Automorphisms of free groups and first-order properties of tuples of elements

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Les Diablerets

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Homogeneity

Definition

Let M be a model and $\bar{a} \in M$ be a tuple. The **type** of $\bar{a}$ in M is $\text{tp}^M(\bar{a}) = \{\phi(\bar{x}) : M \models \phi(\bar{a})\}$.

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M is **homogeneous** if for any $\bar{a}, \bar{b} \in M$, $\text{tp}^M(\bar{a}) = \text{tp}^M(\bar{b})$ implies there is an automorphism of M sending $\bar{a}$ to $\bar{b}$.

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Theorem (Perin-Sklinos, Ould Houcine)

Non-abelian free groups (of any rank) are homogeneous.

$\forall$-homogeneity

Definition

Let M be a model and $\bar{a} \in M$ be a tuple. The $\forall$-**type** of $\bar{a}$ in M is $tp_{\forall}^{M}(\bar{a}) = \{\phi(\bar{x}) : M \models \phi(\bar{a})$ and $\phi(\bar{x})$ is universal$\}$.

M is $\forall$-**homogeneous** if for any $\bar{a}, \bar{b} \in M$, $tp_{\forall}^{M}(\bar{a}) = tp_{\forall}^{M}(\bar{b})$ implies there is an automorphism of M sending $\bar{a}$ to $\bar{b}$.

Theorem (Nies)

The non-abelian free group F_2 of rank 2 is $\forall$-homogeneous.

Theorem

Rigid torsion free hyperbolic groups are $\forall$-homogeneous (in particular fundamental groups of closed hyperbolic n-manifolds, $n \geq 3$). A non-free two generated torsion free hyperbolic groups are $\forall$-homogeneous.

Theorem (K , Natoli)

Non-abelian free groups of finite rank at least 3 or of countable rank are not ∀-homogeneous.

Elementary subgroups

Definition

A substructure M of N is **elementary** in N, denoted $M \preceq N$, if for each $\bar{a} \in M$, we have $\mathrm{tp}^M(\bar{a}) = \mathrm{tp}^N(\bar{a})$.

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Theorem (($\Rightarrow$) K.-Myasnikov, Sela; ($\Leftarrow$) Perin)

Suppose F is a finitely generated free group and $H < F$. Then H is elementary in F iff H is a non-abelian free factor of F.

Existentially closed subgroups

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A substructure M of N is **existentially closed** in N if for each $\bar{a} \in M$, we have $\text{tp}_{\forall}^{M}(\bar{a}) = \text{tp}_{\forall}^{N}(\bar{a})$.

Lemma (K.-Myasnikov-Sklinos)

Suppose L, M are limit groups and $L < M$. Then L is existentially closed in M iff there is an a finite iterated centralizer extension L_n of L such that $M \leq L_n$.

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Definition

L_n is a **finite iterated centralizer extension** of L if $L = L_0 < L_1 < \cdots < L_n$ and each $L_{i+1} = \langle L_i, t_i \mid [C_{L_i}(u_i), t_i] = 1 \rangle$ where $1 \neq u_i \in L_i$ is fixed and t_i is a new letter.

Construction

$$L = \langle a, b \rangle$$

Let $\tilde{x} = x^t$, $b_1 = b^t x^{2nt}$, then $b = h(h^m(b_1)^{-m}\tilde{x}^{-2})^n$; Whitehead graph has cycle $(h^{-1}, h, b_1, b_1^{-1}, \tilde{x}, \tilde{x}^{-1}, h^{-1})$; b is not primitive.

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$$L = \langle a, b \rangle$$

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Let $\tilde{x} = x^t$, $b_1 = b^t x^{2nt}$, then $b = h(h^m (b_1)^{-m} \tilde{x}^{-2})^n$; Whitehead graph has cycle $(h^{-1}, h, b_1, b_1^{-1}, \tilde{x}, \tilde{x}^{-1}, h^{-1})$; b is not primitive.

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Free groups of ranks 3 and 4

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Non-abelian free groups of finite rank at least 3 or countable rank are not ∀-homogeneous.

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$$L = \langle a, b \rangle \underset{\text{existentially closed}}{<} M = \langle a, bx^{2n}, b^t, x^t \rangle < L_2$$

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$$\text{tp}_\forall^M(a,b) = \text{tp}_\forall^L(a,b) = \text{tp}_\forall^{\langle a, bx^{2n}\rangle}(a, bx^{2n}) = \text{tp}_\forall^M(a, bx^{2n})$$

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Rank 3:

$$M_3 = \langle bx^{2n}, b^t, x^t \rangle \qquad \mathrm{tp}_\forall^{M_3}(b) = \mathrm{tp}_\forall^{M_3}(bx^{2n})$$

Free groups of finite rank > 4

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Non-abelian free groups of finite rank at least 3 or countable rank are not $\forall$-homogeneous.

Lemma (Ould Houcine)

If a free group is $\forall$-homogeneous, then every existentially closed subgroup is a free factor.

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Free group of rank ω

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O. Kharlampovich and C. Natoli

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$$\implies \exists \text{ automorphism of } F_\omega \text{ sending } (a, b) \mapsto (a, bx^{2n})$$

$$\implies \mathrm{tp}^M(a, b) = \mathrm{tp}^M(a, bx^{2n})$$

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In a free group of finite rank at least 3 or countable rank, primitive elements are not ∀-type-definable.

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The theory of finitely generated non-abelian free groups does not have quantifier elimination to boolean combinations of ∀-formulas.

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The theory of finitely generated non-abelian free groups does not have quantifier elimination to boolean combinations of ∀-formulas.

Compare:

Theorem (K.-Myasnikov, Sela)

The theory of finitely generated non-abelian free groups has quantifier elimination to boolean combinations of ∀∃-formulas.

Fraïssé limits

Roland Fraïssé in 1954 observed that one can see the class of finite linear orders as approximations of $(\mathbb{Q}, <)$. He constructed it as a direct limit of finite linear orders using amalgamations. Furthermore, his construction implies the countability, the universality and the homogeneity of the limit structure, as well as its uniqueness with respect to those properties.

Examples: The Rado graph, Erdos-Renyi graph, or random graph is a countable infinite graph that can be constructed by choosing independently at random for each pair of its vertices whether to connect the vertices by an edge.

Fraïssé limits

Philip Hall's universal locally finite group is a countable locally finite group, say U, which is uniquely characterized by the following properties:
1) Every finite group G admits a monomorphism to U.
2) All such monomorphisms are conjugate by inner automorphisms of U.
It was defined by Philip Hall in 1959 and has the universal property that all countable locally finite groups embed into it.

Fraïssé class

Definition

Let $\mathcal{K}$ be a countable (with respect to isomorphism types) non-empty class of finitely generated $\mathcal{L}$-structures with the following properties:

- (IP) the class $\mathcal{K}$ is closed under isomorphisms;
- (HP) the class $\mathcal{K}$ is closed under finitely generated substructures;
- (JEP) if $\mathcal{A}_1$, $\mathcal{A}_2$ are in $\mathcal{K}$, then there is $\mathcal{B}$ in $\mathcal{K}$ and embeddings $f_i : \mathcal{A}_i \to \mathcal{B}$ for $i \leq 2$;
- (AP) if $\mathcal{A}_0$, $\mathcal{A}_1$, $\mathcal{A}_2$ are in $\mathcal{K}$ and $f_i : \mathcal{A}_0 \to \mathcal{A}_i$ for $i \leq 2$ are embeddings, then there is $\mathcal{B}$ in $\mathcal{K}$ and embeddings $g_i : \mathcal{A}_i \to \mathcal{B}$ for $i \leq 2$ with $g_1 \circ f_1 = g_2 \circ f_2$.

Then $\mathcal{K}$ is a Fraïssé class.

Fraïssé theorem

Theorem (Fraïssé's theorem)

Let $\mathcal{K}$ be a Fraïssé class. Then there exists a countable $\mathcal{L}$-structure $\mathcal{M}$ such that:

- *the class of finitely generated substructures of $\mathcal{M}$ (age($\mathcal{M}$)) is exactly $\mathcal{K}$;*
- *the $\mathcal{L}$-structure $\mathcal{M}$ is ultrahomogeneous, i.e. every isomorphism between finitely generated substructures of $\mathcal{M}$ extends to an automorphism of $\mathcal{M}$.*

Moreover, any other countable $\mathcal{L}$-structure with the above properties is isomorphic to $\mathcal{M}$.

$\forall$-Fraïssé class

Definition

$f : A \to B$ is a $\forall$-**embedding** if for all $\bar{a} \in A$, for all universal formulas $\phi(\bar{x})$, $A \models \phi(\bar{a})$ implies $B \models \phi(\bar{a})$.

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Definition

Fix a language $\mathcal{L}$. Let $\mathcal{K}$ be a countable non-empty class of finitely generated $\mathcal{L}$-structures with the following properties:

- (IP) the class $\mathcal{K}$ is closed under isomorphisms;
- ($\forall$-HP) the class $\mathcal{K}$ is closed under finitely generated $\forall$-substructures (i.e., existentially closed substructures);
- ($\forall$-JEP) if A_1, A_2 are in $\mathcal{K}$, then there are B in $\mathcal{K}$ and $\forall$-embeddings $f_i : A_i \to_\forall B$ for $i \leq 2$;
- ($\forall$-AP) if A_0, A_1, A_2 are in $\mathcal{K}$ and $f_i : A_0 \to_\forall A_i$ for $i \leq 2$ are $\forall$-embeddings, then there are B in $\mathcal{K}$ and $\forall$-embeddings $g_i : A_i \to_\forall B$ for $i \leq 2$ with $g_1 \circ f_1 = g_2 \circ f_2$.

Then $\mathcal{K}$ is a **universal Fraïssé class** or for short a $\forall$-**Fraïssé class**.

Theorem

[K., Miasnikov, Sklinos] Class $\mathcal{F}$ of non-abelian limit groups is a ∀-Fraïssé class.
In particular there exists a countable group G with the following properties:

- *the ∀-age of G is the class $\mathcal{F}$;*
- *the group G is weakly ∀-homogeneous;*
- *the group G is a union of a ∀-chain of nonabelian limit groups.*

Moreover, any countable group with the above properties is isomorphic to G.

Free groups do not satisfy ∀-AP

Let m, n, p, q be sufficiently large and m, q even. Let $L = \langle a, b \rangle$ be a common subgroup of

$$H = \left\langle a, h, \tilde{b}, \tilde{x} \right\rangle$$

$$b \mapsto h \left(\tilde{x}^8 (\tilde{b}\tilde{x}^{8n})^m h^{-m} \right)^{-n}$$

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$$H = \left\langle a, h, \tilde{b}, \tilde{x} \right\rangle \qquad K = \left\langle a, k, \hat{b}, \hat{y} \right\rangle$$

$$b \mapsto h\left(\tilde{x}^8(\tilde{b}\tilde{x}^{8n})^m h^{-m}\right)^{-n} \qquad b \mapsto k\left(\hat{y}^7(\hat{b}\hat{y}^{7p})^q k^{-q}\right)^{-p}.$$

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Suppose there is a finitely generated free group F satisfying ∀-AP w.r.t L, H, K. Then in F,

$$h\left(h^m(\tilde{x}^{-8n}\tilde{b}^{-1})^m\tilde{x}^{-8}\right)^n\left(\hat{y}^7(\hat{b}\hat{y}^{7p})^q k^{-q}\right)^p k^{-1} = 1.$$

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If in F,

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then either

- for some $h_1, h_2 \in \{h^{\pm 1}, (\tilde{x}^{-8n}\tilde{b}^{-1})^{\pm 1}, \tilde{x}^{\pm 1}\}$, $F \models \exists z[h_1, h_2^z] = 1$, but then $H \models \exists z[h_1, h_2^z] = 1$ (contradiction); or

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- in F, $h^m(\tilde{x}^{-8n}\tilde{b}^{-1})^m\tilde{x}^{-8}$ is a conjugate of $\hat{y}^7(\hat{b}\hat{y}^{7p})^q k^{-q}$, so

$$K \models \exists \check{x}, \check{b}, \check{h}\left(\check{x}^8(\check{b}\check{x}^{8n})^m\check{h}^{-m} = \hat{y}^7(\hat{b}\hat{y}^{7p})^q k^{-q}\right)$$

(contradiction).

Free groups do not satisfy $\forall$-AP

Theorem (K,N)

The class of finitely generated free groups is not a $\forall$-Fraïssé class.

Theorem (K,N)

The class of finitely generated elementary free groups is not a $\forall$-Fraïssé class.

Theorem (K,M,S)

The class of finitely generated elementary free groups is elementary-Fraïssé class.

Countable elementary free groups

Definition

A group G is **elementary free** if $G \equiv F_2$.

All finitely generated elementary free groups are described by K,Miasnikov and Sela.

Question Describe all countable elementary free groups.

Countable elementary free groups

Theorem (Szmielew)

If A, D are torsion-free abelian groups and D is divisible, then $A \equiv A \oplus D$.

E.g., $\mathbb{Z} \equiv \mathbb{Z} \oplus \mathbb{Q}$.

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If A, D are torsion-free abelian groups and D is divisible, then $A \equiv A \oplus D$.

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Theorem (Sela)

If A_1, B_1, A_2, B_2 are groups and $A_1 \equiv A_2$ and $B_1 \equiv B_2$, then $A_1 * B_1 \equiv A_2 * B_2$.

E.g., $F_2 \equiv \mathbb{Z} * (\mathbb{Z} \oplus \mathbb{Q})$.

Countable elementary free groups

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Theorem

$\mathbb{Z} * (\mathbb{Z} \oplus \mathbb{Q})$ cannot be obtained as a union of a chain of finitely generated elementary free groups.