{\mathcal{A}}_{1,2} \rangle_{\Lambda(\underline{\Gamma})}^{\star,s} = \sum_{\underline{\gamma}'} \langle \widetilde{\mathcal{A}} \,|\, \widetilde{\mathcal{A}}_{1,2}(\underline{\gamma}') \rangle_{\Lambda_{\#}(\underline{\Gamma})}^{\star,s} \, \frac{\langle \widetilde{\mathcal{A}}_{1,2}(\underline{\gamma}') \rangle_{\Lambda_{\#}(\underline{\Gamma})}^{\star,s}}{\langle \widetilde{\mathcal{A}}_{1,2} \rangle_{\Lambda_{\#}(\underline{\Gamma})}^{\star,s}} . \tag{7.161}$$

Under the condition $\widetilde{\mathcal{A}}_{1,2}(\underline{\gamma}')$, local events which are $\mathcal{F}_{\mathcal{B}'_j}$-measurable for different j become independent. Since the boxes $\mathcal{B}'_j$ are isometric to $\mathcal{D}(0, L^{\delta'} - 2L^{\delta''})$ there is no condition on the contours inside these boxes. In each box we have a large deviation as in Theorem 7.5.1 with $m = 0$ and $\widetilde{L} = L^{\delta'} - 2L^{\delta''}$ instead of L. Therefore, applying these theorems with $\mathcal{C}$ a Wulff shape in the center of each $\mathcal{B}'_j$,

$$\begin{aligned} \langle \widetilde{\mathcal{A}} \,|\, \widetilde{\mathcal{A}}_{1,2} \rangle_{\Lambda_{\#}(\underline{\Gamma})}^{\star,s} &\geqslant \exp\{-\mathcal{O}(L^{\delta'})\, \mathcal{O}(L^{2-c-2\delta'})\} \\ &\geqslant \exp\{-O(L^{2-c-\delta'})\} . \end{aligned} \tag{7.162}$$

Proposition 5.2.1 and Lemma 4.3.2 imply that $\lim_{L\to\infty} \langle \widetilde{\mathcal{A}}_{1,2} \rangle_{\Lambda_{\#}(\underline{\Gamma})}^{\star,s} = 1$. Indeed, let $\chi(\delta')$ be the event that all contours are $L^{\delta'}$-small and $\chi(\delta'')$ the event that all contours are $L^{\delta''}$-small. Then

$$\begin{aligned} \langle \widetilde{\mathcal{A}}_{1,2} \rangle_{\Lambda_{\#}(\underline{\Gamma})}^{\star,s} &\geqslant \langle \widetilde{\mathcal{A}}_{1,2}\, \chi(\delta'') \rangle_{\Lambda_{\#}(\underline{\Gamma})}^{\star,s} \\ &= \langle \widetilde{\mathcal{A}}_{1,2} \,|\, \chi(\delta'') \rangle_{\Lambda_{\#}(\underline{\Gamma})}^{\star} \, \frac{\langle \chi(\delta'') \rangle_{\Lambda_{\#}(\underline{\Gamma})}^{\star}}{\langle \chi(\delta') \rangle_{\Lambda_{\#}(\underline{\Gamma})}^{\star}} . \end{aligned} \tag{7.163}$$

Lemma 4.3.2 implies that the numerator and denominator of the quotient tend to 1 as $L \to \infty$; Proposition 5.2.1 implies that $\langle \widetilde{\mathcal{A}}_{1,2} \,|\, \chi(\delta'') \rangle_{\Lambda_{\#}(\underline{\Gamma})}^{\star}$ tends to 1 as $L \to \infty$. □

7.6.2 Typical geometry of the polygonal lines

We consider the model in the box Λ_L and scale everything by a factor $1/L$, so that after the operation, the box is the rectangle Q. The rescaled version of the grid $\mathcal{L}(a)$ defines a partition of the box Q. A basic quantity in our approach to the continuum limit is the magnetization profile.

Definition.

(D133) *The* **magnetization profile in** Q **in the configuration** ω *is the function* $\rho_L(\cdot;\omega) : Q \to [-1,1]$ *defined by*

$$\rho_L(x;\omega) \doteqdot m_{\mathcal{C}(x)}(\omega)\,, \tag{7.164}$$

where $\mathcal{C}(x)$ *is the cell of* $\mathcal{L}(a)$ *such that* $Lx \in \mathcal{C}(x)$*, and* $m_{\mathcal{C}}(\omega)$ *is the empirical magnetization in* $\mathcal{C}$*.*

The results of Subsection 7.6.1 show that the magnetization profile is typically almost piecewise-constant, taking values close to $\pm m^*(\beta)$ in the vast majority of cells. To describe the system macroscopically we have to characterize the regions where there is a gradient of magnetization, that is the interfaces. The aim of this subsection is to show that there is a unique huge polygonal line whose shape is close to that predicted by the variational problem. The other polygonal lines have a vanishingly small volume[23]. We have to introduce the set of solutions to the variational problem.

Definition.

(D134) *The* **set of macroscopic droplets** *is defined by*[24]

$$\mathcal{D}(m) \doteqdot \{\mathcal{V} \subset Q \,:\, |\mathcal{V}| = \frac{m^*(\beta)-m}{2m^*(\beta)}|Q|\,, \mathfrak{W}(\partial\mathcal{V}) = \mathfrak{W}^*(m;\beta,h)\}\,.$$

Let $\omega \in E_{1,2,3} \doteqdot E_1 \cap E_2 \cap E_3$ and let $\mathcal{P}(\underline{S}) = \{\mathcal{P}(S_i)(\omega) \,:\, i=1,\ldots,k\}$ be the polygonal lines defined by the configuration ω. Using these polygonal lines scaled by $1/L$ we define a set $V(\underline{S}) \subset Q$ with the following properties (see Theorem 7.5.3)

1. The set $V(\underline{S}) \supset \operatorname{Int}\underline{S}$ and its volume is such that

$$\Big|\,|V(\underline{S})| - \frac{m^*(\beta)-m}{2m^*(\beta)}|Q|\,\Big| \leqslant \frac{1+\eta}{2m^*(\beta)L^c}|Q|\,; \tag{7.165}$$

2. The boundary $\partial V(\underline{S})$ of $V(\underline{S})$ is such that $\partial V(\underline{S}) \subset \bigcup_i \mathcal{P}(S_i)$ and

$$\mathfrak{W}(\partial V(\underline{S})) \leqslant \mathfrak{W}^*(m;\beta,h) + \mathcal{O}(L^{\eta-\delta})\,. \tag{7.166}$$

In the generic case the boundary of the set $V(\underline{S})$ has several connected components. We define an auxiliary connected set $\widehat{V}(\underline{S})$ by translating some of these components so that $\widehat{V}(\underline{S})$ has the same volume as $V(\underline{S})$, its boundary is connected and therefore can be parameterized by a single Lipschitz map $t \mapsto (u(t),v(t))$, and $\mathfrak{W}(\partial V(\underline{S})) = \mathfrak{W}(\partial\widehat{V}(\underline{S}))$[25]. We compare the set $\widehat{V}(\underline{S})$ with the droplets in $\mathcal{D}(m)$. To measure the distance we use the usual

[23] Contrary to what is done in [DKS1, ScSh2], we do not use stability of the variational problem to obtain these results. In fact we do not even need unicity of the solution.

[24] $\mathfrak{W}^*(m;\beta,h)$ is the quantity defined in (7.127).

[25] Since we want this last equality to hold, we don't want that two affine parts of two different components exactly coincide. Therefore, in such a case, we make an arbitrarily small deformation to avoid this problem.

Definition.

(D135) *The* **distance between two subsets of** $\mathbb{R}^2$*, F and G, is defined by*

$$\mathrm{d}(F,G) \doteqdot \max\{\sup_{s\in F}\inf_{t\in G}\|s-t\|_2\,,\, \sup_{t\in G}\inf_{s\in F}\|s-t\|_2\}\,. \tag{7.167}$$

The following lemma, inspired by Corollary 3.2 in [DP], shows that one component of $\widehat{V}(\underline{S})$ is close to a droplet of $\mathcal{D}(m)$ and that the total volume of the other components is small.

Lemma 7.6.4. *Let $\varepsilon > 0$. There exists a function $\delta(\varepsilon)$ with $\lim_{\varepsilon\to 0}\delta(\varepsilon) = 0$ such that if $V \subset Q$ has the properties*

1. *the boundary of V is parameterized by a unit-speed Lipschitz parameterization $t \mapsto (u(t), v(t))$,*
2. *the volume of V is larger than $|Q|(m^*(\beta) - m)/2m^*(\beta) - \varepsilon$,*
3. $\mathfrak{W}(\partial V) \leqslant \mathfrak{W}^*(m;\beta,h) + \varepsilon$,

then

$$\inf_{\mathcal{V}\in\mathcal{D}(m)} \mathrm{d}(\mathcal{V}, V) \leqslant \delta(\varepsilon)\,.$$

Proof. Suppose that there exists $\delta' > 0$, V_n, $n \in \mathbb{N}$, and $\varepsilon_n \downarrow 0$ such that

$$\inf_{\mathcal{V}\in\mathcal{D}(m)} \mathrm{d}(\mathcal{V}, V_n) \geqslant \delta' \;\; \forall n\,. \tag{7.168}$$

Let $t \mapsto (u_n(t), v_n(t))$ be the unit-speed Lipschitz parameterization of the boundary of V_n. We choose the parameterization in such way that

$$|V_n| = \frac{1}{2}\int_{\partial V_n} (v_n'(t)\,u_n(t) - u_n'(t)\,v_n(t))\,\mathrm{d}t\,. \tag{7.169}$$

By our hypothesis the length of the boundary ∂V_n is uniformly bounded, so that we can parameterize all boundaries ∂V_n by maps defined on a single interval $I \subset \mathbb{R}$ (we still denote the parameterizations by $(u_n(t), v_n(t))$). Since the parameterizations are Lipschitz with a Lipschitz constant bounded by one, the maps $t \mapsto (u_n(t), v_n(t))$ are equicontinuous. By Ascoli's Theorem we can extract a uniformly convergent subsequence so that $(u^*(t), v^*(t)) = \lim_k (u_{n_k}(t), v_{n_k}(t))$ is the boundary of a set V^* with volume

$$|V^*| = \lim_{k\to\infty} |V_{n_k}| \geqslant |Q|(m^*(\beta) - m)/2m^*(\beta)\,. \tag{7.170}$$

By the uniform convergence of the sequence we have

$$\liminf_{k\to\infty}\Big[\tau_{\mathrm{bd}}(\beta,h) - \tau((1,0);\beta)\Big]|\partial V_{n_k} \cap w_Q| \geqslant \Big[\tau_{\mathrm{bd}}(\beta,h) - \tau((1,0);\beta)\Big]|\partial V^* \cap w_Q|\,, \tag{7.171}$$

since $\Big[\tau_{\mathrm{bd}}(\beta,h) - \tau((1,0);\beta)\Big] \leqslant 0$. A classical theorem (see e.g. [D] chapter 3) gives

$$\liminf_{k\to\infty}\int_I \tau(\dot{u}_{n_k}(t), \dot{v}_{n_k}(t);\beta)\,\mathrm{d}t \geqslant \int_I \tau(\dot{u}^*(t), \dot{v}^*(t);\beta)\,\mathrm{d}t\,, \tag{7.172}$$

since $\tau(\cdot;\beta)$ is convex. Therefore

$$\mathfrak{W}(\partial V^*) \leqslant \lim_{k\to\infty}\mathfrak{W}(\partial V_{n_k}) \leqslant \mathfrak{W}^*(m;\beta,h)\,, \tag{7.173}$$

thus $V^* \in \mathcal{D}(m)$, which contradicts the existence of δ'. □

Corollary 7.6.1. *Under the hypotheses of Lemma 7.6.4, if ε is small enough, then one connected component of V is at distance at most $\delta(\varepsilon)$ from a droplet of $\mathcal{D}(m)$ and the total volume of the remaining components is at most $\mathcal{O}(\delta(\varepsilon))$.*

Remark. The unicity result we have obtained is rather weak. Indeed, it is expected (and proved at low temperature and $h = 1$[26]) that 1) there is a unique large contour and 2) all other contours are $K \log L$-small for some K. However we think that the present approach has the advantage that it can be used in wide variety of situations for which precise estimates cannot be achieved.

7.6.3 The continuum limit

This is the last step of the proof of the continuum limit. To each macroscopic droplet $\mathcal{V} \in \mathcal{D}(m)$, we can associate a magnetization profile,

$$\rho_{\mathcal{V}}(x) \doteqdot \begin{cases} m^*(\beta)\,, & \text{if } x \in Q \setminus \mathcal{V}\,, \\ -m^*(\beta)\,, & x \in \mathcal{V}\,. \end{cases} \tag{7.174}$$

Before proving convergence, one has to choose a topology. It seems natural to us to use the L^1-norm for functions on Q, that is to measure how far a magnetization profile is from the one given by the solution of the variational problem with the following distance: Let $f : Q \to \mathbb{R}$, we set

$$\mathrm{d}_1(f, \mathcal{D}(m)) \doteqdot \inf_{\mathcal{V} \in \mathcal{D}(m)} \int_Q \mathrm{d}x |f(x) - \rho_{\mathcal{V}}(x)|\,. \tag{7.175}$$

We can state now the main result of this section.

Theorem 7.6.2. *Let $\beta > \beta_c$, $h \in \mathbb{R}$, $-m^*(\beta) < m < m^*(\beta)$ and $c = 1/4 - \delta > 0$. Let $\langle \cdot \,|\, m \rangle^{\beta,h}_{\Lambda_L}$ be the canonical Gibbs state. Then there exists a positive function $\overline{\varepsilon}(L)$, $\lim_{L\to\infty} \overline{\varepsilon}(L) = 0$, and $\kappa > 0$ (see (7.136) such that for L large enough*

$$\mathrm{Prob}[\{\, \mathrm{d}_1(\rho_L(\,\cdot\,;\omega), \mathcal{D}(m)) \leqslant \overline{\varepsilon}(L)\}] \geqslant 1 - \exp\{-\mathcal{O}(L^\kappa)\}\,.$$

Proof. Let $\omega \in E_{1,2,3}$ and let $\mathcal{P}(\underline{S}) = \{\mathcal{P}(S_i)(\omega) \,:\, i = 1, \ldots, k\}$ be the polygonal lines defined by the configuration ω. We define $V(\underline{S}) \subset Q$ with properties (7.165) and (7.166) as above and set

$$\rho_L(x; \underline{S}) \doteqdot \begin{cases} m^*(\beta)\,, & \text{if } x \in Q \setminus V(\underline{S})\,, \\ -m^*(\beta)\,, & \text{if } x \in V(\underline{S})\,. \end{cases} \tag{7.176}$$

There exist two positive numbers μ and η'' (see Lemma 7.6.1),

$$\mu + \eta'' < 1 - a\,, \tag{7.177}$$

such that, if $\omega \in E_{1,2,3}$ and $\mathcal{P}(\underline{S})(\omega) = \mathcal{P}(\underline{S})$, then uniformly in $\omega \in E_{1,2,3}$

$$\int_Q \mathrm{d}x\, |\rho_L(x;\omega) - \rho_L(x;\underline{S})| \leqslant \mathcal{O}(L^{a-1+\mu+\eta''}) + \varepsilon(L)|Q| + \mathcal{O}(L^{a-1})\,. \tag{7.178}$$

[26] Among the results announced by Ioffe and Schonmann, for $h = 1$ and a special shape for the box Λ, this is proved up to T_c. The essential tool required to obtain these precise estimates is a local limit theorem which is the core of their results.

The first term on the right hand side is the contribution coming from the polluted cells, the second term from the phase-cells and the last one from the interface-cells and boundary-cells. We define

$$d_1(L) \doteqdot \sup_{\omega \in E_{1,2,3}} \mathrm{d}_1(\rho_L(\,\cdot\,; \underline{S}(\omega)), \mathcal{D}(m)) \,. \tag{7.179}$$

Then Lemma 7.6.4 and Corollary 7.6.1 imply that $\lim_{L\to\infty} d_1(L) = 0$. The theorem follows by choosing

$$\overline{\varepsilon}(L) \doteqdot \mathcal{O}(L^{a-1+\mu+\eta''}) + \varepsilon(L)|Q| + \mathcal{O}(L^{a-1}) + d_1(L) \,. \tag{7.180}$$

□

Part II

Ashkin–Teller model

Chapter 8

Definitions, notations

We introduce the Ashkin–Teller model on the square lattice and introduce some convenient notations. Since most of the notions introduced have counterparts in the case of the Ising model, we do not make many comments, see Part I if necessary.

8.1 The model

The Ashkin–Teller model is defined on the two-dimensional square lattice $\mathbb{Z}^2$. We use the same geometrical terminology as in part I. The first difference with the Ising model is the spin space.

Definition.

(D136) *The (single)* **spin space** *of the Ashkin–Teller model is* $\mathcal{S}_{\text{AT}} \doteqdot \{-1,1\} \times \{-1,1\}$.

(D137) *The* **configuration space** *of the Ashkin–Teller model is*

$$\Omega \doteqdot (\{-1,1\} \times \{-1,1\})^{\mathbb{Z}^2} \equiv \Omega_\sigma \times \Omega_\tau \,, \tag{8.1}$$

where $\Omega_\sigma \doteqdot \{-1,1\}^{\mathbb{Z}^2}$ *and* $\Omega_\tau \doteqdot \{-1,1\}^{\mathbb{Z}^2}$ *are the* σ*- and* τ*-configuration spaces. The elements* $\omega \in \Omega$ *are the* **configurations***; we can write* $\omega = (\omega_\sigma, \omega_\tau)$*, where* $\omega_\sigma \in \Omega_\sigma$ *and* $\omega_\tau \in \Omega_\tau$ *are the* σ**- and** τ**-configurations**.

(D138) *Let t be some site. The* σ**-spin** *at t is the random variable* $\sigma(t) : \Omega \to \{-1,1\}$, $\sigma(t)(\omega) = \omega_\sigma(t)$. *The* τ**-spin** *at t is the random variable* $\tau(t) : \Omega \to \{-1,1\}$, $\tau(t)(\omega) = \omega_\tau(t)$.

Therefore, we can look at the Ashkin–Teller model as describing the coexistence of two species of spins, σ and τ, living on the same lattice. It is also sometimes convenient to consider that these two species live on two different lattices. In such a case, for each lattice we use the usual terminology, but supplemented by a prefix σ or τ; for example, we can speak of σ-site, τ-edge, and so on...; we also write $\mathcal{E}_\sigma$, $\mathcal{E}_\tau$ for the set of σ- and τ-edges.

Let us construct now the Gibbs measures for this model. We start with the σ-algebras.

Definition.

(D139) *Let* $\Lambda \subset \mathbb{Z}^2$. $\mathcal{F}^\sigma_\Lambda$ *is the σ-algebra generated by the random variables* $\sigma(t)$, $t \in \Lambda$; $\mathcal{F}^\tau_\Lambda$ *is the σ-algebra generated by the random variables* $\tau(t)$, $t \in \Lambda$; $\mathcal{F}_\Lambda$ *is the σ-algebra generated by* $\mathcal{F}^\sigma_\Lambda$ *and* $\mathcal{F}^\tau_\Lambda$. *We set* $\mathcal{F} \doteqdot \mathcal{F}_{\mathbb{Z}^2}$, $\mathcal{F}^\sigma \doteqdot \mathcal{F}^\sigma_{\mathbb{Z}^2}$ *and* $\mathcal{F}^\tau \doteqdot \mathcal{F}^\tau_{\mathbb{Z}^2}$.

(D140) *A function is* **Λ-local** *if it is $\mathcal{F}_\Lambda$-measurable with $\Lambda \subset \mathbb{Z}^2$ finite.*

(D141) *A function is* **Λ^σ-local** *if it is $\mathcal{F}^\sigma_\Lambda$-measurable with $\Lambda \subset \mathbb{Z}^2$ finite.*

(D142) *A function is* **Λ^τ-local** *if it is $\mathcal{F}^\tau_\Lambda$-measurable with $\Lambda \subset \mathbb{Z}^2$ finite.*

We use the same notations as in the first part, since there should be no confusion. However, when we'll have to consider the Ising model in this part, we'll use non-ambiguous notations.

Definition.

(D143) *Let $\Lambda \subset \mathbb{Z}^2$ be a* finite *subset of $\mathbb{Z}^2$ and $\overline{\omega} \in \Omega$ be some configuration. A configuration ω is said to* **satisfy the $\overline{\omega}$-boundary condition in Λ** *, or shortly the* **$\Lambda^{\overline{\omega}}$-b.c.**, *if $\omega(t) = \overline{\omega}(t)$, for all $t \notin \Lambda$.*

(D144) *To each edge $e \in \mathcal{E}$, we associate a triple of real numbers $\underline{J}(e) \doteqdot (J_\sigma(e), J_\tau(e), J_{\sigma\tau}(e))$, the* **coupling constants at** *e.*

(D145) *Let $\Lambda \subset \mathbb{Z}^2$ be a* finite *subset of $\mathbb{Z}^2$. The* **energy in Λ** *, or* **Hamiltonian in Λ** *, is the function $H_\Lambda : \Omega \to \mathbb{R}$ defined by*

$$H_\Lambda(\omega) \doteqdot - \sum_{\substack{e=\langle t,t'\rangle \\ e\cap\Lambda\neq\varnothing}} \big(J_\sigma(e)\sigma(t)(\omega)\sigma(t')(\omega) + J_\tau(e)\tau(t)(\omega)\tau(t')(\omega) + J_{\sigma\tau}(e)\sigma(t)(\omega)\sigma(t')(\omega)\tau(t)(\omega)\tau(t')(\omega)\big)\,.$$

(D146) *The* **Gibbs measure in Λ with $\overline{\omega}$-b.c.** *is the probability measure on $(\Omega, \mathcal{F})$ given by*

$$\mu^{\overline{\omega}}_\Lambda(\omega) \doteqdot \begin{cases} \Xi^{\overline{\omega}}(\Lambda)^{-1}\exp(-H_\Lambda(\omega)) & \textit{if } \omega \textit{ satisfies the } \Lambda^{\overline{\omega}}\textit{-b.c.,} \\ 0 & \textit{otherwise,} \end{cases}$$

where $\Xi^{\overline{\omega}}(\Lambda)$, the normalization constant, is called the **partition function in Λ with $\overline{\omega}$-b.c.**.

(D147) *Let $\Lambda \subset \mathbb{Z}^2$. The* **Gibbs measure in Λ with free-b.c.** *is the probability measure on $((\{-1,1\}\times\{-1,1\})^\Lambda, \mathcal{F}_\Lambda)$ given by*

$$\mu(\omega) \doteqdot \Xi(\Lambda)^{-1} \prod_{e=\langle t,t'\rangle\subset\Lambda} \exp(J_\sigma(e)\sigma(t)(\omega)\sigma(t')(\omega) + J_\tau(e)\tau(t)(\omega)\tau(t')(\omega) + J_{\sigma\tau}(e)\sigma(t)(\omega)\sigma(t')(\omega)\tau(t)(\omega)\tau(t')(\omega))$$

where $\Xi(\Lambda)$, the normalization constant, is called the **partition function in Λ with free-b.c.**.

As in Part I, expectation value with respect to $\mu^{\overline{\omega}}_\Lambda$ is denoted by $P^{\overline{\omega}}_\Lambda[\cdot]$, $P^{\overline{\omega},\underline{J}}_\Lambda[\cdot]$, $\langle\cdot\rangle^{\overline{\omega}}_\Lambda$ or $\langle\cdot\rangle^{\overline{\omega},\underline{J}}_\Lambda$, while expectation value with respect to μ_Λ is denoted by $P_\Lambda[\cdot]$, $P^{\underline{J}}_\Lambda[\cdot]$, $\langle\cdot\rangle_\Lambda$ or $\langle\cdot\rangle^{\underline{J}}_\Lambda$.

We construct some important boundary conditions from the boundary conditions of the Ising model. For example the ++-b.c. is obtained by choosing $\overline{\omega}_\sigma = \overline{\omega}^+$ and $\overline{\omega}_\tau = \overline{\omega}^+$, where $\overline{\omega}^+$ has been defined in part I. In the same way, we define $+-$, $-+$ and $--$ b.c.; we can also construct boundary condition by combining $\overline{\omega}^+$, $\overline{\omega}^-$, or $\overline{\omega}^d$, for example we define $+d$-b.c. by setting $\overline{\omega}_\sigma = \overline{\omega}^+$ and $\overline{\omega}_\tau = \overline{\omega}^d$, where $\overline{\omega}^d$ has been introduced in part I. Expectation values with respect to such boundary conditions are denoted μ_Λ^{++}, μ_Λ^{+-}, μ_Λ^{d-}, μ_Λ^{dd}, and so on...

We see from the above definitions that the Ashkin–Teller model can be seen as two Ising models, with coupling constants $J_\sigma(e)$ and $J_\tau(e)$ respectively, coupled together by four-bodies forces of coupling constant $J_{\sigma\tau}(e)$. This is useful, since this allows us to reduce part of the following constructions to the corresponding ones for the Ising model.

8.2 Contours

The notion of contours in this model is slightly more complicated, but the construction is quite similar. We first define contour in the σ- and τ-plane.

Definition.

(D148) *Let $B_\sigma \subset \mathcal{E}_\sigma$. The contours of the decomposition of B_σ are called* **σ-contours** *and are denoted $(\gamma_1^\sigma, \dots, \gamma_n^\sigma)$.*

(D149) *Let $B_\tau \subset \mathcal{E}_\tau$. The contours of the decomposition of B_τ are called* **τ-contours** *and are denoted $(\gamma_1^\tau, \dots, \gamma_n^\tau)$.*

These objects are identical to those defined in Part I. However, in the contours representation of the Ashkin–Teller model, a given edge can belong at the same time to $\mathcal{E}_\sigma$ and $\mathcal{E}_\tau$ and this induces an *interaction* between the σ- and τ- contours (see Chapter 9). Since we want to use cluster expansion, we need to have independent objects (with only a hard–core condition, i.e. interdiction to have an edge in common); it is therefore very useful to introduce another notion of contours which satisfy such a hard–core condition[1].

Definition.

(D150) *Let $B_\sigma \subset \mathcal{E}_\sigma$ and $B_\tau \subset \mathcal{E}_\tau$. We call* **contours** *of (B_σ, B_τ), the family $(\gamma_1, \dots, \gamma_n)$ given by*

$$\gamma_i = (\gamma_{i,1}^\sigma, \dots, \gamma_{i,n_i^\sigma}^\sigma, \gamma_{i,1}^\tau, \dots, \gamma_{i,n_i^\tau}^\tau) \equiv (\underline{\gamma}_i^\sigma, \underline{\gamma}_i^\tau)$$

where

- *The contours $\gamma_{i,k}^\sigma$, resp. $\gamma_{i,k}^\tau$, are the contours of the decompositions of B_σ, resp. B_τ;*
- *The contours γ_i are minimal families of σ- and τ-contours such that two contours γ_j^σ and γ_k^τ satisfying*

$$[\mathcal{E}(\gamma_j^\sigma) \cap \Delta(\gamma_k^\tau)] \cup [\Delta(\gamma_j^\sigma) \cap \mathcal{E}(\gamma_k^\tau)] \neq \varnothing \quad \text{or} \quad \partial\gamma_j^\sigma \cap \partial\gamma_k^\tau \neq \varnothing$$

necessarily belong to the same γ_i.

We can now state the notion of (Λ-) compatibility.

Definition.

(D151) *A family of contours $(\gamma_1, \dots, \gamma_n)$ is* **compatible** *if the corresponding families of σ- and τ-contours are compatible.*

[1]This definition may seem complicated, but happens to be useful to prove Lemma 12.0.1.

(D152) *Let $\Lambda \subset \mathbb{Z}^2$. A family of contours $(\gamma_1, \dots, \gamma_n)$ is* **Λ-compatible** *if the corresponding families of σ- and τ-contours are Λ-compatible.*

All the geometrical notions introduced for the Ising model have a natural extension in the case of the Ashkin–Teller model.

Definition.

(D153) *The σ-, resp. τ-, boundary of a family of contours is the boundary of the family of its σ-, resp. τ-, contours and is written $\partial_\sigma \underline{\gamma}$, resp. $\partial_\tau \underline{\gamma}$.*

(D154) *A contour γ is closed if $\partial_\sigma \gamma = \partial_\tau \gamma = \varnothing$.*

Since the σ- and τ-contours are Ising contours, we use the terminology of Part I for them, without repeating it.

Chapter 9

Contours representations and duality

The low temperature and high temperature representations introduced in the first Part for the Ising model have natural generalizations in the Ashkin–Teller model. Moreover, duality still holds.

9.1 Low temperature representation

Since a configuration $\omega \in \Omega$ of the Ashkin–Teller model can be decomposed into two Ising-like configurations $\omega = (\omega_\sigma, \omega_\tau)$, the problem of expressing ω in terms of contours reduces to the Ising model case. In particular, when ω_σ is a configuration of the Ising model satisfying the $\Lambda^{\overline{\omega}_\sigma}$-b.c. and ω_τ is a configuration of the Ising model satisfying the $\Lambda^{\overline{\omega}_\tau}$-b.c., then the configuration $\omega = (\omega_\sigma, \omega_\tau)$ satisfies the $\Lambda^{\overline{\omega}}$-b.c.; the converse is also true. Therefore, to construct the contours of the configuration ω, it is enough to construct the contours of the (Ising-like) configurations ω_σ and ω_τ as is done in Part I.

Definition.

(D155) *Let $\omega \in \Omega$. The* **σ-contours of the configuration** ω *are the contours of the configuration ω_σ.*

(D156) *Let $\omega \in \Omega$. The* **τ-contours of the configuration** ω *are the contours of the configuration ω_τ.*

(D157) *Let $\omega \in \Omega$. The* **contours of the configuration** ω *are the contours obtained from the set of its σ- and τ-contours.*

Let us consider first the case of $\overline{\omega}$-b.c. with $\overline{\omega}$ independent of t. We discuss the case of $++$-b.c., but it is immediate to extend the following considerations to $+-$, $-+$ and $--$-b.c.. The contours of the configurations ω_σ and ω_τ are defined in (D56), p. 42. We can state the property of Λ^{++}-compatibility.

Definition.

(D158) *A Λ-compatible family of contours is Λ^{++}-compatible if there exists a configuration ω satisfying the Λ^{++}-b.c. such that $\underline{\gamma}(\omega) = \underline{\gamma}$.*

By the results of Part I, we have the following Lemma.

Lemma 9.1.1. *If Λ is simply connected then $\underline{\gamma}$ is a Λ^*-compatible family of closed contours if and only if $\underline{\gamma}$ is a Λ^{++}-compatible family of contours.*

Let us now introduce, as in Part I, probability measures on the set of Λ^{++}-compatible families of contours.

Definition.

(D159) *Let $\underline{\gamma}$ be a Λ^{++}-compatible family of contours. The probability of $\underline{\gamma}$ is denoted by*

$$P_\Lambda^{++}[\underline{\gamma};\underline{J}] \doteqdot P_\Lambda^{++,\underline{J}}[\omega_{\underline{\gamma}}]\,,$$

where $\omega_{\underline{\gamma}}$ is the unique configuration satisfying Λ^{++}-b.c. which has $\underline{\gamma}$ as its set of contours.

(D160) *The probability that $\underline{\gamma}$ belong to the set of contours of a configuration is denoted by*

$$q_\Lambda^{++}(\underline{\gamma};\underline{J}) \doteqdot P_\Lambda^{++}[\{\underline{\gamma}' \,:\, \underline{\gamma}\subset\underline{\gamma}'\};\underline{J}]\,.$$

As a consequence of the particular form of the coupling between the σ- and τ-spins, these probabilities can be expressed in a simple way in terms of contours. To achieve this, we introduce as before weights for the contours.

Definition.

(D161) *Let B_σ and B_τ be two sets of σ- and τ-edges resp., and $B=(B_\sigma,B_\tau)$. We set*

$$\begin{aligned}
&\mathcal{B}_\sigma(B)\doteqdot B_\sigma\setminus B_\tau\,, &&\mathcal{B}_\tau(B)\doteqdot B_\tau\setminus B_\sigma\,, &&\mathcal{B}_{\sigma\tau}(B)\doteqdot B_\sigma\cap B_\tau\,,\\
&\overline{\mathcal{B}}_\sigma(B)\doteqdot B_\sigma\,, &&\overline{\mathcal{B}}_\tau(B)\doteqdot B_\tau\,, &&\mathcal{B}(B)\doteqdot \overline{\mathcal{B}}_\sigma(B)\cup\overline{\mathcal{B}}_\tau(B)\,.
\end{aligned}$$

We use the same notation for sets of dual edges.

(D162) *The* **weight of a contour** *γ is defined by*

$$w(\gamma)\doteqdot w_\sigma(\gamma)w_\tau(\gamma)w_{\sigma\tau}(\gamma)\,,$$

where

$$\begin{aligned}
w_\sigma(\gamma) &\doteqdot \prod_{e^*\in\mathcal{B}_\sigma(\gamma)} \exp[-2(J_\sigma(e)+J_{\sigma\tau}(e))]\,,\\
w_\tau(\gamma) &\doteqdot \prod_{e^*\in\mathcal{B}_\tau(\gamma)} \exp[-2(J_\tau(e)+J_{\sigma\tau}(e))]\,,\\
w_{\sigma\tau}(\gamma) &\doteqdot \prod_{e^*\in\mathcal{B}_{\sigma\tau}(\gamma)} \exp[-2(J_\sigma(e)+J_\tau(e))]\,.
\end{aligned}$$

(D163) *The* **weight of a compatible family** *of contours $\underline{\gamma}$ is given by $w(\underline{\gamma})\doteqdot\prod_{\gamma\in\underline{\gamma}}w(\gamma)$.*

Then it is easy to check that

$$\Xi^{++}(\Lambda) = C_\Lambda \sum_{\underline{\gamma}\ \Lambda^{++}\text{-comp}} w(\underline{\gamma})\,, \tag{9.1}$$

where $C_\Lambda \doteqdot \prod_{\substack{e\in\mathcal{E}\\ e\cap\Lambda\neq\varnothing}} \exp[J_\sigma(e) + J_\tau(e) + J_{\sigma\tau}(e)]$.

Definition.

(D164) *Let $\underline{\gamma}'$ be some Λ^*-compatible family of closed contours. We set*

$$Z^{++}(\Lambda|\underline{\gamma}';\underline{J}) \doteqdot \sum_{\substack{\underline{\gamma}\,:\,\text{closed}\\ \underline{\gamma}\cup\underline{\gamma}'\ \Lambda^{++}-\text{compatible}}} w(\underline{\gamma})\,.$$

If $\underline{\gamma}' = \varnothing$, we write

$$Z^{++}(\Lambda;\underline{J}) \doteqdot Z^{++}(\Lambda|\varnothing;\underline{J})\,.$$

This last quantity is called the **normalized partition function in Λ with ++-b.c.**.

We can finally state the main result for the low temperature representation of the Ashkin–Teller model with ++-b.c..

Lemma 9.1.2. *Let Λ be a finite subset of $\mathbb{Z}^2$. Then*

$$\Xi^{++}(\Lambda;\underline{J}) = C_\Lambda Z^{++}(\Lambda;\underline{J})\,.$$

Let $\underline{\gamma}$ be a Λ^{++}-compatible family of contours; then

$$P_\Lambda^{++}[\underline{\gamma};\underline{J}] = \frac{w(\underline{\gamma})}{Z^{++}(\Lambda;\underline{J})}\,.$$

Let $\underline{\gamma}$ be a Λ^*-compatible family of closed contours; then

$$q_\Lambda^{++}(\underline{\gamma};\underline{J}) = w(\underline{\gamma})\frac{Z^{++}(\Lambda|\underline{\gamma};\underline{J})}{Z^{++}(\Lambda;\underline{J})}\,.$$

Let us now examine the case of the $d+$-b.c. (the cases of $+d$-, $d-$- and $-d$-b.c. are treated in the same way). The contours of the configurations ω_σ and ω_τ are defined in (D70), p. 47, and (D56), p. 42, respectively. Let $\Lambda_{L,M}$, t_r^* and t_l^* be defined as in Subsection 2.1.2.

Lemma 9.1.3. *Let ω be a configuration satisfying the $\Lambda_{L,M}^{d+}$-b.c.. Then the contours of $\gamma(\omega)$ are all closed except for one contour λ such that $\partial_\sigma\lambda = \{t_l^*, t_r^*\}$ and $\partial_\tau\lambda = \varnothing$.*

As $\Lambda_{L,M}$ is simply connected, it is easy to check that there is a one-to-one correspondence between configurations satisfying $\Lambda_{L,M}^{d+}$-b.c. and families of $\Lambda_{L,M}^*$-compatible contours as in the previous lemma. In particular, the partition function in $\Lambda_{L,M}$ with $d+$-b.c. can be written

$$\Xi^{d+}(\Lambda_{L,M}) = C_{\Lambda_{L,M}} Z^{d+}(\Lambda_{L,M}) \tag{9.2}$$

where we have defined

Definition.

(D165) $$Z^{d+}(\Lambda_{L,M}|\lambda) \doteqdot \sum_{\substack{\underline{\gamma}\,:\,\text{closed}\\ \underline{\gamma}\cup\lambda\ \Lambda^{d+}\text{-comp.}}} w(\underline{\gamma})$$

(D166) $$Z^{d+}(\Lambda_{L,M}) \doteqdot \sum_{\substack{\lambda:\Lambda^*\text{-comp.}\\ \partial_\sigma\lambda=\{t_l^*,t_r^*\}\\ \partial_\tau\lambda=\varnothing}} w(\lambda) Z^{d+}(\Lambda_{L,M}|\lambda).$$

We finally give the corresponding results in the case of the *dd*-boundary condition. The contours of the configurations ω_σ and ω_τ are defined in (D70), p. 47. Let $\Lambda_{L,M}$, t_r^* and t_l^* be defined as in Subsection 2.1.2.

Lemma 9.1.4. *Let ω be a configuration satisfying the $\Lambda_{L,M}^{dd}$-b.c.. Then the contours of $\gamma(\omega)$ are all closed except for one contour λ such that $\partial_\sigma\lambda = \partial_\tau\lambda = \{t_l^*, t_r^*\}$.*

As $\Lambda_{L,M}$ is simply connected, it is easy to check that there is a one-to-one correspondence between configurations satisfying $\Lambda_{L,M}^{dd}$-b.c. and families of $\Lambda_{L,M}^*$-compatible contours as in the previous lemma. In particular, the partition function in $\Lambda_{L,M}$ with *dd*-b.c. can be written

$$\Xi^{dd}(\Lambda_{L,M}) = C_{\Lambda_{L,M}} Z^{dd}(\Lambda_{L,M}) \tag{9.3}$$

where we have defined

Definition.

(D167) $$Z^{dd}(\Lambda_{L,M}|\lambda) \doteqdot \sum_{\substack{\underline{\gamma}\,:\,\text{closed}\\ \underline{\gamma}\cup\lambda\ \Lambda^{dd}\text{-comp.}}} w(\underline{\gamma})$$

(D168) $$Z^{dd}(\Lambda_{L,M}) \doteqdot \sum_{\substack{\lambda:\Lambda^*\text{-comp.}\\ \partial_\sigma\lambda=\partial_\tau\lambda=\{t_l^*,t_r^*\}}} w(\lambda) Z^{dd}(\Lambda_{L,M}|\lambda).$$

9.2 High temperature representation

We show now that the Ashkin–Teller model admits a high temperature representation similar to the one introduced in the first part[1].

As was the case in Part I, it is useful to define partition function on graphs defined by a set of edges. Therefore, we introduce the following

Definition.

(D169) *Let $B \subset \mathcal{E}$ be finite; the* **partition function with free b.c. on the graph $\mathcal{G}(B)$ is defined by**

$$\Xi(\mathcal{G}(B)) \doteqdot \sum_{\omega\in\mathcal{S}_{\mathrm{AT}}{}^{\Lambda(B)}} \prod_{e=\langle t,t'\rangle\subset\Lambda} \exp(J_\sigma(e)\omega_\sigma(t)\omega_\sigma(t') + J_\tau(e)\omega_\tau(t)\omega_\tau(t') + J_{\sigma\tau}(e)\omega_\sigma(t)\omega_\sigma(t')\omega_\tau(t)\omega_\tau(t'))$$

[1] Such a representation for this model has been known for a long time (see [F, B], for example). We follow the straightforward derivation of [PV2]

Clearly, $\Xi(\mathcal{G}(\mathcal{E}(\Lambda))) = \Xi(\Lambda)$.

The method to obtain the high-tempeerature representation is very similar to the corresponding one in Part I. Let $B \subset \mathcal{E}$ be finite.

$$\Xi(\mathcal{G}(B)) = \sum_{\substack{\omega_\sigma \in \{-1,1\}^{\Lambda(B)} \\ \omega_\tau \in \{-1,1\}^{\Lambda(B)}}} \prod_{e=\langle t,t' \rangle \in B} \cosh J_\sigma(e) \cosh J_\tau(e) \cosh J_{\sigma\tau}(e) \times$$
$$\times (1 + \omega_\sigma(t)\omega_\sigma(t') \tanh J_\sigma(e)) \times$$
$$\times (1 + \omega_\tau(t)\omega_\tau(t') \tanh J_\sigma(e)) \times$$
$$\times (1 + \omega_\sigma(t)\omega_\sigma(t')\omega_\tau(t)\omega_\tau(t') \tanh J_{\sigma\tau}(e)) \,. \tag{9.4}$$

It is useful to introduce the following notations

$$s_e \doteqdot \tanh J_\sigma(e)\,,\ t_e \doteqdot \tanh J_\tau(e)\,,\ l_e \doteqdot \tanh J_{\sigma\tau}(e)\,, \tag{9.5}$$

and

$$S_e \doteqdot \frac{s_e + t_e l_e}{1 + s_e t_e l_e}\,,\ T_e \doteqdot \frac{t_e + s_e l_e}{1 + s_e t_e l_e}\,,\ L_e \doteqdot \frac{l_e + s_e t_e}{1 + s_e t_e l_e}\,. \tag{9.6}$$

With these notations,

$$\Xi(\mathcal{G}(B)) = \prod_{e \in B} \cosh J_\sigma(e) \cosh J_\tau(e) \cosh J_{\sigma\tau}(e)(1 + s_e t_e l_e) \times$$
$$\times \sum_{\substack{\omega_\sigma \in \{-1,1\}^{\Lambda(B)} \\ \omega_\tau \in \{-1,1\}^{\Lambda(B)}}} \prod_{e=\langle t,t' \rangle \in B} (1 + S_e \omega_\sigma(t)\omega_\sigma(t') + T_e \omega_\tau(t)\omega_\tau(t') + L_e \omega_\sigma(t)\omega_\sigma(t')\omega_\tau(t)\omega_\tau(t')) \,. \tag{9.7}$$

We want to expand the product over the edges. Each term can be labeled by $\eta = (\eta_\sigma, \eta_\tau) \in (\{0,1\} \times \{0,1\})^B$ in the following way.

1. Each time we take one term 1 in (9.7), we set
$$\eta_\sigma(e) = 0\,,\ \eta_\tau(e) = 0\,, \tag{9.8}$$
2. Each time we take one term $S_e \omega_\sigma(t)\omega_\sigma(t')$ in (9.7), we set
$$\eta_\sigma(e) = 1\,,\ \eta_\tau(e) = 0\,, \tag{9.9}$$
3. Each time we take one term $T_e \omega_\tau(t)\omega_\tau(t')$ in (9.7), we set
$$\eta_\sigma(e) = 0\,,\ \eta_\tau(e) = 1\,, \tag{9.10}$$
4. Each time we take one term $L_e \omega_\sigma(t)\omega_\sigma(t')\omega_\tau(t)\omega_\tau(t')$ in (9.7), we set
$$\eta_\sigma(e) = 1\,,\ \eta_\tau(e) = 1\,. \tag{9.11}$$

For each term η, we introduce now a family of σ-contour $\underline{\gamma}^\sigma$ and a family of τ-contours $\underline{\gamma}^\tau$: $\underline{\gamma}^\sigma$ is the family of contours of the decomposition of the set $\{e \in B \,:\, \eta_\sigma(e) = 1\}$ and

$\underline{\gamma}^\tau$ is the family of contours of the decomposition of the set $\{e \in B : \eta_\tau(e) = 1\}$ [2]. We can now show in exactly the same way as was done in Part I that the summations over ω_σ and ω_τ imply that the only terms which contribute to (9.7) are the configuration composed of only *closed* σ- and τ-contours. Therefore we finally obtain

$$\Xi(\mathcal{G}(B)) = 4^{|\Lambda(B)|} \prod_{e\in B} \cosh J_\sigma(e) \cosh J_\tau(e) \cosh J_{\sigma\tau}(e)(1+s_e t_e l_e) \sum_{\substack{\underline{\gamma}\subset B \\ \text{comp.,closed}}} w^*(\underline{\gamma}) \,, \quad (9.12)$$

where we have introduced

Definition.

(D170) *The* **$*$-weight of a contour** γ *is given by*

$$w^*(\gamma) \doteqdot w^*_\sigma(\gamma) w^*_\tau(\gamma) w^*_{\sigma\tau}(\gamma) \,,$$

where

$$w^*_\sigma(\gamma) \doteqdot \prod_{e\in\mathcal{B}_\sigma(\gamma)} S_e \,,$$
$$w^*_\tau(\gamma) \doteqdot \prod_{e\in\mathcal{B}_\tau(\gamma)} T_e \,,$$
$$w^*_{\sigma\tau}(\gamma) \doteqdot \prod_{e\in\mathcal{B}_{\sigma\tau}(\gamma)} L_e \,.$$

(D171) *The* **$*$-weight of a compatible family** *of contours* $\underline{\gamma}$ *is given by*

$$w^*(\underline{\gamma}) \doteqdot \prod_{\gamma\in\underline{\gamma}} w^*(\gamma) \,.$$

This leads to the following definitions.

Definition.

(D172) *Let* $B \subset \mathcal{E}$ *finite; the* **normalized partition function on $\mathcal{G}(B)$ with free b.c.** *is defined as*

$$Z(\mathcal{G}(B); \underline{J}) \doteqdot \sum_{\substack{\underline{\gamma}\subset B \\ \text{comp., closed}}} w^*(\underline{\gamma}) \,.$$

(D173) *Let* $B \subset \mathcal{E}$ *finite and let* $\underline{\gamma} \subset B$ *be a compatible family of contours; we set*

$$Z(\mathcal{G}(B)|\underline{\gamma}; \underline{J}) \doteqdot \sum_{\substack{\underline{\gamma}'\subset B:\ \text{closed} \\ \underline{\gamma}\cup\underline{\gamma}'\ \text{comp.}}} w^*(\underline{\gamma}) \,.$$

[2] These contours are defined slightly differently from what was done in [PV2], where the σ- and τ-planes were interchanged, see also Section 9.3 below.

(D174) *The* **normalized partition function in Λ with free b.c.** *is defined as*

$$Z(\Lambda; \underline{J}) \doteqdot \sum_{\substack{\underline{\gamma} \\ \Lambda\text{-comp., closed}}} w^*(\underline{\gamma}) .$$

(D175) *Let $\underline{\gamma}$ be a Λ-compatible family of contours; we set*

$$Z(\Lambda|\underline{\gamma}; \underline{J}) \doteqdot \sum_{\substack{\underline{\gamma}'\text{: closed} \\ \underline{\gamma}\cup\underline{\gamma}' \ \Lambda\text{-comp.}}} w^*(\underline{\gamma}) .$$

Again, $Z(\mathcal{G}(\mathcal{E}(\Lambda))) = Z(\Lambda)$ and $Z(\mathcal{G}(\mathcal{E}(\Lambda))|\underline{\gamma}) = Z(\Lambda|\underline{\gamma})$.

In the same way, it is possible to express correlation functions using the high temperature representation,

$$\langle \sigma_A \tau_B \rangle_\Lambda = Z(\Lambda; \underline{J})^{-1} \sum_{\substack{\underline{\gamma}\ \Lambda\text{-comp.} \\ \partial_\sigma \underline{\gamma}=A\,,\, \partial_\tau \underline{\gamma}=B}} w^*(\underline{\gamma}) . \tag{9.13}$$

As in the first part, the quantity which have the greatest interest for us are the 2-point functions, $\langle \sigma(t)\sigma(t') \rangle_\Lambda$, $\langle \tau(t)\tau(t') \rangle_\Lambda$ and $\langle \sigma(t)\tau(t)\sigma(t')\tau(t') \rangle_\Lambda$. Introducing

Definition.

(D176) *Let $\underline{\gamma}$ be a Λ-compatible family of contours; we set*

$$q_\Lambda(\underline{\gamma}; \underline{J}) \doteqdot w^*(\underline{\gamma}) \frac{Z(\Lambda|\underline{\gamma}; \underline{J})}{Z(\Lambda; \underline{J})} ,$$

they can be written in the following very simple form,

$$\langle \sigma(t)\sigma(t') \rangle_\Lambda = \sum_{\substack{\lambda\,:\,\Lambda\text{-comp.} \\ \partial_\sigma \lambda=\{t,t'\} \\ \partial_\tau \lambda=\varnothing}} q_\Lambda(\lambda; \underline{J}) , \tag{9.14}$$

$$\langle \tau(t)\tau(t') \rangle_\Lambda = \sum_{\substack{\lambda\,:\,\Lambda\text{-comp.} \\ \partial_\sigma \lambda=\varnothing \\ \partial_\tau \lambda=\{t,t'\}}} q_\Lambda(\lambda; \underline{J}) , \tag{9.15}$$

$$\langle \sigma(t)\tau(t)\sigma(t')\tau(t') \rangle_\Lambda = \sum_{\substack{\lambda\,:\,\Lambda\text{-comp.} \\ \partial_\sigma \lambda=\{t,t'\} \\ \partial_\tau \lambda=\{t,t'\}}} q_\Lambda(\lambda; \underline{J}) . \tag{9.16}$$

Finally, we introduce a probability measure on the set of Λ-compatible families of *closed contours*.

Definition.

(D177) *Let $\underline{\gamma}$ be a Λ-compatible family of closed contours; we set*

$$P_\Lambda[\underline{\gamma};\underline{J}] \doteqdot \frac{w^*(\underline{\gamma})}{Z(\Lambda;\underline{J})}\,.$$

Observe that we still have $q_\Lambda(\underline{\gamma};\underline{J}) = P_\Lambda[\{\underline{\gamma}' : \underline{\gamma} \subset \underline{\gamma}'\};\underline{J}]$.

9.3 Duality

Looking at (D164), p. 197, and (D174), we see that the Ashkin–Teller model is also self–dual. Indeed, if $\underline{J}^*(e) = (J^*_\sigma(e), J^*_\tau(e), J^*_{\sigma\tau}(e))$ and $J(e) = (J_\sigma(e), J_\tau(e), J_{\sigma\tau}(e))$ are two sets of coupling constants satisfying

$$\begin{aligned} S_e(J^*_\sigma(e), J^*_\tau(e), J^*_{\sigma\tau}(e)) &= \exp[-2(J_\sigma + J_{\sigma\tau})]\,,\\ T_e(J^*_\sigma(e), J^*_\tau(e), J^*_{\sigma\tau}(e)) &= \exp[-2(J_\tau + J_{\sigma\tau})]\,,\\ L_e(J^*_\sigma(e), J^*_\tau(e), J^*_{\sigma\tau}(e)) &= \exp[-2(J_\sigma + J_\tau)]\,, \end{aligned} \tag{9.17}$$

then we have $Z^{++}(\Lambda;\underline{J}) = Z(\Lambda;\underline{J}^*)$.

Definition.

(D178) *Let $\underline{J}(e)$ be a set of coupling constants. The set of coupling constants $\underline{J}^*(e)$ solution to the equations* (9.17) *defines the* **dual coupling constants** *(when they exist).*

It is obviously important to know for which $\underline{J}$ it is possible to find dual coupling constants. The next proposition answers this question.

Proposition 9.3.1. *Let Λ be a simply connected bounded subset of $\mathbb{Z}^2$. Let*

$$\mathcal{D} \doteqdot \{(x,y,z) \in \mathbb{R}^3 \,:\, x \geqslant y \geqslant z, y > 0, \tanh z > -\tanh x \tanh y\}\,,$$

and

$$\mathcal{D}^* \doteqdot \{(x,y,z) \in \mathbb{R}^3 \,:\, y \geqslant x \geqslant z, x > 0, \tanh z > -\tanh x \tanh y\}\,.$$

Let $(J_\sigma, J_\tau, J_{\sigma\tau}) \in \mathcal{D}$ be the coupling constants of the Ashkin–Teller model defined in Λ with ++-boundary conditions. Then equations (9.17) *define a bijection between $\mathcal{D}$ and $\mathcal{D}^*$. On the closure of $\mathcal{D}$, the application is still well-defined, but takes values in $\overline{\mathbb{R}}^3$ and is no more everywhere invertible.*

The following duality relations hold,

$$Z^{++}(\Lambda;\underline{J}) = Z(\Lambda^*;\underline{J}^*)\,.$$

For any Λ^{++}-compatible family of contours $\underline{\gamma}$,

$$P^{++}_\Lambda[\underline{\gamma};\underline{J}] = P_{\Lambda^*}[\underline{\gamma};\underline{J}^*]\,,$$

For any Λ^*-compatible family of closed contours,

$$q^{++}_\Lambda(\underline{\gamma};\underline{J}) = q_{\Lambda^*}(\underline{\gamma};\underline{J}^*)\,.$$

Proof. The proof is straightforward algebra but quite lengthy and can be found in [PV2], so we don't repeat it here (The only difference is that the two planes were interchanged in this work). □

In the case of the Ising model, it was possible to find a value of the coupling constant so that it is equal to its dual, the so-called self–dual coupling constant. Moreover, this self–dual coupling constant was equal to the critical coupling constant. Is there an analogous result for the Ashkin–Teller model? Obviously, since $\mathcal{D} \neq \mathcal{D}^*$, the only case where we can hope to find self-dual coupling constants is when $J_\sigma = J_\tau$. However, using another symmetry of the model, it is possible to extend the self–dual manifold outside this subset. Let $\pi : \Omega \to \Omega$, $(\omega_\sigma, \omega_\tau) \mapsto (\omega_\tau, \omega_\sigma)$, be the application exchanging the σ- and τ-planes. We can look for the fixed point of the application $\pi \circ *$, where $*$ is the duality application defined by (9.17). It is easy to check that $\pi \circ *$ is a bijection from $\mathcal{D}$ to itself such that

$$Z^{++}(\Lambda; \underline{J}) = Z(\Lambda^*; \underline{J}^{\pi\circ *}) \,. \tag{9.18}$$

where $\underline{J} = (J_\sigma, J_\tau, J_{\sigma\tau})$ and $\underline{J}^{\pi\circ *} = (J^*_\tau, J^*_\sigma, J^*_{\sigma\tau})$.
For any Λ^{++}-compatible family of contours $\underline{\gamma}$,

$$P^{++}_\Lambda[\underline{\gamma}; \underline{J}] = P_{\Lambda^*}[\underline{\gamma}; \underline{J}^{\pi\circ *}] \,, \tag{9.19}$$

For any Λ^*-compatible family of closed contours,

$$q^{++}_\Lambda(\underline{\gamma}; \underline{J}) = q_{\Lambda^*}(\underline{\gamma}; \underline{J}^{\pi\circ *}) \,. \tag{9.20}$$

Proposition 9.3.2 shows that there exists a self–dual manifold associated to this application. However, it is known [W, Pf4] that this self–dual manifold *does not* coincide with the critical manifold (even though they do coincide on some parts), see also Section 10.4.1.

Proposition 9.3.2. *The self–dual manifold, i.e. the set of fixed points of the application $\pi \circ *$, is given by*

$$l = \frac{1 - st - s - t}{1 - st + s + t} \,,$$

where $s = \tanh J_\sigma$, $t = \tanh J_\tau$ and $l = \tanh J_{\sigma\tau}$.

Proof. Equations (9.17) can be easily seen to be equivalent to

$$\begin{aligned} \frac{l + ts}{1 + stl} &= \frac{(1-s)(1-t)}{(1+s)(1+t)} \,. \\ \frac{s + tl}{1 + stl} &= \frac{(1-t)(1-l)}{(1+t)(1+l)} \,, \\ \frac{t + sl}{1 + stl} &= \frac{(1-s)(1-l)}{(1+s)(1+l)} \,. \end{aligned} \tag{9.21}$$

where we have used the elementary identity $\exp[-2(x+y)] = \dfrac{(1-\tanh x)(1-\tanh y)}{(1+\tanh x)(1+\tanh y)}$.
Some algebraic manipulations shows that the first of these equations can be rewritten as

$$l = \frac{1 - st - s - t}{1 - st + s + t} \,. \tag{9.22}$$

Moreover, substituting (9.22) into the last two equations of (9.21) show that they are automatically satisfied:

$$\frac{s+tl}{1+stl} = \frac{(1-t)(s+t)}{(1+t)(1-st)} = \frac{(1-t)(1-l)}{(1+t)(1+l)}\,, \tag{9.23}$$

and similarly for the last one. □

Remark. Observe that the symmetries of the Ashkin–Teller model allow us to interchange the coupling constants in the Proposition. For example, if $J_{\sigma\tau} > J_\tau$, then a change of variables $(\omega_\sigma(t), \omega_\tau(t)) \mapsto (\omega_\sigma(t), \omega_\vartheta(t))$, with $\omega_{\sigma\tau}(t) \doteqdot \omega_\sigma(t)\omega_\tau(t)$, results in $J_\vartheta > J_{\sigma\vartheta}$.

Chapter 10

Random–Cluster representation

This chapter is devoted to the introduction of the Random–Cluster (RC) representation of the Ashkin–Teller model, and the derivation of some of its properties. This geometrical representation is somewhat different from the low and high temperature representation in that it is not formulated in terms of contours. Such a representation is well-known in the case of the Ising and Potts models and has been used in many ways, including in the study of large deviations (Ioffe's approach to the phase of small contours in [I2] and Schonman's version of the lower bound in [S1] use extensively this representation for the Ising model; the whole analysis of Pisztora [Pi] is done on the level of the RC measure (including the cases of the percolation, Ising and Potts models.))

It is our conviction that the approach of Part I is more natural than the approach using the RC measure since we work directly with the relevant microscopic object, the phase separation line[1]. Nevertheless since there may be some way to use the RC representation in the case of the Ashkin–Teller model, for which we only have partial results, we think it is interesting to show how the usual RC representation admits a generalization to a much larger class of models containing the Ashkin–Teller model.

In Section 10.1 we define the Random–Cluster model; in Section 10.2 we explain how this model is related to the Ashkin–Teller model; the subject of Section 10.3 is the duality of the Random–Cluster model and the proof of the commutativity of the dualities of the Ashkin–Teller and Random–Cluster models; the validity of the FKG inequalities is shown in Subsection 10.4.1, while comparison inequalities are established in Subsection 10.4.2; finally, we give some applications of this representation in Section 10.4.2[2].

10.1 The Random–Cluster model

Contrarily to the other models considered in this thesis, the Random–Cluster model is not defined on the sites of a lattice, but rather on the edges.

Definition.

(D179) *The (single)* **bond space** *of the Random–Cluster model is* $\{0,1\} \times \{0,1\}$.

[1]However, some estimates appear to be slightly simpler in the Random–Cluster representation.

[2]This chapter follows [PV2]; this representation (formulated in a slightly different way) has also been introduced in [CM] at the same time and has appeared (without being explicitly studied) in [ES] and [SS].

(D180) *The* **configuration space** *of the Random–Cluster model is*

$$\Omega^{\mathrm{RC}} \doteqdot (\{0,1\} \times \{0,1\})^{\mathcal{E}} \equiv \Omega_\sigma^{\mathrm{RC}} \times \Omega_\tau^{\mathrm{RC}}\,, \tag{10.1}$$

where $\Omega_\sigma^{\mathrm{RC}} \doteqdot \{0,1\}^{\mathcal{E}}$ *and* $\Omega_\tau^{\mathrm{RC}} \doteqdot \{0,1\}^{\mathcal{E}}$ *are the* σ*- and* τ*-configuration spaces. The elements* $n \in \Omega^{\mathrm{RC}}$ *are the* **configurations***; we can write* $n = (n_\sigma, n_\tau)$*, where* $n_\sigma \in \Omega_\sigma^{\mathrm{RC}}$ *and* $n_\tau \in \Omega_\tau^{\mathrm{RC}}$ *are the* σ**-** *and* τ**-configurations**.

(D181) *Let* e *be some edge.* $n_\sigma(e)$ *is called the* σ**-bond** *at* e *and* $n_\tau(e)$ *is the* τ**-bond** *at* e.

(D182) *If* $n_\sigma(e) = 1$*, then we say that* e *is* σ**-open in** n*; otherwise, it is* σ**-closed in** n*. If* $n_\tau(e) = 1$*, then we say that* e *is* τ**-open in** n*; otherwise, it is* τ**-closed in** n.

(D183) *Let* $n = (n_\sigma, n_\tau)$ *be some configurations. We denote by* $\overline{n}$ *the configuration such that* $\overline{n}(e) = (1 - n_\sigma(e), 1 - n_\tau(e))$ *for all* e.

(D184) *Let* $\mathcal{B} \subset \mathcal{E}$*.* $\mathcal{F}_{\mathcal{B}}^{\mathrm{RC},\sigma}$ *is the* σ*-algebra generated by the random variables* $n_\sigma(e)$*,* $e \in \mathcal{B}$*;* $\mathcal{F}_{\mathcal{B}}^{\mathrm{RC},\tau}$ *is the* σ*-algebra generated by the random variables* $n_\tau(e)$*,* $e \in \mathcal{B}$*;* $\mathcal{F}_{\mathcal{B}}^{\mathrm{RC}}$ *is the* σ*-algebra generated by* $\mathcal{F}_{\mathcal{B}}^{\mathrm{RC},\sigma}$ *and* $\mathcal{F}_{\mathcal{B}}^{\mathrm{RC},\tau}$*. We set* $\mathcal{F}^{\mathrm{RC}} \doteqdot \mathcal{F}_{\mathcal{E}}^{\mathrm{RC}}$*,* $\mathcal{F}^{\mathrm{RC},\sigma} \doteqdot \mathcal{F}_{\mathcal{E}}^{\mathrm{RC},\sigma}$ *and* $\mathcal{F}^{\mathrm{RC},\tau} \doteqdot \mathcal{F}_{\mathcal{E}}^{\mathrm{RC},\tau}$.

(D185) *A function is* $\mathcal{B}$**-local** *if it is* $\mathcal{F}_{\mathcal{B}}^{\mathrm{RC}}$*-measurable with* $\mathcal{B}$ *finite.*

(D186) *A function is* $\mathcal{B}^\sigma$**-local** *if it is* $\mathcal{F}_{\mathcal{B}}^{\mathrm{RC},\sigma}$*-measurable with* $\mathcal{B}$ *finite.*

(D187) *A function is* $\mathcal{B}^\tau$**-local** *if it is* $\mathcal{F}_{\mathcal{B}}^{\mathrm{RC},\tau}$*-measurable with* $\mathcal{B}$ *finite.*

A fundamental notion in this model is that of connectedness.

Definition.

(D188) *Let* $n \in \Omega^{\mathrm{RC}}$*. Two sites* i *and* $j \in \mathbb{Z}^2$ *are* σ**-connected** *in the configuration* n *if there exists a path containing these two sites and such that all edges of the path are* σ*-open in* n*. Two sites* i *and* $j \in \mathbb{Z}^2$ *are* τ**-connected** *in the configuration* n *if there exists a path containing these two sites and such that all edges of the path are* τ*-open in* n.

(D189) *Two sets of sites* A *and* B *are* σ*-, resp.* τ*-, connected if there exist two sites* $i \in A$ *and* $j \in B$ *which are* σ*-, resp.* τ*-, connected.*

(D190) *The events "*i *is* σ*-connected to* j*" and "*i *is* τ*-connected to* j*" are written*

$$i \overset{\sigma}{\leftrightarrow} j \quad \text{and} \quad i \overset{\tau}{\leftrightarrow} j\,.$$

(D191) *Maximal* σ*-connected components of sites in some configuration* n *are called* σ**-clusters** *of the configuration* n [3]*. The number of* σ*-clusters in* n *which intersects a given set* Λ *is denoted by* $N_\sigma(n|\Lambda)$.

(D192) *Maximal* τ*-connected components of sites in some configuration* n *are called* τ**-clusters** *of the configuration* n*. The number of* τ*-clusters in* n *which intersects a given set* Λ *is denoted by* $N_\tau(n|\Lambda)$.

[3] We emphasize the fact that each isolated sites is a cluster.

We define now the *a priori* measure, which generalizes the usual percolation measure. We consider a family $(\lambda_e)_{e\in\mathcal{E}}$ of probability measures on $\{-1,1\}\times\{-1,1\}$. We write

$$\lambda_e((0,0)) = a_0(e)\,,\quad \lambda_e((1,1)) = a_{\sigma\tau}(e)\,, \tag{10.2}$$

$$\lambda_e((1,0)) = a_\sigma(e)\,,\quad \lambda_e((0,1)) = a_\tau(e)\,. \tag{10.3}$$

Definition.

(D193) *Let $\mathcal{B}\subset\mathcal{E}$ be finite. The* **2-bonds percolation measure in $\mathcal{B}$** [4] *is a probability measure on $([\{0,1\}\times\{0,1\}]^{\mathcal{B}},\mathcal{F}^{\mathrm{RC}}_{\mathcal{B}})$, defined by*

$$\lambda_{\mathcal{B}}(n) \doteqdot \prod_{\substack{e\in\mathcal{B}\\ n(e)=(0,0)}} a_0(e) \prod_{\substack{e\in\mathcal{B}\\ n(e)=(1,0)}} a_\sigma(e) \prod_{\substack{e\in\mathcal{B}\\ n(e)=(0,1)}} a_\tau(e) \prod_{\substack{e\in\mathcal{B}\\ n(e)=(1,1)}} a_{\sigma\tau}(e)\,.$$

We use the following convenient convention: If $n\in\Omega^{\mathrm{RC}}$, then $\lambda_{\mathcal{B}}(n)\doteqdot\lambda_{\mathcal{B}}(n_{\mathcal{B}})$, where $n_{\mathcal{B}}$ is the restriction of the configuration n to the set $\mathcal{B}$.

We are now in measure to define the Random–Cluster measures. We first have to define the boundary conditions. Let $\Lambda\subset\mathbb{Z}^2$; we have already introduced a set of edges associated to Λ, $\mathcal{E}(\Lambda)$, see (D73), p. 48. We now define a new set of edges associated to Λ.

Definition.

(D194) *Let $\Lambda\subset\mathbb{Z}^2$. The set $\mathcal{E}^+(\Lambda)$ is defined by*

$$\mathcal{E}^+(\Lambda)\doteqdot\{e\in\Lambda\,:\,e\cap\Lambda\neq\varnothing\}\,.$$

We introduce two kinds of boundary condition in Λ.

Definition.

(D195) *A configuration $n\in\Omega^{\mathrm{RC}}$ satisfies the* **++-boundary condition in Λ** *if*

$$n(e)=(1,1)\quad\forall e\notin\mathcal{E}^+(\Lambda)\,.$$

(D196) *A configuration $n\in\Omega^{\mathrm{RC}}$ satisfies the* **free-boundary condition in Λ** *if*

$$n(e)=(0,0)\quad\forall e\notin\mathcal{E}(\Lambda)\,.$$

Remark. 1) Observe that the sets appearing in these two definitions are different.
2) The corresponding boundary conditions for the usual Random–Cluster measure are usually called *wired* and *free* boundary conditions.
3) It is of course possible to define various other kind of boundary conditions by imposing the required configuration outside $\mathcal{E}(\Lambda)$ or $\mathcal{E}^+(\Lambda)$.

[4] The "2" is here to remind us that there are two types of bonds; generalization of these constructions are mentioned at the end of this chapter.

To lighten somewhat the notations, we write from now on

$$\sum_{+,\Lambda} \doteqdot \sum_{\substack{n:\\ n \text{ satisf. } ++\text{-b.c. in } \Lambda}} , \tag{10.4}$$

$$\sum_{f,\Lambda} \doteqdot \sum_{\substack{n:\\ n \text{ satisf. free-b.c. in } \Lambda}} . \tag{10.5}$$

Definition.

(D197) *Let q_σ, q_τ be two strictly positive real numbers. The (q_σ, q_τ)*-**Random–Cluster measure with ++-b.c. in** Λ *is the probability measure on $(\Omega^{\mathrm{RC}}, \mathcal{F}^{\mathrm{RC}})$ defined by*

$$\mu_\Lambda^{\mathrm{RC},++}(n|q_\sigma, q_\tau) \doteqdot \begin{cases} (\Xi^{\mathrm{RC},++}(\Lambda|q_\sigma, q_\tau))^{-1} \lambda_{\mathcal{E}^+(\Lambda)}(n) {q_\sigma}^{N_\sigma(n|\Lambda)} {q_\tau}^{N_\tau(n|\Lambda)} & \text{if } n \text{ satisfies } ++\text{-b.c. in } \Lambda, \\ 0 & \text{otherwise,} \end{cases}$$

where the normalization constant $\Xi^{\mathrm{RC},++}(\Lambda|q_\sigma, q_\tau)$ is the **partition function with ++-b.c. in** Λ.

(D198) *Let q_σ, q_τ be two strictly positive real numbers. The (q_σ, q_τ)*-**Random–Cluster measure with free-b.c. in** Λ *is the probability measure on $(\Omega^{\mathrm{RC}}, \mathcal{F}^{\mathrm{RC}})$ defined by*

$$\mu_\Lambda^{\mathrm{RC}}(n|q_\sigma, q_\tau) \doteqdot \begin{cases} (\Xi^{\mathrm{RC}}(\Lambda|q_\sigma, q_\tau))^{-1} \lambda_{\mathcal{E}(\Lambda)}(n) {q_\sigma}^{N_\sigma(n|\Lambda)} {q_\tau}^{N_\tau(n|\Lambda)} & \text{if } n \text{ satisfies free-b.c. in } \Lambda, \\ 0 & \text{otherwise,} \end{cases}$$

where the normalization constant $\Xi^{\mathrm{RC}}(\Lambda|q_\sigma, q_\tau)$ is the **partition function with free-b.c. in** Λ.

Before proceeding to the study of the relationship between this model and the Ashkin–Teller model, we make some remarks on its relation to the usual Random–Cluster model. We first recall the basic definitions for this model[5].

Definition.

(D199) *Let $\mathcal{B} \subset \mathcal{E}$ and $\underline{p} \doteqdot (p(e))_{e\in\mathcal{E}}$ be a family of real numbers such that $0 \leqslant p(e) \leqslant 1$. The (1-bond)* **percolation measure** *in $\mathcal{B}$ with probabilities $(p(e))_{e\in\mathcal{B}}$ is the probability measure on $\{0,1\}^{\mathcal{B}}$ defined by*

$$\zeta(n|\underline{p}) \doteqdot \prod_{\substack{e\in\mathcal{B}\\ n(e)=1}} p(e) \prod_{\substack{e\in\mathcal{B}\\ n(e)=0}} (1-p(e)) .$$

We use the following convenient convention: If $n \in \{0,1\}^{\mathcal{E}}$, then $\zeta_{\mathcal{B}}(n) \doteqdot \lambda_{\mathcal{B}}(n_{\mathcal{B}})$, where $n_{\mathcal{B}}$ is the restriction of the configuration n to the set $\mathcal{B}$.

[5] There is a vast literature on this model and its applications, to which we refer for its discussion; see for example [FK, Fo1, Fo2, Gr1, Gr2, ACCN, H].

(D200) *Let $n \in \{0,1\}^{\mathcal{E}}$. We say that two sites $i, j \in \mathbb{Z}^2$ are* **connected in** n *if there exists a path containing these two sites such that for each edge e of the path, $n(e) = 1$. Maximal connected components are called* **clusters**. *Let $\Lambda \subset \mathbb{Z}^2$; the number of clusters intersecting Λ in n is denoted by $N(n|\Lambda)$.*

(D201) *Let $q > 0$ and let Λ be a finite subset of $\mathbb{Z}^2$. The* **Random–Cluster measure in Λ with wired-b.c.** *is the probability measure on $\{0,1\}^{\mathcal{E}}$ defined by*

$$\rho^{\mathrm{w}}_{\Lambda}(n|\underline{p}, q) \doteqdot \begin{cases} (\Xi^{\mathrm{RC,w}}(\Lambda))^{-1}\zeta_{\mathcal{E}^+(\Lambda)}(n|\underline{p})q^{N(n|\Lambda)} & \text{if } n(e) = 1\,,\ \forall e \notin \mathcal{E}^+(\Lambda)\,, \\ 0 & \text{otherwise,} \end{cases}$$

where the normalization constant $\Xi^{\mathrm{RC,w}}(\Lambda)$ is the **partition function with wired-b.c. in Λ**.

(D202) *Let $q > 0$ and let Λ be a finite subset of $\mathbb{Z}^2$. The* **Random–Cluster measure in Λ with free-b.c.** *is defined by*

$$\rho_{\Lambda}(n|\underline{p}, q) \doteqdot \begin{cases} (\Xi^{\mathrm{RC,f}}(\Lambda))^{-1}\zeta_{\mathcal{E}(\Lambda)}(n|\underline{p})q^{N(n|\Lambda)} & \text{if } n(e) = 0\,,\ \forall e \notin \mathcal{E}(\Lambda)\,, \\ 0 & \text{otherwise,} \end{cases}$$

where the normalization constant $\Xi^{\mathrm{RC,f}}(\Lambda)$ is the **partition function with free-b.c. in Λ**.

Lemma 10.1.1. *Let f be a $\mathcal{F}^{\mathrm{RC},\sigma}$-measurable function. Then*

1. $\lambda_{\mathcal{B}}(f) = \zeta_{\mathcal{B}}(f|p(e) = a_\sigma(e) + a_{\sigma\tau}(e))$.
2. $\mu^{\mathrm{RC},++}_{\Lambda}(f|q_\sigma, 1) = \rho^{\mathrm{w}}_{\Lambda}(f|p(e) = a_\sigma(e) + a_{\sigma\tau}(e), q_\sigma)$.
3. $\mu^{\mathrm{RC}}_{\Lambda}(f|q_\sigma, 1) = \rho_{\Lambda}(f|p(e) = a_\sigma(e) + a_{\sigma\tau}(e), q_\sigma)$.

Let f be a $\mathcal{F}^{\mathrm{RC},\tau}$-measurable function. Then

1. $\lambda_{\mathcal{B}}(f) = \zeta_{\mathcal{B}}(f|p(e) = a_\tau(e) + a_{\sigma\tau}(e))$.
2. $\mu^{\mathrm{RC},++}_{\Lambda}(f|1, q_\tau) = \rho^{\mathrm{w}}_{\Lambda}(f|p(e) = a_\tau(e) + a_{\sigma\tau}(e), q_\tau)$.
3. $\mu^{\mathrm{RC}}_{\Lambda}(f|1, q_\tau) = \rho_{\Lambda}(f|p(e) = a_\tau(e) + a_{\sigma\tau}(e), q_\tau)$.

Proof. The proof is straightforward. We have (omitting the dependence on the edges)

$$\begin{aligned}
\lambda_{\mathcal{B}}(f) &= \sum_{n_\sigma} f(n_\sigma) \sum_{n_\tau} \prod_{e\in\mathcal{B}} a_0^{\overline{n}_\sigma(e)\overline{n}_\tau(e)} {a_\sigma}^{n_\sigma(e)\overline{n}_\tau(e)} {a_\tau}^{\overline{n}_\sigma(e)n_\tau(e)} {a_{\sigma\tau}}^{n_\sigma(e)n_\tau(e)} \\
&= \sum_{n_\sigma} f(n_\sigma) \prod_{e\in\mathcal{B}} \sum_{n_\tau(e)=\pm1} a_0^{\overline{n}_\sigma(e)\overline{n}_\tau(e)} {a_\sigma}^{n_\sigma(e)\overline{n}_\tau(e)} {a_\tau}^{\overline{n}_\sigma(e)n_\tau(e)} {a_{\sigma\tau}}^{n_\sigma(e)n_\tau(e)} \\
&= \sum_{n_\sigma} f(n_\sigma) \prod_{e\in\mathcal{B}} \left(a_0^{\overline{n}_\sigma(e)} {a_\sigma}^{n_\sigma(e)} + {a_\tau}^{\overline{n}_\sigma(e)} {a_{\sigma\tau}}^{n_\sigma(e)} \right) \\
&= \sum_{n_\sigma} f(n_\sigma) \prod_{\substack{e\in\mathcal{B} \\ n_\sigma(e)=1}} (a_\sigma + a_{\sigma\tau}) \prod_{\substack{e\in\mathcal{B} \\ n_\sigma(e)=0}} (a_0 + a_\tau)\,.
\end{aligned} \tag{10.6}$$

From this, the other statements follow easily. □

10.2 Relation to the Ashkin–Teller model

It is well-known that the usual Random–Cluster measure is related to the Ising model when $q = 2$ (and to the q-states Potts model for larger integer values of q). We show now that the same is true for the (q_σ, q_τ)-Random–Cluster measure which has been introduced in the previous section and the Ashkin–Teller model.

Proposition 10.2.1. *Let $(\underline{J}(e))_{e\in\mathcal{E}}$ be a set of coupling constants of the Ashkin–Teller model. Set*

$$
\begin{aligned}
a_0(e) &= \mathrm{e}^{-2(J_\sigma(e)+J_\tau(e))}\,,\\
a_\sigma(e) &= \mathrm{e}^{-2J_\tau(e)}\,(\,\mathrm{e}^{-2J_{\sigma\tau}(e)} - \mathrm{e}^{-2J_\sigma(e)}\,)\,,\\
a_\tau(e) &= \mathrm{e}^{-2J_\sigma(e)}\,(\,\mathrm{e}^{-2J_{\sigma\tau}(e)} - \mathrm{e}^{-2J_\tau(e)}\,)\,,\\
a_{\sigma\tau}(e) &= 1 - \mathrm{e}^{-2(J_\sigma(e)+J_{\sigma\tau}(e))} - \mathrm{e}^{-2(J_\tau(e)+J_{\sigma\tau}(e))} + \mathrm{e}^{-2(J_\sigma(e)+J_\tau(e))}\,.
\end{aligned}
$$

Then $a_0(e), a_\sigma(e), a_\tau(e), a_{\sigma\tau}(e)$ define a probability measure on $\{0,1\}\times\{0,1\}$ if and only if

$$
\begin{gathered}
J_\sigma \geqslant J_{\sigma\tau}\,,\ J_\tau \geqslant J_{\sigma\tau}\,,\\
J_\sigma \geqslant 0\,,\ J_\tau \geqslant 0\,,\ \tanh J_{\sigma\tau} \geqslant -\tanh J_\sigma \tanh J_\tau\,.
\end{gathered}
$$

Moreover, in such a case, if Λ is a finite, simply connected subset of $\mathbb{Z}^2$, then

1.

$$
\begin{aligned}
\Xi^{++}(\Lambda) &= C_1\,\Xi^{\mathrm{RC},++}(\Lambda|2,2)\,,\\
\Xi(\Lambda) &= C_2\,\Xi^{\mathrm{RC}}(\Lambda|2,2)\,.
\end{aligned}
$$

2.

$$
\begin{aligned}
\langle\sigma_A\sigma_B\rangle_\Lambda^{++} &= \mu_\Lambda^{\mathrm{RC},++}(\kappa_A^\sigma\kappa_B^\tau|2,2)\,,\\
\langle\sigma_A\sigma_B\rangle_\Lambda &= \mu_\Lambda^{\mathrm{RC}}(\kappa_A^\sigma\kappa_B^\tau|2,2)\,,
\end{aligned}
$$

where $C_1 \doteqdot \prod_{e\in\mathcal{E}^+(\Lambda)} \mathrm{e}^{J_\sigma+J_\tau+J_{\sigma\tau}}$, $C_2 \doteqdot \prod_{e\in\mathcal{E}(\Lambda)} \mathrm{e}^{J_\sigma+J_\tau+J_{\sigma\tau}}$ and

$$
\kappa_A^\sigma(n) \doteqdot \begin{cases} 1\,, & \text{if no finite } \sigma\text{-cluster of } n \text{ contains an odd number of sites of } A,\\ 0\,, & \text{otherwise.}\end{cases}
$$

$$
\kappa_B^\tau(n) \doteqdot \begin{cases} 1\,, & \text{if no finite } \tau\text{-cluster of } n \text{ contains an odd number of sites of } B,\\ 0\,, & \text{otherwise.}\end{cases}
$$

Proof. We prove the first statement. By definition $a_0(e) + a_\sigma(e) + a_\tau(e) + a_{\sigma\tau}(e) = 1$, so it only remains to check the positivity of these four numbers.
Clearly, a_0 is not negative. Moreover,

$$a_\sigma \geqslant 0 \Leftrightarrow J_\sigma \geqslant J_{\sigma\tau}\,, \tag{10.7}$$

$$a_\tau \geqslant 0 \Leftrightarrow J_\tau \geqslant J_{\sigma\tau}\,. \tag{10.8}$$

Finally,

$$a_{\sigma\tau} \geqslant 0 \Leftrightarrow \mathrm{e}^{-2J_{\sigma\tau}} \leqslant \frac{1+\mathrm{e}^{-2(J_\sigma+J_\tau)}}{\mathrm{e}^{-2J_\sigma}+\mathrm{e}^{-2J_\tau}} \Leftrightarrow \frac{1-\mathrm{e}^{-2J_{\sigma\tau}}}{1+\mathrm{e}^{-2J_{\sigma\tau}}} \geqslant -\frac{1-\mathrm{e}^{-2J_\sigma}}{1+\mathrm{e}^{-2J_\sigma}}\frac{1-\mathrm{e}^{-2J_\tau}}{1+\mathrm{e}^{-2J_\tau}} \tag{10.9}$$

which is just $\tanh J_{\sigma\tau} \geqslant -\tanh J_\sigma \tanh J_\tau$. Now it is enough to observe that

$$J_\sigma \geqslant J_{\sigma\tau}\,,\; J_\tau \geqslant J_{\sigma\tau}\,,\; \tanh J_{\sigma\tau} \geqslant -\tanh J_\sigma \tanh J_\tau \implies J_\sigma \geqslant 0\,,\; J_\tau \geqslant 0\,. \tag{10.10}$$

Let us prove the second statement. This is proved in a similar way as what was done for the high temperature representation; the basic idea is to write down the Boltzmann weight in the following form:

$$\begin{aligned}
&\exp\{J_\sigma\sigma(t)\sigma(t') + J_\tau\tau(t)\tau(t') + J_{\sigma\tau}\sigma(t)\sigma(t')\tau(t)\tau(t')\} \\
&\qquad = K\exp\{(J_\sigma + J_{\sigma\tau})(\sigma(t)\sigma(t')-1) + (J_\tau + J_{\sigma\tau})(\tau(t)\tau(t')-1) + \\
&\qquad\qquad + J_{\sigma\tau}(\sigma(t)\sigma(t')-1)(\tau(t)\tau(t')-1)\} \\
&\qquad\qquad = K(a_0 + a_\sigma\delta_{\sigma(t)\sigma(t')} + a_\tau\delta_{\tau(t)\tau(t')} + a_{\sigma\tau}\delta_{\sigma(t)\sigma(t')}\delta_{\tau(t)\tau(t')})\,,
\end{aligned} \tag{10.11}$$

where $K = \exp\{J_\sigma + J_\tau + J_{\sigma\tau}\}$. Therefore, expanding the product over edges and indexing the terms with configurations of edges, the partition function in Λ with free-b.c. of the Ashkin–Teller model can be written as

$$\begin{aligned}
\Xi(\Lambda) &= C_2 \sum_{f,\Lambda} \lambda_{\mathcal{E}(\Lambda)}(n) \sum_{\substack{\omega:\\ \text{free b.c. in } \Lambda}} \prod_{\substack{e=\langle t,t'\rangle\in\mathcal{E}(\Lambda):\\ n_\sigma(e)=1}} \delta_{\sigma(t)(\omega)\sigma(t')(\omega)} \prod_{\substack{e=\langle t,t'\rangle\in\mathcal{E}(\Lambda):\\ n_\tau(e)=1}} \delta_{\tau(t)(\omega)\tau(t')(\omega)} \\
&= C_2 \sum_{f,\Lambda} \lambda_{\mathcal{E}(\Lambda)}(n)\, 2^{N_\sigma(n|\Lambda)} 2^{N_\tau(n|\Lambda)} \\
&= C_2\, \Xi^{\mathrm{RC}}(\Lambda|2,2)\,.
\end{aligned} \tag{10.12}$$

The ++-boundary condition is treated in the same way.
We finally prove the last statements concerning the correlation functions. To prove them, we make the same expansion as before; we then obtain,

$$\begin{aligned}
\langle\sigma_A\tau_B\rangle_\Lambda = (\Xi^{\mathrm{RC}}(\Lambda|2,2))^{-1} \sum_{f,\Lambda} \lambda_{\mathcal{E}(\Lambda)}(n) \sum_{\substack{\omega:\\ \text{free b.c. in } \Lambda}} \sigma_A(\omega)\tau_B(\omega) \prod_{\substack{e=\langle t,t'\rangle\in\mathcal{E}(\Lambda):\\ n_\sigma(e)=1}} \delta_{\omega_\sigma(t)\omega_\sigma(t')} \times \\
\times \prod_{\substack{e=\langle t,t'\rangle\in\mathcal{E}(\Lambda):\\ n_\tau(e)=1}} \delta_{\omega_\tau(t)\omega_\tau(t')}\,.
\end{aligned} \tag{10.13}$$

The conclusion follows from the observation that the only configurations n which contribute are those which have the property that $\sigma_A(\omega) = \tau_B(\omega) = 1$, for every configuration ω. But this is only possible if the intersection of A, resp. B, and any σ-cluster, resp. τ-cluster, contains an even number of sites.
The case of the ++-b.c. is proved similarly, the only new fact to take into account being that all sites belonging to the infinite cluster have the fixed value $(1,1)$. □

Remark. 1) Observe that the condition on the coupling constants are the same as in Proposition 9.3.1.
2) As already noted after Proposition 9.3.1, the symmetries of the Ashkin–Teller model allow us to interchange the roles of the coupling constants above. As an important example, suppose that $J_\sigma = J_\tau \leqslant J_{\sigma\tau}$; then the change of variables $(\omega_\sigma(t), \omega_\tau(t)) \mapsto (\omega_\vartheta(t), \omega_\tau(t))$ in the Ashkin–Teller model results in $J_\vartheta \geqslant J_\tau = J_{\vartheta\tau}$ to which we can apply Proposition 10.2.1. In such a case, the (q_σ, q_τ)-Random–Cluster measure has the nice property that $a_\tau = 0$ and therefore that the ϑ-bonds define a (random) on which the τ-bonds are constrained to "live". This situation is considered again in Section 10.4.2.

10.3 Duality

In this section, we show that the Random–Cluster model also satisfies a duality relation (which is stronger in some sense than the corresponding duality in the spin system); moreover, this duality, the duality of the Ashkin–Teller model and the transformation from the Ashkin–Teller model to the corresponding Random–Cluster representation commute. We first need to recall some geometrical results.

In the case of spins systems, the dual of a set Λ is constructed by considering all the plaquettes dual to sites of Λ. This is a natural way to proceed, since the variables (the spins) are located on the sites of the lattice. However, in the Random–Cluster model, it is the edges which support the variables. Therefore, it is reasonable to introduce another notion of dual of a set Λ starting not from the sites of Λ but from its edges.

Definition.

(D203) *Let Λ be a bounded, simply connected subset of $\mathbb{Z}^2$. The* **edge-dual of** Λ *is the set $\Lambda(\widehat{B})$, where $\widehat{B} \subset \mathcal{E}^*$ is defined by*

$$\widehat{B} \doteqdot \{e^* \in \mathcal{E}^* \, : \, e \in \mathcal{E}(\Lambda)\} .$$

The duality of the Ashkin–Teller model relates the partition functions with free and ++-b.c.. However, it is not possible to associate to a given configuration ω a dual configuration $\widehat{\omega}$. In particular, given a set $\mathcal{J} \subset \Omega$ of configurations satisfying the free b.c. in Λ, it is not possible to associate a set $\widehat{\mathcal{J}} \subset \Omega$ of dual configurations satisfying the ++-b.c. in Λ^* so that the probability of $\mathcal{J}$ computed with the Gibbs measure with free b.c. coincides with the probability of $\widehat{\mathcal{J}}$ computed with the Gibbs measure with ++-b.c.. In the Random–Cluster model this much stronger notion of duality holds and is one of the properties of this model which make it very useful.

Definition.

(D204) *Let $n \in \{0,1\}^{\mathcal{E}}$ be some configuration of bonds. The* **dual of the configuration** *n is the configuration $\widehat{n} \in \{0,1\}^{\mathcal{E}^*}$ given by*

$$\widehat{n}(e^*) \doteqdot 1 - n(e) \, , \; \forall e^* \in \mathcal{E}^* .$$

The proof of this duality relies on the two following graph-theoretical relations. Let n be some configuration of bonds. We write $\mathcal{G}_\Lambda(n) \doteqdot (\Lambda, \mathcal{B}_\Lambda(n))$ the graph with Λ as set of vertices and $\mathcal{B}_\Lambda(n) \doteqdot \{e \in \mathcal{E}(\Lambda) : n(e) = 1\}$ as set of edges. Let us write $\mathcal{N}_\Lambda(n)$ and $\mathcal{L}_\Lambda(n)$ the number of maximal connected components and the cyclomatic number of the graph $\mathcal{G}_\Lambda(n)$ [6]. Then

$$\mathcal{N}_\Lambda(n) = |\Lambda| - |\mathcal{B}_\Lambda(n)| + \mathcal{L}_\Lambda(n)\,, \tag{10.14}$$

$$\mathcal{N}_\Lambda(n) = \mathcal{L}_\Lambda(\widehat{n}) + 1\,. \tag{10.15}$$

Relation (10.14) is just the well-known Euler formula for the graph $\mathcal{G}_\Lambda(n)$ and can be easily proved (see, for example, Theorem 1 in [Be]). Relation (10.15) becomes clear once we use the fact that the cyclomatic number of a planar graph also corresponds to the number of bounded connected components of $\mathbb{R}^2$ delimited by the edges of the graph (which are called *finite faces* in [Be]; see, for example, Theorem 2 therein). Then (10.15) amounts to saying that to each finite cluster of n corresponds one and only one such finite component of $\widehat{n}$, which is straightforward to prove.

Below, we will have to use these relations in the case of infinite graphs $(\mathbb{Z}^2, \mathcal{B}_\Lambda(n))$, where n is a configuration satisfying the ++-boundary condition. In such a case, it is still possible to make sense of the above formulae by applying them to the restriction of this graph to the graph (V, B), where $V \doteqdot \{t \in \mathbb{Z}^2 : d_1(t, \Lambda) \leqslant 1\}$ and $B = \mathcal{E}(V) \cap \mathcal{B}_\Lambda(n)$. This will give us the relation we require, up to some constant independent of n.

Remark. We emphasize the fact that we supposed that Λ was simply connected; if it is not the case, then the last of two relations above may not be valid. Let, for example, Λ be some finite non-simply connected subset of $\mathbb{Z}^2$ and consider the Ashkin–Teller model in Λ with ++-b.c.. Then a result analogous to Proposition 10.2.1 holds, but, if we want to keep the same definition for the number of clusters, then it is necessary to add some edges to $\mathcal{E}$; more precisely, we have to connect the different maximal connected components by additional edges and define the corresponding ++-b.c. as before but imposing moreover that these new edges carry open bonds. The important point here is that this makes the corresponding graph *non-planar* and therefore relation (10.15) does not hold anymore. This makes simple connectedness an essential property, similarly to what we have in the case of duality in the spins systems. Notice however that it is possible to consider more general situations using an appropriate formalism (see for example [LMR]).

[6] An **elementary cycle** of an oriented graph (V, E) (i.e. a graph whose edges have an orientation) is a sequence of distinct edges $(e_1, \ldots, e_n)$ such that every e_k is connected to e_{k-1} by one of its extremities and to e_{k+1} by the other one ($e_0 \doteqdot e_n$, $e_{n+1} := e_1$) and no vertex of the graph belongs to more than two of the edges of the family. To each cycle one can associate a vector $\underline{c}$ in $\mathbb{R}^{|E|}$ by

$$c_e \doteqdot \begin{cases} 0 & \text{if the edge } e \text{ does not belong to the cycle,} \\ 1 & \text{if the edge } e \text{ belongs to the cycle and is positively oriented,} \\ -1 & \text{if the edge } e \text{ belongs to the cycle and is not positively oriented.} \end{cases}$$

A family of elementary cycles is **independent** if the corresponding vectors are linearly independent. The **cyclomatic number** of the graph is the maximal number of independent elementary cycles of the graph; it is independent of the orientation.

10.3.1 Duality in the Random–Cluster model

The aim of this subsection is to establish a duality relation for the Random–Cluster model. We first introduce the notion of dual configuration[7].

Definition.

(D205) *Let $n = (n_\sigma, n_\tau) \in \Omega^{\mathrm{RC}}$ be some configuration of σ and τ bonds. The configuration $\widehat{n} \in \Omega^{\mathrm{RC},*}$* **dual** *to n is defined by $\widehat{n} \doteqdot (\widehat{n}_\sigma, \widehat{n}_\tau)$.*

Proposition 10.3.1. *Let Λ be a bounded, simply connected subset of $\mathbb{Z}^2$. Let $\mu_\Lambda^{\mathrm{RC},++}(\cdot|2,2)$ be the (q_σ, q_τ)-Random–Cluster measure with ++-b.c. in Λ corresponding to the set of parameters $a_0(e)$, $a_\sigma(e)$, $a_\tau(e)$, $a_{\sigma\tau}(e)$, q_σ, q_τ, and $\widehat{\mu}_\Lambda^{\mathrm{RC}}(\cdot|2,2)$ be the (q_σ, q_τ)-Random–Cluster measure with free b.c. in Λ^* corresponding to the set of parameters $\widehat{a}_0(e)$, $\widehat{a}_\sigma(e)$, $\widehat{a}_\tau(e)$ and $\widehat{a}_{\sigma\tau}(e)$ with*

$$\begin{aligned}
\widehat{a}_0(e^*) &= C' a_{\sigma\tau}(e)\,, \\
\widehat{a}_\sigma(e^*) &= C' q_\sigma a_\tau(e)\,, \\
\widehat{a}_\tau(e^*) &= C' q_\tau a_\sigma(e)\,, \\
\widehat{a}_{\sigma\tau}(e^*) &= C' q_\sigma q_\tau a_0(e)\,.
\end{aligned}$$

where $C' \doteqdot a_{\sigma\tau} + q_\tau a_\sigma + q_\sigma a_\tau + q_\sigma q_\tau a_0$. Then the following duality relation holds: For any function $f : \Omega^{\mathrm{RC}} \to \mathbb{R}$,

$$\mu_\Lambda^{\mathrm{RC},++}(f|2,2) = \widehat{\mu}_\Lambda^{\mathrm{RC}}(\widehat{f}|2,2)\,,$$

where the function $\widehat{f} : \Omega^{\mathrm{RC},*} \to \mathbb{R}$ is defined by $\widehat{f}(\widehat{n}) \doteqdot f(n)$.

Proof. Observe first that the configuration $\widehat{n}$ dual to some configuration n satisfying the ++-b.c. in Λ necessarily satisfies the free b.c. in Λ^*. We check that $\widehat{\mathcal{E}^+}(\Lambda) = \mathcal{E}(\Lambda^*)$. Indeed, let us examine what happens to a single site of Λ during the process of going the Random–Cluster representation and then to its dual (see also Fig. 10.1).

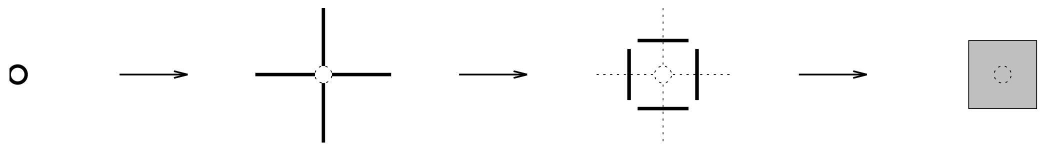

FIGURE 10.1. Lattice transformations (only one site shown)

We begin with the Ashkin–Teller model in Λ with ++-b.c. and consider a site $t \in \Lambda$. The bonds of the Random–Cluster representation of this model whose value is not fixed are located on the edges of $\mathcal{E}^+(\Lambda)$; in particular among them are the four edges with an endpoint at t. In the dual of this Random–Cluster model, the bonds whose value are not fixed are located on all edges dual to some edge of $\mathcal{E}^+(\Lambda)$ (i.e. on $\widehat{\mathcal{E}^+}(\Lambda)$); in particular, the four edges dual to the edges incident on t belong to this set. But this four bonds form the boundary of the plaquette $p^*(t)$ dual to t. Hence to each $t \in \Lambda$ this process associates a plaquette $p^*(t)$; the set obtained by taking the union of all such plaquettes is what we

[7]This definition is slightly different from that of [PV2].

defined as Λ^*. Consequently, these four edges belong to $\mathcal{E}(\Lambda^*)$. Doing this for all sites $t \in \Lambda$ completes the argument.
Using (10.14), (10.15) and the preceding observations, we can write

$$\begin{aligned}
&\sum_{+,\Lambda} f(n)\lambda_{\mathcal{E}^+(\Lambda)}(n){q_\sigma}^{N_\sigma(n|\Lambda)}{q_\tau}^{N_\tau(n|\Lambda)} = \\
&\quad= \sum_{\substack{n:\\ ++\text{-b.c. in }\Lambda}} f(n)\Big(\prod_{\substack{e\in\mathcal{E}^+(\Lambda):\\ n(e)=(0,0)}} a_0(e) \prod_{\substack{e\in\mathcal{E}^+(\Lambda):\\ n(e)=(1,0)}} a_\sigma(e) \prod_{\substack{e\in\mathcal{E}^+(\Lambda):\\ n(e)=(0,1)}} a_\tau(e) \prod_{\substack{e\in\mathcal{E}^+(\Lambda):\\ n(e)=(1,1)}} a_{\sigma\tau}(e)\Big)\times \\
&\qquad\qquad\times {q_\sigma}^{N_\sigma(n|\Lambda)}{q_\tau}^{N_\tau(n|\Lambda)} \\
&\quad= C \sum_{\substack{\widehat{n}:\\ \text{free b.c. in }\Lambda^*}} \widehat{f}(\widehat{n})\Big(\prod_{\substack{e^*\in\mathcal{E}(\Lambda^*):\\ \widehat{n}(e^*)=(1,1)}} a_0(e) \prod_{\substack{e^*\in\mathcal{E}(\Lambda^*):\\ \widehat{n}(e^*)=(0,1)}} a_\sigma(e) \prod_{\substack{e^*\in\mathcal{E}(\Lambda^*):\\ \widehat{n}(e^*)=(1,0)}} a_\tau(e) \prod_{\substack{e^*\in\mathcal{E}(\Lambda^*):\\ \widehat{n}(e^*)=(0,0)}} a_{\sigma\tau}(e)\Big)\times \\
&\qquad\qquad\times {q_\sigma}^{N_\sigma(\widehat{n}|\Lambda)+|\widehat{n}_\sigma|}{q_\tau}^{N_\tau(\widehat{n}|\Lambda)+|\widehat{n}_\tau|} \\
&\quad= C \sum_{\substack{\widehat{n}:\\ \text{free b.c. in }\Lambda^*}} \widehat{f}(\widehat{n})\Big(\prod_{\substack{e^*\in\mathcal{E}(\Lambda^*):\\ \widehat{n}(e^*)=(0,0)}} \widehat{a}_0(e) \prod_{\substack{e^*\in\mathcal{E}(\Lambda^*):\\ \widehat{n}(e^*)=(1,0)}} \widehat{a}_\sigma(e) \prod_{\substack{e^*\in\mathcal{E}(\Lambda^*):\\ \widehat{n}(e^*)=(0,1)}} \widehat{a}_\tau(e) \prod_{\substack{e^*\in\mathcal{E}(\Lambda^*):\\ \widehat{n}(e^*)=(1,1)}} \widehat{a}_{\sigma\tau}(e)\Big)\times \\
&\qquad\qquad\times {\widehat{q}_\sigma}^{N_\sigma(\widehat{n}|\Lambda)}{\widehat{q}_\tau}^{N_\tau(\widehat{n}|\Lambda)},
\end{aligned} \tag{10.16}$$

where C is some constant independent of f and $|\widehat{n}|$ is the number of open bonds in $\widehat{n}$. The conclusion follows easily. □

Remark. Observe that there are no other ways to distribute the factors q_σ and q_τ in (10.16).

10.3.2 Commutativity of the dualities

One can wonder if the dualities introduced in the Ashkin–Teller model and in the Random–Cluster model commute. This is known to hold in the case of the usual Random–Cluster model and this property plays an important role in some proofs relying on the Random–Cluster representation. The next proposition answers this question.

Proposition 10.3.2. *Let Λ be a bounded, simply connected subset of $\mathbb{Z}^2$. Let us write $\mathcal{FK}$ for the application from the Ashkin–Teller model to its Random–Cluster representation and $*$ for the dualities. Then the following diagram is commutative:*

$$\begin{array}{ccc}
AT & \xrightarrow{\ *\ } & AT^* \\
{\scriptstyle \mathcal{FK}}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{FK}} \\
RC & \xrightarrow{\ *\ } & RC^*
\end{array}$$

Proof. We start with the Ashkin–Teller model in Λ with ++-b.c.. We have already checked that the model obtained by first going to the Random–Cluster representation and then to its dual, or by first taking the dual of the Ashkin–Teller model and then going to its Random–Cluster representation are defined on the same set of edges, i.e. $\widehat{\mathcal{E}^+}(\Lambda) = \mathcal{E}(\Lambda^*)$.

It is therefore sufficient to check that the parameters of the Random–Cluster models obtained in these two ways are the same.
For the dual of the Random–Cluster representation of the Ashkin–Teller model, the parameters are given by (see Propositions 10.2.1 and 10.3.1)

$$\begin{aligned}\widehat{a}_0 &= \frac{1 - j_{\sigma\tau}(j_\sigma + j_\tau) + j_\sigma j_\tau}{1 + j_{\sigma\tau}(j_\sigma + j_\tau) + j_\sigma j_\tau}\,,\\ \widehat{a}_\sigma &= \frac{2j_\sigma(j_{\sigma\tau} - j_\tau) + j_\sigma j_\tau}{1 + j_{\sigma\tau}(j_\sigma + j_\tau) + j_\sigma j_\tau}\,,\\ \widehat{a}_\tau &= \frac{2j_\tau(j_{\sigma\tau} - j_\sigma)}{1 + j_{\sigma\tau}(j_\sigma + j_\tau) + j_\sigma j_\tau}\,,\\ \widehat{a}_{\sigma\tau} &= 1 - \widehat{a}_0 - \widehat{a}_\sigma - \widehat{a}_\tau\,,\end{aligned} \tag{10.17}$$

where we introduced the notations $j_\sigma = \mathrm{e}^{-2J_\sigma}$, $j_\tau = \mathrm{e}^{-2J_\tau}$ and $j_{\sigma\tau} = \mathrm{e}^{-2J_{\sigma\tau}}$.
For the Random–Cluster representation of the dual of the Ashkin–Teller model, they are given by (see (9.17) and Proposition 10.2.1)

$$\begin{aligned}{a_0}^* &= \frac{l + st}{1 + stl}\,,\\ {a_\sigma}^* &= \frac{(s - l)(1 - t)}{1 + stl}\,,\\ {a_\tau}^* &= \frac{(t - l)(1 - s)}{1 + stl}\,,\\ {a_{\sigma\tau}}^* &= 1 - {a_0}^* - {a_\sigma}^* - {a_\tau}^*\,,\end{aligned} \tag{10.18}$$

where s, t and l have been defined in (9.5).
Using the elementary relations

$$s = \frac{1 - j_\sigma}{1 + j_\sigma}\,,\; t = \frac{1 - j_\tau}{1 + j_\tau}\,,\; l = \frac{1 - j_{\sigma\tau}}{1 + j_{\sigma\tau}}\,, \tag{10.19}$$

it is easy to see that (10.17) and (10.18) define the same set of parameters. □

10.4 Basic properties of the Random–Cluster measure

This section is devoted to the proof of some basic properties of the (q_σ, q_τ)-Random–Cluster measure, which also hold for the usual Random–Cluster measure, and which play an important role in many of the proofs based on such a representation.

10.4.1 FKG inequalities

The validity of FKG inequalities is certainly the most important property of the usual Random–Cluster measure. We prove now that by suitably defining the notions of monotonous events, it is possible to establish such inequalities for the Random–Cluster measure considered in this chapter. The basic idea is to partition the set of edges into two classes depending on the probability measure associated to them.

Definition.

(D206) *An edge e is* **positive** *if* $a_0(e)a_{\sigma\tau}(e) \geqslant a_\sigma(e)a_\tau(e)$*; it is* **negative** *if it not positive.*

(D207) *The set of positive, resp. negative, edges is denoted by* $\mathfrak{B}_>$*, resp.* $\mathfrak{B}_<$*.*

We introduce two partial orders on the (single) bond space at edge e depending on whether e is positive or not.

Definition.

(D208) *Let* $e \in \mathfrak{B}_>$*. Then the single bond space at e is (partially) ordered by*

$$(0,0)\preccurlyeq(0,1)\preccurlyeq(1,1)\,, \quad (0,0)\preccurlyeq(1,0)\preccurlyeq(1,1)\,.$$

(D209) *Let* $e \in \mathfrak{B}_<$*. Then the single bond space at e is (partially) ordered by*

$$(0,1)\preccurlyeq(1,1)\preccurlyeq(1,0)\,, \quad (0,1)\preccurlyeq(0,0)\preccurlyeq(1,0)\,.$$

Remark. For the Random–Cluster representation of the Ashkin–Teller model, the condition $e \in \mathfrak{B}_>$ is equivalent to $J_{\sigma\tau}(e) \geqslant 0$.

The above partial orders induce a natural partial ordering of the set of configurations of bonds.

Definition.

(D210) *Let* $m, n \in \Omega^{\mathrm{RC}}$*. We say that m* **dominates** *n if* $n(e)\preccurlyeq m(e)$*, for all edges e; in such a case, we write* $n\preccurlyeq m$*.*

(D211) *A function* $f : \Omega^{\mathrm{RC}} \to \mathbb{R}$ *is* **increasing** *if* $m\succcurlyeq n \implies f(m) \geqslant f(n)$*. A function f such that $-f$ is increasing is said to be* **decreasing**.

(D212) *An event is* **increasing** *if the corresponding characteristic function is increasing.*

Example: The functions $N_\sigma(n|\Lambda)$ and $N_\tau(n|\Lambda) + \sum_{e\in\mathfrak{B}_<} n_\tau(e)$ are decreasing, while the functions $N_\sigma(n|\Lambda) + \sum_{e\in\mathcal{E}} n_\sigma(e)$ and $N_\tau(n|\Lambda) + \sum_{e\in\mathfrak{B}_>} n_\tau(e)$ are increasing. Let us consider the case of the function $N_\tau(n|\Lambda)+\sum_{e\in\mathfrak{B}_>} n_\tau(e)$; it is sufficient to prove the result for two configuration $n\preccurlyeq m$ which differ only at one edge e. There are two cases: Either $e \in \mathfrak{B}_>$ and therefore the τ-bond at e is open in m and closed in n, or $e \in \mathfrak{B}_<$ and therefore the τ-bond at e is closed in m and open in n. In the first case,

$$\begin{aligned} N_\tau(n|\Lambda) + \sum_{e\in\mathfrak{B}_>} n_\tau(e) &= N_\tau(m|\Lambda) + \sum_{e\in\mathfrak{B}_>} m_\tau(e) + (N_\tau(n|\Lambda) - N_\tau(m|\Lambda)) - 1 \\ &\leqslant N_\tau(m|\Lambda) + \sum_{e\in\mathfrak{B}_>} m_\tau(e)\,, \end{aligned} \tag{10.20}$$

since $N_\tau(n|\Lambda) - N_\tau(m|\Lambda) \leqslant 1$. In the second case,

$$\begin{aligned} N_\tau(n|\Lambda) + \sum_{e\in\mathfrak{B}_>} n_\tau(e) &= N_\tau(m|\Lambda) + \sum_{e\in\mathfrak{B}_>} m_\tau(e) + (N_\tau(n|\Lambda) - N_\tau(m|\Lambda)) \\ &\leqslant N_\tau(m|\Lambda) + \sum_{e\in\mathfrak{B}_>} m_\tau(e)\,, \end{aligned} \tag{10.21}$$

since either the τ-bond links two sites belonging to the same cluster and $N_\tau(n|\Lambda) = N_\tau(m|\Lambda)$, or it links two different clusters and $N_\tau(n|\Lambda) - N_\tau(m|\Lambda) = -1$.

Definition.

(D213) *A measure μ has the* **FKG property** *if $\mu(fg) \geqslant \mu(f)\mu(g)$, for all pairs of increasing functions f and g.*

Lemma 10.4.1. *Let $\mathcal{B} \subset \mathcal{E}$. The 2-bonds percolation measure $\lambda_{\mathcal{B}}$ has the FKG property.*

Proof. It is sufficient to prove that, for all pairs of configurations of bonds m and n, (see [FKG])

$$\lambda_{\mathcal{B}}(n \vee m)\lambda_{\mathcal{B}}(n \wedge m) \geqslant \lambda_{\mathcal{B}}(n)\lambda_{\mathcal{B}}(m)\,, \tag{10.22}$$

where $a \vee b$ denotes the least upper bound of a and b, while $a \wedge b$ denotes their greatest lower bound. Since the 2-bonds percolation measure is a product measure, it is enough to check the above inequality for each edge separately, which is straightforward. There are only two non-trivial inequalities:

$$\lambda_e((1,1))\lambda_e((0,0)) \geqslant \lambda_e((1,0))\lambda_e((0,1))\,, \tag{10.23}$$

for bonds $e \in \mathfrak{B}_>$ and

$$\lambda_e((1,0))\lambda_e((0,1)) \geqslant \lambda_e((1,1))\lambda_e((0,0))\,, \tag{10.24}$$

for bonds $e \in \mathfrak{B}_<$. But these two inequalities reduce to

$$a_{\sigma\tau}(e)a_0(e) \geqslant a_\sigma(e)a_\tau(e)\,, \tag{10.25}$$

and

$$a_{\sigma\tau}(e)a_0(e) \leqslant a_\sigma(e)a_\tau(e)\,, \tag{10.26}$$

respectively, which are satisfied by definition of the sets $\mathfrak{B}_>$ and $\mathfrak{B}_<$. □

Proposition 10.4.1. *Suppose that $q_\sigma \geqslant 1$ and $q_\tau \geqslant 1$. Then the (q_σ, q_τ)-Random–Cluster measure has the FKG property.*

Proof. It is sufficient to prove that, for all pairs of configurations of bonds m and n, (see [FKG])

$$q_\sigma{}^{N_\sigma(n\wedge m|\Lambda)+N_\sigma(n\vee m|\Lambda)} q_\tau{}^{N_\tau(n\wedge m|\Lambda)+N_\tau(n\vee m|\Lambda)} \geqslant q_\sigma{}^{N_\sigma(n|\Lambda)+N_\sigma(m|\Lambda)} q_\tau{}^{N_\tau(n|\Lambda)+N_\tau(m|\Lambda)}\,. \tag{10.27}$$

This can be proved exactly in the same way as for the usual Random–Cluster measure, see [ACCN]). Observe that the unusual order for edges in $\mathfrak{B}_<$ does not play any role since it only interchanges the operators $\vee$ and $\wedge$ and these appear symmetrically in the above equation. □

Let us state an interesting corollary to this proposition.

Corollary 10.4.1. *Let $\Lambda \subset \mathbb{Z}^2$ finite. Let $J_\sigma(e) \geqslant J_\tau(e) \geqslant 0$ and $0 \geqslant \tanh J_{\sigma\tau}(e) \geqslant -\tanh J_\sigma(e) \tanh J_\tau(e)$ be the coupling constant of the Ashkin–Teller model in Λ with $++$ boundary condition. Then the following inequalities hold*

$$\begin{aligned}
\langle \sigma_A \sigma_B \rangle_\Lambda^{++,\underline{J}} &\geqslant \langle \sigma_A \rangle_\Lambda^{++,\underline{J}} \langle \sigma_B \rangle_\Lambda^{++,\underline{J}}\,, \\
\langle \tau_A \tau_B \rangle_\Lambda^{++,\underline{J}} &\geqslant \langle \tau_A \rangle_\Lambda^{++,\underline{J}} \langle \tau_B \rangle_\Lambda^{++,\underline{J}}\,, \\
\langle \sigma_A \tau_B \rangle_\Lambda^{++,\underline{J}} &\leqslant \langle \sigma_A \rangle_\Lambda^{++,\underline{J}} \langle \tau_B \rangle_\Lambda^{++,\underline{J}}\,,
\end{aligned}$$

The same inequalities also hold for free boundary condition.

Proof. This is a straightforward consequence of Propositions 10.2.1 and 10.4.1. Just notice that $\{i \overset{\sigma}{\leftrightarrow} j\}$ is an increasing event and $\{i \overset{\tau}{\leftrightarrow} j\}$ is a decreasing event. □

The inequalities proved in the last corollary extend GKS inequalities for the Ashkin–Teller model out of the ferromagnetic region: These inequalities also hold for negative (but not too much) values of the coupling constant $J_{\sigma\tau}$.

10.4.2 Comparison inequalities

One of the reasons the usual Random–Cluster representation is interesting is that it gives a unified representation of models as different from each others as the percolation, Ising and Potts models as a one–parameter family of models (with parameter q). This is even more so with the (q_σ, q_τ)-Random–Cluster model since it contains all these previous models and several other ones (Ashkin–Teller model, partially symmetric Potts models, cubic models, see the end of the chapter). Since all these models can be represented using a two-parameters family of measures, one can hope that it is possible to compare them (i.e. to compare models corresponding to different q_σ and q_τ)[8]. This happens to be an easy consequence of the FKG property of the (q_σ, q_τ)-Random–Cluster measure.

Let $\mu_\Lambda^{\mathrm{RC},\star}(\cdot|q_\sigma, q_\tau)$ be the (q_σ, q_τ)-Random–Cluster measure with ++- or free boundary condition and parameters $a_0(e)$, $a_\sigma(e)$, $a_\tau(e)$, $a_{\sigma\tau}(e)$, and let $\tilde{\mu}_\Lambda^{\mathrm{RC},\star}(\cdot|\tilde{q}_\sigma, \tilde{q}_\tau)$ be the $(\tilde{q}_\sigma, \tilde{q}_\tau)$-Random–Cluster measure with the same boundary condition on Λ as $\mu_\Lambda^{\mathrm{RC},\star}(\cdot|q_\sigma, q_\tau)$ and with parameters $\tilde{a}_0(e)$, $\tilde{a}_\sigma(e)$, $\tilde{a}_\tau(e)$ and $\tilde{a}_{\sigma\tau}(e)$ respectively.
We introduce the following notations

$$\rho_\sigma \doteqdot \frac{q_\sigma}{\tilde{q}_\sigma},\ \rho_\tau \doteqdot \frac{q_\tau}{\tilde{q}_\tau}\,, \tag{10.28}$$

$$\alpha_0 \doteqdot \frac{a_0}{\tilde{a}_0}\,,\ \alpha_\sigma \doteqdot \frac{a_\sigma}{\tilde{a}_\sigma}\,,\ \alpha_\tau \doteqdot \frac{a_\tau}{\tilde{a}_\tau}\,,\ \alpha_{\sigma\tau} \doteqdot \frac{a_{\sigma\tau}}{\tilde{a}_{\sigma\tau}}\,. \tag{10.29}$$

We can now prove the following result about comparison inequalities between the two preceding measures.

Proposition 10.4.2. *Suppose $\tilde{q}_\sigma, \tilde{q}_\tau \geqslant 1$. Suppose that any one the following set of conditions is satisfied,*

[8] Such comparison inequalities are well-known for the usual Random–Cluster model, see for example [Gr4].

1.

$$\begin{gathered}\rho_\sigma \leqslant 1\,,\ \rho_\tau \leqslant 1\,,\\ \alpha_{\sigma\tau} \geqslant \max(\alpha_\sigma,\alpha_\tau) \geqslant \min(\alpha_\sigma,\alpha_\tau) \geqslant \alpha_0\,,\ \forall e\in\mathfrak{B}_>\,,\\ \rho_\tau\alpha_\sigma \geqslant \max(\alpha_{\sigma\tau},\rho_\tau\alpha_0) \geqslant \min(\alpha_{\sigma\tau},\rho_\tau\alpha_0) \geqslant \alpha_\tau\,,\ \forall e\in\mathfrak{B}_<\,;\end{gathered}$$

2.

$$\begin{gathered}\rho_\sigma \geqslant 1\,,\ \rho_\tau \geqslant 1\,,\\ \alpha_{\sigma\tau} \geqslant \max(\rho_\tau\alpha_\sigma,\rho_\sigma\alpha_\tau) \geqslant \min(\rho_\tau\alpha_\sigma,\rho_\sigma\alpha_\tau) \geqslant \rho_\sigma\rho_\tau\alpha_0\,,\ \forall e\in\mathfrak{B}_>\,,\\ \alpha_\sigma \geqslant \max(\alpha_{\sigma\tau},\rho_\sigma\alpha_0) \geqslant \min(\alpha_{\sigma\tau},\rho_\sigma\alpha_0) \geqslant \rho_\sigma\alpha_\tau\,,\ \forall e\in\mathfrak{B}_<\,;\end{gathered}$$

3.

$$\begin{gathered}\rho_\sigma \geqslant 1\,,\ \rho_\tau \leqslant 1\,,\\ \alpha_{\sigma\tau} \geqslant \max(\alpha_\sigma,\rho_\sigma\alpha_\tau) \geqslant \min(\alpha_\sigma,\rho_\sigma\alpha_\tau) \geqslant \rho_\sigma\alpha_0\,,\ \forall e\in\mathfrak{B}_>\,,\\ \rho_\tau\alpha_\sigma \geqslant \max(\alpha_{\sigma\tau},\rho_\sigma\rho_\tau\alpha_0) \geqslant \min(\alpha_{\sigma\tau},\rho_\sigma\rho_\tau\alpha_0) \geqslant \rho_\sigma\alpha_\tau\,,\ \forall e\in\mathfrak{B}_<\,;\end{gathered}$$

4.

$$\begin{gathered}\rho_\sigma \leqslant 1\,,\ \rho_\tau \geqslant 1\,,\\ \alpha_{\sigma\tau} \geqslant \max(\rho_\tau\alpha_\sigma,\alpha_\tau) \geqslant \min(\rho_\tau\alpha_\sigma,\alpha_\tau) \geqslant \rho_\tau\alpha_0\,,\ \forall e\in\mathfrak{B}_>\,,\\ \alpha_\sigma \geqslant \max(\alpha_{\sigma\tau},\alpha_0) \geqslant \min(\alpha_{\sigma\tau},\alpha_0) \geqslant \alpha_\tau\,,\ \forall e\in\mathfrak{B}_<\,;\end{gathered}$$

then, for any increasing function f,

$$\mu_\Lambda^{\mathrm{RC},\star}(f|q_\sigma,q_\tau) \geqslant \tilde\mu_\Lambda^{\mathrm{RC},\star}(f|\tilde q_\sigma,\tilde q_\tau)\,.$$

Proof. We first consider the first set of conditions. Assume these conditions hold. The idea is to prove that the Radon-Nikodym density $\chi(n)$ of $\mu_\Lambda^{\mathrm{RC},\star}(\cdot|q_\sigma,q_\tau)$ relative to $\tilde\mu_\Lambda^{\mathrm{RC},\star}(\cdot|\tilde q_\sigma,\tilde q_\tau)$ is an increasing function. We can write

$$\begin{aligned}\chi(n) &\doteqdot \frac{\mu_\Lambda^{\mathrm{RC},\star}(n|q_\sigma,q_\tau)}{\tilde\mu_\Lambda^{\mathrm{RC},\star}(n|\tilde q_\sigma,\tilde q_\tau)}\\ &= C\Big\{\prod_{e\in\mathfrak{B}_>}\Big(\frac{a_0}{\tilde a_0}\Big)^{\overline{n}_\sigma(e)\overline{n}_\tau(e)}\Big(\frac{a_\sigma}{\tilde a_\sigma}\Big)^{n_\sigma(e)\overline{n}_\tau(e)}\Big(\frac{a_\tau}{\tilde a_\tau}\Big)^{\overline{n}_\sigma(e)n_\tau(e)}\Big(\frac{a_{\sigma\tau}}{\tilde a_{\sigma\tau}}\Big)^{n_\sigma(e)n_\tau(e)}\Big\}\times\\ &\times\Big\{\prod_{e\in\mathfrak{B}_<}\Big(\frac{q_\tau a_0}{\tilde q_\tau\tilde a_0}\Big)^{\overline{n}_\sigma(e)\overline{n}_\tau(e)}\Big(\frac{q_\tau a_\sigma}{\tilde q_\tau\tilde a_\sigma}\Big)^{n_\sigma(e)\overline{n}_\tau(e)}\Big(\frac{a_\tau}{\tilde a_\tau}\Big)^{\overline{n}_\sigma(e)n_\tau(e)}\Big(\frac{a_{\sigma\tau}}{\tilde a_{\sigma\tau}}\Big)^{n_\sigma(e)n_\tau(e)}\Big\}\times\\ &\times\Big(\frac{q_\sigma}{\tilde q_\sigma}\Big)^{N_\sigma(n|\Lambda)}\Big(\frac{q_\tau}{\tilde q_\tau}\Big)^{N_\tau(n|\Lambda)+\sum_{e\in\mathfrak{B}_<}n_\tau(e)}\,,\end{aligned} \tag{10.30}$$

where C is some positive constant. Since $N_\sigma(n|\Lambda)$ and $N_\tau(n|\Lambda)+\sum_{e\in\mathfrak{B}_<}n_\tau(e)$ are decreasing functions, the function χ is necessarily increasing and therefore the conclusion follows from Proposition 10.4.1.

The other cases are proved in the same way, using also that $N_\sigma(n|\Lambda)+\sum_{e\in\mathcal{E}}n_\sigma(e)$ and $N_\tau(n|\Lambda)+\sum_{e\in\mathfrak{B}_>}n_\tau(e)$ are increasing. □

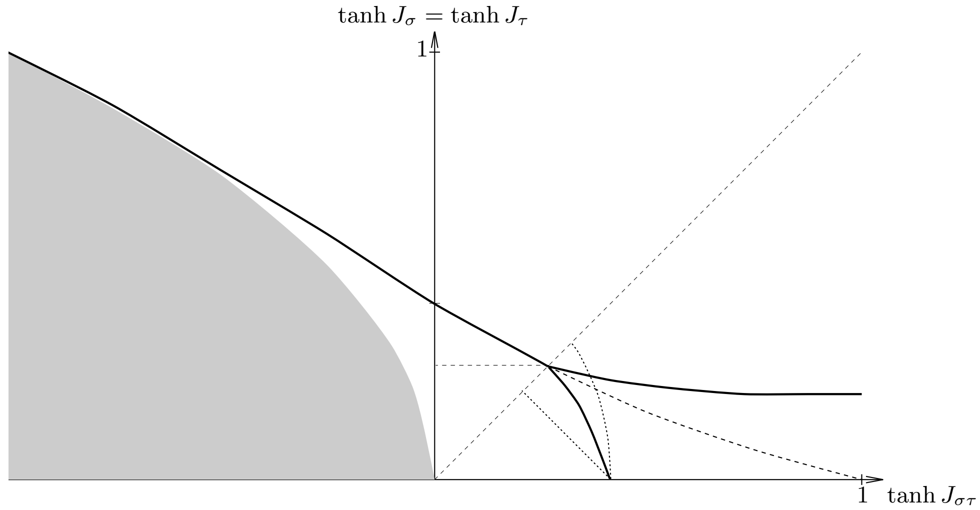

FIGURE 10.2. Schematic representation of the phase diagram in the plane $\tanh J_{\sigma\tau}$ versus $\tanh J_\sigma = \tanh J_\tau$. We have indicated the critical lines (solid lines), and the self–dual line (which coincides with the solid line up to the splitting and then follows the dashed line). The shaded region corresponds to the set of parameters at which the Random–Cluster representation is not available. The estimates of the location of the critical line are also shown (dotted lines).

Proposition 10.4.2 provides tools to compare expectation values for different models. As an interesting example, we use it to compare the usual Random–Cluster model with the (q_σ, q_τ)-Random–Cluster model and as a particular case we obtain inequalities relating expectation values in the Ising and Ashkin–Teller models, which give us some information on the phase diagram of the Ashkin–Teller model.

Corollary 10.4.2. *Suppose that $q_\sigma, q_\tau \geqslant 1$ and f is an increasing, $\mathcal{F}^{\mathrm{RC},\sigma}$-measurable function. Let $\mu_\Lambda^{\mathrm{RC},++}(f|q_\sigma,q_\tau)$ be the (q_σ,q_τ)-Random–Cluster measure with parameters $a_0(e)$, $a_\sigma(e)$, $a_\tau(e)$, $a_{\sigma\tau}(e)$. Then*

$$\rho_\Lambda^{\mathrm{w}}(f|\underline{p}_1, q_\sigma) \leqslant \mu_\Lambda^{\mathrm{RC},++}(f|q_\sigma,q_\tau) \leqslant \rho_\Lambda^{\mathrm{w}}(f|\underline{p}_2, q_\sigma)\,,$$

where, for all edges,

$$\begin{aligned} p_1(e) &\doteqdot \frac{q_\tau a_\sigma(e) + a_{\sigma\tau}(e)}{q_\tau(a_0(e)+a_\sigma(e)) + a_\tau(e) + a_{\sigma\tau}(e)}\,,\\ p_2(e) &\doteqdot a_\sigma(e) + a_{\sigma\tau}(e)\,. \end{aligned}$$

The same inequalities hold for the measures with free b.c. and similar results are also true for $\mathcal{F}^{\mathrm{RC},\tau}$-measurable functions.

Proof. Using Lemma 10.1.1 we can replace the Random–Cluster measure by the $(q_\sigma,1)$-Random–Cluster measure and using Proposition 10.4.2 we can compare it with the (q_σ,q_τ)-Random–Cluster measure. □

We give a consequence of the preceding corollary to the phase diagram of the Ashkin–Teller model. We consider the Ashkin–Teller model in Λ with ++-boundary condition. Suppose $J_{\sigma\tau} \geqslant J_\sigma = J_\tau \equiv J$ and the coupling constants are the same at each bond. This region of the phase diagram is particularly interesting since the critical manifold splits into two dual parts (and in particular is different from the self-dual manifold); the associated order parameters are given by $\langle\sigma(t)\rangle^{++} = \langle\tau(t)\rangle^{++}$ and $\langle\sigma(t)\tau(t)\rangle^{++}$. We give an estimate on the location of the corresponding lines. We first make the change of variables $(\sigma(t),\tau(t)) \mapsto (\vartheta(t),\tau(t))$, with $\vartheta(t) \doteqdot \sigma(t)\tau(t)$. Applying the previous corollary to the Random–Cluster representation of the resulting Ashkin–Teller model and to the characteristic function of the event $\{t \overset{\vartheta}{\leftrightarrow} \Lambda^c\}$, we find

$$\langle\vartheta(t)\rangle_\Lambda^{\text{Ising},+,J_1} \leqslant \langle\vartheta(t)\rangle_\Lambda^{++,\underline{J}} \leqslant \langle\vartheta(t)\rangle_\Lambda^{\text{Ising},+,J_2}\,, \tag{10.31}$$

where $\langle\cdot\rangle_\Lambda^{\text{Ising},+,J_1}$ and $\langle\cdot\rangle_\Lambda^{\text{Ising},+,J_1}$ denote expectation values in the Ising models with coupling constants given by $J_1 \doteqdot J_{\sigma\tau}+\frac{1}{2}\log\cosh 2J$ and $J_2 \doteqdot J+J_{\sigma\tau}$[9]. Since these inequalities hold for any simply connected set Λ, and the limit $\Lambda \nearrow \mathbb{Z}^2$ of the expectation values exist (by monotonicity), they also hold in the thermodynamic limit. From this, denoting by $J_c^{\text{Ising}} \doteqdot \frac{1}{2}\text{argsinh}1$ the critical coupling of the 2D Ising model, we can deduce the following bounds on the location of one of the critical lines,

$$J_{\sigma\tau} \leqslant J_c^{\text{Ising}} - J \implies \langle\sigma(t)\tau(t)\rangle^{++} = 0\,, \tag{10.32}$$
$$J_{\sigma\tau} > J_c^{\text{Ising}} - \tfrac{1}{2}\log\cosh 2J \implies \langle\sigma(t)\tau(t)\rangle^{++} > 0\,. \tag{10.33}$$

Observe that such a behaviour cannot take place in the sector $J_\sigma = J_\tau \geqslant J_{\sigma\tau} \geqslant 0$ since

$$\begin{aligned}\langle\sigma(t)\rangle_\Lambda^{++} &= \mu_\Lambda^{\text{RC},++}(\{t \overset{\sigma}{\leftrightarrow} \Lambda^c\}|2,2) \geqslant \mu_\Lambda^{\text{RC},++}(\{t \overset{\sigma}{\leftrightarrow} \Lambda^c\} \cap \{t \overset{\tau}{\leftrightarrow} \Lambda^c\}|2,2) = \langle\sigma(t)\tau(t)\rangle_\Lambda^{++}\\ &\geqslant \langle\sigma(t)\rangle_\Lambda^{++}\langle\tau(t)\rangle_\Lambda^{++} = (\langle\sigma(t)\rangle_\Lambda^{++})^2\,,\end{aligned} \tag{10.34}$$

which implies $\langle\sigma(t)\tau(t)\rangle^{++} = 0 \Leftrightarrow \langle\sigma(t)\rangle^{++} = \langle\tau(t)\rangle^{++} = 0$.

10.5 Some remarks

We make some comments about possible extensions of the results discussed in this chapter. First, we restricted our attention to the model defined on the 2D square lattice. However, all the above, except the duality, can be shown to hold in much greater generality (we can replace the square lattice by any simple graph using exactly the same techniques), since we made no use of the structure of $\mathbb{Z}^2$. Moreover, more complicated types of boundary conditions can be treated.

Proposition 10.2.1 which shows the relation between the Ashkin–Teller model and the $(2,2)$-Random–Cluster model can be extended in the same way to establish a relationship between the (q_σ, q_τ)-Random–Cluster model with positive, integer parameters q_σ and q_τ and some spins systems. More precisely, we can introduce the following family of Hamiltonians,

$$H_\Lambda(\omega) \doteqdot \sum_{e=\langle t,t'\rangle\cap\Lambda\neq\varnothing} -2(J_\sigma(e) - J_{\sigma\tau}(e))\delta_{\omega_\sigma(t)\omega_\sigma(t')} - 2(J_\tau(e) - J_{\sigma\tau}(e))\delta_{\omega_\tau(t)\omega_\tau(t')} - 4J_{\sigma\tau}(e)\delta_{\omega_\sigma(t)\omega_\sigma(t')}\delta_{\omega_\tau(t)\omega_\tau(t')}\,, \tag{10.35}$$

[9] The upper bound could also have been obtained easily using GKS inequalities for the Ashkin–Teller model; this does not seem to be the case for the lower bound.

with single spin space $\{1,\dots,q_\sigma\}\times\{1,\dots,q_\tau\}$, which obviously generalizes the Ashkin–Teller model. Models of this family are called (q_σ,q_τ)-*cubic models*, see [DR]; in the case $J_\tau = J_{\sigma\tau}$, they are also known as *partially symmetric Potts models* [DLMMR, LMR]. They can be thought of as resulting from two coupled Potts models. Then it can be proved in the same way as in Proposition 10.2.1 that the (q_σ,q_τ)-cubic model admits the (q_σ,q_τ)-Random–Cluster model as Random–Cluster representation.

Equation 10.35 suggests a natural generalization. Indeed, we may consider more than two coupled Potts models. A natural Hamiltonian would be[10]

$$H_\Lambda(\omega) = -\sum_{e=\langle t,t'\rangle\cap\Lambda\neq\varnothing}\{\sum_{k=1}^{N}\sum_{r_1<\dots<r_k} J_k^{(r_1,\dots,r_k)}(e)\prod_{i=1}^{k}(\delta_{\omega_i(t)\omega_i(t')}-1)\}\,, \tag{10.36}$$

with $\omega(t) \doteqdot (\omega_1,\dots,\omega_N)\in\{1,\dots,q_1\}\times\cdots\times\{1,\dots,q_N\}$, $N\in\mathbb{N}$, $1<q_i\in\mathbb{N}$, $i=1,\dots,N$. Then we introduce a N-percolation measure,

$$\lambda_{\mathcal{B}}(n) \doteqdot \prod_{A\subset\{1,\dots,N\}}\ \prod_{\substack{e\in\mathcal{B}\\ n_k(e)=1\,,\forall k\in A\\ n_k(e)=0\,,\forall k\notin A}} a^A\,, \tag{10.37}$$

with $n \doteqdot (n_1,\dots,n_N)$, $n_i\in\{0,1\}^{\mathcal{E}}(\Lambda)$,$i=1,\dots,N$ and

$$a^A \doteqdot \exp\Big(\sum_{k=1}^{N}\sum_{\substack{r_1<\dots<r_k:\\ r_i\notin A,\forall i}}(-1)^k J_k^{(r_1,\dots,r_k)}\Big) - \sum_{\substack{B\subset A\\ B\neq A}} a^B\,, \tag{10.38}$$

for all $A\subset\{1,\dots,N\}$. Under suitable assumptions on the coupling constants, the coefficients a^A become positive and can therefore be interpreted as probabilities. The corresponding $(q_1,\dots,q_N)$-Random–Cluster measure can then be defined as

$$\mu_\Lambda^{\mathrm{RC},\star}(n|q_1,\dots,q_N) \doteqdot \begin{cases}(\Xi^{\mathrm{RC},\star}(\Lambda|q_1,\dots,q_N))^{-1}\lambda_{\mathcal{E}^+(\Lambda)}(n)\prod_{i=1}^N q_i^{N_i(n|\Lambda)} & \text{if } n \text{ satisfies } \star\text{-b.c. in } \Lambda,\\ 0 & \text{otherwise,}\end{cases} \tag{10.39}$$

where $N_i(n|\Lambda)$ is the number of clusters of the configuration of bonds n_i intersecting Λ and $\star$ denotes the boundary condition (free or $+$). As in the case we have considered this $(q_1,\dots,q_N)$-Random–Cluster model can be seen to be related to the spins system with Hamiltonian given by (10.36). Moreover, partitioning the set of edges into a suitable number of classes and introducing a convenient partial order on the single bond space at all edges of each class, it is possible to prove FKG inequalities and comparison inequalities.

[10]The Potts models with many-body interactions introduced in [Gr3] cannot be put into the form (10.36).

Chapter 11

Surface tensions and massgaps

In this chapter, we introduce the three relevant surface tensions and massgaps of the corresponding 2-point functions. We give some basic properties of these quantities and show how they are related by duality. We are mostly interested in the fully ferromagnetic case ($J_\sigma, J_\tau, J_{\sigma\tau} \geqslant 0$) but we also make some comments about the case of negative four-body interaction (more precisely, $0 > \tanh J_{\sigma\tau} \geqslant -\tanh J_\sigma \tanh J_\tau$).

11.1 The surface tensions

In this section, the coupling constants are supposed to be independent of the edges, i.e. we consider a fixed triple $\underline{J} \doteqdot (J_\sigma, J_\tau, J_{\sigma\tau})$ for all edges $e \in \mathcal{E}$. We set

$$\Lambda_{L,M} \doteqdot \{t \in \mathbb{Z}^2 \,:\, -L < t(1) \leqslant L\,,\, -M < t(2) \leqslant M\}\,. \tag{11.1}$$

We refer the reader to Chapter 3 for motivations of the definitions of the present section. It should be clear that the following three quantities are natural generalizations of the surface tension introduced in Chapter 3 for the 2D Ising model.

Let $\boldsymbol{n}$ be a unit vector in $\mathbb{R}^2$. We write $d(\boldsymbol{n})$ the straight line containing $(\frac{1}{2}, \frac{1}{2})$ and with normal $\boldsymbol{n}$. Moreover, let t_l^* and t_r^* be the two points defined in Section 9.1.

Definition.

(D214) *The* σ**-surface tension in the direction** $\boldsymbol{n}$ *is defined as*

$$\tau_\sigma(\boldsymbol{n}; \underline{J}) \doteqdot \lim_{\substack{L\to\infty \\ M\to\infty}} \tau_\sigma(\boldsymbol{n}|\Lambda_{L,M}; \underline{J})\,,$$

with

$$\tau_\sigma(\boldsymbol{n}|\Lambda_{L,M}; \underline{J}) \doteqdot -\frac{1}{\|t_r^* - t_l^*\|_2} \log \frac{\Xi^{d(\boldsymbol{n})+}(\Lambda_{L,M})}{\Xi^{++}(\Lambda_{L,M})}\,.$$

(D215) *The* τ**-surface tension in the direction** $\boldsymbol{n}$ *is defined as*

$$\tau_\tau(\boldsymbol{n}; \underline{J}) \doteqdot \lim_{\substack{L\to\infty \\ M\to\infty}} \tau_\tau(\boldsymbol{n}|\Lambda_{L,M}; \underline{J})\,,$$

with

$$\tau_\tau(\boldsymbol{n}|\Lambda_{L,M}; \underline{J}) \doteqdot -\frac{1}{\|t_r^* - t_l^*\|_2} \log \frac{\Xi^{+d(\boldsymbol{n})}(\Lambda_{L,M})}{\Xi^{++}(\Lambda_{L,M})}\,.$$

(D216) *The* $\sigma\tau$**-surface tension in the direction** $\boldsymbol{n}$ *is defined as*

$$\tau_{\sigma\tau}(\boldsymbol{n};\underline{J}) \doteqdot \lim_{\substack{L\to\infty \\ M\to\infty}} \tau_{\sigma\tau}(\boldsymbol{n}|\Lambda_{L,M};\underline{J})\,,$$

with

$$\tau_{\sigma\tau}(\boldsymbol{n}|\Lambda_{L,M};\underline{J}) \doteqdot -\frac{1}{\|t_r^* - t_l^*\|_2} \log \frac{\Xi^{d(\boldsymbol{n})d(\boldsymbol{n})}(\Lambda_{L,M})}{\Xi^{++}(\Lambda_{L,M})}\,.$$

By symmetry, replacing any $+$-b.c. by $-$-b.c. in the above definitions does not change the value of the corresponding surface tension.

Existence of these quantities follows, as in the Ising model case, from the existence of the massgap of the corresponding 2-point function.

11.2 The massgaps

As in the first Part, it is possible to relate the surface tensions to the massgap of the relevant 2-point functions in the dual model. In the case of the Ashkin–Teller model, there are three surface tensions and therefore three corresponding 2-point functions.

Let $\boldsymbol{n}$ be a unit vector in $\mathbb{R}^2$ such that the straight line $d^*(\boldsymbol{n})$ through $t_0^* \doteqdot (\frac{1}{2},\frac{1}{2})$ in the direction $\boldsymbol{n}$ has a rational slope. Let $t \in \mathbb{Z}^2$ such that $t_0^* + t \in d^*(\boldsymbol{n})$ and $\|t\|_2$ is minimal.

Definition.

(D217) *The massgap of the* σ**-2-point function in the direction** $\boldsymbol{n}$ *is defined by*

$$\alpha_\sigma(\boldsymbol{n};\underline{J}) \doteqdot -\lim_{k\to\infty} \frac{1}{\|kt\|_2} \log \langle \sigma(t_0^*)\sigma(t_0^*+kt)\rangle^{\underline{J}}\,.$$

(D218) *The massgap of the* τ**-2-point function in the direction** $\boldsymbol{n}$ *is defined by*

$$\alpha_\tau(\boldsymbol{n};\underline{J}) \doteqdot -\lim_{k\to\infty} \frac{1}{\|kt\|_2} \log \langle \tau(t_0^*)\tau(t_0^*+kt)\rangle^{\underline{J}}\,.$$

(D219) *The massgap of the* $\sigma\tau$**-2-point function in the direction** $\boldsymbol{n}$ *is defined by*

$$\alpha_{\sigma\tau}(\boldsymbol{n};\underline{J}) \doteqdot -\lim_{k\to\infty} \frac{1}{\|kt\|_2} \log \langle \sigma(t_0^*)\tau(t_0^*)\sigma(t_0^*+kt)\tau(t_0^*+kt)\rangle^{\underline{J}}\,.$$

Lemma 11.2.1. *Suppose that* $J_\sigma, J_\tau, J_{\sigma\tau} \geqslant 0$ *and that* $\boldsymbol{n}$ *is as above. Then the three quantities defined in* (D217), (D218) *and* (D219) *exist and the convergence is uniform in* $\boldsymbol{n} \in S^1$. *Moreover*

$$\langle \sigma(t)\sigma(t')\rangle^{\underline{J}} \leqslant \exp(-\|t'-t\|_2\alpha_\sigma(\boldsymbol{n}_{t'-t};\underline{J}))\,,$$
$$\langle \tau(t)\tau(t')\rangle^{\underline{J}} \leqslant \exp(-\|t'-t\|_2\alpha_\tau(\boldsymbol{n}_{t'-t};\underline{J}))\,,$$
$$\langle \sigma(t)\tau(t)\sigma(t')\tau(t')\rangle^{\underline{J}} \leqslant \exp(-\|t'-t\|_2\alpha_{\sigma\tau}(\boldsymbol{n}_{t'-t};\underline{J}))\,,$$

where $\boldsymbol{n}_{t'-t}$ *is the unit vector in* $\mathbb{R}^2$ *given by* $\boldsymbol{n}_{t'-t} \doteqdot \dfrac{t'-t}{\|t'-t\|_2}$.
If $\boldsymbol{n}_\perp$ *is a unit vector normal to* $\boldsymbol{n}$*, then*

$$\begin{aligned}\tau_\sigma(\boldsymbol{n};\underline{J}) &= \alpha_\sigma(\boldsymbol{n}_\perp;\underline{J}^*)\,,\\ \tau_\tau(\boldsymbol{n};\underline{J}) &= \alpha_\tau(\boldsymbol{n}_\perp;\underline{J}^*)\,,\\ \tau_{\sigma\tau}(\boldsymbol{n};\underline{J}) &= \alpha_{\sigma\tau}(\boldsymbol{n}_\perp;\underline{J}^*)\,.\end{aligned}$$

Proof. The proof is identical to the corresponding one for the Ising model (which uses only GKS inequalities). □

Remark. A similar statement should also hold in the non-ferromagnetic case, $0 > \tanh J_{\sigma\tau} \geqslant -\tanh J_\sigma \tanh J_\tau$, $J_\sigma, J_\tau \geqslant 0$. However, even though the Random–Cluster representation provides inequalities similar to the GKS inequalities, it is not possible to use them to prove convergence of expectation values in the thermodynamic limit. This is due to the fact that they do not imply monotonicity in the volume, since $\langle\sigma_A\sigma_B\rangle_\Lambda \geqslant \langle\sigma_A\rangle_\Lambda\langle\sigma_B\rangle_\Lambda$, $\langle\tau_A\tau_B\rangle_\Lambda \geqslant \langle\tau_A\rangle_\Lambda\langle\tau_B\rangle_\Lambda$, but $\langle\sigma_A\tau_B\rangle_\Lambda \leqslant \langle\sigma_A\rangle_\Lambda\langle\tau_B\rangle_\Lambda$. It is however not difficult to show that these limits are well-defined perturbatively[1].

11.3 Properties of the surface tensions and massgaps

The following proposition gives some basic properties of these three quantities in the ferromagnetic case.

Proposition 11.3.1. *Let* $\underline{J}$ *be such that* J_σ*,* J_τ *and* $J_{\sigma\tau}$ *are all non-negative. We write* $\tau_\star(\,\cdot\,;\underline{J})$ *to denote any of the three surface tensions* $\tau_\sigma(\,\cdot\,;\underline{J})$*,* $\tau_\tau(\,\cdot\,;\underline{J})$ *and* $\tau_{\sigma\tau}(\,\cdot\,;\underline{J})$*. Then*

1. $\tau_\star(\boldsymbol{n};\underline{J})$ *can be extended to positively homogeneous, Lipschitz, convex functions* $\tau_\star(x;\underline{J})$ *on* $\mathbb{R}^2$.
2. $\tau_\star(x;\underline{J}) = \tau_\star(-x;\underline{J}) = \tau_\star(x_\perp;\underline{J}) = \tau_\star(-x_\perp;\underline{J})$.
3. $\tau_\star(x;\underline{J})$ *is a non-negative increasing function of* J_σ*,* J_τ*, and* $J_{\sigma\tau}$.
4. *If* β *is large enough, then* $\tau_\star(x;\beta\underline{J}) > 0$*, for all* $x \neq 0$.
5. *If* $J_\sigma \geqslant J_\tau$ *then*
$$\tau_\sigma(x;\underline{J}) \geqslant \tau_\tau(x;\underline{J})\,.$$
If $\min(J_\sigma, J_\tau) \geqslant J_{\sigma\tau}$ *then*
$$\tau_\sigma(x;\underline{J}) + \tau_\tau(x;\underline{J}) \geqslant \tau_{\sigma\tau}(x;\underline{J}) \geqslant \max(\tau_\sigma(x;\underline{J}), \tau_\tau(x;\underline{J}))\,.$$

Proof. The proof of the first three statements is identical to the corresponding one for the Ising model. Notice that it only uses GKS inequalities (which hold for the Ashkin–Teller model) and not FKG, or GHS inequalities (which do not).

To prove the third statement, we use GKS inequalities to compare expectation values of 2-point functions in the Ashkin–Teller model and Ising models. In the limit $J^*_{\sigma\tau} \to \infty$,

[1] For example, a very simple way to do so is to use Kunz and Souillard's trick (see the proof of Lemma 13.3.1).

the only configurations contributing to the expectation values are those satisfying the constraint $\omega_\sigma(t)\omega_\sigma(t') = \omega_\tau(t)\omega_\tau(t')$, for all t, t' nearest neighbours. This implies that

$$\begin{aligned} \langle\sigma(t)\sigma(t')\rangle^{\underline{J}^*} &\leqslant \langle\sigma(t)\sigma(t')\rangle^{\text{Ising},J^*_\sigma+J^*_\tau}\,, \\ \langle\tau(t)\tau(t')\rangle^{\underline{J}^*} &\leqslant \langle\tau(t)\tau(t')\rangle^{\text{Ising},J^*_\sigma+J^*_\tau}\,. \end{aligned} \tag{11.2}$$

By first performing a change of variables $(\omega_\sigma, \omega_\tau) \mapsto (\omega_\vartheta, \omega_\tau)$, $\omega_\vartheta(t) \doteqdot \omega_\sigma(t)\omega_\tau(t)$, and then taking the limit $J^*_\sigma \to \infty$, we show that

$$\langle\vartheta(t)\vartheta(t')\rangle^{\underline{J}^*} \leqslant \langle\vartheta(t)\vartheta(t')\rangle^{\text{Ising},J^*_{\sigma\tau}+J^*_\tau}\,. \tag{11.3}$$

The above inequalities implies that the massgaps of the 2-point functions in the Ashkin–Teller model are strictly positive when the coupling constants $\underline{J}^*$ are small enough.

Let us prove the last statement; we consider only the case $J_\sigma \geqslant J_\tau \geqslant J_{\sigma\tau}$. We introduce new variables, in a similar way as in the proof of Lemma 4.2.1:

$$\omega_q(t) \doteqdot \omega_\tau(t) - \omega_\sigma(t)\,, \qquad \omega_r(t) \doteqdot \omega_\tau(t) + \omega_\sigma(t)\,. \tag{11.4}$$

It is easy to check that the following relations hold:

$$\begin{aligned} \omega_\sigma(t)\omega_\sigma(t') &= \tfrac14(\omega_q(t)\omega_q(t') + \omega_r(t)\omega_r(t') - \omega_q(t)\omega_r(t') - \omega_r(t)\omega_q(t'))\,, \\ \omega_\tau(t)\omega_\tau(t') &= \tfrac14(\omega_q(t)\omega_q(t') + \omega_r(t)\omega_r(t') + \omega_q(t)\omega_r(t') + \omega_r(t)\omega_q(t'))\,, \\ \omega_\sigma(t)\omega_\sigma(t')\omega_\tau(t)\omega_\tau(t') &= \tfrac14\omega_r(t)^2\omega_r(t')^2 + \tfrac12(\omega_q(t)^2 + \omega_q(t')^2) - 3\,. \end{aligned} \tag{11.5}$$

Therefore we can write the Hamiltonian of the Ashkin–Teller model with free b.c. in Λ^* as

$$\begin{aligned} H(\omega) = -\tfrac14 \sum_{e=\langle t,t'\rangle\subset\mathcal{E}(\Lambda^*)} \Big(&(J^*_\sigma + J^*_\tau)\big(\omega_q(t)\omega_q(t') + \omega_r(t)\omega_r(t')\big) + (J^*_\tau - J^*_\sigma)\omega_q(t)\omega_r(t') \\ &+ J^*_{\sigma\tau}\big(\omega_r(t)^2\omega_r(t')^2 + 2(\omega_q(t)^2 + \omega_q(t')^2) - 12\big)\Big)\,. \end{aligned} \tag{11.6}$$

This Hamiltonian is ferromagnetic since $J^*_\sigma \leqslant J^*_\tau$. Therefore the same argument as in the proof of Lemma 4.2.1 yields, for example,

$$\langle q(t)r(t')\rangle^{\underline{J}^*}_{\Lambda^*} \geqslant 0\,, \quad \forall t, t' \in \Lambda^*\,. \tag{11.7}$$

But this is equivalent to

$$\langle\sigma(t)\sigma(t')\rangle^{\underline{J}^*}_{\Lambda^*} \leqslant \langle\tau(t)\tau(t')\rangle^{\underline{J}^*}_{\Lambda^*}\,, \tag{11.8}$$

which by duality implies that $\tau_\tau(x;\underline{J}) \leqslant \tau_\sigma(x;\underline{J})$.

The other inequalities are easily proven using the following two elementary observations:

$$\begin{aligned} &\langle\sigma(t)\sigma(t')\tau(t)\tau(t')\rangle^{\underline{J}^*}_{\Lambda^*} \geqslant \langle\sigma(t)\sigma(t')\rangle^{\underline{J}^*}_{\Lambda^*}\langle\tau(t)\tau(t')\rangle^{\underline{J}^*}_{\Lambda^*}\,, \\ &\langle\sigma(t)\sigma(t')\tau(t)\tau(t')\rangle^{\underline{J}^*}_{\Lambda^*} = \mu^{\text{RC}}_{\Lambda^*}(\{t \overset{\sigma}{\leftrightarrow} t'\} \cap \{t \overset{\tau}{\leftrightarrow} t'\}) \leqslant \min(\langle\sigma(t)\sigma(t')\rangle^{\underline{J}^*}_{\Lambda^*}, \langle\tau(t)\tau(t')\rangle^{\underline{J}^*}_{\Lambda^*})\,. \end{aligned} \tag{11.9}$$

□

Remark.

1. In particular, when $J_\sigma = J_\tau$, we have that $\tau_{\sigma\tau} = 0 \Leftrightarrow \tau_\sigma = \tau_\tau = 0$.
2. Of course, similar statements should also be true in the non-ferromagnetic case, however the absence of monotonicity in volume prevents us to do the proof. It is nevertheless interesting to notice that the inequalities of Corollary 10.4.1 imply that

$$\langle \sigma(t)\sigma(t')\tau(t)\tau(t')\rangle_\Lambda^{J^*} \leqslant \langle \sigma(t)\sigma(t')\rangle_\Lambda^{J^*} \langle \tau(t)\tau(t')\rangle_\Lambda^{J^*}, \tag{11.10}$$

which can be written

$$-\frac{1}{\|t'-t\|_2} \log \langle \sigma(t)\sigma(t')\tau(t)\tau(t')\rangle_\Lambda^{J^*} \geqslant \\ -\Big(\frac{1}{\|t'-t\|_2} \log \langle \sigma(t)\sigma(t')\rangle_\Lambda^{J^*} + \frac{1}{\|t'-t\|_2} \log \langle \tau(t)\tau(t')\rangle_\Lambda^{J^*}\Big). \tag{11.11}$$

Assuming the existence of the massgaps and the duality relation, we obtain the following interesting inequality[2]

$$\tau_{\sigma\tau}(\boldsymbol{n}; \underline{J}) \geqslant \tau_\sigma(\boldsymbol{n}; \underline{J}) + \tau_\tau(\boldsymbol{n}; \underline{J}). \tag{11.12}$$

Existence and the duality relation[3] can be proved at low temperature using cluster expansion techniques.

[2] Observe that the inequality is reversed in the ferromagnetic case.

[3] The only thing non-trivial to prove to establish the duality relation between massgaps and surface tensions in this case is the analogous to Lemma 3.1.3.

Chapter 12

The 2-point functions

In chapter 9, we introduce three different kinds of 2-point functions, which can be written as

$$\langle\sigma(t)\sigma(t')\rangle_\Lambda^{\underline{J}} = \sum_{\substack{\lambda\,:\,\Lambda\text{-comp.}\\ \partial_\sigma\lambda=\{t,t'\}\\ \partial_\tau\lambda=\varnothing}} q_\Lambda(\lambda;\underline{J})\,, \tag{12.1}$$

$$\langle\tau(t)\tau(t')\rangle_\Lambda^{\underline{J}} = \sum_{\substack{\lambda\,:\,\Lambda\text{-comp.}\\ \partial_\sigma\lambda=\varnothing\\ \partial_\tau\lambda=\{t,t'\}}} q_\Lambda(\lambda;\underline{J})\,, \tag{12.2}$$

$$\langle\sigma(t)\tau(t)\sigma(t')\tau(t')\rangle_\Lambda^{\underline{J}} = \sum_{\substack{\lambda\,:\,\Lambda\text{-comp.}\\ \partial_\sigma\lambda=\{t,t'\}\\ \partial_\tau\lambda=\{t,t'\}}} q_\Lambda(\lambda;\underline{J})\,, \tag{12.3}$$

where $q_\Lambda(\underline{\gamma};\underline{J}) \doteqdot w^*(\underline{\gamma})\dfrac{Z(\Lambda|\underline{\gamma};\underline{J})}{Z(\Lambda;\underline{J})}$.

Our aim in this chapter is to study these 2-point functions. We first establish some basic properties of the weight $q_\Lambda(\underline{\gamma};\underline{J})$ (Section 12.1); we then study the typical contours contributing to the high-temperature expansion of the 2-point functions (Section 12.2); finally, we prove statements analogous to the box propositions of the first Part (Section 12.3). The results of Section 12.1 are proved in the same way as the corresponding ones for the Ising model and are therefore entirely non-perturbative; on the other hand, those of Sections 12.2 and 12.3 are obtained using cluster expansions. The main difficulty in the Ashkin–Teller model is that the existence of cutting-points (i.e. the places at which a contour can be broken) is a non-trivial problem which we could only solve by perturbative arguments.

In this chapter, we consider the high-temperature representation of the Ashkin–Teller model on the lattice $\mathbb{Z}^2$. The coupling constants are supposed to be *ferromagnetic*, i.e. such that

$$J_\sigma(e) \geqslant 0\,,\quad J_\tau(e) \geqslant 0\,,\quad J_{\sigma\tau}(e) \geqslant 0\,, \tag{12.4}$$

for all edges $e \in \mathcal{E}$. We first introduce some terminology similar to that of Chapter 4.

Definition.

(D220) *Let γ be some contour. The* **edge-boundary** *of a pair $A = (A_\sigma, A_\tau)$ of set of edges is defined by*

$$\Delta(A) \doteqdot \Delta(A_\sigma) \cup \Delta(A_\tau)\,.$$

(D221) *Let $\underline{\gamma}$ be a Λ-compatible family of contours. The graph $\mathcal{G}(\underline{\gamma})$ is defined by the set of edges $\mathcal{E}(\Lambda) \setminus \Delta(\underline{\gamma})$.*

The definition of contours and of edge-boundary are such that the following Lemma holds.

Lemma 12.0.1. *Let $\underline{\gamma}$ be a Λ-compatible family of contours. Then*

$$Z(\Lambda|\underline{\gamma}) = Z(\mathcal{G}(\underline{\gamma}))\,.$$

12.1 Properties of the 2-point functions

We first state monotonicity properties of the 2-point functions, corresponding to those of Lemma 4.2.1.

Lemma 12.1.1. *Suppose that $\underline{J}$ are translation invariant and ferromagnetic. Let $u, v \in \mathbb{Z}^2$ and let l be the half-line containing u and v with endpoint u. Then, if $\overline{u}$ satisfies one the following conditions*

- $|\overline{u}(1) - u(1)| = 1$, $\overline{u}(2) = u(2)$ *and the vertical line separating u and $\overline{u}$ does not intersect l,*
- $\overline{u}(1) = u(1)$, $|\overline{u}(2) - u(2)| = 1$ *and the horizontal line separating u and $\overline{u}$ does not intersect l,*
- $|\overline{u}(1) - u(1)| = 1$, $|\overline{u}(2) - u(2)| = 1$ *and the diagonal line separating u and $\overline{u}$ does not intersect l,*

the following inequalities holds,

$$\langle \sigma(u)\sigma(v)\rangle^\beta \geqslant \langle \sigma(\overline{u})\sigma(v)\rangle^\beta\,,$$
$$\langle \tau(u)\tau(v)\rangle^\beta \geqslant \langle \tau(\overline{u})\tau(v)\rangle^\beta\,,$$
$$\langle \sigma(u)\tau(u)\sigma(v)\tau(v)\rangle^\beta \geqslant \langle \sigma(\overline{u})\tau(\overline{u})\sigma(v)\tau(v)\rangle^\beta\,.$$

Proof. We prove only the first statement. The proof is very similar to that of Lemma 4.2.1. We only point out the differences. Let m be the vertical line separating u and $\overline{u}$ and let Λ be any box containing u and v and invariant under a reflection of axis m. If t is some site of the lattice, let us write $\overline{t}$ for the site obtained by a reflection of axis m. We define $\omega^1_\sigma(t) \doteqdot \omega_\sigma(t) + \omega_\sigma(\overline{t})$ and $\omega^2_\sigma(t) \doteqdot \omega_\sigma(t) - \omega_\sigma(\overline{t})$. In the same way, we define $\omega^1_\tau(t)$ and $\omega^2_\tau(t)$. The following relations hold: Let t and t' be two nearest neighbours sites.

- If $t' \neq \overline{t}$, then

$$\omega_\sigma(t)\omega_\sigma(t') + \omega_\sigma(\overline{t})\omega_\sigma(\overline{t}') = \tfrac{1}{2}(\omega^1_\sigma(t)\omega^1_\sigma(t') + \omega^2_\sigma(\overline{t})\omega^2_\sigma(\overline{t}'))\,,$$
$$\omega_\tau(t)\omega_\sigma(t') + \omega_\tau(\overline{t})\omega_\tau(\overline{t}') = \tfrac{1}{2}(\omega^1_\tau(t)\omega^1_\tau(t') + \omega^2_\tau(\overline{t})\omega^2_\tau(\overline{t}'))\,. \tag{12.5}$$

- If $t' = \overline{t}$, then

$$\omega_\sigma(t)\omega_\sigma(t') = \tfrac{1}{2}(\omega^1_\sigma(t)^2 - 1)\,, \tag{12.6}$$
$$\omega_\tau(t)\omega_\tau(t') = \tfrac{1}{2}(\omega^1_\tau(t)^2 - 1)\,. \tag{12.7}$$

It is easy to check that the Hamiltonian of the new model has only positive coupling constants, except for the terms corresponding to $\omega_\sigma(t)\omega_\sigma(\bar{t})\omega_\tau(t)\omega_\tau(\bar{t})$ which becomes $\frac{1}{4}(\omega_\sigma^1(t)^2\omega_\tau^1(t)^2 - \omega_\sigma^1(t)^2 - \omega_\tau^1(t)^2 + 1)$. However this term is constant for each partial sum. Therefore GKS inequalities still apply and we have ($\sigma_i(t)$ and $\tau_i(t)$ being defined similarly as in the proof of Lemma 4.2.1)

$$\begin{aligned} \langle \sigma_1(u)\sigma_2(v)\rangle_\Lambda^\beta &\geqslant 0\,, \\ \langle \tau_1(u)\tau_2(v)\rangle_\Lambda^\beta &\geqslant 0\,, \end{aligned} \tag{12.8}$$

which are equivalent to the first two statements. The last one is proved in the same way, by first performing the change of variables $(\omega_\sigma, \omega_\tau) \mapsto (\omega_\vartheta, \omega_\tau)$, with $\omega_\vartheta(t) \doteqdot \omega_\sigma(t)\omega_\tau(t)$, $t \in \Lambda$. □

12.1.1 Monotonicity properties of the weight

Lemma 12.1.2. *Suppose $\underline{J}'(e) \geqslant \underline{J}(e) \geqslant 0$ [1] for all edges e, then*

1. *$B_1 \subset B_2$ implies that*
$$\frac{Z(\mathcal{G}(B_1); \underline{J}')}{Z(\mathcal{G}(B_2); \underline{J}')} \leqslant \frac{Z(\mathcal{G}(B_1); \underline{J})}{Z(\mathcal{G}(B_2); \underline{J})}\,;$$
2. *If $\underline{\gamma}$ is a Λ-compatible family of contours, then*
$$\frac{Z(\Lambda|\underline{\gamma}; \underline{J}')}{Z(\Lambda; \underline{J}')} \leqslant \frac{Z(\Lambda|\underline{\gamma}; \underline{J})}{Z(\Lambda; \underline{J})}\,.$$

Proof. This is proved in the same way as Lemmas 4.2.2 and 4.2.3. □

As a consequence of the previous lemma, we can state the first set of properties of the weight $q_\Lambda(\underline{\gamma})$.

Lemma 12.1.3. *Suppose that the coupling constants $\underline{J}$ are ferromagnetic. Then*

1. *Let $\mathcal{M}_\sigma, \mathcal{M}_\tau, \mathcal{M}_{\sigma\tau} \subset \mathcal{E}$, and let $\underline{\gamma}$ be a Λ-compatible family of contours such that*
$$\mathcal{B}_\sigma(\underline{\gamma}) \cap \mathcal{M}_\sigma = \mathcal{B}_\tau(\underline{\gamma}) \cap \mathcal{M}_\tau = \mathcal{B}_{\sigma\tau}(\underline{\gamma}) \cap \mathcal{M}_{\sigma\tau} = \varnothing\,.$$
Then $q_\Lambda(\underline{\gamma}; \underline{J})$ is a decreasing function of
   - *$J_\sigma(e)$, for edges $e \in \mathcal{M}_\sigma$;*
   - *$J_\tau(e)$, for edges $e \in \mathcal{M}_\tau$;*
   - *$J_{\sigma\tau}(e)$, for edges $e \in \mathcal{M}_{\sigma\tau}$.*
2. *Let $\underline{\gamma}$ be a Λ-compatible family of contours. Then, for all $\Lambda' \supset \Lambda$,*
$$q_{\Lambda'}(\underline{\gamma}; \underline{J}) \leqslant q_\Lambda(\underline{\gamma}; \underline{J})\,.$$
3. *The limit $q(\underline{\gamma}; \underline{J}) \doteqdot \lim_{\Lambda \nearrow \mathbb{Z}^2} q_\Lambda(\underline{\gamma}; J)$ exists and satisfies, for any Λ-compatible family of contours $\underline{\gamma}$,*
$$q(\underline{\gamma}; \underline{J}) \leqslant q_\Lambda(\underline{\gamma}; \underline{J})\,.$$

[1] We write $\underline{J}'(e) \geqslant \underline{J}(e)$ instead of $J'_\sigma(e) \geqslant J_\sigma(e)$, $J'_\tau(e) \geqslant J_\tau(e)$ and $J'_{\sigma\tau}(e) \geqslant J_{\sigma\tau}(e)$.

12.1.2 Some results about decomposition of contours

We first have to define special sites along a given contour at which it is possible to cut it; we want to do this in such a way as to ensure that the different parts of the contour obtained after the operation can be easily decoupled.

Definition.

(D222) *Let γ be some contour and λ be some closed σ-contour of γ. Let $\lambda_1, \ldots, \lambda_n$ be a decomposition of λ with cutting-points $t_1, \ldots, t_n$ (see* (D104)*, p. 78) and let e_i, $i = 1, \ldots, n$, be the edge of λ_i incident on the site t_i. We denote by γ' the set of σ- and τ-contours belonging to γ different from λ. Suppose that the following conditions hold:*

- $\mathrm{i}(t_i, \gamma') = 0$, $i = 1, \ldots, n$;
- *the decomposition into contours of $(\overline{\mathcal{B}}_\sigma(\gamma) \setminus \bigcup_i \{e_i\}, \overline{\mathcal{B}}_\tau(\gamma))$ gives n compatible contours $\gamma'_1, \ldots, \gamma'_n$.*

In such a case we say that the family of contours $\gamma_1, \ldots, \gamma_n$, where γ_i is the contour obtained by adding the σ-edge e_i to γ'_i ($i = 1, \ldots, n$), is the **λ-decomposition of γ with cutting-points** $t_1, \ldots, t_n$.

We define similarly the λ-decomposition when λ is an open σ-contour or a τ-contour.

(D223) *Let γ be some contour and λ^σ, λ^τ be some closed σ- and τ-contours of γ. Suppose $t_1, \ldots, t_n$ belong to both λ^σ and λ^τ and suppose that*

$$\{e \in \lambda^\sigma \,:\, \mathrm{i}(t_i, e) \neq 0\} = \{e \in \lambda^\sigma \,:\, \mathrm{i}(t_i, e) \neq 0\}\,.$$

Let $\lambda^\sigma_1, \ldots, \lambda^\sigma_n$, resp. $\lambda^\tau_1, \ldots, \lambda^\tau_n$, be the decomposition of λ^σ, resp. λ^τ, with cutting-points $t_1, \ldots, t_n$ and let e_i, $i = 1, \ldots, n$, be the edge of λ^σ_i (or λ^τ_i) incident on the site t_i. We denote by γ' the set of σ- and τ-contours belonging to γ different from λ_σ and λ_τ. Suppose that the following conditions hold:

- $\mathrm{i}(t_i, \gamma') = 0$, $i = 1, \ldots, n$;
- *The decomposition into contours of $(\overline{\mathcal{B}}_\sigma(\gamma) \setminus \bigcup_i \{e_i\}, \overline{\mathcal{B}}_\tau(\gamma) \setminus \bigcup_i \{e_i\})$ gives n compatible contours $\gamma'_1, \ldots, \gamma'_n$.*

In such a case we say that the family of contours $\gamma_1, \ldots, \gamma_n$, where γ_i is the contour obtained by adding the σ- and τ-edges e_i to γ'_i ($i = 1, \ldots, n$), is the **$(\lambda^\sigma, \lambda^\tau)$-decomposition of γ with cutting-points** $t_1, \ldots, t_n$.

We define similarly the $(\lambda^\sigma, \lambda^\tau)$-decomposition when λ^σ and λ^τ are two open contour with the same boundary.

The cutting-points are good locations to cut a contour. A useful property is given in the following lemma, which is proven similarly as Lemma 4.1.2, point 2.

Lemma 12.1.4. *Let γ be some open contour and let γ_1 and γ_2 be the decomposition of γ with cutting point t. The graph $\mathcal{G}_t(\gamma_2)$, which is defined by the set of edges obtained by adding the edge $\overline{e}$ of $\Delta(\gamma_2) \setminus \mathcal{E}(\gamma_2)$ which is adjacent to t to $\mathcal{E}(\mathcal{G}(\gamma_2))$, is such that*

$$Z(\Lambda|\gamma) = Z(\mathcal{G}_t(\gamma_2)|\gamma_1)\,.$$

We can now state the following important results related to the decomposition of contours.

Lemma 12.1.5. *Let $\underline{J}$ be ferromagnetic.*

1. *Let γ be some contour such that $\partial_\sigma \gamma = \{t_1, t_2\}$, $\partial_\tau \gamma = \varnothing$. If λ_1, λ_2 is a decomposition with cutting-point of γ, then*

$$q_\Lambda(\gamma; \underline{J}) \geqslant q_\Lambda(\lambda_1; \underline{J}) q_\Lambda(\lambda_2; \underline{J}) .$$

Similar results hold if $\partial_\sigma \gamma_2 = \varnothing$ and $\partial_\tau \gamma_2 = \{t_2, t_3\}$, or $\partial_\sigma \gamma_2 = \{t_2, t_3\}$ and $\partial_\tau \gamma_2 = \{t_2, t_3\}$.

2. *Let $t, t', t_1, \ldots, t_n$ be $n+2$ disjoint points in $\mathbb{Z}^2$ (n may be equal to zero); we introduce the set $\mathcal{Q}_\sigma \equiv \mathcal{Q}_\sigma(t, t', t_1, \ldots, t_n)$ of all contours γ with $\partial_\sigma \gamma = \{t, t'\}$ and $\partial_\tau \gamma = \varnothing$ which admit a λ-decomposition with cutting-points $t_1, \ldots, t_n$, where λ is the corresponding open σ-contour; the sets $\mathcal{Q}_\tau$ and $\mathcal{Q}_{\sigma\tau}$ are defined similarly. Then*

$$\sum_{\gamma \in \mathcal{Q}_\sigma} q_\Lambda(\gamma; \underline{J}) \leqslant \prod_{i=0}^{n} \langle \sigma(t_i) \sigma(t_{i+1}) \rangle_\Lambda^{\underline{J}} \leqslant \prod_{i=0}^{n} \exp\{-\tau_\sigma(t_{i+1} - t_i; \underline{J}^*)\} ,$$

$$\sum_{\gamma \in \mathcal{Q}_\tau} q_\Lambda(\gamma; \underline{J}) \leqslant \prod_{i=0}^{n} \langle \tau(t_i) \tau(t_{i+1}) \rangle_\Lambda^{\underline{J}} \leqslant \prod_{i=0}^{n} \exp\{-\tau_\tau(t_{i+1} - t_i; \underline{J}^*)\} ,$$

$$\sum_{\gamma \in \mathcal{Q}_{\sigma\tau}} q_\Lambda(\gamma; \underline{J}) \leqslant \prod_{i=0}^{n} \langle \sigma(t_i) \tau(t_i) \sigma(t_{i+1}) \tau(t_{i+1}) \rangle_\Lambda^{\underline{J}} \leqslant \prod_{i=0}^{n} \exp\{-\tau_{\sigma\tau}(t_{i+1} - t_i; \underline{J}^*)\} ,$$

where we have set $t_0 \equiv t$ and $t_{n+1} \equiv t'$. A similar statement holds for contours γ with $\partial_\sigma \gamma = \partial_\tau \gamma = \varnothing$ with respect to any of their σ- or τ-contours.

3. *Let $t_1, t_2, t_3 \in \Lambda \subset \mathbb{Z}^2$ and let γ_2 be some Λ-compatible contour such that $\partial_\sigma \gamma_2 = \{t_2, t_3\}$ and $\partial_\tau \gamma_2 = \varnothing$. Writing $\mathcal{Q}'$ the set of all contours γ_1 such that*
   - $\partial_\sigma \gamma_1 = \{t_1, t_2\}$,
   - $\partial_\tau \gamma_1 = \varnothing$,
   - *$\gamma \doteqdot \gamma_1 \cup \gamma_2$ is a Λ-compatible contour,*
   - *γ_1, γ_2 is a λ-decomposition of γ with cutting-point t_2, where λ is the open σ-contour of γ,*

   the following inequality holds,

$$\sum_{\gamma_1 \in \mathcal{Q}'} q_\Lambda(\gamma_1 \cup \gamma_2; \underline{J}) \leqslant 8\, q_\Lambda(\gamma_2; \underline{J}) \sum_{\substack{\gamma_1 : \\ \partial_\sigma \gamma_1 = \{t_1, t_2\},\, \partial_\tau \gamma_1 = \varnothing}} q_\Lambda(\gamma_1; \underline{J}) .$$

Similar results hold if $\partial_\sigma \gamma_2 = \varnothing$ and $\partial_\tau \gamma_2 = \{t_2, t_3\}$, or $\partial_\sigma \gamma_2 = \{t_2, t_3\}$ and $\partial_\tau \gamma_2 = \{t_2, t_3\}$.

Proof. Point 1. is proved exactly as the corresponding statement of Lemma 4.2.4.
We prove the second statement. Let us introduce the set $\mathcal{Q}_{t,t'}(t_1, \ldots, t_n)$ of contours γ satisfying the following conditions

- $\partial_\sigma \gamma = \{t, t'\}$,
- $\partial_\tau \gamma = \varnothing$,
- $t_1, \ldots, t_n$ are cutting-points of some λ-decomposition of γ, where λ is the open σ-contour of γ.

We also introduce the set $\mathcal{Q}_1^c$ of contours γ_1^c satisfying the conditions

- $\partial_\sigma \gamma_1^c = \{t_1, t'\}$,
- $\partial_\tau \gamma_1^c = \varnothing$,
- $t_2, \ldots, t_n$ are cutting-points of some λ_1^c-decomposition of γ_1^c, where λ_1^c is the open σ-contour of γ_1^c,
- $\mathrm{i}(t_1, \lambda_1^c) = 1$,
- $\mathrm{i}(t_1, \gamma_1^c \setminus \lambda_1^c) = 0$,

and the set $\mathcal{Q}_1$ of contours γ_1 satisfying the conditions

- $\partial_\sigma \gamma_1 = \{t, t_1\}$,
- $\partial_\tau \gamma_1 = \varnothing$,
- $\mathrm{i}(t_1, \gamma_1 \setminus \lambda_1) = 0$, where λ_1 is the corresponding σ-contour.

With these notations, we can proceed similarly to the proof of Lemma 4.2.6,

$$\begin{aligned}\sum_{\gamma \in \mathcal{Q}_{t,t'}(t_1,\ldots,t_n)} q_\Lambda(\gamma) &= \sum_{\gamma_1^c \in \mathcal{Q}_1^c} w^*(\gamma_1^c) \sum_{\substack{\gamma_1 \in \mathcal{Q}_1 \\ \gamma_1 \cup \gamma_1^c \in \mathcal{Q}_{t,t'}(t_1,\ldots,t_n)}} w^*(\gamma_1) \frac{Z(\mathcal{G}(\gamma_1 \cup \gamma_1^c))}{Z(\Lambda)} \\ &= \sum_{\gamma_1^c \in \mathcal{Q}_1^c} w^*(\gamma_1^c) \frac{Z(\mathcal{G}_{t_1}(\gamma_1^c))}{Z(\Lambda)} \sum_{\substack{\gamma_1 \in \mathcal{Q}_1 \\ \gamma_1 \cup \gamma_1^c \in \mathcal{Q}_{t,t'}(t_1,\ldots,t_n)}} w^*(\gamma_1) \frac{Z(\mathcal{G}_{t_1}(\gamma_1^c) | \gamma_1)}{Z(\mathcal{G}_{t_1}(\gamma_1^c))} .\end{aligned} \tag{12.9}$$

This last sum is easily estimated,

$$\begin{aligned}\sum_{\substack{\gamma_1 \in \mathcal{Q}_1 \\ \gamma_1 \cup \gamma_1^c \in \mathcal{Q}_{t,t'}(t_1,\ldots,t_n)}} w^*(\gamma_1) \frac{Z(\mathcal{G}_{t_1}(\gamma_1^c) | \gamma_1)}{Z(\mathcal{G}_{t_1}(\gamma_1^c))} &\leqslant \sum_{\substack{\gamma_1 \\ \partial_\sigma \gamma_1 = \{t, t_1\} \\ \partial_\tau \gamma_1 = \varnothing}} w^*(\gamma_1) \frac{Z(\mathcal{G}_{t_1}(\gamma_1^c) | \gamma_1)}{Z(\mathcal{G}_{t_1}(\gamma_1^c))} \\ &\leqslant \langle \sigma(t) \sigma(t_1) \rangle_\Lambda^J .\end{aligned} \tag{12.10}$$

We therefore have to obtain an estimate for

$$\sum_{\gamma_1^c \in \mathcal{Q}_1^c} w^*(\gamma_1^c) \frac{Z(\mathcal{G}_{t_1}(\gamma_1^c))}{Z(\Lambda)} . \tag{12.11}$$

The set $\mathcal{Q}_1^c$ is only a subset of the set $\mathcal{Q}_{t_1,t'}(t_2, \ldots, t_n)$, which is defined as above. The additional conditions are

- $\mathrm{i}(t_1, \lambda_1^c) = 1$,
- $\mathrm{i}(t_1, \gamma_1^c \setminus \lambda_1^c) = 0$.

The presence of additional constraints on the sum is not a problem since we are looking for an upper bound, however we must be sure that no configurations of contours appear more than once (the contours appearing in $Z(\mathcal{G}_{t_1}(\gamma_1^c))$ being not necessarily compatible with γ_1^c, it may be necessary to glue one of them with γ_1^c, which may cause the corresponding configuration of contours to appear twice). Let $\underline{\gamma}' \subset \mathcal{E}(\Lambda)$ be some compatible family of contours contributing to $Z(\mathcal{G}_{t_1}(\gamma_1^c))$. If $\underline{\gamma}'$ is compatible with γ_1^c then there is no problem. Suppose this is not the case. If $\mathrm{i}(t_1, \underline{\gamma}'^\tau) \neq 0$, then this term cannot appear in

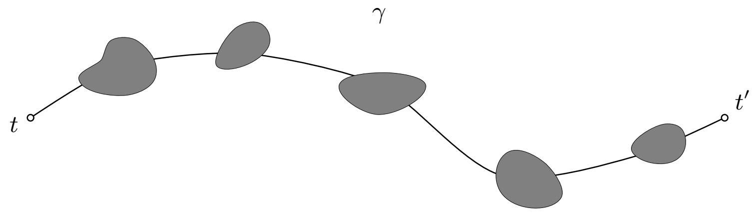

FIGURE 12.1. Sketch of a typical contour γ contributing to the 2-point function $\langle\sigma(t)\sigma(t')\rangle^{\underline{J}}_{-}$. The line represents the open σ-contour λ, while the patches represent the contours of the decomposition of $(\underline{\gamma}^{\sigma},\underline{\gamma}^{\tau})$ (see the text).

the sum (12.11), because of the second condition above and therefore the corresponding configuration of contours appears only once. The last possible case is $\mathrm{i}(t_1,\underline{\gamma}'^{\sigma})\neq 0$ and $\mathrm{i}(t_1,\underline{\gamma}'^{\tau})=0$, which can be treated as in the proof of Lemma 4.2.6. It is therefore possible to iterate the procedure to obtain the claimed statement.

The last statement is proved in the same way as Lemma 4.2.9. We start as in the proof of the previous point to obtain

$$\sum_{\gamma_1\in\mathcal{Q}'} q_\Lambda(\gamma_1\cup\gamma_2;\underline{J}) \leqslant w^*(\gamma_2)\frac{Z(\mathcal{G}(\gamma_2))}{Z(\Lambda)}\frac{Z(\mathcal{G}_{t_2}(\gamma_2))}{Z(\mathcal{G}(\gamma_2))}\sum_{\substack{\gamma_1:\\ \partial_\sigma\gamma_1=\{t_1,t_2\},\,\partial_\tau\gamma_1=\varnothing}} q_\Lambda(\gamma_1;\underline{J})\,. \qquad (12.12)$$

The second quotient of partition functions being always smaller than 8, the result follows easily. □

Since not every point of a contour can be a cutting-point, it is not possible to derive Simon's inequality from Point 2. of the preceding lemma, as was the case in the first Part. We derive a weaker perturbative form of this inequality later on, after having studied typical high-temperature contours.

12.2 Typical high-temperature contours

In this section, we suppose that the coupling constants are translation invariant, i.e. $\underline{J}(e)$ is independent of e.

Let γ be some contour contributing to the 2-point function $\langle\sigma(t)\sigma(t')\rangle^{\underline{J}}_{-}$. We would like to study the geometry of typical such contours; in particular, we need to obtain some informations on the presence and location of cutting-points. This is a difficult problem, which we do not know how to solve non-perturbatively. On the other hand, using cluster expansion techniques, it is possible to give an answer to this problem quite easily.

The main result of this section is Lemma 12.2.1 which shows, if the temperature is high enough, that γ looks like a "necklace"; more precisely, typical contours γ are of the form $\gamma=(\lambda,\underline{\gamma}^{\sigma},\underline{\gamma}^{\tau})$, where

- λ is an open σ-contour with boundary $\{t,t'\}$;
- for any site $u\in\gamma$, there exists a site $v\in\gamma$ with $\|v-u\|_1\leqslant\log\|t'-t\|_1$ such that v is a λ-cutting point of γ;
- the family of contours $\underline{\gamma}'$ of the decomposition of the family $(\underline{\gamma}^{\sigma},\underline{\gamma}^{\tau})$ satisfies

$$|\mathcal{B}(\gamma_i')|\leqslant\log\|t'-t\|_1,\ \forall i\,. \qquad (12.13)$$

We write $\mathcal{T}^\sigma(t,t')$ for the set of such contours (see Fig. 12.1 for a sketch of such a contour); we similarly define the set $\mathcal{T}^\tau(t,t')$. We also define the set $\mathcal{T}^{\sigma\tau}(t,t')$ of typical contours γ contributing to the 2-point function $\langle\sigma(t)\tau(t)\sigma(t')\tau(t')\rangle^{\underline{J}}$, when $J_{\sigma\tau} > 0$. These contours are of the form $\gamma = (\lambda^\sigma, \lambda^\tau, \underline{\gamma}^\sigma, \underline{\gamma}^\tau)$ where

- λ^σ is an open σ-contour with boundary $\{t,t'\}$;
- λ^τ is an open τ-contour with boundary $\{t,t'\}$;
- for any site $u \in \gamma$, there exists a site $v \in \gamma$ with $\|v-u\|_1 \leqslant \log\|t'-t\|_1$ such that v is a $(\lambda^\sigma,\lambda^\tau)$-cutting point of γ;
- the family of contours $\underline{\gamma}'$ of the decomposition of the family $(\underline{\gamma}^\sigma, \underline{\gamma}^\tau)$ satisfies

$$|\mathcal{B}(\gamma'_i)| \leqslant \log\|t'-t\|_1,\ \forall i\,. \tag{12.14}$$

The two open contours are therefore glued together on "most of their length" (this result is similar to Lemma 4.4.8).

Lemma 12.2.1. *Suppose $S,T,L \in (0,1)$.*

1. *Suppose $TL < S$. Let $t,t' \in \mathbb{Z}^2$. There exists a constant $\beta_0 > 0$ such that, $\forall\beta < \beta_0$,*

$$\langle\sigma(t)\sigma(t')\rangle_\Lambda^{\beta\underline{J}} = \sum_{\substack{\gamma:\\ \partial_\sigma\gamma=\{t,t'\},\,\partial_\tau\gamma=\varnothing}} q_\Lambda(\gamma) = \big(1+(\tanh\mathcal{O}(\beta))^{\log\|t'-t\|_1}\big)\sum_{\gamma\in\mathcal{T}^\sigma(t,t')} q_\Lambda(\gamma)\,.$$

2. *Suppose $SL < T$. Let $t,t' \in \mathbb{Z}^2$. There exists a constant β_0 such that, $\forall\beta < \beta_0$,*

$$\langle\tau(t)\tau(t')\rangle_\Lambda^{\beta\underline{J}} = \sum_{\substack{\gamma:\\ \partial_\sigma\gamma=\varnothing,\,\partial_\tau\gamma=\{t,t'\}}} q_\Lambda(\gamma) = \big(1+(\tanh\mathcal{O}(\beta))^{\log\|t'-t\|_1}\big)\sum_{\gamma\in\mathcal{T}^\tau(t,t')} q_\Lambda(\gamma)\,.$$

3. *Suppose $ST < L$. Let $t,t' \in \mathbb{Z}^2$. There exists a constant β_0 such that, $\forall\beta < \beta_0$,*

$$\begin{aligned}\langle\sigma(t)\tau(t)\sigma(t')\tau(t')\rangle_\Lambda^{\beta\underline{J}} &= \sum_{\substack{\gamma:\\ \partial_\sigma\gamma=\{t,t'\},\,\partial_\tau\gamma=\{t,t'\}}} q_\Lambda(\gamma)\\ &= \big(1+(\tanh\mathcal{O}(\beta))^{\log\|t'-t\|_1}\big)\sum_{\gamma\in\mathcal{T}^{\sigma\tau}(t,t'))} q_\Lambda(\gamma)\,.\end{aligned}$$

Proof. The beginning of the proof is the same for all cases; it consists in constructing, for every contour γ, an optimal open contour and some sets of edges, which are local deformations of this open contour.

Let $g = g(\gamma)$ be the shortest path in $\mathcal{B}(\gamma)$ with boundary $\{t,t'\}$ and let ρ_i, $i=1,\dots,n$, denote the maximal connected components of the set $\mathcal{B}(\gamma)\setminus g$; to each edge is associated its type, i.e. σ, τ or $\sigma\tau$ depending on its belonging to $\mathcal{B}_\sigma(\gamma)$, $\mathcal{B}_\tau(\gamma)$ or $\mathcal{B}_{\sigma\tau}(\gamma)$. We are now going to glue together some of the components ρ_i, so that, when we will use cluster expansion, the constraint between the objects will be local. We consider a unit–speed parameterization of g, $s \mapsto g(s)$ and define

$$\begin{aligned}&\partial_g\rho_i \doteqdot (g(s_1), g(s_2)),\text{ where } s_1 = \min\{t\,:\,g(t)\in\rho_i\},\ s_2=\max\{t\,:\,g(t)\in\rho_i\}\,,\\ &\qquad\operatorname{supp}\rho_i \doteqdot \{g(t)\,:\,t\in[t_1-1,t_2+1],\ \partial_g\rho_i=\{g(t_1),g(t_2)\}\}\,.\end{aligned} \tag{12.15}$$

Let us introduce new sets of edges

$$\hat{\rho}_i \doteqdot \rho_i \cup \operatorname{supp} \rho_i \,. \tag{12.16}$$

The maximal connected components of $\bigcup_i \hat{\rho}_i$ are denoted by $\tilde{\rho}_1, \ldots, \tilde{\rho}_{\tilde{n}}$. We define the following equivalence relation

$$\rho_k \sim \rho_l \Leftrightarrow \exists j \text{ s.t. } \hat{\rho}_k \subset \tilde{\rho}_j \text{ and } \hat{\rho}_l \subset \tilde{\rho}_j \,. \tag{12.17}$$

The unions of the members of a same equivalence class define the *excitations* of g and are denoted by $\zeta_1, \ldots, \zeta_p$. The support of ζ_i is defined by

$$\operatorname{supp} \zeta_i \doteqdot \bigcup_{\rho_j \subset \zeta_i} \operatorname{supp} \rho_j \,. \tag{12.18}$$

We write $\underline{\zeta} \equiv (\zeta_1, \ldots, \zeta_p)$, and $\gamma = (g, \underline{\zeta})$.

We prove now the first statement. Introducing the following weights for the excitations[2],

$$\hat{w}(\zeta_i|g) \doteqdot \prod_{e \in \zeta_i} w^*(e) \prod_{e \in \operatorname{supp} \zeta_i} \frac{w^*(e)}{S} \,, \tag{12.19}$$

we can write the 2-point function in the following way,

$$\begin{aligned} \sum_{\substack{\gamma: \\ \partial_\sigma \gamma = \{t,t'\}, \partial_\tau \gamma = \varnothing}} q_\Lambda(\gamma) &= \sum_g w_\sigma^*(g) \sum_{\substack{\gamma: \\ g(\gamma)=g}} \prod_i \hat{w}(\zeta_i|g) \frac{Z(\Lambda|\gamma)}{Z(\Lambda)} \\ &= \sum_g w_\sigma^*(g) \sum_{\substack{\underline{\zeta}: \\ (g,\underline{\zeta})=\gamma \\ g(\gamma)=g}} \sum_{\substack{\underline{\gamma}': \\ (\gamma,\underline{\gamma}')\Lambda\text{-comp.}}} \prod_{\zeta \in \underline{\zeta}} \hat{w}(\zeta|g) \prod_{\gamma' \in \underline{\gamma}'} w(\gamma') \,. \end{aligned} \tag{12.20}$$

The last two sums can be seen as a sum over polymers with the following local compatibility conditions:

- $\zeta_1, \ldots, \zeta_p$ are disjoint two by two, as well as their support;
- The decomposition into contours of the set of edges $(g, \underline{\zeta})$ is a single contour γ such that $\partial_\sigma \gamma = \{t, t'\}$ and $\partial_\tau \gamma = \varnothing$;
- $g(\gamma) = g$, in particular $|\zeta| \geqslant |\operatorname{supp} \zeta|$, for all $\zeta \in \underline{\zeta}$;
- $(\gamma, \underline{\gamma}')$ form a Λ-compatible family of contours.

We can therefore use a cluster expansion to evaluate it; this will conclude the proof if we can show that the polymers satisfy a Peierls condition. This is what we do now.

There are two kinds of polymers in this sum. First, there are the usual contours which obviously satisfy a Peierls condition; second, there are the sets of edges with a weight modified by the presence of g. The polymers belonging to this second class are the dangerous ones; indeed, the part of the weight of the contour coming from its support may be larger than 1! However, since $|\zeta| \geqslant |\operatorname{supp} \zeta|$, it is possible to show that they still behave nicely.

[2] The weight $w^*(e)$ of an edge e is S if e is of type σ, T if it is of type τ, and L otherwise (see (9.6)).

There are several cases, in which the maximal weight of such an excitation is realized for different set of edges. We are going to study these less expensive excitations and show that they already satisfy a Peierls condition. First, observe that every pair $(\zeta, \operatorname{supp}\zeta)$ contains some connected part of the open σ-contour of γ, as well as some closed contours. From this we can conclude that, of these $|\zeta| + |\operatorname{supp}\zeta|$ edges, at least $|\operatorname{supp}\zeta|$ belong to the open σ-contour. Indeed, suppose that, starting from t, the path g does not follow the open σ-contour; then g must meet this contour again at least once (possibly only at t'). Once g and the σ contour have coincided, each time an excitation is met g has to end on the σ-contour again.

Let us consider the different possible cases; we write x and y the boundary of $\operatorname{supp}\zeta$.

1. $T \geqslant S \geqslant L$
   Given the total number of edges in the excitation and its support, $|\zeta|$ and $|\operatorname{supp}\zeta|$, the maximal weight possible is
   $$\hat{w}(\zeta) \leqslant \left(\frac{S}{S}\right)^{|\operatorname{supp}\zeta|} T^{|\zeta|} = T^{|\zeta|}\,. \tag{12.21}$$
   Indeed, without any constraints (except the number of edges), the best solution would be to choose only τ-edges. However, the excitation must satisfy the constraint $\partial_\sigma(\zeta \cup \operatorname{supp}\zeta) = \{x, y\}$ and $\partial_\tau(\zeta \cup \operatorname{supp}\zeta) = \varnothing$. The minimal number of edges we have to modify to satisfy this constraint is $\operatorname{supp}\zeta$ (since this is the length of the shortest path from x to y).
2. $T \geqslant L \geqslant S$
   With a similar argument, we obtain
   $$\hat{w}(\zeta) \leqslant \left(\frac{L}{S}\right)^{|\operatorname{supp}\zeta|} T^{|\zeta|} = \left(\frac{LT}{S}\right)^{|\zeta|}. \tag{12.22}$$
3. $L \geqslant S \geqslant T$
   With a similar argument, we obtain
   $$\hat{w}(\zeta) \leqslant \left(\frac{S}{S}\right)^{|\operatorname{supp}\zeta|} L^{|\zeta|} = T^{|\zeta|}\,. \tag{12.23}$$
4. $L \geqslant T \geqslant S$
   With a similar argument, we obtain
   $$\hat{w}(\zeta) \leqslant \left(\frac{T}{S}\right)^{|\operatorname{supp}\zeta|} L^{|\zeta|} \leqslant \left(\frac{LT}{S}\right)^{|\zeta|}. \tag{12.24}$$

In the other cases, the Peierls condition is trivially satisfied, since the weight along the support is always smaller than 1.

The rest of the proof is now straightforward. We divide and multiply (12.20) by the same sum over polymers, but with the additional constraint that no excitation has a length larger than $\log\|t' - t\|_1$. The ratio is then easily estimated by standard cluster expansion argument, providing thus the desired result.

The two other statements are proved in the same way. □

12.3 Box propositions

In this section, we suppose that the coupling constants are translation invariant, i.e. $\underline{J}(e)$ is independent of e. We first prove some perturbative version of Simon's inequality.

Lemma 12.3.1. *Suppose that $\underline{J}$ are translation invariant and ferromagnetic. Let $t, t' \in \mathbb{Z}^2$. Let $A \subset \mathbb{Z}^2$ such that $t \in A$ and $t' \notin A$ and let $A' \doteqdot \{u \in A : \min_{v \in \partial A} \|v - u\|_1 \leqslant \log\|t' - t\|_1\}$. Then, there exists $\beta_0 > 0$ such that, for all $\beta < \beta_0$,*

$$\langle \sigma(t)\sigma(t')\rangle_\Lambda^{\beta\underline{J}} \leqslant (1 + (\tanh \mathcal{O}(\beta))^{\log\|t'-t\|_1}) \sum_{u \in A'} \langle \sigma(t)\sigma(u)\rangle_\Lambda^{\beta\underline{J}} \, \langle \sigma(u)\sigma(t')\rangle_\Lambda^{\beta\underline{J}} \,,$$

$$\langle \tau(t)\tau(t')\rangle_\Lambda^{\beta\underline{J}} \leqslant (1 + (\tanh \mathcal{O}(\beta))^{\log\|t'-t\|_1}) \sum_{u \in A'} \langle \tau(t)\tau(u)\rangle_\Lambda^{\beta\underline{J}} \, \langle \tau(u)\tau(t')\rangle_\Lambda^{\beta\underline{J}} \,,$$

$$\begin{aligned}\langle \sigma(t)\tau(t)\sigma(t')\tau(t')\rangle_\Lambda^{\beta\underline{J}} &\leqslant (1 + (\tanh \mathcal{O}(\beta))^{\log\|t'-t\|_1}) \times \\ &\times \sum_{u \in A'} \langle \sigma(t)\tau(t)\sigma(u)\tau(u)\rangle_\Lambda^{\beta\underline{J}} \, \langle \sigma(u)\tau(u)\sigma(t')\tau(t')\rangle_\Lambda^{\beta\underline{J}} \,.\end{aligned}$$

Proof. We discuss only the first statement; the other ones are treated similarly. Let γ be some contour contributing to $\langle \sigma(t)\sigma(t')\rangle_\Lambda^{\beta\underline{J}}$ and let λ be the corresponding open σ-contour. Lemma 12.2.1 implies that γ has typically a λ-cutting point in A'. Lemma 12.1.5 part 2 can then be used to obtain the desired result. □

The preceding lemma and the various tools introduced in the previous sections allow us to prove some results similar to the box Proposition 4.4.1 of Part I.
Let $t \in \mathbb{Z}^2$; we suppose, without loss of generality, that t is such that $t(1) \geqslant t(2) \geqslant 0$. Let

$$\mathcal{B}_t \doteqdot \{u \in \mathbb{Z}^2 \,:\, 0 \leqslant u(1) \leqslant t(1)\,,\, \frac{t(2) - t(1)}{2} \leqslant u(2) \leqslant \frac{t(2) + t(1)}{2}\} \tag{12.25}$$

be the same box as in Section 4.4. We introduce the sets

$$\begin{aligned}\mathfrak{L}_t^\sigma &\doteqdot \{\gamma \,:\, \partial_\sigma\gamma = \{0, t\},\, \partial_\tau\gamma = \varnothing\}\,,\\ \mathfrak{L}_t^\tau &\doteqdot \{\gamma \,:\, \partial_\sigma\gamma = \varnothing,\, \partial_\tau\gamma = \{0, t\}\}\,,\\ \mathfrak{L}_t^{\sigma\tau} &\doteqdot \{\gamma \,:\, \partial_\sigma\gamma = \{0, t\},\, \partial_\tau\gamma = \{0, t\}\}\,.\end{aligned} \tag{12.26}$$

The aim of the present section is to compare the sums over contours connecting the points 0 and t with and without the constraint $\gamma \subset \mathcal{B}_t$. The procedure is very similar to what is done in Section 4.4, the only real modification being related to the problem of finding cutting-points for the contour, which we solve using Lemma 12.2.1.
It is useful to introduce the following quantities, similar to those introduced in Section 4.4,

$$\chi^\sigma(a, 0) \doteqdot \sum_{\substack{t \in \mathbb{Z}^2: \\ t(1) = a}} \langle \sigma(0)\sigma(t)\rangle^{\underline{J}}, \quad \chi^\sigma(a, a) \doteqdot \sum_{\substack{t \in \mathbb{Z}^2: \\ t(1)+t(2) = a}} \langle \sigma(0)\sigma(t)\rangle^{\underline{J}}. \tag{12.27}$$

We define similarly the quantities $\chi^\tau(a, 0)$, $\chi^\tau(a, a)$, $\chi^{\sigma\tau}(a, 0)$ and $\chi^{\sigma\tau}(a, a)$.

Lemma 12.3.2. *Let $\underline{J}$ be independent of e. Then*

$$\begin{aligned}\lim_{a\to\infty} -\tfrac{1}{a} \log \chi^\sigma(a, 0) &= \tau_\sigma((1, 0); \underline{J}^*)\,,\\ \lim_{a\to\infty} -\tfrac{1}{a} \log \chi^\sigma(a, a) &= \tau_\sigma(\tfrac{1}{\sqrt{2}}(1, 1); \underline{J}^*)\,.\end{aligned}$$

Similar results hold for the quantities $\chi^{\tau}(a,0)$, $\chi^{\tau}(a,a)$, $\chi^{\sigma\tau}(a,0)$ *and* $\chi^{\sigma\tau}(a,a)$.

As in the first part, we first consider the simpler problem of a sum over contours constrained to remain on the right-hand side of the vertical line ($a \in \mathbb{N}$ being a strictly positive number)

$$l_a \doteqdot \{s \in \mathbb{Z}^2 \,:\, s(1) = -a\}\,. \tag{12.28}$$

Lemma 12.3.3. *Let* $\underline{J}$ *be translation invariant. Let* $t \in \mathbb{Z}^2$ *such that* $t(1) \geqslant 0$ *and let* $\mathfrak{E}^{\sigma}_a(t)$ *be the set*

$$\mathfrak{E}^{\sigma}_a(t) \doteqdot \{\gamma \in \mathfrak{L}^{\sigma}_t \,:\, \gamma \cap l_a \neq \varnothing\}\,.$$

Let $a' = a - \log\|t\|_1$. *Suppose that* $\overline{a} \doteqdot \min\{2a' + t(1) - |t(2)|, 2a'\} - \log\|t\|_1$ *is strictly positive and* $TL < S$. *Then there exists* $\beta_0 > 0$ *such that for all* $\beta < \beta_0$,

$$\sum_{\gamma \in \mathfrak{E}^{\sigma}_a(t)} q(\gamma;\beta\underline{J}) \leqslant \left((8\chi^{\sigma}(\overline{a},\overline{a})|t(2)|\log\|t\|_1^2 + 1)(\tanh\mathcal{O}(\beta))^{\log\|t\|_1} + \chi^{\sigma}(a',0)\right)\langle\sigma(0)\sigma(t)\rangle^{\beta\underline{J}}\,.$$

Similar results hold for contours γ *in* $\mathfrak{L}^{\tau}_t$ *(if* $SL < T$*) or* $\mathfrak{L}^{\sigma\tau}_t$ *(if* $ST < L$*).*

Proof. Let λ be the open σ-contour of some contour $\gamma \in \mathfrak{L}^{\sigma}_t$ intersecting the line l_a. The proof is essentially the same as that of Lemma 4.4.2, the main difference being that the point of intersection between a contour γ and the line l_a is not necessarily a λ-cutting-point of γ. However, as a consequence of Lemma 12.2.1, we know that every typical γ has a λ-cutting-point in the set

$$\mathcal{V}^{\sigma}_a \doteqdot \{t \in \mathbb{Z}^2 \,:\, -a \leqslant t \leqslant -a + \log\|t\|_1\}\,. \tag{12.29}$$

Therefore the sum over all contours γ in $\mathfrak{L}^{\sigma}_t$ can be bounded above by

$$\sum_{u \in \mathcal{V}_a} \sum_{\substack{\gamma \in \mathfrak{E}^{\sigma}_a(t):\\ u\ \lambda\text{-cut.-pt of } \gamma}} q(\gamma) + (\tanh\mathcal{O}(\beta))^{\log\|t\|_1}\langle\sigma(0)\sigma(t)\rangle^{\beta\underline{J}}\,. \tag{12.30}$$

The first sum can be estimated as in the proof of Lemma 4.4.2, using Lemma 12.3.1. □

Using this lemma, it is possible to prove the following proposition.

Proposition 12.3.1. *Let* $\underline{J}(e)$ *be independent of* e. *Suppose* $t \in \mathbb{Z}^2$ *is such that* $0 \leqslant t(2) \leqslant t(1)$. *Let* $a \in \mathbb{N}$ *with* $2a < t(1)$. *If* $\mathcal{B}_t$ *is the box* (12.25), *then there exists* $\beta_0 > 0$ *such that for all* $\beta < \beta_0$,

$$\sum_{\substack{\gamma \in \mathfrak{L}^{\sigma}_t:\\ \gamma \text{ inside } \mathcal{B}_t}} q(\gamma;\beta\underline{J}) \geqslant$$
$$\langle\sigma(0)\sigma(t)\rangle^{\beta\underline{J}}\left[1 - \mathcal{O}(\|t\|_1\log\|t\|_1\exp\{-\mathcal{O}(a)\} + (\tanh\mathcal{O}(\beta))^{\log\|t\|_1})\right]\exp\{-\mathcal{O}(a)\}\,,$$

where γ *inside* $\mathcal{B}_t$ *means* $\gamma \subset \mathcal{B}_t$ *and* $\gamma \cap \partial\mathcal{B}_t = \{0,t\}$. *Similar results hold for* γ *in* $\mathfrak{L}^{\tau}_t$ *or* $\mathfrak{L}^{\sigma\tau}_t$.

Proof. This is proved in the same way as Proposition 4.4.1, we only have to check that the result corresponding to Lemma 4.4.3 can be established for the Ashkin–Teller model.

We explain here the modification to the proof of Lemma 4.4.3 needed in the present case. Let $\gamma' \subset \mathcal{B}_t$ be some contour with $\partial_\sigma \gamma' = \{u_a, v_a\}$, $\partial_\tau \gamma' = \varnothing$, and let λ' be the corresponding open σ-contour. We would like to break γ' into three pieces as in the proof of Lemma 4.4.3; however in the present case, it is not obvious that the points t_1 and t_2 of that proof are λ'-cutting-points of γ'. Nevertheless, Lemma 12.2.1[3] and a simple cluster expansion estimate provides the following result:

$$\sum_{\gamma'} q(\gamma') = (1 + \tanh \mathcal{O}(\beta)) \sum_{\substack{\gamma': \\ t_1, t_2 \text{ cut.-pts of } \gamma'}} q(\gamma') \,, \tag{12.31}$$

where the summation is over contours γ' as described above (from Lemma 12.2.1, we know that we can take $\gamma' \in \mathcal{T}^\sigma(u_a, v_a)$; it is then enough to forbid those contours for which t_1 and t_2 are not cutting points.). We can then proceed with the proof of Lemma 4.4.3, up to simple modifications: we glue together the part of λ' between t_1 and t_2 and two shortest paths of σ-edges from 0 to t_1 and from t_2 to t to obtain a contour γ such that $\partial_\sigma \gamma = \{0, t\}$, $\partial_\tau \gamma = \varnothing$. □

[3] Notice that the proof of the lemma is unchanged if we add a constraint like $\gamma \subset \mathcal{B}_t$.

Chapter 13

Large deviations

Let $\Lambda_L \doteqdot \{t \in \mathbb{Z}^2 : \|t\|_\infty \leqslant L\}$ and let $\underline{J}(e)$ be translation-invariant, ferromagnetic couplings. We introduce the spontaneous magnetizations in the σ- and τ-planes,

$$m^*_\sigma(\beta\underline{J}) \doteqdot \lim_{L\to\infty} \langle\sigma(0)\rangle^{+,\beta\underline{J}}_{\Lambda_L}\,, \tag{13.1}$$

$$m^*_\tau(\beta\underline{J}) \doteqdot \lim_{L\to\infty} \langle\tau(0)\rangle^{+,\beta\underline{J}}_{\Lambda_L}\,. \tag{13.2}$$

GKS inequalities give immediately that $m^*_\sigma(\beta\underline{J}) > 0$ and $m^*_\tau(\beta\underline{J}) > 0$ if β is large enough and J_σ and J_τ are strictly positive (it is sufficient to set $J_{\sigma\tau}(e) = 0$ for all edges).

In this chapter, our aim is to study the large deviations of the magnetization in the Ashkin–Teller model. More precisely, we are interested in the probability of the events

$$\mathcal{A}_\sigma(m_\sigma, c) \doteqdot \{\omega : |\sum_{t\in\Lambda_L} \sigma(t)(\omega) - m_\sigma|\Lambda_L|| \leqslant |\Lambda_L|L^{-c}\}\,, \tag{13.3}$$

$$\mathcal{A}_\tau(m_\sigma, c) \doteqdot \{\omega : |\sum_{t\in\Lambda_L} \tau(t)(\omega) - m_\tau|\Lambda_L|| \leqslant |\Lambda_L|L^{-c}\}\,, \tag{13.4}$$

and

$$\mathcal{A}_{\sigma\tau}(m_\sigma, m_\tau, c) \doteqdot \mathcal{A}_\sigma(m_\sigma, c) \cap \mathcal{A}_\tau(m_\tau, c)\,, \tag{13.5}$$

where $|m_\sigma| < m^*_\sigma(\beta\underline{J})$ and $|m_\tau| < m^*_\tau(\beta\underline{J})$. In fact, we only consider the cases where the magnetizations m_σ and m_τ are close enough to the corresponding spontaneous magnetizations, so that the solution of the unconstrained variational problems (see below) can always be placed inside the box[1,2].

[1]This is not a necessary restriction, since the techniques used in Part I to take these effects into account still apply. However, in this chapter, we try to simplify the problem as much as possible so that the new difficulties, those which were not present in the Ising model case, are not hidden among technicalities; it is not our aim to obtain the best estimates in this case, but rather to give a flavour of what are the new problems arising when more complicated models are considered).

[2]Observe that the wall surface tension is not smaller than the bulk one. Indeed GKS inequalities imply that if $x, y \in \Sigma$, then $\langle\sigma(x)\sigma(y)\rangle_\Lambda \leqslant \langle\sigma(x)\sigma(y)\rangle = \exp\{-\tau_\sigma(y-x)\}$, and therefore the massgap of the boundary 2-point function is never smaller than the massgap of the corresponding bulk 2-point function (the same was true for the Ising model, since $\tau_{\mathrm{bd}}(\beta, 1) \leqslant \tau((1,0);\beta)$, see Proposition 3.1.1).

In Section 13.3, we give some lower bounds on the probability of these three events. We prove the corresponding upper bounds for the first two events in Section 13.4; we do not know how to prove the corresponding upper bound for $\mathcal{A}_{\sigma\tau}(m_\sigma, m_\tau, c)$.

The proofs are very similar to the corresponding ones in Part I. For this reason, we only sketch them, insisting on the new difficulties which arise in the present case. Most of the tools we need have been introduced in the preceding Chapters, however we have discussed neither lower bounds on the 2-point functions, nor the large deviations in the phase of small contours.

General lower bounds on 2-point functions are known in the Ising model only thanks to the exact solution, so it is not surprising that we cannot do anything better in the present case. Nevertheless, it is possible to make a perturbative study, combining the techniques of [DKS1] with arguments similar to those appearing in the proof of Lemma 12.2.1. We do not want to do that here, and therefore the proofs in this chapter do not use any lower bounds on 2-point functions, but only the uniform convergence to the limits, which provide slightly weaker results.

The methods used to study the large deviations in the phase of small contours in the first Part fail in the present case. There are several reasons, the main one being the non-validity of FKG inequalities for the Ashkin–Teller model. The other techniques used in [I2], which rely on the Random–Cluster representation of the Ising model, also seem to fail (the reason being that the presence of a dual σ-cluster separating the lattice into two components does *not* decouple the subsystems in these two regions since τ-clusters can transmit information from one to the other). The perturbative analysis done in [Pf1] can however be extended in a straightforward way to the Ashkin–Teller model. The corresponding results are stated in the next section.

13.1 The phase of small contours

This section deals with some basic results about the phase of small contours, which play an important role in the rest of this chapter. Contrarily to what is done in Chapter 5, the techniques used in the present section rely on cluster expansion estimates and therefore are perturbative. The statements made here are straightforward generalizations of the corresponding ones in Chapter 4 of [Pf1]; the main point is that Theorem 4.1 of that paper can be easily extended to cover the case of the Ashkin–Teller model (for ferromagnetic couplings). We do not give the proof here, since it would essentially follow the one given there.

As before, we write, for any $t \in \mathbb{Z}^2$ and $\delta > 0$,

$$\mathcal{D}(t,\delta) \doteqdot \{t' \in \mathbb{Z}^2 \,:\, \|t'-t\|_\infty\} \leqslant \delta/2\,. \tag{13.6}$$

Definition.

(D224) *A contour $\gamma = (\underline{\gamma}^\sigma, \underline{\gamma}^\tau)$ is s-**small** if there exists $t \in \mathbb{Z}^2$ such that $\overline{\mathrm{int}}\gamma^\sigma \subset \mathcal{D}(t,s)$, for all $\gamma^\sigma \in \underline{\gamma}^\sigma$ and $\overline{\mathrm{int}}\gamma^\tau \subset \mathcal{D}(t,s)$, for all $\gamma^\tau \in \underline{\gamma}^\tau$.*

(D225) *A contour is s-**large** if it is not s-small.*

We sometimes omit to write down explicitly the size s if there is no ambiguity. The phase of s-small contours in $\Lambda \subset \mathbb{Z}^2$ is described by the following conditional probability,

$$P_\Lambda^{++,s}[\,\cdot\,] \doteqdot P_\Lambda^{++}[\,\cdot\,|\{\text{All contours are } s\text{-small}\}]\,. \tag{13.7}$$

The corresponding expectation values are denoted by $\langle\,\cdot\,\rangle_\Lambda^{++,s}$ or $\langle\,\cdot\,\rangle_\Lambda^{++,\underline{J},s}$. The first result states that expectation values of local observables in the phase of small contours are close to the corresponding expectation values in the unconstrained phase.

Lemma 13.1.1. *Suppose that the coupling constants are ferromagnetic, and that $J_\sigma(e)$ and $J_\tau(e)$ are bounded below by strictly positive constants independent of e. Let $\Lambda \subset \mathbb{Z}^2$ be a simply connected set and let A, B be two finite subsets of Λ. Then there exists β_0 such that, for all $\beta > \beta_0$,*

$$\left|\langle\sigma_A\tau_B\rangle_\Lambda^{++,s,\beta\underline{J}} - \langle\sigma_A\tau_B\rangle_\Lambda^{++,\beta\underline{J}}\right| \leqslant |A\cup B|\,\mathrm{e}^{-\mathcal{O}(\beta s)}\,\langle\sigma_A\tau_B\rangle_\Lambda^{++,\beta\underline{J}}\,.$$

Proof. The proof is a straightforward generalization of the corresponding Lemma 4.3 in [Pf1]. □

The second result of this Section is the analogue of Proposition 5.2.1, which provides an estimate of the probability of large deviations in the phase of small contours.

Proposition 13.1.1. *Suppose that the coupling constants are ferromagnetic, and that $J_\sigma(e)$ and $J_\tau(e)$ are bounded below by strictly positive constants independent of e. Let $\delta > c > 0$, $C' > 0$, $C'' > 0$ and set $s = L^\delta$. Let $\Lambda \subset \mathbb{Z}^2$ be finite and simply connected, such that $|\Lambda| = CL^2$, $C > 0$. For each of the connected components of Λ, the boundary conditions are either $++$, or $+-$, or $-+$, or $--$. We write $P_\Lambda^{\star,\beta\underline{J}}[\,\cdot\,]$ and $\langle\,\cdot\,\rangle_\Lambda^{\star,\beta\underline{J}}$ for the probability measure and the expectation value in the phase of small contours with these boundary conditions. Then there exists β_0, independent of s and Λ, such that, for all $\beta > \beta_0$,*

$$\begin{aligned} P_\Lambda^{\star,\beta\underline{J}}[\,\{\big|\sum_{t\in\Lambda}(\sigma(t) - \langle\sigma(t)\rangle_\Lambda^{\star,\beta\underline{J}})\big| \geqslant C'|\Lambda|L^{-c}\}\,] &\leqslant \exp\{-\mathcal{O}(\beta L^{2-c-\delta})\}\,,\\ P_\Lambda^{\star,\beta\underline{J}}[\,\{\big|\sum_{t\in\Lambda}(\tau(t) - \langle\tau(t)\rangle_\Lambda^{\star,\beta\underline{J}})\big| \geqslant C''|\Lambda|L^{-c}\}\,] &\leqslant \exp\{-\mathcal{O}(\beta L^{2-c-\delta})\}\,. \end{aligned}$$

Proof. This is proved in the same way as Theorem 5.1 in [Pf1]. □

13.2 The variational problem

We first discuss the variational problems associated to the three events introduced at the beginning of this chapter.

Let $(\mathcal{C}_1^\sigma \ldots, \mathcal{C}_{n_\sigma}^\sigma, \mathcal{C}_1^\tau \ldots, \mathcal{C}_{n_\tau}^\tau)$ be a family of closed rectifiable curves, which are the boundaries of some open sets; we call σ-curves the curves $\mathcal{C}_1^\sigma \ldots, \mathcal{C}_{n_\sigma}^\sigma$, and τ-curves the curves $\mathcal{C}_1^\tau \ldots, \mathcal{C}_{n_\tau}^\tau$. We introduce

$$\begin{aligned} C_{\sigma\tau} &\doteqdot (\bigcup_i \mathcal{C}_i^\sigma) \cap (\bigcup_i \mathcal{C}_i^\tau)\,, \\ C_\sigma &\doteqdot (\bigcup_i \mathcal{C}_i^\sigma) \setminus C_{\sigma\tau}\,, \\ C_\tau &\doteqdot (\bigcup_i \mathcal{C}_i^\tau) \setminus C_{\sigma\tau}\,. \end{aligned} \tag{13.8}$$

We introduce the following functional on the set of such families.

$$\begin{aligned} \mathfrak{F}(\mathcal{C}_1^\sigma \ldots, \mathcal{C}_{n_\sigma}^\sigma, \mathcal{C}_1^\tau \ldots, \mathcal{C}_{n_\tau}^\tau) \doteqdot \int \tau_\sigma((-\dot v_\sigma(s), \dot u_\sigma(s))\,\mathrm{d}s + \int \tau_\tau((-\dot v_\tau(s), \dot u_\tau(s))\,\mathrm{d}s \\ + \int \tau_{\sigma\tau}((-\dot v_{\sigma\tau}(s), \dot u_{\sigma\tau}(s))\,\mathrm{d}s\,, \end{aligned} \tag{13.9}$$

where $(u_\sigma(s), v_\sigma(s))$, $(u_\tau(s), v_\tau(s))$ and $(u_{\sigma\tau}(s), v_{\sigma\tau}(s))$ are unit-speed parameterizations of C_σ, C_τ and $C_{\sigma\tau}$, respectively.

We consider three variational problems corresponding to the events $\mathcal{A}_\sigma(m_\sigma, c)$, $\mathcal{A}_\tau(m_\tau, c)$ and $\mathcal{A}_{\sigma\tau}(m_\sigma, m_\tau, c)$ (in this order).

Variational Problem I : Find the minimum of the functional $\mathfrak{F}$ among all rectifiable closed σ-curves, which are the boundary of an open set of Lebesgue measure V^σ.

Variational Problem II : Find the minimum of the functional $\mathfrak{F}$ among all rectifiable closed τ-curves, which are the boundary of an open set of Lebesgue measure V^τ.

Variational Problem III : Find the minimum of the functional $\mathfrak{F}$ among all families of rectifiable closed σ- and τ-curves $(\mathcal{C}_1^\sigma, \ldots, \mathcal{C}_{n_\sigma}^\sigma, \mathcal{C}_1^\tau \ldots, \mathcal{C}_{n_\tau}^\tau)$ such that

- The closed subsets $\overline{\operatorname{int}\mathcal{C}_i^\sigma}$ of $\mathbb{R}^2$ with boundary $\mathcal{C}_i^\sigma$ are disjoint;
- The closed subsets $\overline{\operatorname{int}\mathcal{C}_i^\sigma}$ of $\mathbb{R}^2$ with boundary $\mathcal{C}_i^\tau$ are disjoint;
- The Lebesgue measure of $\bigcup_i \operatorname{int}\mathcal{C}_i^\sigma$ is V^σ;
- The Lebesgue measure of $\bigcup_i \operatorname{int}\mathcal{C}_i^\tau$ is V^τ.

The solutions to the variational problems I and II are given by the Wulff construction with surface tension $\tau_\sigma(\cdot)$ and $\tau_\tau(\cdot)$, respectively. Unfortunately, the solution of the variational problem associated to the event $\mathcal{A}_{\sigma\tau}(m_\sigma, m_\tau, c)$ is not known. We could try to make an analysis similar to that of Subsection 7.5.1. We do not want to do that here. Instead, we show that any such family of curves provides us with a lower bound. The best one is obtained with the family which solves the variational problem, if it exists. We suppose that this is the case; for this reason, we do not try to obtain *uniform* estimates which would be much more complicated.

Let $\mathcal{M}(V^\sigma, V^\tau)$ be the set of all families $(\mathcal{C}_1^\sigma \ldots, \mathcal{C}_{n_\sigma}^\sigma, \mathcal{C}_1^\tau \ldots, \mathcal{C}_{n_\tau}^\tau)$ such that

- Each curve is the boundary of an open set in $\mathbb{R}^2$;
- The closed subsets $\overline{\operatorname{int}\mathcal{C}_i^\sigma}$ of $\mathbb{R}^2$ with boundary $\mathcal{C}_i^\sigma$ are disjoint;
- The closed subsets $\overline{\operatorname{int}\mathcal{C}_i^\sigma}$ of $\mathbb{R}^2$ with boundary $\mathcal{C}_i^\tau$ are disjoint;
- The Lebesgue measure of $\bigcup_i \operatorname{int}\mathcal{C}_i^\sigma$ is V^σ;
- The Lebesgue measure of $\bigcup_i \operatorname{int}\mathcal{C}_i^\tau$ is V^τ.

13.3 The lower bounds

Before considering the main question of this section, we state a simple useful lemma.

Lemma 13.3.1. *Let $\underline{J}(e)$ be independent of e and ferromagnetic; J_σ and J_τ are supposed to be strictly positive. Let A and B be two subsets of Λ_L and let*

$$d \doteqdot \min\{\|t' - t\|_1 \,:\, t \in A \cup B\,,\, t' \notin \Lambda_L\}\,.$$

There exists β_0 such that, for all $\beta > \beta_0$,

$$|\langle \sigma_A \tau_B \rangle_{\Lambda_L}^{++,\beta \underline{J}} - \langle \sigma_A \tau_B \rangle_{\Lambda_L}^{++,\beta \underline{J}}| \leqslant \mathrm{e}^{-|A\cup B|\,\mathrm{e}^{-\mathcal{O}(\beta d)}}\ \langle \sigma_A \tau_B \rangle_{\Lambda_L}^{++,\beta \underline{J}}$$

Proof. This is very easy to show using the following trick, which is due to Kunz and Souillard. Defining new weights for contours by

$$w'(\gamma) = \sigma_A(\omega_\gamma)\tau_B(\omega_\gamma)\, w(\gamma)\,, \tag{13.10}$$

where ω_γ is the unique configuration such that $\omega_\gamma(t) = (1,1)$ for all but a finite number of sites t, and such that γ is the unique contour of ω_γ. Notice that $|w'(\gamma)| = w(\gamma)$; in particular, the cluster expansion with weights ω' converges absolutely if β is large enough. With these weights, it is possible to write the numerator of the expectation value as

$$Z_{\Lambda_L}^{++,\beta \underline{J}}\, \langle \sigma_A \tau_B \rangle_{\Lambda_L}^{++,\beta \underline{J}} = 1 + \sum_{n\geqslant 1} \frac{1}{n!} \sum_{\substack{\gamma_1,\ldots,\gamma_n \\ \text{compatible}}} \prod_{k=1}^{n} w'(\gamma_k)\,. \tag{13.11}$$

Therefore

$$\langle \sigma_A \tau_B \rangle_{\Lambda_L}^{++,\beta \underline{J}} = \exp\Big\{ \sum_{n\geqslant 1} \frac{1}{n!} \sum_{\gamma_1,\ldots,\gamma_n} \varphi_n^{\mathrm{T}}(\gamma_1,\ldots,\gamma_n) \big(\prod_{k=1}^{n} w'(\gamma_k) - \prod_{k=1}^{n} w(\gamma_k) \big) \Big\}\,, \tag{13.12}$$

so that only clusters containing at least one contour γ with $w'(\gamma) \neq w(\gamma)$ contribute; this implies that these clusters contain some site of $A \cup B$ in the interior of one of their σ- or τ-contours. The lemma follows by considering a set $\Lambda_{L'}$, $L' > L$, and taking the ratio of the expectation values in Λ_L and $\Lambda_{L'}$, since the only contributing clusters will contain simultaneously some sites of $A \cup B$ and $\Lambda_{L'} \setminus \Lambda_L$ (observe that this gives an estimate uniform in $L' > L$). □

We can now state the main result of this section, which gives a lower bound on the probability of the three large deviations events.

Theorem 13.3.1. *Let $\underline{J}(e)$ be independent of e and ferromagnetic; J_σ and J_τ are supposed to be strictly positive. Let $m_\sigma < m_\sigma^*(\beta \underline{J})$ be close enough to $m_\sigma^*(\beta \underline{J})$ and $m_\tau < m_\tau^*(\beta \underline{J})$ be close enough to $m_\tau^*(\beta \underline{J})$; we write $V^\sigma \doteqdot (m_\sigma^* - m_\sigma)/2m_\sigma^*$ and $V^\tau \doteqdot (m_\tau^* - m_\tau)/2m_\tau^*$. Let $1/2 > c > 0$.*

1. *Let $\mathcal{C}$ be the σ-curve which is the boundary of the Wulff shape of volume $V^\sigma(\beta; \underline{J})$ corresponding to the surface tension $\tau_\sigma(x; \beta \underline{J})$. Then, for any $\varepsilon > 0$, there exists β_0 independent of ε and $L_0 = L_0(\varepsilon)$ such that, for all $\beta > \beta_0$ and all $L \geqslant L_0$,*

$$P_{\Lambda_L}^{++,\beta \underline{J}}[\mathcal{A}_\sigma(m_\sigma, c)] \geqslant \exp\{-L\mathfrak{F}(\mathcal{C})(1+\varepsilon)\}\,.$$

2. *Let $\mathcal{C}$ be the τ-curve which is the boundary of the Wulff shape of volume $V^\tau(\beta;\underline{J})$ corresponding to the surface tension $\tau_\tau(x;\beta\underline{J})$. Then, for any $\varepsilon > 0$, there exists β_0 independent of ε and $L_0 = L_0(\varepsilon)$ such that, for all $\beta > \beta_0$ and all $L \geqslant L_0$,*
$$P_{\Lambda_L}^{++,\beta\underline{J}}[\mathcal{A}_\tau(m_\tau, c)] \geqslant \exp\{-L\mathfrak{F}(\mathcal{C})(1+\varepsilon)\}\,.$$
3. *Let $(\underline{\mathcal{C}}^\sigma, \underline{\mathcal{C}}^\tau) \in \mathcal{M}(V^\sigma(\beta;\underline{J}), V^\tau(\beta;\underline{J}))$. Then, for any $\varepsilon > 0$, there exists β_0 independent of ε and $L_0 = L_0(\varepsilon, (\underline{\mathcal{C}}^\sigma, \underline{\mathcal{C}}^\tau))$ such that, for all $\beta > \beta_0$ and all $L \geqslant L_0$,*
$$P_{\Lambda_L}^{++,\beta\underline{J}}[\mathcal{A}_{\sigma\tau}(m_\sigma, m_\tau, c)] \geqslant \exp\{-L\mathfrak{F}(\underline{\mathcal{C}}^\sigma, \underline{\mathcal{C}}^\tau)(1+\varepsilon)\}\,.$$

Proof. We only prove the third statement, since the other ones can be proved in the same way (in fact their proof is easier). Since the case $J_{\sigma\tau} = 0$ can be readily reduced to the case of the Ising model of Part I, we suppose that $J_{\sigma\tau} \neq 0$. We follow the proof of Theorem 7.5.1.

Step 1. Polygonal approximation of the family.

Let
$$\delta_L \doteqdot L^{-1/2}\log L\,. \tag{13.13}$$
For each (maximal) connected component of C_σ, C_τ and $C_{\sigma\tau}$, we construct a polygonal approximation, similarly to what is done in Chapter 7. Let us denote by $\mathcal{C}_i$ such a component. The polygonal approximation of $\mathcal{C}_i$ is the polygonal curve with vertices $x_1^i, \ldots, x_{n_i}^i$ given by the following construction. Let $s \mapsto \mathcal{C}_i(s)$, $s \in [0,1]$, be a parameterization of $\mathcal{C}_i$,

1. $x_1 \doteqdot \mathcal{C}_i(0)$;
2. x_{n+1} is the first point on the curve $\mathcal{C}_i$ which does not belong to the disk of center x_n and radius δ_L;
3. the last vertex is $t_{n_i} \doteqdot \mathcal{C}_i(1)$.

The collection of all these polygonal curves is the polygonal approximation of the family, which we write $\mathcal{P}_L$. It is convenient to decompose $\mathcal{P}_L$ into a family of closed polygonal σ- and τ-curves, $\mathcal{P}_L \equiv (\underline{\mathcal{P}}_L^\sigma, \underline{\mathcal{P}}_L^\tau)$. $\mathcal{P}_L$ has the following properties,

- $\mathfrak{F}(\mathcal{P}_L) \leqslant \mathfrak{F}((\underline{\mathcal{C}}^\sigma, \underline{\mathcal{C}}^\tau))$.
- The sum of the Lebesgue measures of the sets delimited by the curves in $\underline{\mathcal{P}}_L^\sigma$ differs from V^σ by at most $\pi\delta_L^2\mathcal{O}(\delta_L^{-1}) = \mathcal{O}(\delta_L)$.
- The sum of the Lebesgue measures of the sets delimited by the curves in $\underline{\mathcal{P}}_L^\tau$ differs from V^τ by at most $\pi\delta_L^2\mathcal{O}(\delta_L^{-1}) = \mathcal{O}(\delta_L)$.

The first statement follows from Jensen's inequality, while the two others are a consequence of the construction of $\mathcal{P}_L$.

For each pair of successive vertices such that their distance is δ_L, we construct the square box with the two vertices on its boundary, with vertical and horizontal sides, and which is divided into two equal parts by the straight line segment connecting these two vertices. Any such box which has an intersection with another one is removed. The total length of the sides of $\mathcal{P}_L$ which have no box associated to them goes to zero when L is large enough[3].

Step 2. Scaling and definition of the set of contours.

Let $L\mathcal{P}_L \equiv (L\underline{\mathcal{P}}_L^\sigma, L\underline{\mathcal{P}}_L^\tau)$ be the family of closed polygonal lines obtained by scaling the

[3]Remember that we are not looking for *uniform* estimates!

family $\mathcal{P}_L$ by a factor L, shifting it by $(0, -1/2)$, and modifying, if necessary, the position of the vertices so that they belong to Λ_L^*. If m_σ and m_τ are close enough to m_σ^* and m_τ^*, then all vertices of $L\mathcal{P}_L$ are at a distance of order $\mathcal{O}(L)$ from the boundary of Λ_L^*.

We define now a set $\mathfrak{G}_L$ of Λ_L^*-compatible families of contours. We construct the configuration by gluing together several open σ- and τ-contours; notice however that, at the end, the resulting contours are closed.

- Let t_1 and t_2 be two successive vertices of some maximal connected component of C_σ. If there is a box associated to them, then there is an open contour γ such that $\partial_\sigma\gamma = \{t_1, t_2\}$, $\partial_\tau\gamma = \varnothing$ and $\gamma \cap \partial\mathcal{B}(t_1, t_2) = \{t_1, t_2\}$, where $\mathcal{B}(t_1, t_2)$ is a translate (which has possibly been rotated by $\pi/2$) of the box (12.25) chosen so that t_1 and t_2 play the roles of 0 and t. Otherwise, there is a shortest path of σ-edges with endpoints t_1 and t_2.
- We make the corresponding construction for the maximal connected components of C_τ and $C_{\sigma\tau}$.

The contours obtained by gluing together these open contours contain some number of large σ- and τ-contours (those that are not contained inside one of the box); we call them the *large σ- and τ-contours*. The other (smaller) contours obtained when we remove the large ones are called the *decorations*. We evaluate the probability of $\mathcal{A}_{\sigma\tau}(m_\sigma, m_\tau, c)$ by conditioning on the presence of a given Λ_L^*-compatible family $\underline{\gamma} \in \mathfrak{G}_L$. More precisely, let $s = L^{-\delta}$; introducing

$$\mathfrak{G}_L^s(\underline{\gamma}) \doteqdot \{\underline{\gamma}' : \underline{\gamma} \subset \underline{\gamma}',\ \underline{\gamma} \in \mathfrak{G}_L,\ \text{all other contours are } s\text{-small}\}, \tag{13.14}$$

we write this probability as

$$\sum_{\underline{\gamma}\in\mathfrak{G}_L} P_{\Lambda_L}^{++,\beta\underline{J}}[\mathcal{A}_{\sigma\tau}(m_\sigma, m_\tau, c)|\mathfrak{G}_L^s(\underline{\gamma})]\, P_{\Lambda_L}^{++,\beta\underline{J}}[\mathfrak{G}_L^s(\underline{\gamma})]\,. \tag{13.15}$$

Step 3. Estimation of $P_{\Lambda_L}^{++,\beta\underline{J}}[\mathcal{A}_{\sigma\tau}(m_\sigma, m_\tau, c)|\mathfrak{G}_L^s(\underline{\gamma})]$.

The first thing to realize is that the family of contours $\underline{\gamma}$ divides Λ_L in several parts with different boundary conditions and a remaining region corresponding to the interior of the decorations and which has a volume of order $\mathcal{O}(L^2\delta_L)$. The mean value of the magnetizations in the σ and τ planes, conditioned on the presence of $\underline{\gamma}$, are given by (see Lemma 13.3.1)

$$\langle \sum_{t\in\Lambda_L} \sigma(t)|\mathfrak{G}_L^s(\underline{\gamma})\rangle_{\Lambda_L}^{++,\beta\underline{J}} = m_\sigma \pm \mathcal{O}(L^2\delta_L)\,, \tag{13.16}$$

$$\langle \sum_{t\in\Lambda_L} \tau(t)|\mathfrak{G}_L^s(\underline{\gamma})\rangle_{\Lambda_L}^{++,\beta\underline{J}} = m_\tau \pm \mathcal{O}(L^2\delta_L)\,. \tag{13.17}$$

Therefore the conditional probability of the event complementary to $\mathcal{A}_{\sigma\tau}(m_\sigma, m_\tau, c)$ is bounded above by the probability of having a deviation of order $\mathcal{O}(L^{2-c})$ in Λ_L, which can be estimated by Proposition 13.1.1. We therefore obtain

$$P_{\Lambda_L}^{++,\beta\underline{J}}[\mathcal{A}_{\sigma\tau}(m_\sigma, m_\tau, c)|\mathfrak{G}_L^s(\underline{\gamma})] \geqslant 1 - \exp\{-\mathcal{O}(\beta L^{2-c-\delta})\}\,. \tag{13.18}$$

Step 4. Estimation of $P^{++,\beta\underline{J}}_{\Lambda_L}[\mathfrak{G}^s_L(\underline{\gamma})]$.

We can easily remove the constraint on the size of the contours using a cluster expansion estimate. We obtain

$$P^{++,\beta\underline{J}}_{\Lambda_L}[\mathfrak{G}^s_L(\underline{\gamma})] \geqslant (1 - |\Lambda_L| \exp\{-\mathcal{O}(\beta L^{\delta})\})\, q_{\Lambda^*_L}(\underline{\gamma})\,. \tag{13.19}$$

Step 5. Estimation of $P^{++,\beta\underline{J}}_{\Lambda_L}[\mathcal{A}_{\sigma\tau}(m_\sigma, m_\tau, c)]$.

The rest of the proof is done as in Part I, using the corresponding results of Chapter 12. As in the proof of Theorem 7.5.2, we use the uniform convergence of the massgaps instead of a lower bound on the 2-point functions. □

13.4 The upper bounds

We prove now optimality of the lower bounds obtained in the preceding section for the events $\mathcal{A}_\sigma(m_\sigma, c)$ and $\mathcal{A}_\tau(m_\tau, c)$. Unfortunately, we do not know how to handle the event $\mathcal{A}_{\sigma\tau}(m_\sigma, m_\tau, c)$. The proof relies heavily on perturbative arguments; the proof of existence of cutting-points is much more complicated than in the previous sections. The main difficulty is to construct convenient polygonal approximations for the large contours, which are suitable for the application of Lemma 12.1.5 point 2.. More precisely, the most difficult part is to establish a statement similar to (7.64). We restrict our attention to the event $\mathcal{A}_\sigma(m_\sigma, c)$, since $\mathcal{A}_\tau(m_\tau, c)$ can be handled in the same way.

Let $1/2 > c > 0$, $-m^*_\sigma < m_\sigma < m^*_\sigma$ and $s \doteqdot L^a$, where $a \doteqdot 1 - c$; we set $V^\sigma \doteqdot (m^*_\sigma - m_\sigma)/2m^*_\sigma$. We denote by $\underline{\Gamma}^\sigma(\omega)$ the family of all s-large external σ-contours of the configuration ω. The first result states that the total length of these contours is of order L and the total volume is close to $V^\sigma L^2$. Let C_1 and C_2 be some strictly positive constants; we denote by E^σ_1 the set of all Λ^*_L-compatible families of external σ-contours $\underline{\Gamma}^\sigma$ which satisfy the following set of conditions

- $\sum_{\Gamma^\sigma \in \underline{\Gamma}^\sigma} |\Gamma^\sigma| \leqslant C_1 L$;
- $\sum_{\Gamma^\sigma \in \underline{\Gamma}^\sigma} \operatorname{vol} \Gamma^\sigma \geqslant V^\sigma L^2 - C_2 L^{2-c}$.

Conditioned on $\mathcal{A}_\sigma(m_\sigma, c)$, the event E^σ_1 is typical, as is shown in the next lemma.

Lemma 13.4.1. *Let $\underline{J}(e)$ be independent of e and ferromagnetic; we suppose that $J_\sigma > 0$ and $J_\tau > 0$. Let C_1 and C_2 be sufficiently large. There exists $\beta_0 = \beta_0(C_1, C_2)$ and $L_0 = L_0(C_1, C_2)$ such that for all $\beta > \beta_0$ and $L > L_0$,*

$$P^{++,\beta\underline{J}}_{\Lambda_L}[E^\sigma_1 | \mathcal{A}_\sigma(m_\sigma, c)] \geqslant 1 - \exp\{-\mathcal{O}(L)\}\,.$$

Proof. The conditional probability that the total length of the large external σ-contours exceeds $C_1 L$ can be estimated by

$$\begin{aligned} P^{++,\beta\underline{J}}_{\Lambda_L}[\{\sum_{\Gamma^\sigma \in \underline{\Gamma}^\sigma} |\Gamma^\sigma| > C_1 L\} | \mathcal{A}_\sigma(m_\sigma, c)] &\leqslant \frac{P^{++,\beta\underline{J}}_{\Lambda_L}[\{\sum_{\Gamma^\sigma \in \underline{\Gamma}^\sigma} |\Gamma^\sigma| > C_1 L\}]}{P^{++,\beta\underline{J}}_{\Lambda_L}[\mathcal{A}_\sigma(m_\sigma, c)]} \\ &\leqslant \exp\{-\mathcal{O}(L)\}\,, \end{aligned} \tag{13.20}$$

where we have used Theorem 13.3.1 and the fact that C_1 is large enough.
It remains to find an upper bound on the conditional probability of the event

$$E' \doteqdot \{ \sum_{\Gamma^\sigma \in \underline{\Gamma}^\sigma} \mathrm{vol}\,\Gamma^\sigma < V^\sigma L^2 - C_2 L^{2-c} \}\,. \tag{13.21}$$

Let $\underline{\Gamma}^\sigma$ be the family of external large σ-contours of a configuration in E'. We denote by $[\underline{\Gamma}^\sigma]$ the set of all configurations with this family of external large σ-contours. Let $\omega \in [\underline{\Gamma}^\sigma]$ and let $(\hat{\underline{\Gamma}}^\sigma, \hat{\underline{\Gamma}}^\tau)$ be the family of all s-large σ- and τ-contours of ω. Let E'' be the intersection of the events E' and the event that the total length of these contours is bounded above by $C'L$, for some constant $C' > 0$. The same argument as above implies that

$$P_{\Lambda_L}^{++,\beta\underline{J}}[E' | \mathcal{A}_\sigma(m_\sigma, c)] \leqslant P_{\Lambda_L}^{++,\beta\underline{J}}[E'' | \mathcal{A}_\sigma(m_\sigma, c)] + \mathrm{e}^{-\mathcal{O}(L)}\,. \tag{13.22}$$

It is not difficult to estimate the conditional probability of E''. Let $(\hat{\underline{\Gamma}}^\sigma, \hat{\underline{\Gamma}}^\tau)$ be the large σ- and τ-contours of $\omega \in E''$. Since all other contours are s-small, and thanks to Lemmas 13.1.1 and 13.3.1, we can write

$$\begin{aligned} \langle \sum_{t \in \Lambda_L} \sigma(t) | (\hat{\underline{\Gamma}}^\sigma, \hat{\underline{\Gamma}}^\tau) \rangle_{\Lambda_L}^{++,\beta\underline{J}} &\leqslant m_\sigma - \mathcal{O}(L^{-c}) + \mathcal{O}(L^{-1}\log L) + \mathcal{O}(L^{a-1}) \\ &\leqslant m_\sigma - \mathcal{O}(L^{-c})\,, \end{aligned} \tag{13.23}$$

using $c = 1 - a$. Consequently, we have, by Proposition 13.1.1,

$$\begin{aligned} P_{\Lambda_L}^{++,\beta\underline{J}}[E'' | \mathcal{A}_\sigma(m_\sigma, c)] &< \frac{P_{\Lambda_L}^{++,\beta\underline{J}}[\mathcal{A}_\sigma(m_\sigma, c) | E'']}{P_{\Lambda_L}^{++,\beta\underline{J}}[\mathcal{A}_\sigma(m_\sigma, c)]} \\ &\leqslant \exp\{-\mathcal{O}(L)\}\,, \end{aligned} \tag{13.24}$$

if C_2 is large enough, which concludes the proof. □

Since ($E_1^{\sigma c}$ is the complementary event to E_1^σ)

$$P_{\Lambda_L}^{++,\beta\underline{J}}[\mathcal{A}_\sigma(m_\sigma, c)] = P_{\Lambda_L}^{++,\beta\underline{J}}[\mathcal{A}_\sigma(m_\sigma, c) \cap E_1^\sigma] + P_{\Lambda_L}^{++,\beta\underline{J}}[\mathcal{A}_\sigma(m_\sigma, c) \cap E_1^{\sigma c}]\,, \tag{13.25}$$

we can write

$$\begin{aligned} P_{\Lambda_L}^{++,\beta\underline{J}}[\mathcal{A}_\sigma(m_\sigma, c)] &\leqslant \frac{P_{\Lambda_L}^{++,\beta\underline{J}}[E_1^\sigma]}{1 - P_{\Lambda_L}^{++,\beta\underline{J}}[E_1^{\sigma c} | \mathcal{A}_\sigma(m_\sigma, c)]} \\ &\leqslant (1 + \mathrm{e}^{-\mathcal{O}(L)})\, P_{\Lambda_L}^{++,\beta\underline{J}}[E_1^\sigma]\,. \end{aligned} \tag{13.26}$$

Therefore we need to estimate the probability (without constraint) of the event E_1^σ. This is the difficult part of the proof. We would like to construct a coarse-grained version of the contours containing the large external σ-contours. However, since they can behave quite wildly, it is not obvious that we can find cutting-points along them. The main difference with the analysis done for the box Proposition is that we have a constraint on the total volume of the large external σ-contours. For this reason, it is not possible to simply remove all large excitations, since in such a case we would generally lose volume. The next proposition shows that it is fortunately still possible to prove existence of a sufficient number of cutting-points for the typical configurations.

Proposition 13.4.1. *Let $\underline{J}(e)$ be independent of e and ferromagnetic; we suppose that $J_\sigma > 0$ and $J_\tau > J_{\sigma\tau}$. Let $\mathfrak{F}^*(m_\sigma) \doteqdot \mathfrak{F}(\mathcal{C}(m_\sigma))$, where $\mathcal{C}(m_\sigma)$ is the boundary of the Wulff shape of volume V^σ associated to the surface tension $\tau_\sigma(x;\beta\underline{J})$. For any $\varepsilon > 0$, there exists β_0 independent of ε and $L_0 = L_0(\varepsilon)$ such that, for all $\beta > \beta_0$ and all $L > L_0$,*

$$P_{\Lambda_L}^{++,\beta\underline{J}}[E_1^\sigma] \leqslant \exp\{-\mathfrak{F}^*(m_\sigma)L(1-\varepsilon)\}\,.$$

Proof. Let $\omega \in E_1^\sigma$ and let $\underline{\Gamma}^\sigma$ be the family of its external large σ-contours; We write

$$\mathfrak{E}_1^\sigma \doteqdot \{\underline{\Gamma}^\sigma \,:\, \forall\omega \in [\underline{\Gamma}^\sigma],\, \omega \in E_1^\sigma\}\,. \tag{13.27}$$

The complete set of contours of the configuration ω can be written as $(\underline{\Gamma}^\sigma, \underline{\zeta})$, where $\underline{\zeta}$ is the family of all contours of ω except those of $\underline{\Gamma}^\sigma$. We use the short-hand notations (see (D162), p. 196)

$$w_\sigma \doteqdot \mathrm{e}^{-2(J_\sigma+J_{\sigma\tau})}\,, \quad w_\tau \doteqdot \mathrm{e}^{-2(J_\tau+J_{\sigma\tau})}\,, \quad w_{\sigma\tau} \doteqdot \mathrm{e}^{-2(J_\sigma+J_\tau)}\,. \tag{13.28}$$

We consider the family $\underline{\Gamma}^\sigma$ as being build of σ-edges and consequently associate to these edges the weight w_σ. Since, in general some of these edges will also be used by some τ-contours of $\underline{\zeta}$, it is necessary to take the correction into account; for this reason, we define new weights for the contours $\zeta \in \underline{\zeta}$,

$$w(\zeta|\underline{\Gamma}^\sigma) \doteqdot w(\zeta) \prod_{e\in\mathcal{B}(\zeta)\cap\mathcal{B}(\underline{\Gamma}^\sigma)} \frac{w_{\sigma\tau}}{w_\sigma}\,. \tag{13.29}$$

The important observation is that these weights still satisfy a Peierls condition (since $J_\tau > J_{\sigma\tau}$). The main idea of the proof is to simplify step by step the structure of the family $\underline{\Gamma}^\sigma$. We divide the proof in several steps in each of which a different part of this surgery is done.

Step 1. The contours of $\underline{\zeta}$ are $\log L$-small.

An elementary estimate using cluster expansion gives

$$\begin{aligned}\Xi^{++}(\Lambda_L)\, P_{\Lambda_L}^{++,\beta\underline{J}}[E_1^\sigma] &= \sum_{\underline{\Gamma}^\sigma\in\mathfrak{E}_1^\sigma} w_\sigma(\underline{\Gamma}^\sigma) \sum_{\substack{\underline{\zeta}\\ (\underline{\Gamma}^\sigma,\underline{\zeta})\text{comp.}}} \prod_{\zeta\in\underline{\zeta}} w(\zeta|\underline{\Gamma}^\sigma)\\ &= (1+L^{-\mathcal{O}(\beta)}) \sum_{\underline{\Gamma}^\sigma\in\mathfrak{E}_1^\sigma} w_\sigma(\underline{\Gamma}^\sigma) \sum_{\substack{\underline{\zeta}\ \text{small}\\ (\underline{\Gamma}^\sigma,\underline{\zeta})\text{comp.}}} \prod_{\zeta\in\underline{\zeta}} w(\zeta|\underline{\Gamma}^\sigma)\,.\end{aligned} \tag{13.30}$$

Therefore,

$$P_{\Lambda_L}^{++,\beta\underline{J}}[E_1^\sigma] \leqslant (1+L^{-\mathcal{O}(\beta)})\, P_{\Lambda_L}^{++,\beta\underline{J}}[E_1^\sigma \cap E_{\mathrm{s}}^\sigma]\,, \tag{13.31}$$

where E_{s}^σ is the event that all the contours of $\underline{\zeta}$ are $\log L$-small.

Step 2. The contours of $\underline{\Gamma}^\sigma$ are far from one another.

This is the first kind of pathology that may prevent the existence of cutting-points. If two of the large σ-contours are sufficiently close so that some of the contours of $\underline{\zeta}$ can intersect

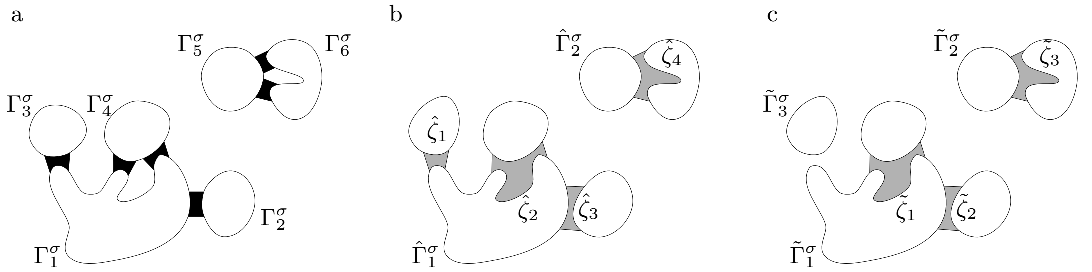

FIGURE 13.1. Surgery: removing the bad-points of type I. (a) We connect the contours which are too close from one another by straight shortest paths (the black parts). (b) By taking the exterior envelope of the contours and the shortest paths, we define a new family $\underline{\hat{\Gamma}}^\sigma$; we also introduce a family $\underline{\hat{\zeta}}$ of closed contours (the boundaries of the shaded regions). (c) The contour $\hat{\zeta}_1$ was too small and has therefore been removed, separating the contour Γ_3^σ from $\hat{\Gamma}_1^\sigma$; the new family of contours is called $\underline{\tilde{\Gamma}}^\sigma$, and the subset of $\underline{\hat{\zeta}}$ is denoted by $\underline{\tilde{\zeta}}$.

both of them at the same time, then the two contours cannot be considered separately. Fortunately, the total number of such "bad points" is small compared to L.

Before giving a precise definition of these "bad points", it is useful to introduce a notion of straight shortest path. A set of edges g is a *straight shortest path between x and y* if the following conditions are verified

- $\partial g = \{x, y\}$;
- its length is minimal;
- the distance (with the norm $\|\cdot\|_1$) between any point of g and the straight line segment joining x and y is at most 1.

Let $D_1 > 1$ be some constant to be chosen later on. We call *D_1-bad points of type I*, the set of all sites $t \in \Gamma_i^\sigma$ such that there exists $t' \in \Gamma_j^\sigma$, $j \neq i$, with $\|t' - t\|_1 < D_1 \log L$.
Let us consider the following set of edges,

$$\hat{\mathcal{E}}(\underline{\Gamma}^\sigma) \doteqdot \mathcal{E}(\underline{\Gamma}^\sigma) \cup \bigcup_g \mathcal{E}(g)\,, \tag{13.32}$$

where g runs through all straight shortest paths between corresponding pairs of D_1-bad points of type I belonging to two different contours (see Fig. 13.1 (a)). We then construct (see Fig. 13.1 (b)) a family $\underline{\hat{\Gamma}}^\sigma$ of large σ-contours by decomposing into contours the exterior envelope[4] of $\hat{\mathcal{E}}(\underline{\Gamma}^\sigma)$. We also define a set of closed contours $\underline{\hat{\zeta}}$ by decomposing into contours the set of edges $\mathcal{E}(\underline{\Gamma}^\sigma) \triangle \mathcal{E}(\underline{\hat{\Gamma}}^\sigma)$ [5]. To each $\hat{\zeta} \in \underline{\hat{\zeta}}$, we associate two numbers: The total number of "bad edges", $b(\hat{\zeta}) \doteqdot |\mathcal{E}(\hat{\zeta}) \cap \mathcal{E}(\underline{\hat{\Gamma}}^\sigma)|$, and the number $\mathcal{N}(\hat{\zeta})$ of maximal connected components of such edges (for example, in Fig. 13.1 (b), $\mathcal{N}(\hat{\zeta}_i) = 2$, $i = 1, \ldots, 4$, and in Fig. 13.2, $\mathcal{N}(\hat{\zeta}_1) = 3$, $\mathcal{N}(\hat{\zeta}_2) = 0$ and $\mathcal{N}(\hat{\zeta}_3) = \mathcal{N}(\hat{\zeta}_4) = 2$); we call $\mathcal{N}(\hat{\zeta}_i)$ the number of *contacts* of $\hat{\zeta}_i$ with the corresponding $\hat{\Gamma}^\sigma$.

We partition the contours $\hat{\zeta} \in \underline{\hat{\zeta}}$ into two classes: the large ones and the small ones. Let $\underline{\tilde{\zeta}} \subset \underline{\hat{\zeta}}$ be the set of all $\hat{\zeta} \in \underline{\hat{\zeta}}$ such that

$$|\hat{\zeta}| > \max\{\log L, 4b(\hat{\zeta})\}\,. \tag{13.33}$$

[4]We call *exterior envelope* of a finite set of edges A the boundary of the infinite component of $\mathbb{R}^2 \setminus A$, seen as a set of edges.

[5]We recall that $A \triangle B \doteqdot (A \setminus B) \cup (B \setminus A)$.

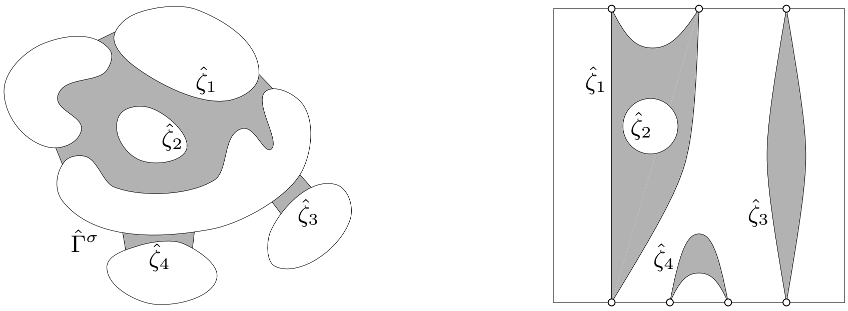

FIGURE 13.2. Left: A contour $\hat\Gamma^\sigma$ and the associated family $\underline{\hat\zeta}$ (notice that the boundary of the large central shaded area has two components). Right: The equivalent representation; the dots correspond to the different contacts of the contours $\hat\zeta$ with the exterior envelope; the shaded component does not correspond to the interior of some contour.

We write $\underline{\tilde\Gamma}^\sigma$ the family of σ-contours obtained in the decomposition into contours of the set of edges (see Fig. 13.1 (c))

$$\mathcal{E}(\underline{\hat\Gamma}^\sigma)\triangle\Big[\bigcup_{\hat\zeta\in\underline{\hat\zeta}\setminus\underline{\tilde\zeta}}\mathcal{E}(\hat\zeta)\Big]\,. \tag{13.34}$$

We introduce the following weight for the contours in $\underline{\tilde\zeta}$,

$$\tilde w(\tilde\zeta|\underline{\tilde\Gamma}^\sigma)=\frac{\prod_{e\in\mathcal{E}(\tilde\zeta)\setminus\mathcal{E}(\underline{\tilde\Gamma}^\sigma)}w_\sigma}{\prod_{e\in\mathcal{E}(\underline{\tilde\Gamma}^\sigma)\cap\mathcal{E}(\tilde\zeta)}w_\sigma}\,. \tag{13.35}$$

Observe that $w_\sigma(\underline{\tilde\Gamma}^\sigma)\tilde w(\underline{\tilde\zeta}|\underline{\tilde\Gamma}^\sigma)=w_\sigma(\underline{\Gamma}^\sigma)$. These weights satisfy a Peierls condition for all $\tilde\zeta\in\underline{\tilde\zeta}$. Indeed, by definition of $\underline{\tilde\zeta}$,

$$w(\tilde\zeta|\underline{\tilde\Gamma}^\sigma)=w_\sigma^{|\tilde\zeta|-2b(\tilde\zeta)}\leqslant w_\sigma^{|\tilde\zeta|/2}\,. \tag{13.36}$$

Let us state some properties of the contours $\underline{\tilde\Gamma}^\sigma$ and $\underline{\tilde\zeta}$.

- $\underline{\tilde\Gamma}^\sigma$ is a Λ_L^*-compatible family of L^a-large, closed, external σ-contours;
- $\sum_{\tilde\Gamma^\sigma\in\underline{\tilde\Gamma}^\sigma}\operatorname{vol}\tilde\Gamma^\sigma>V^\sigma L^2-C_2L^{2-c}$;
- $\sum_{\tilde\Gamma^\sigma\in\underline{\tilde\Gamma}^\sigma}|\tilde\Gamma^\sigma|<C_1L$;
- The total number of $\frac{D_1}{6}$-bad points of type I in the family $\underline{\tilde\Gamma}^\sigma$ is bounded above by $2C_1D_1L^{1-a}\log L$.

Proof of the statements. The first three statements follow from the construction. We prove the last statement.

Let us consider one contour $\hat\Gamma^\sigma$ and the contours $\underline{\zeta}$ which are associated to it. It is not difficult to check that the set of edges $\mathcal{E}(\hat\Gamma)$ can be deformed continuously into a rectangle and that the maximal connected components of $\mathcal{E}(\underline{\hat\zeta})\setminus\mathcal{E}(\hat\Gamma)$ can be mapped inside this rectangle on a family of lines, either closed or with endpoints on the boundary of the rectangle, which are self-avoiding and disjoint; we contract each part of $\hat\zeta$ in contact with $\hat\Gamma^\sigma$ into a point (see figure 13.2 for an example of this mapping). Using this representation (which preserves $\mathcal{N}(\hat{\zeta})$, for all $\hat{\zeta} \in \hat{\underline{\zeta}}$), it is easy to see that the total number of points of contact of the lines with the boundary is at most equal to twice the number of connected components in the rectangle, since from each such point at least two lines depart, and each line creates a new component. Moreover, the total number of such components is at most equal to twice the number of contours of $\underline{\Gamma}^\sigma$ contained inside $\hat{\underline{\Gamma}}^\sigma$, since any component which does not correspond to some contour (e.g. the shaded components in the right-hand picture in Fig. 13.2) only touches components which do correspond to some contour (indeed, the boundary of these components are made from part of contours). From this, we can conclude that

$$\sum_{\hat{\zeta} \in \hat{\underline{\zeta}}} \mathcal{N}(\hat{\zeta}) \leqslant 4\,M\,, \tag{13.37}$$

where M is the number of contours in $\underline{\Gamma}^\sigma$. Notice that we have the following bound:

$$M \leqslant \frac{C_1 L}{2L^a} = \tfrac{1}{2} C_1 L^{1-a}\,. \tag{13.38}$$

The total number l' of D_1-bad points of type I which have not been removed in the above construction, because the corresponding contour $\hat{\zeta}$ was too small, can be estimated as follows,

$$\begin{aligned} l' \leqslant \sum_{\hat{\zeta} \in \hat{\underline{\zeta}} \setminus \tilde{\underline{\zeta}}} \max\{\log L, 4b(\hat{\zeta})\} &\leqslant \sum_{\hat{\zeta} \in \hat{\underline{\zeta}} \setminus \tilde{\underline{\zeta}}} 4\mathcal{N}(\hat{\zeta}) D_1 \log L \\ &\leqslant 4 M D_1 \log L \leqslant 2 C_1 D_1 L^{1-a} \log L\,. \end{aligned} \tag{13.39}$$

Finally, it may happen that some of the points that we have added to $\underline{\Gamma}^\sigma$, by gluing some straight shortest paths to it, contain some new bad points of type I. However, it is possible to show that we cannot have added $D_1/6$-bad points of type I in this way. Indeed, let g be some straight shortest path of the above construction. The extremities x and y of g are such that $\|y - x\|_1 \leqslant D_1 \log L$ and $x, y \in \hat{\Gamma}_i^\sigma$. Let z be any point of another contour $\hat{\Gamma}_j^\sigma$, $j \neq i$, which is not a D_1-bad point of type I. We know that $\|z - x\|_1 > D_1 \log L$ and $\|z - y\|_1 > D_1 \log L$. Using this and the definition of straight shortest path, it is not difficult to show that $\|z - t\|_1 > D_1/3 \log L$ for any $t \in g$. From this the conclusion follows. □

The set of all families of σ-contours satisfying the above set of conditions is denoted by $\mathfrak{C}_2^\sigma$ and $[\tilde{\underline{\Gamma}}^\sigma] \doteqdot \{\underline{\Gamma}^\sigma \,:\, \tilde{\underline{\Gamma}}^\sigma(\underline{\Gamma}^\sigma) = \tilde{\underline{\Gamma}}^\sigma\}$. We have

$$\begin{aligned} \Xi^{++}(\Lambda_L) P_{\Lambda_L}^{++,\beta J}[E_1^\sigma] &\leqslant \sum_{\tilde{\underline{\Gamma}}^\sigma \in \mathfrak{C}_2^\sigma} w_\sigma(\tilde{\underline{\Gamma}}^\sigma) \sum_{\substack{\tilde{\underline{\zeta}} \\ (\tilde{\underline{\Gamma}}^\sigma, \tilde{\underline{\zeta}}) \in [\tilde{\underline{\Gamma}}^\sigma]}} \prod_{\tilde{\zeta} \in \tilde{\underline{\zeta}}} \tilde{w}(\tilde{\zeta} | \tilde{\underline{\Gamma}}^\sigma) \sum_{\substack{\underline{\zeta} \text{ small} \\ (\underline{\zeta}, (\tilde{\underline{\Gamma}}^\sigma, \tilde{\underline{\zeta}})) \text{ comp.}}} \prod_{\zeta \in \underline{\zeta}} w(\zeta | (\tilde{\underline{\Gamma}}^\sigma, \tilde{\underline{\zeta}})) \\ &\leqslant (1 + L^{-\mathcal{O}(\beta)}) \sum_{\tilde{\underline{\Gamma}}^\sigma \in \mathfrak{C}_2^\sigma} w_\sigma(\tilde{\underline{\Gamma}}^\sigma) \sum_{\substack{\underline{\zeta} \text{ small} \\ (\underline{\zeta}, \tilde{\underline{\Gamma}}^\sigma) \text{ comp.}}} \prod_{\zeta \in \underline{\zeta}} w(\zeta | \tilde{\underline{\Gamma}}^\sigma)\,, \end{aligned} \tag{13.40}$$

where we used the definition of $\tilde{\zeta}$, the fact that the weights satisfy a Peierls condition and we have glued together the contours ζ and $\tilde{\zeta}$ intersecting each other. Notice that

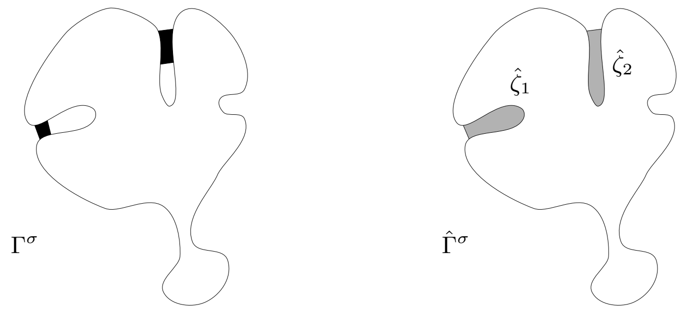

FIGURE 13.3. Surgery: removing the bad-points of type II. Left: We connect the sites of the contour Γ^σ which are too close from one another by straight shortest paths (the black parts). Right: By taking the exterior envelope of the contour and the shortest paths, we define a new contour $\hat\Gamma^\sigma$; we also introduce a family $\underline{\hat\zeta}$ of closed contours (the boundaries of the shaded regions). Notice that the straight shortest paths which are *inside* Γ^σ, as there would be some at the bottleneck in the lower part of Γ^σ, do not give rise to modifications.

the constraints on the contours $\underline{\tilde\zeta}$ in the second sum of the second expression are *not* local! However, since we are interested in an upper bound, we can remove the non-local constraints and keep only the local ones about the size, the hard-core, etc...
Consequently, we have proved that

$$P^{++,\beta J}_{\Lambda_L}[E^\sigma_1] \leqslant (1 + L^{-\mathcal{O}(\beta)})\, P^{++,\beta J}_{\Lambda_L}[E^\sigma_2 \cap E^\sigma_{\rm s}]\,, \tag{13.41}$$

where $E^\sigma_2 \doteqdot \{\omega \,:\, \underline{\Gamma}^\sigma(\omega) \in \mathfrak{E}^\sigma_2\}$.

Step 3. The contours of $\underline{\Gamma}^\sigma$ have only few "gulfs".

Our aim now is to simplify the internal structure of the large external σ-contours. Several kinds of pathologies can appear; they are related to the presence of sites of a contour which are spatially close, but are far from one another "along the contour". To make this idea precise, let us introduce a notion of distance along a contour. For all $t, t' \in \underline{\Gamma}^\sigma$, we define

$$\mathrm{d}_{\Gamma^\sigma}(t,t') \doteqdot \min\{|g| \,:\, \delta g = \{t,t'\}\,,\, \mathcal{E}(g) \subset \mathcal{E}(\Gamma^\sigma)\}\,. \tag{13.42}$$

Let D_2 be some constant such that $D_1/18 > D_2 > 0$ and $\underline{\Gamma}^\sigma \in \mathfrak{E}^\sigma_2$. We call D_2*-bad points of type II* any pair of sites t, t' belonging to a same $\Gamma^\sigma \in \underline{\Gamma}^\sigma$ and such that

- $\|t' - t\|_1 \leqslant D_2 \log L$,
- $\mathrm{d}_{\Gamma^\sigma}(t,t') \geqslant 4D_2 \log L$,
- there exists a straight shortest path $g \not\subset \mathrm{int}\Gamma^\sigma$ connecting t and t'.

We construct a new family $\underline{\hat\Gamma}^\sigma$ of contour by decomposing the exterior envelope of the set of edges (see Fig. 13.3)

$$\mathcal{E}(\underline{\Gamma}^\sigma) \cup \bigcup_g \mathcal{E}(g)\,, \tag{13.43}$$

where g runs through all straight shortest paths connecting pairs of corresponding D_2-bad points of type II. We also define a family $\underline{\hat\zeta}$ of closed contours by decomposing the set of edges

$$\mathcal{E}(\underline{\Gamma}^\sigma) \triangle \mathcal{E}(\underline{\hat\Gamma}^\sigma)\,. \tag{13.44}$$

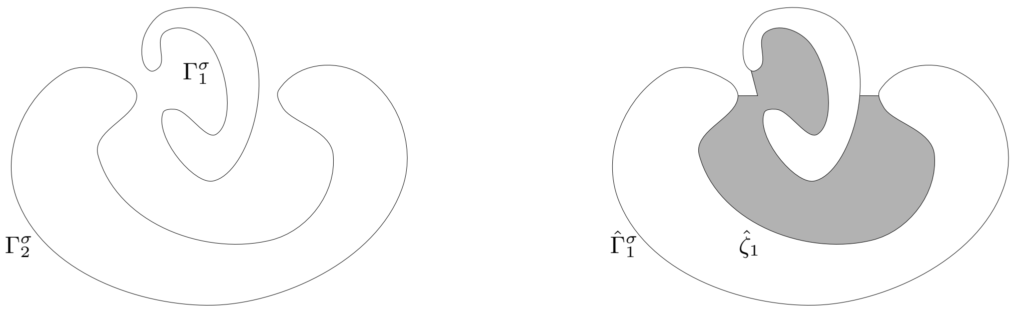

FIGURE 13.4. An example of how the procedure may glue together two different contours.

We associate to the contours of $\underline{\hat\zeta}$ a weight defined in the same way as $\tilde w(\,\cdot\,|\underline{\tilde\Gamma}^\sigma)$ in Step 2. For these contours, the weight satisfies a Peierls condition.
Let us state some properties of the family $\underline{\hat\Gamma}^\sigma$.

- $\underline{\hat\Gamma}^\sigma$ is a Λ_L^*-compatible family of L^a-large, closed, external σ-contours;
- $\sum_{\hat\Gamma^\sigma\in\underline{\hat\Gamma}^\sigma} \operatorname{vol}\tilde\Gamma^\sigma > V^\sigma L^2 - C_2L^{2-c}$;
- $\sum_{\hat\Gamma^\sigma\in\underline{\hat\Gamma}^\sigma} |\tilde\Gamma^\sigma| < C_1L$;
- The total number of $\frac{D_1}{12}$-bad points of type I in the family $\underline{\hat\Gamma}^\sigma$ is bounded above by $2C_1D_1D_2L^{1-a}(\log L)^2$.
- The total number of D_2-bad points of type II of the family $\underline{\hat\Gamma}^\sigma$ is bounded above by $2C_1D_1D_2L^{1-a}(\log L)^2$.

Proof of the statements. The first three statements follow from the construction.
We prove the fourth statement. Let $t\in\hat\Gamma_i^\sigma$ and $t'\in\hat\Gamma_j^\sigma$, $i\neq j$. If t and t' are at a distance at most $\frac{D_2}{2}\log L$ from two points $\bar t$ and $\bar t'$ which are not $\frac{D_1}{6}$-bad points of type I for $\underline{\Gamma}^\sigma$, then t and t' are not $\frac{D_1}{12}$-bad points of type I for $\underline{\hat\Gamma}^\sigma$, since

$$\|t'-t\|_1 \geqslant \|\bar t'-\bar t\|_1 - \|\bar t - t\|_1 - \|\bar t' - t'\|_1 \geqslant (\frac{D_1}{6} - D_2)\log L \geqslant \frac{D_1}{12}\log L\,. \tag{13.45}$$

Therefore, only sites of $\underline{\hat\Gamma}^\sigma$ close to $\frac{D_1}{6}$-bad points of type I for $\underline{\Gamma}^\sigma$ can be $\frac{D_1}{12}$-bad points of type I for $\underline{\hat\Gamma}^\sigma$. The conclusion follows from the definition of $\mathfrak{E}_2^\sigma$ (fourth condition).

We prove the last statement. Creation of bad points of type II may happen when two contours of $\underline{\Gamma}^\sigma$ are glued together by the above procedure (see Fig. 13.4). Let Γ_1^σ and Γ_2^σ two contours of $\underline{\Gamma}^\sigma$ which are glued together during the procedure. Let $t\in\Gamma_1^\sigma$ and $t'\in\Gamma_2^\sigma$. If t and t' are at a distance at most $\frac{D_2}{2}\log L$ from two points $\bar t$ and $\bar t'$ which are not $\frac{D_1}{6}$-bad points of $\underline{\Gamma}^\sigma$, then t and t' are not D_2-bad points of type II for $\underline{\hat\Gamma}^\sigma$, since

$$\|t'-t\|_1 \geqslant \|\bar t'-\bar t\|_1 - D_2\log L \geqslant \frac{D_1}{9}\log L \geqslant 2D_2\log L\,. \tag{13.46}$$

The conclusion follows as before. □

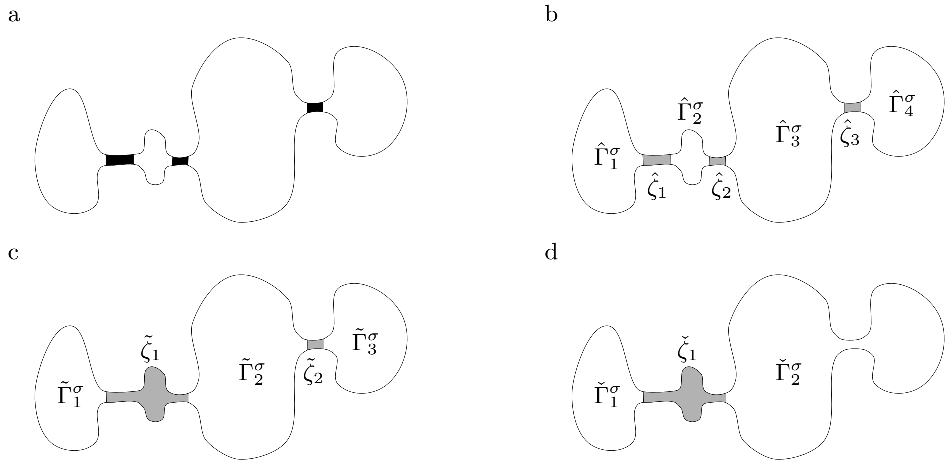

FIGURE 13.5. Surgery: removing the bad-points of type III. (a) We connect the sites of the contour Γ^σ which are too close from one another by shortest paths (the black parts). (b) We thus obtain two families of closed σ-contours $\hat{\underline{\Gamma}}^\sigma$ and $\hat{\underline{\zeta}}$. (c) Some of the contours of $\hat{\underline{\Gamma}}^\sigma$ are too small; for this reason we glue them to the incident contours of $\hat{\underline{\zeta}}$. This defines two families of closed contours, $\tilde{\underline{\Gamma}}^\sigma$ and $\tilde{\underline{\zeta}}$. (d) The contours of $\tilde{\underline{\zeta}}$ which are too small are removed, and we restore the corresponding part of the original contours. This gives the final families of closed contours, $\check{\underline{\Gamma}}^\sigma$ and $\check{\underline{\zeta}}$.

The set of all families of σ-contours satisfying the above set of conditions is denoted by $\mathfrak{C}_3^\sigma$. We can prove as in Step 2 that

$$P_{\Lambda_L}^{++,\beta J}[E_1^\sigma] \leqslant (1 + L^{-\mathcal{O}(\beta)})\, P_{\Lambda_L}^{++,\beta J}[E_3^\sigma \cap E_s^\sigma]\,, \tag{13.47}$$

where $E_3^\sigma \doteqdot \{\omega : \underline{\Gamma}^\sigma(\omega) \in \mathfrak{C}_3^\sigma\}$.

Step 4. The contours of $\underline{\Gamma}^\sigma$ have only few "pipes".

We have not dealt yet with all sites which are spatially close but far along the contours. As shown in Figure 13.3, the procedure of Step 3 does not take care of sites which are linked by straight shortest paths contained inside the contours of $\underline{\Gamma}^\sigma$. These are the last kind of pathologies we have to analyze.

Let D_3 be some constant and $\underline{\Gamma}^\sigma \in \mathfrak{C}_3^\sigma$. We call D_3-*bad points of type III* any pair of sites t, t' belonging to a same $\Gamma^\sigma \in \underline{\Gamma}^\sigma$ and such that

- $\|t' - t\|_1 \leqslant D_3 \log L$,
- $\mathrm{d}_{\Gamma^\sigma}(t, t') \geqslant 4 D_3 \log L$,
- all straight shortest paths connecting t and t' are such that $g \subset \mathrm{int}\Gamma^\sigma$.

We construct a new family $\hat{\underline{\Gamma}}^\sigma$ of contours. We describe the construction for one contour (see Fig. 13.5). Let $\Gamma^\sigma \in \underline{\Gamma}^\sigma$. We remove from Γ^σ all edges both endpoints of which are D_3-bad points of type III. This results in a family of open contours. We close these open contours with disjoint straight shortest paths[6]. The family of closed contours thus obtained is denoted by $\hat{\underline{\Gamma}}^\sigma$. We also define a family of closed contours $\hat{\underline{\zeta}}$ by decomposing the set of edges

$$\mathcal{E}(\Gamma^\sigma) \triangle \mathcal{E}(\hat{\underline{\Gamma}}^\sigma)\,. \tag{13.48}$$

[6]We do this using a fixed algorithm, for example by setting a complete order on the set of straight shortest paths.

Some of the contours of the family $\underline{\hat{\Gamma}}^\sigma$ may be quite small, this will be a problem for defining a coarse-graining procedure. For this reason, we define a new family $\underline{\tilde{\Gamma}}^\sigma$ of closed contours containing only the sufficiently large ones; the remaining contours are glued to the contours of the family $\underline{\hat{\zeta}}$. More precisely, let $a > b > 0$ and define

$$\underline{\tilde{\Gamma}}^\sigma \doteqdot \{\hat{\Gamma}^\sigma \in \underline{\hat{\Gamma}}^\sigma \,:\, \hat{\Gamma}^\sigma \; L^b\text{-large}\}\,. \tag{13.49}$$

The family $\underline{\tilde{\zeta}}$ is obtained by the decomposition of the set of edges

$$\mathcal{E}(\underline{\hat{\zeta}}) \triangle \bigcup_{\hat{\Gamma}^\sigma \in \underline{\hat{\Gamma}}^\sigma \setminus \underline{\tilde{\Gamma}}^\sigma} \mathcal{E}(\hat{\Gamma}^\sigma)\,. \tag{13.50}$$

To apply cluster expansion, it is necessary that the contours in $\underline{\tilde{\zeta}}$ be large enough. Therefore, we have to remove those that are too small. To each $\tilde{\zeta}$, we associate two numbers: The total number of "bad edges", $b(\tilde{\zeta}) \doteqdot |\mathcal{E}(\tilde{\zeta}) \cap \mathcal{E}(\underline{\hat{\Gamma}}^\sigma)|$ and the number $\mathcal{N}(\tilde{\zeta})$ of maximal connected components of such edges. We define a new family of closed contours,

$$\underline{\check{\zeta}} \doteqdot \{\tilde{\zeta} \in \underline{\tilde{\zeta}} \,:\, |\tilde{\zeta}| \geqslant \max\{\log L, 4b(\tilde{\zeta})\}\}\,. \tag{13.51}$$

We introduce finally the family of closed contours $\underline{\check{\Gamma}}^\sigma$ as the decomposition of the set of edges

$$\mathcal{E}(\underline{\tilde{\Gamma}}^\sigma) \triangle \bigcup_{\tilde{\zeta} \in \underline{\tilde{\zeta}} \setminus \underline{\check{\zeta}}} \mathcal{E}(\tilde{\zeta})\,. \tag{13.52}$$

Doing this for each contour $\Gamma^\sigma \in \underline{\Gamma}^\sigma$ defines two family of closed contours which we still denote by $\underline{\check{\Gamma}}^\sigma$ and $\underline{\check{\zeta}}$.

To each contour $\check{\zeta} \in \underline{\check{\zeta}}$, we associate a weight defined as in Steps 2 and 3. The Peierls condition is still fulfilled.

Let us state some properties of the family $\underline{\check{\Gamma}}^\sigma$.

- $\underline{\check{\Gamma}}^\sigma$ is a Λ_L^*-compatible family of L^b-large, closed, external σ-contours;
- $\sum_{\check{\Gamma}^\sigma \in \underline{\check{\Gamma}}^\sigma} \operatorname{vol} \check{\Gamma}^\sigma > V^\sigma L^2 - 2C_2 L^{2-c}$;
- $\sum_{\check{\Gamma}^\sigma \in \underline{\check{\Gamma}}^\sigma} |\check{\Gamma}^\sigma| < C_1 L$;
- The total number of D_2-bad points of type I in the family $\underline{\check{\Gamma}}^\sigma$ is bounded above by $20 D_1 D_2 L^{1-b} \log L$.
- The total number of D_2-bad points of type II of the family $\underline{\check{\Gamma}}^\sigma$ is bounded above by $2 C_1 D_1 D_2 L^{1-a} (\log L)^2$.
- The total number of D_3-bad points of type III in the family $\underline{\check{\Gamma}}^\sigma$ is bounded above by $16 C_1 D_3 L^{1-b} \log L$.

Proof of the statements. The first, third and fifth statements follow from the construction.

We prove the second statement. The total volume of the contours $\underline{\hat{\zeta}}$ is at most $C_1 D_3 L \log L$, since the length of the family $\underline{\Gamma}^\sigma$ is at most $C_1 L$ and the length of the straight shortest paths is at most $C_3 \log L$. The total volume of the contours $\hat{\Gamma}^\sigma \in \underline{\hat{\Gamma}}^\sigma \setminus \underline{\tilde{\Gamma}}^\sigma$ is at most $\frac{C_1}{4} L^{1+b}$, since we can partition them into families of volume between L^{2b} and $2L^{2b}$ (except possibly for the last family) and the number of such families is not larger than $\frac{C_1}{4} L^{1-b}$.

Therefore, the total volume of the contours $\underline{\tilde{\zeta}}$ is at most $C_1 L^{1+b}$ (if L is large enough); clearly the same bound also holds for $\underline{\check{\zeta}}$. Since $1-c=a>b$, the conclusion follows when L is large enough.

We prove the last statement. The key point is to observe that under the relation "is in contact with (through a contour $\check{\zeta} \in \underline{\check{\zeta}}$)", the family $\underline{\check{\Gamma}}^\sigma$ has the structure of a tree. This follows from the fact that the contours of $\underline{\Gamma}^\sigma$ are closed and external.

With this observation, it is easy to compute the number of D_3-bad points of type III remaining at the end of the procedure. Indeed, these points belong to the contours $\tilde{\zeta} \in \underline{\tilde{\zeta}} \setminus \underline{\check{\zeta}}$, the total length of which is at most $16 C_1 D_3 L^{1-b} \log L$, by definition of these contours and the tree-graph structure[7] (there are at most $C_1 L^{1-b}$ contours in $\underline{\tilde{\Gamma}}^\sigma$).

We prove the fourth statement. Let us consider two different contours $\check{\Gamma}_1^\sigma$ and $\check{\Gamma}_2^\sigma$ which arise from the same contour $\Gamma^\sigma \in \underline{\Gamma}^\sigma$. Let $t_1 \in \check{\Gamma}_1^\sigma$ and $t_2 \in \check{\Gamma}_2^\sigma$ be two sites such that $\|t_2 - t_1\|_1 \leqslant D_2 \log L$. If any one of these two sites is at a distance larger than $4D_2 \log L$ from the contour connecting $\check{\Gamma}_1^\sigma$ and $\check{\Gamma}_2^\sigma$, then $\mathrm{d}_{\Gamma^\sigma}(t_1,t_2) \geqslant 4D_2 \log L$ and therefore t_1 and t_2 are D_2-bad points of type II, which we already have under control.

The number of remaining sites, which are at a distance at most $4D_2 \log L$ from a contour of $\underline{\check{\zeta}}$ connecting two contours as above, is bounded above by $16 C_1 D_2 L^{1-b} \log L$, since there is at most $C_1 L^{1-b}$ such contours. □

The set of all families of σ-contours satisfying the above set of conditions is denoted by $\mathfrak{C}_4^\sigma$. We can prove as in Step 2 that

$$P_{\Lambda_L}^{++,\beta J}[E_1^\sigma] \leqslant (1 + L^{-\mathcal{O}(\beta)})\, P_{\Lambda_L}^{++,\beta J}[E_4^\sigma \cap E_{\mathrm{s}}^\sigma]\,, \tag{13.53}$$

where $E_4^\sigma \doteqdot \{\omega : \underline{\Gamma}^\sigma(\omega) \in \mathfrak{C}_4^\sigma\}$.

Step 5. The coarse-graining of $\underline{\Gamma}^\sigma$.

We denote by E_5^σ the set of all configurations $\omega \in E_4^\sigma$ such that there is no contours $\zeta \in \underline{\zeta}$ intersecting D_2-bad points of type I, II or III of the family $\underline{\Gamma}^\sigma \in \mathfrak{C}_4^\sigma$ of L^b-large external σ-contours of the configuration ω (we set $D_3 = D_2$). Then

$$P_{\Lambda_L}^{++,\beta J}[E_1^\sigma] \leqslant \mathrm{e}^{\mathcal{O}(L^{1-b} \log L\, \mathrm{e}^{-\mathcal{O}(\beta)})}\; P_{\Lambda_L}^{++,\beta J}[E_5^\sigma \cap E_{\mathrm{s}}^\sigma]\,, \tag{13.54}$$

as a simple cluster expansion estimate yields, since the total number of such bad points is $\mathcal{O}(L^{1-b}\log L)$.

Let $\omega \in E_5^\sigma$ and $\underline{\Gamma}^\sigma$ be its family of L^b-large external σ-contours. The structure of $\underline{\Gamma}^\sigma$ makes it possible to construct a polygonal approximation of the corresponding L^b-large contours of ω, suitable for an application of Lemma 12.1.5; notice in particular that each external L^b-large contours Γ contains exactly one of the external L^b-large σ-contours, which we write Γ^σ. The nice property is the following one: there exists a constant $K' > 0$ such that, if t belongs to some external L^b-large contours Γ of ω, then there exists a site t' of the same contour, with $\|t' - t\|_1 < K' \log L$, which is a Γ^σ-cutting point of Γ, by definition of E_5^σ.

Let us now define the polygonal approximation of the large external contours of ω. Let $b > \delta > 0$, let Γ be one of these large contours and let $s \mapsto \Gamma^\sigma(s)$ be a unit-speed

[7] Notice that the contours $\tilde{\zeta}$ which touch only one contour $\tilde{\Gamma}^\sigma$ necessarily belong to $\underline{\check{\zeta}}$.

parameterization of the corresponding large σ-contour. The following procedure associates to each large contour a sequence of its sites[8].

1. Let $s_0 \geqslant 0$ be the first time such that $\Gamma^\sigma(s_0)$ is a Γ^σ-cutting point of Γ; we set $t_0 \doteqdot \Gamma^\sigma(s_0)$.
2. Let $s_1 > s_0$ be the first time such that $\Gamma^\sigma(s_1)$ does not belong to $\mathcal{D}(t_0, L^\delta)$ and is a Γ^σ-cutting point of Γ; we set $t_1 \doteqdot \Gamma^\sigma(s_1)$.
3. We iterate this procedure until it stops.

With this procedure, we associate to each external large contour Γ_i of ω a sequence of points $S_i \doteqdot (t_{i0}, \dots, t_{in_i})$ and the corresponding polygonal line $\mathcal{P}_i$. Since $\omega \in E_5^\sigma$,

$$\Gamma_i \subset \bigcup_{k=0}^{n_i} \mathcal{D}(t_{ik}, L^\delta + K' \log L)\,, \tag{13.55}$$

where $K' > 0$ is some constant independent of ω. Defining the volume of S by

$$\operatorname{vol} S \doteqdot |\mathrm{int}\Gamma^\sigma \setminus \bigcup_{k=0}^{n} \mathcal{D}(t_{ik}, L^\delta + K' \log L)|\,, \tag{13.56}$$

where Γ^σ is the large σ-contour of any large contour Γ of some configuration $\omega \in E_5^\sigma$ such that $\mathcal{P}(\Gamma) = \mathcal{P}(S)$. We have the following bounds on the volume of the family $\underline{S}$ and on the total number of its vertices.

- $\sum_i n_i < 2C_1 L^{1-\delta}$;
- $\sum_i \operatorname{vol} S_i \geqslant V^\sigma L^2 - 3C_2 L^{2-c}$.

The first assertion follows from the construction and the fact that $\sum_{\Gamma^\sigma \in \underline{\Gamma}^\sigma} |\Gamma^\sigma| < C_1 L$. The second follows from the fact that

$$\sum_i \sum_j |\mathcal{D}(t_{ij}, L^\delta + K' \log L)| \leqslant 2C_1 L^{1+\delta}\,, \tag{13.57}$$

and $\delta < b < a = 1 - c$. Consequently, an application of Lemma 12.1.5 yields for any $\varepsilon > 0$

$$P_{\Lambda_L}^{++,\beta\underline{J}}[E_1^\sigma] \leqslant \exp\{-\mathfrak{F}^*(m_\sigma) L (1-\varepsilon)\}\,, \tag{13.58}$$

as soon as L is large enough (see the proof of Lemma 7.4.1 for the details). □

The previous proposition and the comments preceding it imply the desired upper bound. Together with the lower bound, this yields the following theorem.

Theorem 13.4.1. *Let $\underline{J}(e)$ be independent of e and ferromagnetic; we suppose that $J_\sigma > 0$ and $J_\tau > J_{\sigma\tau}$. Let $\mathfrak{F}^*(m_\sigma) \doteqdot \mathfrak{F}(\mathcal{C}(m_\sigma))$, where $\mathcal{C}(m_\sigma)$ is the boundary of the Wulff shape of volume V^σ associated to the surface tension τ_σ. There exists β_0 such that, for all $\beta > \beta_0$,*

$$\lim_{L\to\infty} -\frac{1}{L} \log P_{\Lambda_L}^{++,\beta\underline{J}}[\mathcal{A}_\sigma(m_\sigma, c)] = \mathfrak{F}^*(m_\sigma)\,.$$

[8] We recall that $\mathcal{D}(t,d) \doteqdot \{t' \in \mathbb{Z}^2 : \|t'-t\|_\infty\} \leqslant d/2$.

Appendix A

Correlation inequalities

In this appendix, we give a quick exposition of one of the basic tools of rigorous statistical mechanics, correlation inequalities. Their main interests are their non-perturbative character and the fact that they often yield rather simple proofs when they can be applied. There is however some drawback, namely to be able to apply such inequalities to a model, it must have some quite particular properties (ferromagnetic interactions for example). We only state the inequalities we use in the main body of this work, although sometimes in a slightly more general setting, and we do not give any proof since they can be found in many places (references are indicated). We also give some basic examples of applications of these inequalities. Nice surveys of this subject are [Sy, Sh2, Br], see also Section 3 of [FP3]. This appendix is partitioned into several sections, in each of which one particular inequality is discussed.

We state the results only for models in the following class (several of the results are valid in much greater generality!). Let $\Lambda \subset \mathbb{Z}^d$ be some finite set. The product space $\Omega_\Lambda \doteqdot \{-1,1\}^\Lambda$ is called the *configuration space* and the elements $\omega \in \Omega_\Lambda$ are called *configurations*. To each element t of Λ (each *site*), we associate a random variable $\sigma(t) : \Omega_\Lambda \to \{-1,1\}$, the *spin at* t, defined by $\sigma(t)(\omega) \doteqdot \omega(t)$.

The *Hamiltonian* H_Λ of the system is some polynomial on the configuration space,

$$H_\Lambda(\omega) \doteqdot - \sum_{A \in \mathcal{S}(\Lambda)} J_A \sigma_A(\omega) \,, \tag{A.1}$$

where $\omega \in \Omega$, $\mathcal{S}(\Lambda)$ is the set of all finite families of sites of Λ with possible repetitions, $J_A \in \mathbb{R}$ are the *couplings*, and $\sigma_A \doteqdot \prod_{t\in A} \sigma(t)$. If $J_A \geqslant 0$, for all $A \in \mathcal{S}(\Lambda)$, we say that the model has *ferromagnetic* couplings. We call $J_{\{t\}}$ the *magnetic field* at site t and sometimes write it $h(t) \equiv J_{\{t\}}$. Couplings of the form $J_{\{t,t'\}}$, $t \neq t'$, are called *pair interactions*.

The *Gibbs measure* in Λ is the probability measure on Ω_Λ defined by

$$\mu_\Lambda(\omega) \doteqdot \frac{1}{\Xi(\Lambda)} \exp\{-H_\Lambda(\omega)\} \,. \tag{A.2}$$

Expectation values with respect to the measure μ_Λ is denoted by $\langle \,\cdot\, \rangle_\Lambda$.

A.1 GKS inequalities

GKS (Griffiths-Kelly-Sherman, [Gr, KS]) inequalities hold for ferromagnetic systems and together with FKG inequalities are the most often used. They can be stated in the following form[1]

Lemma A.1.1 (GKS inequalities). *Suppose that the couplings are ferromagnetic. For any $A, B \in \mathcal{S}(\Lambda)$, the following inequalities hold*

$$\langle \sigma_A \rangle_\Lambda \geqslant 0\,,$$
$$\langle \sigma_A \sigma_B \rangle_\Lambda \geqslant \langle \sigma_A \rangle_\Lambda \langle \sigma_B \rangle_\Lambda\,.$$

As a direct application of these inequalities, we can prove the existence of the thermodynamic limit for the n-point functions of the Ising model.

Let us first consider the case of free b.c. which corresponds to having only pair interactions. Let $\Lambda_n \nearrow \mathbb{Z}^d$ be some sequence of finite subsets of $\mathbb{Z}^d$. The existence of the limit follows from the fact that the correlation functions are bounded and

$$\langle \sigma_A \rangle_\Lambda \leqslant \langle \sigma_A \rangle_{\Lambda'}\,, \tag{A.3}$$

if $\Lambda \subset \Lambda'$. Indeed, it is sufficient to observe that the second Griffiths inequality can be written as

$$\frac{\partial}{\partial J_B} \langle \sigma_A \rangle_\Lambda \geqslant 0\,, \tag{A.4}$$

and that expectation values with respect to μ_Λ can be written as expectation values with respect to $\mu_{\Lambda'}$ by setting to zero all pair interactions $J_{\{t,t'\}}$ with $\{t,t'\} \cap \Lambda' \neq \varnothing$.

The case of the $+$-b.c., which corresponds to having only pair interactions and a magnetic field $h(t)$ given by the number of nearest-neighbours of t which are not inside Λ, is treated similarly. One proves that

$$\langle \sigma_A \rangle^+_\Lambda \geqslant \langle \sigma_A \rangle^+_{\Lambda'}\,, \tag{A.5}$$

if $\Lambda \subset \Lambda'$ by adding a magnetic field $h'(t)$ to each site $t \in \Lambda' \setminus \Lambda$ and letting h' go to infinity.

The existence of these thermodynamic limits imply the existence of the corresponding limiting Gibbs measures, since $(\sigma_A)_{A \subset \mathbb{Z}^d}$ forms a total set of functions for the local functions on $\Omega_{\mathbb{Z}^d}$. Clearly, the case of the Ashkin–Teller model can be handled in the same way.

A.2 GHS inequality

GHS (Griffiths-Hurst-Sherman, [GHS]) inequality holds for models with positive magnetic field and ferromagnetic pair interactions. While the second Griffiths inequality implies that the magnetization is an increasing function of the magnetic field, GHS inequality shows that it is in fact a concave function of h.

[1] The first of these inequalities is often referred to as the *first Griffiths inequality* and the second one as the *second Griffiths inequality*.

Lemma A.2.1 (GHS inequality)**.** *Suppose that the magnetic fields are positive and that the pair correlation functions are ferromagnetic. Let $t, u, v \in \Lambda$, then*

$$\frac{\partial^2}{\partial h(u)\partial h(v)}\langle\sigma(t)\rangle_\Lambda \leqslant 0\,.$$

A.3 FKG inequalities

FKG (Fortuin-Kasteleyn-Ginibre, [FKG]) inequalities have a large domain of validity and are extremely useful. We only state them in the case of the Ising model. In this case, they state that increasing functions (see (D26), p. 34) are positively correlated.

Lemma A.3.1 (FKG inequalities)**.** *Suppose that the only non-zero couplings are an arbitrary magnetic field and ferromagnetic pair interactions. Let f and g be two increasing functions. Then*

$$\langle fg\rangle_\Lambda \geqslant \langle f\rangle_\Lambda\langle g\rangle_\Lambda\,.$$

As an application of these inequalities, let us show that the two Ising measures μ^+ and μ^- are extremal Gibbs states, and that $\langle\sigma(t)\rangle^+ = \langle\sigma(t)\rangle^-$, for all t, implies that the limit

$$\lim_{n\to\infty}\langle\,\sigma_A\,\rangle_{\Lambda_n}^{\overline{\omega}_n}\,, \tag{A.6}$$

where $\Lambda_n \nearrow \mathbb{Z}^d$ and $\overline{\omega}_n \in \Omega$ for all n, exists and is equal to $\langle\,\cdot\,\rangle^+ = \langle\,\cdot\,\rangle^-$ (therefore the Gibbs state is unique).

Let $n(t) \doteqdot \frac{1}{2}(1+\sigma(t))$. Let Λ be a finite subset of $\mathbb{Z}^2$. Any boundary conditions $\overline{\omega}$ can be replaced by a magnetic field $h(t)$ different from 0 only for sites $t \in \partial\Lambda$. Since $\sigma(t)$ and $n_A \doteqdot \prod_{t\in A} n(t)$ are increasing, it follows by differentiating with respect to this magnetic field that

$$\langle n_A\rangle_\Lambda^+ \geqslant \langle n_A\rangle_\Lambda^{\overline{\omega}} \geqslant \langle n_A\rangle_\Lambda^-\,. \tag{A.7}$$

Since the function $\sum_{t\in A} n(t) - n_A$ is increasing, FKG inequalities yield

$$0 \leqslant \langle n_A\rangle_\Lambda^+ - \langle n_A\rangle_\Lambda^- \leqslant \sum_{t\in A}(\langle n(t)\rangle_\Lambda^+ - \langle n(t)\rangle_\Lambda^-)\,. \tag{A.8}$$

The statements follow easily from (A.7) and (A.8)[2].

A.4 BLP inequality

BLP (Bricmont-Lebowitz-Pfister, [BLP1]) inequality holds for the 2D Ising model with translation invariant couplings and no magnetic field. It is quite different from the previous ones. It states that the effect of the boundary on n-point correlation functions decays exponentially with the distance between the support and the boundary.

[2]Observe that the quantities σ_A are finite linear combination of quantities of the form n_B. It is therefore sufficient to study these functions.

Lemma A.4.1 (BLP inequality)**.** *Consider the 2D Ising model with coupling constants* $J(e) = \beta > \beta_c$*, for all edges. Let* Λ_1 *and* Λ_2 *be two subsets of* $\mathbb{Z}^2$ *and* $A \subset \Lambda_1 \cap \Lambda_2$*. Then there exists* $\overline{a}(\beta) > 0$ *and* K *such that*

$$|\langle \sigma_A \rangle^{+,\beta}_{\Lambda_1} - \langle \sigma_A \rangle^{+,\beta}_{\Lambda_2}| \leqslant K \sum_{t \in A} \sum_{t' \in \Lambda_1 \triangle \Lambda_2} \exp\{-\overline{a}(\beta) \|t' - t\|_1\} \,,$$

where $\Lambda_1 \triangle \Lambda_2 \doteqdot (\Lambda_1 \setminus \Lambda_2) \cup (\Lambda_2 \setminus \Lambda_1)$.

Appendix B

Elements of convex analysis

This appendix is dedicated to some elementary results in convex analysis which we use in the rest of this thesis[1]. We restrict our attention to the 2-dimensional case, however most of what follows can be easily extended to higher dimensions. In the following, if x and y are two vectors in $\mathbb{R}^2$, we write $\langle x, y \rangle$ their scalar product.

We first give some terminology and basic results related to the geometry of a convex body (Section B.1). Then we introduce a geometrical notion of curvature for convex bodies and discuss positive stiffness (Section B.2). Finally, we state the Sharp Triangle Inequality, which plays a major role in the first part of this thesis, and show its equivalence with positive stiffness (Section B.3).

B.1 Geometry of a convex body

Definition.

(D226) *A* **convex body** *is a compact convex subset of* $\mathbb{R}^2$*, with non-empty interior.*

Let $\mathcal{W} \subset \mathbb{R}^2$ be a convex body. We always suppose that the origin is inside $\mathcal{W}$. We denote by $\partial\mathcal{W}$ its boundary. There exists another, dual characterization of $\mathcal{W}$ by a function on $\mathbb{R}^2$, which is known as its support function.

Definition.

(D227) *The* **support function** $\tau_{\mathcal{W}}$ *of a convex body* $\mathcal{W}$ *is defined by*

$$\tau_{\mathcal{W}}(x) \doteq \sup_{y \in \mathcal{W}} \langle x, y \rangle \,, \quad \forall y \in \mathbb{R}^2 \,.$$

Lemma B.1.1. *1. $\tau_{\mathcal{W}}$ is a positively homogeneous, convex function on $\mathbb{R}^2$. Moreover, if $\mathcal{W}$ has a central symmetry, i.e. $x \in \mathcal{W} \implies -x \in \mathcal{W}$, then $\tau_{\mathcal{W}}$ is a norm on $\mathbb{R}^2$.*

2. If $\mathcal{W}$ has zero has an interior point, then $\tau_{\mathcal{W}}$ is strictly positive at $x \neq 0$.

3. For any $0 \neq x \in \mathbb{R}^2$, there exists $\tilde{x} \in \partial\mathcal{W}$ such that $\tau_{\mathcal{W}}(x) = \langle \tilde{x}, x \rangle$.

[1] The exposition follows [PV1] and [PV4]; the results of Sections B.2 and B.3 have appeared in [PV4].

Proof. The first and second statements follow from the definition.
The compactness of $\mathcal{W}$ implies the existence of $\tilde{x} \in \mathcal{W}$ such that $\tau_{\mathcal{W}}(x) = \langle \tilde{x}, y \rangle$. Suppose that $\tilde{x} \in \mathcal{W} \setminus \partial\mathcal{W}$, then there exists $\varepsilon > 0$ such that $\tilde{x} + \varepsilon x \in \mathcal{W}$ and $\langle \tilde{x} + \varepsilon x, x \rangle = \langle \tilde{x}, x \rangle + \varepsilon \|x\|_2^2 > \langle \tilde{x}, x \rangle$. □

The function $\tau_{\mathcal{W}}$ contains all the information about the convex body $\mathcal{W}$. Indeed, Lemma B.1.3 implies that it is possible to reconstruct $\mathcal{W}$ from its support function.

Definition.
(D228) *Let $\tau : \mathbb{R}^2 \to \mathbb{R}$. The* **dual function of** τ, τ^*, *is defined as its Legendre transform,*

$$\tau^*(x) \doteqdot \sup_{y \in \mathbb{R}^2} \big(\langle x, y \rangle - \tau(y) \big) .$$

Lemma B.1.2. *Let $\tau : \mathbb{R}^2 \to \mathbb{R}$ be a convex function. Then $(\tau^*)^* = \tau$.*

Proof. The proof of this statement is standard and can be found in most books on convex geometry. □

Lemma B.1.3. *Let $\tau : \mathbb{R}^2 \to \mathbb{R}$ be a positively homogeneous, convex function. Then there exists a unique convex body $\mathcal{W}$ such that $\tau = \tau_{\mathcal{W}}$, i.e. such that τ is the support function of $\mathcal{W}$. This convex body $\mathcal{W}$ can be reconstructed from the following relation:*

$$\tau^*_{\mathcal{W}}(x) = \begin{cases} 0 & \text{if } x \in \mathcal{W}\,, \\ \infty & \text{otherwise.} \end{cases}$$

Proof. If $x \in \mathcal{W}$ then $\tau^*_{\mathcal{W}}(x) \leqslant 0$, for all x. Since $\tau^*_{\mathcal{W}}(x) \geqslant \langle x, 0 \rangle - \tau_{\mathcal{W}}(0) = 0$, $\tau^*_{\mathcal{W}}(x) = 0$.
If $x \notin \mathcal{W}$, then there exist y and $\tilde{y}$ such that $\tau_{\mathcal{W}}(y) = \langle \tilde{y} | y \rangle$ and $\langle x - \tilde{y}, y \rangle > 0$. Therefore $\tau^*_{\mathcal{W}}(x) \geqslant \langle x - \tilde{y}, \lambda y \rangle$, for all $\lambda > 0$. □

Definition.
(D229) *Let $\mathcal{W}$ be a convex body. The function $\tau^*_{\mathcal{W}}$ is the* **indicator function of** $\mathcal{W}$.
(D230) *Two points $x \in \mathbb{R}^2$ and $x^* \in \mathbb{R}^2$* **are in duality** *if*

$$\tau_{\mathcal{W}}(x) + \tau^*_{\mathcal{W}}(x^*) = \langle x, x^* \rangle .$$

Since $\tau^*_{\mathcal{W}}(x) = 0$ for all $x \in \partial\mathcal{W}$, it follows that $\tilde{y}$ is in duality with y, for all $y \in \mathbb{R}^2$ (see Lemma B.1.1). The next lemma shows that the converse is true.
Lemma B.1.4. *Let $0 \neq x \in \mathbb{R}^2$ and x^* be two points in duality. Then*

1. *$x^* \in \partial\mathcal{W}$, and therefore $\tau_{\mathcal{W}}(x) = \langle x, x^* \rangle$.*
2. *x^* is in duality with λx for all $\lambda > 0$.*

Proof. We prove 1.
Suppose $x^* \in \mathcal{W} \setminus \partial\mathcal{W}$, then $\tau^*_{\mathcal{W}}(x^*) = 0$ and $\tau_{\mathcal{W}}(x) = \langle x, x^* \rangle$, which contradicts Lemma B.1.1.
Suppose $x^* \notin \mathcal{W}$, then $\tau^*_{\mathcal{W}}(x^*) = \infty$, and therefore x and x^* cannot be in duality.
The second statement is an immediate consequence of the first one. □

In view of the previous lemma, we can restrict our attention to $\widehat{x} \in \mathbb{R}^2$ with $\|\widehat{x}\|_2 = 1$ and to $x^* \in \partial\mathcal{W}$. Geometrically, x^* can be interpreted as a point on the boundary of the convex body, while $\widehat{x}$ can be seen as the (outward unit) normal to a support plane of $\mathcal{W}$ at x^*.

Definition.

(D231) *Let $\widehat{x}$ and x^* be in duality. Let $A(\widehat{x}) \doteqdot \{y \in \mathbb{R}^2 : \langle y, \widehat{x}\rangle = \tau_{\mathcal{W}}(\widehat{x})\}$. $A(\widehat{x})$ is a* **support plane for $\mathcal{W}$ at** *x^*, i.e. $x^* \in A(\widehat{x})$ and $\mathcal{W} \subset H(\widehat{x})$, where*

$$H(\widehat{x}) \doteqdot \{y \in \mathbb{R}^2 : \langle y, \widehat{x}\rangle \leqslant \tau_{\mathcal{W}}(\widehat{x})\}\,.$$

Lemma B.1.1 implies that for any $\widehat{x}$ there exists at least one $x^* \in \partial\mathcal{W}$ such that x^* and $\widehat{x}$ are in duality. The next lemma states that the same is true for x^*.

Lemma B.1.5. *For any $x^* \in \partial\mathcal{W}$, there exists at least one $\widehat{x}$ such that x^* and $\widehat{x}$ are in duality.*

Proof. The separating hyperplane theorem (see [E]) states that if O is an open convex set and L is an affine set such that $O \cap L = \varnothing$, then there exists a hyperplane H with the properties $L \subset H$ and $O \cap H = \varnothing$. This implies in our case that for any $x^* \in \partial\mathcal{W}$ there exists a support plane for $\mathcal{W}$ at x^*. The outward normal to this support plane is in duality with x^*. □

A natural question now is to determine when there is a single point in duality with a given $\widehat{x}$, or a given x^*. This motivates the following terminology.

Definition.

(D232) *A point $x^* \in \partial\mathcal{W}$ is a* **regular point** *if there is a single support plane containing x^*.*

(D233) *A support plane $A(\widehat{x})$ for $\mathcal{W}$ is a* **regular support plane** *if $A(\widehat{x}) \cap \mathcal{W}$ is 0-dimensional.*

Therefore x^* is regular if and only if there is a unique $\widehat{x} \in \mathbb{R}^2$, $\|\widehat{x}\|_2 = 1$, such that x^* and $\widehat{x}$ are in duality. Similarly, for any $\widehat{x} \in \mathbb{R}^2$, $\|\widehat{x}\|_2 = 1$, the support plane $A(\widehat{x})$ is regular if and only if there is a unique $x^* \in \partial\mathcal{W}$ such that $\widehat{x}$ and x^* are in duality.

Definition.

(D234) *Let $x^* \in \partial\mathcal{W}$; if $A(\widehat{x})$ is a support plane for $\mathcal{W}$ at x^* which is not regular then we say that $\mathcal{W}$ has a* **facet** *of (unit) normal $\widehat{x}$.*

(D235) *If $x^* \in \partial\mathcal{W}$ is not regular, then $\mathcal{W}$ has a* **corner** *at x^*.*

Remark. Observe that if x_1^* and x_2^* are two distinct points in duality with $\widehat{x}$, then any point of the form $\alpha x_1^* + (1-\alpha)x_2^*$, with $0 \leqslant \alpha \leqslant 1$, is also in duality with $\widehat{x}$.

There is a characterization of the notions of facet and corner in terms of the support function. We first introduce the notion of subdifferential of a convex function and show how it is related to duality.

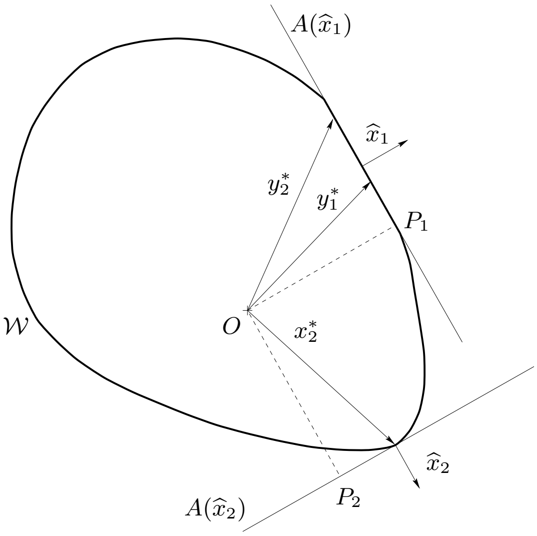

FIGURE B.1. A convex body $\mathcal{W}$ and two of its support planes $A(\widehat{x}_1)$, $A(\widehat{x}_2)$. x_2^* is a regular point and $A(\widehat{x}_2)$ is a regular support plane. y_1^* and y_2^* are regular points and $A(\widehat{x}_1)$ is not regular. The vectors y_1^* and y_2^* are in the subdifferential of τ at $\widehat{x}_1$. $\mathrm{dist}(O, P_1) = \tau(\widehat{x}_1)$; $\mathrm{dist}(O, P_2) = \tau(\widehat{x}_2)$.

Definition.
(D236) *Let $\tau : \mathbb{R}^2 \to \mathbb{R}$ be a convex function. The* **subdifferential of** τ **at** x *is the set*

$$\partial\tau(x) \doteqdot \{y \in \mathbb{R}^2 \,:\, \tau(z + x) \geqslant \tau(x) + \langle y, z\rangle \,,\, \forall z \in \mathbb{R}^2\}\,.$$

Remark. The subdifferential of τ at x is unique if and only if τ is differentiable at x.

Lemma B.1.6. *1. Let $x \in \mathbb{R}^2$. Then*

$$\partial\tau_{\mathcal{W}}(x) = \{x^* \in \partial\mathcal{W} \,:\, x \text{ and } x^* \text{are in duality}\}\,.$$

Proof. We suppose without loss of generality that $x = \widehat{x}$, $\|\widehat{x}\|_2 = 1$.
Suppose that $y \in \partial\tau_{\mathcal{W}}(\widehat{x})$,

$$\tau_{\mathcal{W}}(z + \widehat{x}) \geqslant \tau_{\mathcal{W}}(\widehat{x}) + \langle y, z\rangle \quad \forall z \in \mathbb{R}^2\,. \tag{B.1}$$

This can be rewritten as

$$\langle z + \widehat{x}, y\rangle - \tau_{\mathcal{W}}(z + \widehat{x}) \leqslant \langle \widehat{x}, y\rangle - \tau_{\mathcal{W}}(\widehat{x}) \quad \forall z \in \mathbb{R}^2\,, \tag{B.2}$$

and therefore

$$\tau^*_{\mathcal{W}}(y) = \sup_{v \in \mathbb{R}^2} \Big\{ \langle y, v\rangle - \tau_{\mathcal{W}}(v) \Big\} = \langle \widehat{x}, y\rangle - \tau_{\mathcal{W}}(\widehat{x})\,, \tag{B.3}$$

so that $\widehat{x}$ and y are in duality.
Conversely, suppose that $\widehat{x}$ and y are in duality; in particular, $y \in \mathcal{W}$. This implies that

$$\tau_{\mathcal{W}}(\widehat{x} + z) = \sup_{v \in \mathcal{W}} \langle v, \widehat{x} + z\rangle \geqslant \langle y, \widehat{x} + z\rangle = \tau_{\mathcal{W}}(\widehat{x}) + \langle y, z\rangle \quad \forall z \in \mathbb{R}^2\,, \tag{B.4}$$

and therefore $y \in \partial\tau_{\mathcal{W}}(\widehat{x})$. $\square$

Lemma B.1.7. *1. $\mathcal{W}$ has a facet of (unit) normal $\widehat{x}$ if and only if $\tau_{\mathcal{W}}$ is not differentiable at $\widehat{x}$.*

2. $\mathcal{W}$ has a corner at x^* if and only there exists a segment $[\widehat{x}_1, \widehat{x}_2] \doteqdot \{x : x = \widehat{x}_1 + t(\widehat{x}_2 - \widehat{x}_1)\,,\ t \in [0,1]\}$, $\widehat{x}_1 \neq \widehat{x}_2$, in duality with x^*, such that $\tau_{\mathcal{W}}$ is affine on $[\widehat{x}_1, \widehat{x}_2]$.

Proof. The first statement follows from Lemma B.1.6.
We prove 2.
Suppose that there is a corner at y^*. Then it is possible to find $\widehat{x}_1$ and $\widehat{x}_2$, $\widehat{x}_1 \neq \widehat{x}_2$, both in duality with y^*. Therefore,

$$\begin{aligned} \tau_{\mathcal{W}}((1-t)\widehat{x}_1 + t\widehat{x}_2) &= \sup_{z^* \in \mathcal{W}} \langle z^*, \widehat{x}_1 + t(\widehat{x}_2 - \widehat{x}_1)\rangle \\ &\geqslant \langle y^*, \widehat{x}_1 + t(\widehat{x}_2 - \widehat{x}_1)\rangle \\ &= (1-t)\langle y^*, \widehat{x}_1\rangle + t\langle y^*, \widehat{x}_2\rangle \\ &= (1-t)\tau_{\mathcal{W}}(\widehat{x}_1) + t\tau_{\mathcal{W}}(\widehat{x}_2)\,. \end{aligned} \tag{B.5}$$

Since $\tau_{\mathcal{W}}$ is convex, $\tau_{\mathcal{W}}((1-t)\widehat{x}_1 + t\widehat{x}_2) \leqslant (1-t)\tau_{\mathcal{W}}(\widehat{x}_1) + t\tau_{\mathcal{W}}(\widehat{x}_2)$, and therefore we have equality in (B.5).
Suppose that $\tau_{\mathcal{W}}$ is affine on $[\widehat{x}_1, \widehat{x}_2]$. Let $x_{1/2} \doteqdot \frac{1}{2}(\widehat{x}_1 + \widehat{x}_2)$. If $y^* \in \partial\tau_{\mathcal{W}}(x_{1/2})$, then $y^* \in \partial\tau_{\mathcal{W}}(\widehat{x}_k)$, $k = 1, 2$. Indeed, for all z

$$\tau_{\mathcal{W}}(z) - \tau_{\mathcal{W}}(x_{1/2}) - \langle y^*, z - x_{1/2}\rangle \geqslant 0\,. \tag{B.6}$$

But, if $\tau_{\mathcal{W}}$ is affine on $[\widehat{x}_1, \widehat{x}_2]$, then

$$\tfrac{1}{2}\sum_{k=1}^{2}\{\, \tau_{\mathcal{W}}(\widehat{x}_k) - \tau_{\mathcal{W}}(x_{1/2}) - \langle y^*, \widehat{x}_k - x_{1/2}\rangle \,\} = 0\,. \tag{B.7}$$

Therefore

$$\tau_{\mathcal{W}}(\widehat{x}_k) = \tau_{\mathcal{W}}(x_{1/2}) + \langle y^*, \widehat{x}_k - x_{1/2}\rangle\,. \tag{B.8}$$

From this it follows that $y^* \in \partial\tau_{\mathcal{W}}(\widehat{x}_k)$,

$$\tau_{\mathcal{W}}(z) \geqslant \tau_{\mathcal{W}}(x_{1/2}) + \langle y^*, z - x_{1/2}\rangle = \tau_{\mathcal{W}}(\widehat{x}_k) + \langle y^*, z - \widehat{x}_k\rangle \quad \forall z\,, \tag{B.9}$$

which implies in our case that y^* and $\widehat{x}_k$ are in duality. □

We state a last result concerning the convex body $\mathcal{W}$.
Lemma B.1.8. *Let τ be a positively homogeneous, convex function. Then τ is the support function of the convex body $\mathcal{W}$ given by*

$$\mathcal{W} = \{x \in \mathbb{R}^2 : \langle x, y\rangle \leqslant \tau_{\mathcal{W}}(y)\,,\ \forall y \neq 0\} = \bigcap_{y \neq 0} H(y)\,.$$

Proof. It is clear that

$$\mathcal{W} \subset \bigcap_{\widehat{x}\,:\,\|\widehat{x}\|_2 = 1} \{y \in \mathbb{R}^2 : \langle y, \widehat{x}\rangle \leqslant \tau_{\mathcal{W}}(\widehat{x})\}\,. \tag{B.10}$$

Suppose that $y \notin \mathcal{W}$. We can separate strictly a closed convex set B and a compact convex set K by a hyperplane, when they are disjoint [E]. Therefore there exists a hyperplane

$$H = \{x \in \mathbb{R}^2 : \langle x, \widehat{u} \rangle = \delta\}, \tag{B.11}$$

$\|\widehat{u}\|_2 = 1$, such that for all $x \in \mathcal{W}$ we have $\langle x, \widehat{u} \rangle < \delta$ and at the same time $\langle y, \widehat{u} \rangle > \delta$. Therefore

$$\sup_{x \in \mathcal{W}} \langle x, \widehat{u} \rangle = \tau_{\mathcal{W}}(\widehat{u}) \leqslant \delta, \tag{B.12}$$

and consequently

$$y \notin \bigcap_{\widehat{x} : \|\widehat{x}\|_2 = 1} \{y \in \mathbb{R}^2 : \langle y, \widehat{x} \rangle \leqslant \tau_{\mathcal{W}}(\widehat{x})\}. \tag{B.13}$$

□

Remark. In the language of Statistical Mechanics, when τ is the surface tension, the set $\mathcal{W}$ thus obtained is called the *Wulff shape* and the construction of Lemma B.1.8 is known as the *Wulff construction* (see also Chapter 7).

B.2 Curvature of a convex body

In this subsection, we introduce a purely geometrical notion of curvature for convex bodies and prove a useful result about convex bodies satisfying the so-called positive-stiffness property.

Let τ be a positively homogeneous convex function. We would like to avoid using any smoothness hypothesis on τ, since they would have implications on the kind of convex body which can be considered (for example, if we suppose τ differentiable, then Lemma B.1.7 shows that the corresponding convex body has no facet). We first introduce a notion of curvature which does not depend on any smoothness assumption on neither τ nor the associated convex body and which is equivalent to the usual definition when this latter applies.

Definition.

(D237) *Two convex bodies $\mathcal{W}_1$ and $\mathcal{W}_2$ are* **tangent at** x^* *if they have a common support plane at x^*.*

(D238) *A point is* **tangent** *to a convex body if it belongs to its boundary; a half plane H is* **tangent** *to a convex body $\mathcal{W}$ if the boundary of H is a support plane for $\mathcal{W}$.*

Let $\mathcal{W}$ be a convex body, $x^* \in \partial\mathcal{W}$ and U an open neighbourhood of x^*. Let $\mathcal{T}_i(x^*, U)$ be the family of discs $\mathcal{D}$ with the following properties:

1. $\mathcal{D}$ is tangent to $\mathcal{W}$ at x^*;
2. $\mathcal{W} \cap U \supset \mathcal{D} \cap U$.

The degenerate cases when $\mathcal{D}$ is equal to $\{x^*\}$ or to a half plane are allowed[2]. By definition, $\mathcal{T}_i(x^*, U) \neq \varnothing$. We denote by $\rho(\mathcal{D})$ the radius of the disc $\mathcal{D}$ and set

$$\underline{\rho}(x^*, U) \doteqdot \sup\{\rho(\mathcal{D}) : \mathcal{D} \in \mathcal{T}_i(x^*, U)\}. \tag{B.14}$$

[2]Notice that the definition implies that $\mathcal{D} \subset \bigcap_{\substack{\widehat{x}: \\ \widehat{x} \text{ dual to } x^*}} H(\widehat{x})$.

Clearly, we have the following monotonicity property:

$$U_1 \supset U_2 \implies \underline{\rho}(x^*, U_1) \leqslant \underline{\rho}(x^*, U_2)\,. \tag{B.15}$$

Therefore the following quantity is well defined,

Definition.
(D239) *The* **lower radius of curvature of** $\mathcal{W}$ **at** x^* *is defined by*

$$\underline{\rho}(x^*) \doteqdot \sup\{\underline{\rho}(x^*, U) \,:\, U \text{ open neighbourhood of } x^*\}\,.$$

Similarly, we introduce the set $\mathcal{T}_s(x^*, U) \neq \varnothing$ of discs with the properties

1. $\mathcal{D}$ is tangent to $\mathcal{W}$ at x^*;
2. $\mathcal{W} \cap U \subset \mathcal{D} \cap U$.

As before, the degenerate cases are allowed. We set

$$\overline{\rho}(x^*, U) \doteqdot \inf\{\rho(\mathcal{D}) \,:\, \mathcal{D} \in \mathcal{T}_s(x^*, U)\}\,. \tag{B.16}$$

Definition.
(D240) *The* **upper radius of curvature of** $\mathcal{W}$ **at** x^* *is defined by*

$$\overline{\rho}(x^*) \doteqdot \inf\{\overline{\rho}(x^*, U) \,:\, U \text{ open neighbourhood of } x^*\}\,.$$

We can now define the radius of curvature:

Definition.
(D241) *If* $\underline{\rho}(x^*) = \overline{\rho}(x^*)$ *then the* **radius of curvature of** $\mathcal{W}$ **at** x^* *is defined by*

$$\rho(x^*) \doteqdot \underline{\rho}(x^*) = \overline{\rho}(x^*)\,.$$

Let us verify that this definition is equivalent to the usual notion of curvature. Let us first recall this standard notion.

Let $x^*, y^* \in \partial W$, $x^* \neq y^*$ and suppose x^* is regular. We denote by $\mathcal{D}(x^*, \rho_{y^*})$ the disk of radius ρ_{y^*} which is tangent to $\mathcal{W}$ at x^* and such that $y^* \in \partial\mathcal{D}(x^*, \rho_{y^*})$. Then the curvature of $\mathcal{W}$ at x^* is given by $\rho'(x^*) \doteqdot \lim_{y^* \to x^*} \rho_{y^*}$ if this limit exists.

Lemma B.2.1. *Let* $x^* \in \mathcal{W}$ *be a regular point of* $\mathcal{W}$*. Then*

$$\underline{\rho}(x^*) = \overline{\rho}(x^*) \Leftrightarrow \lim_{y^* \to x^*} \rho_{y^*} \text{ exists.}$$

Moreover, in this case, $\rho(x^*) = \rho'(x^*)$.

Proof. If $y^* \in U$, then

$$\underline{\rho}(x^*, U) \leqslant \rho_{y^*} \leqslant \overline{\rho}(x^*, U)\,. \tag{B.17}$$

Indeed, since x^* is regular, there is a unique support plane for $\mathcal{W}$ at x^*. Consequently, the disk $\mathcal{D}(x^*, \rho_{y^*})$ and every disks in $\mathcal{T}_i(x^*, U)$ or $\mathcal{T}_s(x^*, U)$ have the same support plane at x^* and are contained into $H(\widehat{x})$, where $\widehat{x}$ is the unique point in duality with x^*. Such a family of disks can be totally ordered by inclusion. Every disks of radius smaller than $\underline{\rho}(x^*, U)$ belong to $\mathcal{T}_i(x^*, U)$, and every disks of radius larger than $\overline{\rho}(x^*, U)$ belong to $\mathcal{T}_s(x^*, U)$. The statement follows from the fact that either $\mathcal{D}(x^*, \rho_{y^*})$ does not belong to $\mathcal{T}_i(x^*, U) \cup \mathcal{T}_s(x^*, U)$, or it belongs to $\mathcal{T}_i(x^*, U) \cap \mathcal{T}_s(x^*, U)$ if this set is not empty.

From (B.17) it follows that $\underline{\rho}(x^*) = \overline{\rho}(x^*) \implies \lim_{y^* \to x^*} \rho_{y^*}$ exists and $\rho(x^*) = \rho'(x^*)$.

Conversely, suppose that $\rho'(x^*) = \lim_{y^* \to x^*} \rho_{y^*}$ exists. Then, for every $\varepsilon > 0$, there exists a neighbourhood U of x^* such that $|\rho_{y^*} - \rho'| \leqslant \varepsilon$, for all $y \in U$. Therefore $\widehat{\rho} \doteqdot \inf_{y^* \in U} \rho_{y^*} \geqslant \rho' - \varepsilon$. Let us denote by $\widehat{\mathcal{D}}$ the disk of radius $\widehat{\rho}$ tangent to $\mathcal{W}$ at x^*. By definition, $\widehat{\mathcal{D}} \cap U \subset \mathcal{W} \cap U$, so that $\widehat{\mathcal{D}} \in \mathcal{T}_i(x^*, U)$. This shows that

$$\underline{\rho}(x^*, U) \geqslant \widehat{\rho} \geqslant \rho' - \varepsilon\,. \tag{B.18}$$

Similarly, we prove that

$$\overline{\rho}(x^*, U) \leqslant \rho' + \varepsilon\,. \tag{B.19}$$

□

Remark. If we have a corner, then clearly our definition implies that $\underline{\rho}(x^*, U) = 0$ for any open neighbourhood $U \ni x^*$, and we can take $\overline{\rho}(x^*, U)$ as small as we wish provided U is small enough. Consequently, $\rho(x^*) = 0$. However, the converse is not true: $\rho(x^*)$ may be equal to zero even if x^* is not corner! This is shown in the following example: Consider the convex body whose boundary is given by

$$\partial\mathcal{W} = \{z \in \mathbb{R}^2 \,:\, z(1) = \cos t |\cos t|^{.6}\,,\ z(2) = \sin t |\sin t|^{.6}\,,\ t \in [0, 2\pi]\}\,. \tag{B.20}$$

It is elementary to check that $\rho(x^*) = 0$, at the four points given by $t = k\pi/2$, $k = 0, \ldots, 3$, however there is a unique support plane at any point of the boundary.

We may wonder if it is possible to obtain different values for $\underline{\rho}(x^*)$ and $\overline{\rho}(x^*)$. A convex curve exhibiting such a behaviour can be constructed by integrating twice the function $1 + \sin 1/x$ on a neighbourhood of the origin.

We are now ready to define positive stiffness.

Definition.

(D242) *Let $K > 0$. A support function $\tau_{\mathcal{W}}$ has the* **positive stiffness** *property with constant K if the corresponding convex body has its lower radius of curvature bounded below uniformly by K.*

Remark. We emphasize the fact that this property is stronger than strict convexity of $\tau_{\mathcal{W}}$. Indeed, the support function of the convex body defined by (B.20) is strictly convex (since $\mathcal{W}$ does not have a corner), but the lower radius of curvature of $\mathcal{W}$ is not bounded below.

The following result is important for the next subsection.

Lemma B.2.2. *Let $K_0 > 0$ and let $\mathcal{W}$ be a convex body such that its support function satisfies the positive stiffness property for the constant K_0. Then for any $\rho < K_0$ and any $x^* \in \partial\mathcal{W}$, the disk $\mathcal{D}(x^*, \rho)$ of radius ρ and tangent to $\mathcal{W}$ at x^* satisfies*

$$\mathcal{D}(x^*, \rho) \subset \mathcal{W}\,.$$

Proof. The first observation is that $\mathcal{W}$ has no corner, since the lower radius of curvature is bounded below by K_0 at every x^*. Consequently, for any $x^* \in \partial\mathcal{W}$, there exists a unique $\widehat{x}$ in duality with x^*.

The second observation is that the hypothesis implies that at every $x^* \in \partial\mathcal{W}$ there is a disk $\mathcal{D}(x^*)$ of radius $\rho(\mathcal{D}(x^*))$ with the properties:

- $\rho(\mathcal{D}(x^*)) \neq 0$;
- $\mathcal{D}(x^*) \subset \mathcal{W}$;
- $\mathcal{D}(x^*)$ is tangent to $\mathcal{W}$ at x^*.

Indeed, let U and $\mathcal{D}' \in \mathcal{T}_i(x^*, U)$ be such that $\mathcal{D}'$ has a strictly positive radius, then the set $\mathcal{D}' \cap U$ is not empty and it is always possible to put a disk inside it.

Since $\mathcal{W}$ is convex, the convex envelope of all these disks is a subset of $\mathcal{W}$. Therefore, by compactness of $\mathcal{W}$, we can find $\delta > 0$ such that $\rho(\mathcal{D}(x^*)) \geqslant \delta$ for any x^*.

Let $x^* \in \partial\mathcal{W}$ and $\widehat{y}$ be given[3]. Let $\mathcal{D}(x^*, \widehat{y}) \subset H(\widehat{x}) \cap H(\widehat{y})$ be the largest disk, which is tangent to $A(\widehat{x})$ at x^*. If $\widehat{x} = \widehat{y}$, then the radius $r(x^*, \widehat{x})$ of $\mathcal{D}(x^*, \widehat{y})$ is infinite, otherwise it is finite. Since $\tau_{\mathcal{W}}$ is continuous (it is convex), $r(x^*, \widehat{y})$ is a continuous function of $\widehat{y}$ at any $\widehat{y} \neq \widehat{x}$. We set

$$r(x^*) \doteqdot \inf_{\widehat{y}} r(x^*, \widehat{y})\,. \tag{B.21}$$

Let $(\widehat{y}_n)$ be a minimizing sequence such that $\lim_n r(x^*, \widehat{y}_n) = r(x^*)$ and $\lim_n \widehat{y}_n =: \widehat{y}$. There are two cases: $\widehat{y} = \widehat{x}$ and $\widehat{y} \neq \widehat{x}$.

If $\widehat{y} = \widehat{x}$, then $r(x^*) \geqslant K_0$. Suppose this is not true, $r(x^*) < K_0$. Then, for any n such that $r(x^*, \widehat{y}_n) < K_0$, we can find a disk $\mathcal{D}_n$ and a neighbourhood U_n of x^* such that $\mathcal{D}_n$ is tangent to $\mathcal{W}$ at x^* and

$$\mathcal{W} \cap U_n \supset \mathcal{D}_n \cap U_n \supset \mathcal{D}(x^*, \widehat{y}_n) \cap U_n\,, \tag{B.22}$$

[3]The idea now is to take the largest disk with the three properties above at each point, to consider the smallest of them and then to prove that it cannot have a radius smaller than ρ. Of course these disks may not exist (i.e. the supremum may not be a maximum, or the infimum a minimum) so that the proof requires some care.

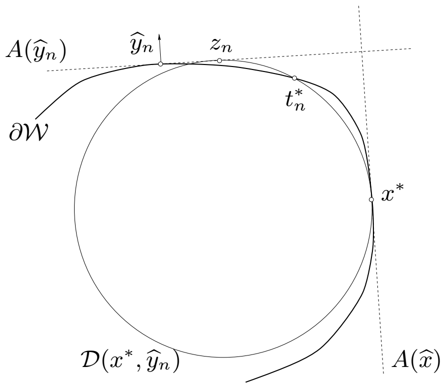

FIGURE B.2. The points $\widehat{y}_n$, x^*, ... of the proof of Lemma B.2.2.

since $\underline{\rho}(x^*) \geqslant K_0$. Let z_n be the point of contact of $\mathcal{D}(x^*, \widehat{y}_n)$ with $A(\widehat{y}_n)$ (there must be one by definition of $\mathcal{D}(x^*, \widehat{y}_n)$). Since $\mathcal{W}$ is convex, $\partial\mathcal{W}$ intersects $\partial\mathcal{D}(x^*, \widehat{y}_n)$ at some point t_n^* belonging to the circle arc of $\partial\mathcal{D}(x^*, \widehat{y}_n)$ from x^* to z_n (see Fig. B.2). Since $r(x^*, \widehat{y}_n) < K_0$ and $\widehat{x} = \lim_n \widehat{y}_n$, we also have $\lim_n z_n = x^*$ and thus $\lim_n t_n^* = x^*$. But this contradicts (B.17). Thus $r(x^*) \geqslant K_0$ and for any $\rho < K_0$ there is a disk $\mathcal{D}(x^*, \rho)$ of radius ρ, tangent to $\mathcal{W}$ at $x^* \in \partial\mathcal{W}$; Lemma B.1.8 implies that $\mathcal{D}(x^*, \rho) \subset \mathcal{W}$.

If $\widehat{y} \neq \widehat{x}$, then $r(x^*, \widehat{y}) = r(x^*) \geqslant \delta$ and the disk $\mathcal{D}(x^*, \widehat{y}) \subset \mathcal{W}$ by Lemma B.1.8. Let $r \doteqdot \inf_{x^*} r(x^*)$; we claim that $r \geqslant K_0$. Suppose this is not true, $r < K_0$. Let (z_n^*) be a minimizing sequence such that $r(z_n^*) < K_0$, $\lim_n r(z_n^*) = r$ and $\lim_n z_n^* =: z^*$. For every n there exists $\widehat{y}_n$ such that $r(z_n^*, \widehat{y}_n) = r(z_n^*)$ and $\mathcal{D}(z_n^*, \widehat{y}_n)$ is the largest disk in $\mathcal{W}$ which is tangent to $\partial\mathcal{W}$ at z_n^*. Since $r < K_0$, there exists $\widehat{y} \neq \widehat{z}$, so that

$$r = r(z^*, \widehat{y}) . \tag{B.23}$$

Indeed, if $\mathcal{D}(z^*) \subset \mathcal{W}$ is a disk tangent to $\partial\mathcal{W}$ at z^*, then by convexity the convex envelope of $\mathcal{D}(z^*)$ and $\mathcal{D}(z_n^*, \widehat{y}_n)$ is a subset of $\mathcal{W}$. If $\rho(\mathcal{D}(z^*)) > r$, then the disks $\mathcal{D}(z_n^*, \widehat{y}_n)$ are not the largest disks in $\mathcal{W}$ which are tangent to $\mathcal{W}$ at z_n^*, when n is sufficiently large. The existence of $\mathcal{D}(z^*, \widehat{y})$ and the convexity of $\mathcal{W}$ imply the existence of an open set V such that

$$\partial\mathcal{W} \cap V = \partial\mathcal{D}(z^*, \widehat{y}) \cap V . \tag{B.24}$$

Indeed, $\mathcal{D}(z^*, \widehat{y})$ is tangent to $\partial\mathcal{W}$ at z^* and also at some y^* in duality with $\widehat{y}$; moreover, at any point $x^* \in \partial\mathcal{W}$, there exists a disk of radius r contained in $\mathcal{W}$, tangent to $\partial\mathcal{W}$ at x^*. (B.24) implies that the radius of curvature at any point $v \in V \cap \partial\mathcal{W}$ is exactly r. But this contradicts $r < K_0$. □

B.3 The Sharp Triangle Inequality

A very important property of the surface tension in the first part of this thesis is the sharp Triangle Inequality. It is related to some property of the curvature of the corresponding Wulff shape. We discuss this point in a more general setting.

Definition.

(D243) *Let* $K > 0$. *A function* τ *satisfies the* **Sharp Triangle Inequality** *with constant* K *if*

$$\tau(x) + \tau(y) - \tau(x+y) \geqslant K(\|x\|_2 + \|y\|_2 - \|x+y\|_2)\,,$$

for any $x, y \in \mathbb{R}^2$.

Remark. If τ is a norm, then the Sharp Triangle Inequality provides an interesting comparison inequality between the behaviour of the triangle inequalities of the norm τ and of the Euclidean norm.

The next Proposition states that this very useful property is in fact equivalent to the positive stiffness property[4].

Proposition B.3.1. *Let* τ *be a positively homogeneous, convex function. Then the following statements are equivalent.*

1. τ *satisfies the positive stiffness property with a constant* K_0.
2. *There exists a constant* $K_1 > 0$ *such that, for any* $\widehat{x}$ *and* $\widehat{y}$ *in duality with* x^* *and* y^*,
$$\langle x^* - y^*, \widehat{x} \rangle \geqslant K_1 \|\widehat{x} - \widehat{y}\|_2^2\,. \tag{B.25}$$
3. *There exists a constant* $K_2 > 0$ *such that* τ *satisfies the Sharp Triangle Inequality with constant* K_2.

Moreover, if the corresponding convex body has its curvature bounded above everywhere by κ, *then 1. holds with* $K_0 = 1/\kappa$, *1. implies 2. with* $K_1 = 1/2\kappa$ *and 2. implies 3. with* $K_2 = 1/\kappa$.

Proof. We denote by $\mathcal{W}$ the convex body whose support function is τ.
We prove 1 $\implies$ 2. Notice that, by hypothesis, the convex body has no angles. Let $x^*, y^* \in \partial\mathcal{W}$, $x^* \neq y^*$ and $0 < 2K_1 < K_0$. The disk $\mathcal{D}(x^*, 2K_1)$ of radius $2K_1$, tangent to $\mathcal{W}$ at x^* is a subset of $\mathcal{W}$ by Lemma B.2.2; we denote by c its center. If

$$\langle x^* - y^*, \widehat{x} \rangle \geqslant 4K_1\,, \tag{B.26}$$

then

$$\langle x^* - y^*, \widehat{x} \rangle \geqslant K_1 \|\widehat{x} - \widehat{y}\|_2^2\,, \tag{B.27}$$

since $\|\widehat{x} - \widehat{y}\|_2 \leqslant 2$.
We suppose that

$$\langle x^* - y^*, \widehat{x} \rangle < 4K_1\,. \tag{B.28}$$

We can find $z \in \mathcal{D}(x^*, 2K_1)$ such that, if $\widehat{v} \doteqdot (z-c)/\|z-c\|_2$ and ϕ is the angle between the unit vectors $\widehat{x}$ and $\widehat{v}$, then $\langle \widehat{x}, \widehat{v} \rangle \geqslant 0$ and

$$\langle x^* - y^*, \widehat{x} \rangle = 2K_1(1 - \cos\phi) = K_1 \|\widehat{x} - \widehat{v}\|_2^2\,. \tag{B.29}$$

The key observation is that the vector $\widehat{y}$ cannot "turn" faster than $\widehat{v}$:

$$\langle \widehat{x}, \widehat{y} \rangle \geqslant \langle \widehat{x}, \widehat{v} \rangle\,. \tag{B.30}$$

[4] A partial result in that direction has already been proved by Ioffe in [I1]. He proved that the positive stiffness property implies the validity of the Sharp Triangle Inequality, under some smoothness assumptions.

Suppose this is not true. We proceed in two steps. First suppose that $\langle x^* - y^*, \widehat{x}\rangle \leqslant 2K_1$. Let $\mathcal{D}(y^*, 2K_1) \subset \mathcal{W}$ be the disk of radius $2K_1$, tangent at y^* and let $\mathcal{D}'(y^*, 2K_1)$ be the disk of radius $2K_1$, whose center c' is on the segment from z to y^* and such that $y^* \in \mathcal{D}'(y^*, 2K_1)$. Then (B.28) and convexity of $\mathcal{W}$ imply that c' is in the tube delimited by $A(\widehat{x})$ and its translate going through c. Moreover, if (B.30) does not hold, then the support plane for $\mathcal{D}(x^*, 2K_1)$ at z and the support plane for $\mathcal{D}'(y^*, 2K_1)$ at y^* intersects inside $H(\widehat{x})$. Since $\mathcal{D}(y^*, 2K_1)$ can be obtained from $\mathcal{D}'(y^*, 2K_1)$ by rotating it around y^* in such a way as to make these two support planes coincide, $\mathcal{D}(y^*, 2K_1)$ cannot be contained inside $H(\widehat{x})$. Since $\mathcal{W}$ is convex, this implies that $\mathcal{D}(y^*, 2K_1) \not\subset \mathcal{W}$, which is impossible by Lemma B.2.2.

We have thus shown that for any $y^* \in \partial\mathcal{W}$ such that $\langle x^* - y^*, \widehat{x}\rangle \leqslant 2K_1$, equation (B.30) is true. We consider now the case of the points $y^* \in \partial\mathcal{W}$ such that $2K_1 < \langle x^* - y^*, \widehat{x}\rangle < 4K_1$. It is enough to observe that we can make the same argument replacing x^* by the two points $u_i^* \in \partial\mathcal{W}$, $i = 1, 2$, such that $\langle x^* - u_i^*, \widehat{x}\rangle = 2K_1$. Indeed all the points y^* satisfying $2K_1 < \langle x^* - y^*, \widehat{x}\rangle < 4K_1$ also satisfy $\langle u_i^* - y^*, \widehat{u}_i\rangle < 2K_1$, for one of these two points u_i^*. From (B.30) we have

$$\|\widehat{x} - \widehat{y}\|_2 \leqslant \|\widehat{x} - \widehat{v}\|_2 \tag{B.31}$$

and

$$\langle x^* - y^*, \widehat{x}\rangle = K_1 \|\widehat{x} - \widehat{v}\|_2^2 \geqslant K_1 \|\widehat{x} - \widehat{y}\|_2^2\,. \tag{B.32}$$

We prove $2 \implies 1$. Again the hypothesis implies the absence of corners (otherwise we could find $\widehat{x}_1 \neq \widehat{x}_2$ both in duality with x^*). Suppose that

$$\langle x^* - y^*, \widehat{x}\rangle \geqslant K_1 \|\widehat{x} - \widehat{y}\|_2^2\,. \tag{B.33}$$

Let $\mathcal{D}(x^*, \rho_{y^*})$ be the disk of radius ρ_{y^*} which is tangent to $\mathcal{W}$ at x^* and such that $y^* \in \partial\mathcal{D}(x^*, \rho_{y^*})$; we denote by c its center and write $\widehat{u} \doteqdot (y^* - c)/\|y^* - c\|_2$. Assume furthermore that $\langle \widehat{x}, \widehat{u}\rangle \geqslant 0$, which is certainly true when y^* is close enough to x^*. Let $v \doteqdot \widehat{x} + \widehat{u}$ and $\widehat{v} \doteqdot v/\|v\|_2$. Then

$$\langle x^* - y^*, \widehat{x}\rangle = \frac{\rho_{y^*}}{2} \|\widehat{x} - \widehat{u}\|_2^2\,. \tag{B.34}$$

Since $\langle x^* - y^*, \widehat{v}\rangle = 0$ and $\partial\mathcal{W}$ is convex there exists $z^* \in \partial\mathcal{W}$ "between" x^* and y^* such that $\widehat{z} = \widehat{v}$ and

$$\|\widehat{x} - \widehat{z}\|_2 \leqslant \|\widehat{x} - \widehat{y}\|_2\,. \tag{B.35}$$

On the other hand,

$$\|\widehat{x} - \widehat{u}\|_2 \leqslant \|\widehat{x} - \widehat{v}\|_2 + \|\widehat{v} - \widehat{u}\|_2 \tag{B.36}$$

and, by construction,

$$\|\widehat{x} - \widehat{v}\|_2 = \|\widehat{v} - \widehat{u}\|_2 = \|\widehat{x} - \widehat{z}\|_2\,. \tag{B.37}$$

If $\widehat{x} = \widehat{y}$, then $\rho_{y^*} = \infty$; otherwise, using the preceding equations, we can write

$$\begin{aligned} 2\rho_{y^*} \|\widehat{x} - \widehat{v}\|_2^2 &\geqslant \frac{\rho_{y^*}}{2} \|\widehat{x} - \widehat{u}\|_2^2 \\ &= \langle x^* - y^*, \widehat{x}\rangle \\ &\geqslant K_1 \|\widehat{x} - \widehat{y}\|_2^2 \\ &\geqslant K_1 \|\widehat{x} - \widehat{v}\|_2^2\,, \end{aligned} \tag{B.38}$$

and therefore $\rho_{y^*} \geqslant \frac{1}{2}K_1$. Since this holds for any y^* in a neighbourhood of x^*, we have $\underline{\rho}(x^*) \geqslant \frac{1}{2}K_1$.
We prove 2 $\Longrightarrow$ 3. We set

$$z \doteqdot x + y\,, \quad z^* \doteqdot (x+y)^* \text{ and } \widehat{z} = \frac{x+y}{\|x+y\|_2}\,. \tag{B.39}$$

With these notations, we can write

$$\begin{aligned}\tau(x) + \tau(y) - \tau(z) &= \langle x^*, x\rangle + \langle y^*, y\rangle - \langle z^*, z\rangle \\ &= \langle x^* - z^*, x\rangle + \langle y^* - z^*, y\rangle \\ &= \|x\|_2\langle x^* - z^*, \widehat{x}\rangle + \|y\|_2\langle y^* - z^*, \widehat{y}\rangle\,.\end{aligned} \tag{B.40}$$

By elementary trigonometry,

$$\|x\|_2 + \|y\|_2 - \|z\|_2 = \tfrac{1}{2}\big(\|x\|_2\,\|\widehat{x} - \widehat{z}\|_2^2 + \|y\|_2\,\|\widehat{y} - \widehat{z}\|_2^2\big)\,. \tag{B.41}$$

The conclusion follows easily by comparison between these two equations.
We prove 3 $\Longrightarrow$ 2. Let x^* and y^* be given. We set $z \doteqdot \widehat{x} + \widehat{y}$. Using (B.41), $\|\widehat{x} - \widehat{z}\|_2 = \|\widehat{y} - \widehat{z}\|_2$ and $\|\widehat{x} - \widehat{y}\|_2 \leqslant \|\widehat{x} - \widehat{z}\|_2 + \|\widehat{y} - \widehat{z}\|_2$, we have

$$\begin{aligned}\langle x^* - y^*, \widehat{x}\rangle &= \langle x^*, \widehat{x}\rangle + \langle y^*, \widehat{y}\rangle - \langle y^*, \widehat{x} + \widehat{y}\rangle \\ &\geqslant \tau(\widehat{x}) + \tau(\widehat{y}) - \tau(z) \\ &\geqslant K_2(\|\widehat{x}\|_2 + \|\widehat{y}\|_2 - \|z\|_2) \\ &= K_2\|\widehat{x} - \widehat{z}\|_2^2 \\ &\geqslant \tfrac{1}{4}K_2\|\widehat{x} - \widehat{y}\|_2^2\,.\end{aligned} \tag{B.42}$$

□

Appendix C

Cluster expansion

In this appendix, we state without proof one version of the Theorem about the convergence of the cluster expansion. Proofs can be found for example in [KoPr, Pf1, Do2]. We state it in its most explicit formulation, which can be found in [Pf1]. We restrict ourselves to the settings used in this thesis.

Let $\mathfrak{C} = \{\eta_1, \eta_2 \dots\}$ be a countable set, whose elements are pairs $\eta_i \equiv (\eta_i^s, \eta_i^c)$ with η_i^s a connected set of edges and $\eta_i^c : \mathcal{E} \supset \eta_i^s \to C$ is an application associating to each edge of η_i a value, which we call its *color*, in some finite set C. There is a reflexive, symmetric relation on $\mathfrak{C}$, which we write ι; if $\eta \,\iota\, \eta'$, we say that η and η' are *incompatible*. There is a function $z : \mathfrak{C} \to \mathbb{C}$; $z(\eta)$ is the *weight* of η. We suppose that there exists a function $w : \mathfrak{C} \to \mathbb{R}$ such that

$$|z(\eta)| \leqslant w(\eta)\,, \quad \forall \eta \in \mathfrak{C}\,, \tag{C.1}$$

and $w(\eta_i) = w(\eta_j')$ if η_i^s is a translate of η_j^s and they have the same colors. We also suppose that there is a function $b : \mathfrak{C} \to \mathcal{P}(\mathbb{Z}^2)$ such that

$$\eta \,\iota\, \eta' \implies \eta^s \cap b(\eta') \neq \varnothing\,. \tag{C.2}$$

We then have the following

Theorem C.0.1. *If*

$$D \doteqdot \sum_{\substack{\eta \in \mathfrak{C} \\ \eta \ni t}} w(\eta) \exp(|b(\eta)|) < 1\,,$$

for any site t, then

$$1 + \sum_{n \geqslant 1} \frac{1}{n!} \sum_{\substack{\eta_1, \dots, \eta_n \\ \eta_i \not\iota \eta_j, \forall i \neq j}} \prod_i z(\eta_i) = \exp\Big(\sum_{n \geqslant 1} \frac{1}{n!} \sum_{\eta_1, \dots, \eta_n} \varphi_n^{\mathrm{T}}(\eta_1, \dots, \eta_n) \prod_i z(\eta_i) \Big)\,,$$

where $\varphi_n^{\mathrm{T}}(\eta_1, \dots, \eta_n)$ is some combinatorial function which is zero if the graph of the relation ι restricted to $\eta_1, \dots \eta_n$ is not connected (when $\varphi_n^{\mathrm{T}}(\eta_1, \dots, \eta_n) \neq 0$, we say that the family $\eta_1, \dots, \eta_n$ forms a cluster*); moreover, we have the following bound*

$$\sum_{\eta_1 \ni t} \sum_{\eta_2, \dots \eta_n} |\varphi_n^{\mathrm{T}}(\eta_1, \dots, \eta_n)| \prod_i |z(\eta_i)| \leqslant (n-1)!\, D^n\,.$$

We give now an example of how this theorem is used in practice. We consider the Ashkin–Teller model in some finite simply connected subset Λ of $\mathbb{Z}^2$, with +-b.c. and with coupling constants $\beta\underline{J}$. Let $\mathfrak{C}$ be the set of all contours contributing to the partition function (see (D164), p. 197). Notice that these objects can be written in the same way as above, by specifying their support (i.e. which edges they use) and the color of each edge (i.e. if it is a σ, τ or $\sigma\tau$-edge). The compatibility relation is simply Λ-compatibility. The weight is defined in (D162), p. 196, and we can take $b(\gamma) = \Delta(\gamma)$, the edge-boundary of the contour γ. The hypotheses of the Theorem are easily seen to be satisfied when β is large enough and therefore we know that it is possible to expand the logarithm of the partition function.

The main use of the theorem is to estimate a ratio of partition functions, where the contours appearing in the numerator (for example) must satisfy some supplementary constraints, for example that they are $\log \Lambda$-small. Then both the numerator and denominator can be expanded, and all families of contours $\gamma_1, \ldots, \gamma_n$ which contain only small contours cancel and there only remain terms corresponding to clusters containing at least one large contour. Since the constant D can then be taken as $|\Lambda|^{-\mathcal{O}(\beta)}$. This implies that the probability that there are only $\log|\Lambda|$-small contours is $1 - |\Lambda|^{-\mathcal{O}(\beta)}$.

List of definitions

List of definitions

Part I: Ising model

Part II: Ashkin–Teller model

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Teaching

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Publications

Mathematical Theory of the Wetting Phenomenon in the 2D Ising Model, C.-E. Pfister and Y. Velenik, Helv. Phys. Acta **69** (1996), 949–973.
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