p-adic limits of Betti numbers

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Lück's approximation theorem

Let X be a finite connected CW-complex with $\Gamma = \pi_1(X)$ and let $\widetilde{X} \to X$ be its universal covering.

Theorem (Lück 1994)

Let $\Gamma_1 \supseteq \Gamma_2 \supseteq \Gamma_3 \supseteq \dots$ be a decreasing chain of finite index normal subgroups of Γ such that $\bigcap_{n\in\mathbb{N}} \Gamma_n = \{1\}$. Then

$$\lim_{n\to\infty} \frac{b_j(\widetilde{X}/\Gamma_n, \mathbb{Q})}{|\Gamma : \Gamma_n|} = b_j^{(2)}(X) \in \mathbb{R}_{\geq 0}.$$

Is there a p-adic analog?

Recall: $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ w.r.t.

$$\left|p^r \frac{a}{b}\right|_p = p^{-r} \qquad (a, b, r \in \mathbb{Z} \text{ with } p \nmid a, b)$$

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Problem:
Let $X = S^1$, $\widetilde{X} = \mathbb{R}$ and $\Gamma = \mathbb{Z}$. The sequence

$$\frac{b_0(\mathbb{R}/p^n\mathbb{Z}, \mathbb{Q})}{|\mathbb{Z} : p^n\mathbb{Z}|} = \frac{1}{p^n} \to \infty$$

diverges in $\mathbb{Q}_p$.

A p-adic approximation theorem

Let (G, φ) be a *virtual pro-p completion* of Γ; i.e.

- a profinite group G which is virtually pro-p and
- $\varphi\colon \Gamma \to G$ with dense image.

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Theorem (K. 2018)

Let $G_1 \supseteq G_2 \supseteq G_3 \supseteq \dots$ be a decreasing chain of open normal subgroups in G with $\bigcap_{n\in\mathbb{N}} G_n = \{1\}$. Then

- $\lim_{n\to\infty} b_j(\widetilde{X}/\varphi^{-1}(G_n), \mathbb{Q}) = b_j^{[p]}(X; \varphi, \mathbb{Q}) \in \mathbb{Z}_p,$

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- $\lim_{n\to\infty} \left|\operatorname{tors} H^j(\widetilde{X}/\varphi^{-1}(G_n), \mathbb{Z}[1/p])\right| = t_j^{[p]}(X; \varphi, \mathbb{Z}[1/p]) \in \mathbb{Z}_p$.

Basic properties of p-adic Betti numbers

Euler characteristic:

$$\chi^{[p]}(X;\varphi) = \sum_{j=0}^{\dim(X)} (-1)^j b_j^{[p]}(X;\varphi,\mathbb{Q})$$

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Further properties:

- Künneth formula
- Poincaré duality
- a formula for wedge sums

Examples

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T^r the r-torus and (G, φ) a virtual pro-p completion of $\mathbb{Z}^r$:

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Σ_g closed orientable surface of genus $g \geq 1$ and (G, φ) an infinite virtual pro-p completion of $\pi_1(\Sigma_g)$:

$$b_0^{[p]}(\Sigma_g; \varphi) = 1, \qquad b_1^{[p]}(\Sigma_g; \varphi) = 2, \qquad b_2^{[p]}(\Sigma_g; \varphi) = 1$$

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Free groups

(G, φ) an infinite virtual pro-p completion of the free group F_r:

$$b_0^{[p]}(F_r; \varphi) = 1, \qquad b_1^{[p]}(F_r; \varphi) = 1$$

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Theorem (K. 2018)

If G is abelian, then $b_j^{[p]}(X;\varphi,k) \in \mathbb{Z}$.

Definition of p-adic Betti numbers

X finite connected CW-complex and (G, φ) a virtual pro-p completion of $\Gamma = \pi_1(X)$.

$$\bar{H}^j(X; \varphi, \mathbb{Q}) = \varinjlim_{N \trianglelefteq_o G} H^j(\widetilde{X}/\varphi^{-1}(N), \mathbb{Q})$$

is a vector space with a smooth admissible representation of G.

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Theorem

There is a unique continuous $\mathbb{Z}_p$-linear function

$$p\text{-}\dim_{\mathbb{Q}}^G \colon \mathsf{K}_0^{\mathsf{adm}}(\mathbb{Q}[\![G]\!]) \to \mathbb{Z}_p$$

such that $p\text{-}\dim_{\mathbb{Q}}^G(V) = \dim_{\mathbb{Q}} V$ for every finite dimensional representation of G.