L^2-Betti numbers of groups

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1 Motivation for L^2-cohomology

Let X be a finite complex. The *Betti numbers*

$$b_i(X) = \dim H^i(X, \mathbf{R})$$

are homotopy invariants of X. The *Euler characteristic* is

$$\chi(X) = \sum_i (-1)^i b_i(X).$$

Let $\hat{X}$ be a d-sheeted covering of X. Then

$$\chi(\hat{X}) = d \cdot \chi(X).$$

However: in general $b_i(\hat{X}) \neq d \cdot b_i(X)$.

Example 1. *Take $X = \hat{X} = S^1$, the unit circle in $\mathbf{C}$. The map $\hat{X} \to X : z \mapsto z^d$ is a d-sheeted covering. But $b_i(X) = b_i(\hat{X}) = 1$ for $i = 0, 1$.*

Goal: By means of the universal cover $\tilde{X}$ of X, construct L^2-Betti numbers $b_i^{(2)}(X) \geq 0$ such that:

B1: $b_i^{(2)}(X)$ is a homotopy invariant of X.

B2: $\chi(X) = \sum_i (-1)^i b_i^{(2)}(X)$.

B3: If $\hat{X}$ is a d-sheeted cover of X, then $b_i^{(2)}(\hat{X}) = d \cdot b_i^{(2)}(X)$.

B4: (Lück approximation) Suppose $G = \pi_1(X)$, and if $(G_j)_{j\geq 1}$ is a family of finite index normal subgroups that decreases to $\{1\}$ for $j \to \infty$, then with $X_j = \tilde{X}/G_j$ we have:

$$b_i^{(2)}(X) = \lim_{j\to\infty} \frac{b_i(X_j)}{[G : G_j]}.$$

THIS CAN BE DONE!

2 A few successes

2.1 The Hopf conjecture

Conjecture 1. *(Hopf 1931, Chern 1955) Let M^{2n} be a closed manifold carrying a Riemannian metric of negative sectional curvature. Then $(-1)^n\chi(M) > 0$.*

The conjecture is known in dimensions 2 (Gauss-Bonnet formula) and 4 (Milnor 1955); it is open in general. Using L^2-Betti numbers, Gromov proved in 1991:

Theorem 2.1. *The Conjecture holds for Kähler manifolds.*

2.2 Deficiency of groups

For G a finitely presented group, the *deficiency* of G is:

$def(G) = \max\{g - r : G \text{ admits a presentation on } g \text{ generators and } r \text{ relations}\}$.

It is a measure of the "complexity" of G.

Let BG be the *classifying space* of G: a complex such that $\pi(BG) = G$ and $\widetilde{BG}$ is contractible (BG is unique up to homotopy). Set $b_i^{(2)}(G) =: b_i^{(2)}(BG)$.

Theorem 2.2. *(B. Eckmann) $def(G) \leq 1 + b_1^{(2)}(G)$. Moreover, if $def(G) = 1$ and $b_1^{(2)}(G) = 0$, then G admits a 2-dimensional model for BG.* $\square$

2.3 Structure of groups

In 1993 Gromov conjectured that, if

$$1 \to N \to G \to G/N \to 1$$

is a short exact sequence of infinite groups which are fundamental groups of finite aspherical complexes, then $b_1^{(2)}(G) = 0$. Actually a much stronger statement holds!

Theorem 2.3. *(Gaboriau 2000; Lück 1995 under extra assumption $\mathbf{Z} \subset G/N$). Let*

$$1 \to N \to G \to G/N \to 1$$

be a short exact sequence of infinite groups. If N is finitely generated (as a group), then $b_1^{(2)}(G) = 0$.

Corollary 2.4. *(Schreier 1930) Any (non-trivial) finitely generated normal subgroup of the free group $\mathbf{F}_n$, has finite index.*

Proof: Assume $N \triangleleft \mathbf{F}_n$, with $n \geq 2$, and N finitely generated. Since $b_1^{(2)}(\mathbf{F}_n) > 0$, we either have N finite (hence trivial) or G/N finite. ■

2.4 Cost and measurable group theory (Gaboriau 2002)

Theorem 2.5. *$\mathcal{C}(G) - 1 \geq b_1^{(2)}(G) - b_0^{(2)}(G)$, where $\mathcal{C}$ denotes the cost.* □

Theorem 2.6. *Assume G admits a free, treeable action. Then $b_k^{(2)}(G) = 0$ for $k > 1$. Moreover $b_1^{(2)}(G) = 0$ if and only if G is amenable.* □

3 G-dimension

For G a countable group, let $L(G)$ be the *group von Neumann algebra*, i.e. the commutant of the left regular representation λ of G on $\ell^2(G)$. The map

$$L(G) \to \ell^2(G) : S \mapsto S(\delta_1)$$

is an embedding with dense image. The functional

$$\tau(S) = \langle S(\delta_1)|\delta_1\rangle$$

defines a *trace* on $L(G)$ (i.e. $\tau(ST) = \tau(TS)$), which is *positive* ($\tau(S^*S) \geq 0$) and *faithful* ($\tau(S^*S) = 0 \Leftrightarrow S = 0$).
Let $\mathcal{H}$ be a Hilbert space. We extend τ to a densely defined trace on $L(G) \otimes B(\mathcal{H})$ by:

$$\tau(S \otimes T) =: \tau(S) \cdot Tr(T).$$

Let G act on $\ell^2(G) \otimes \mathcal{H}$ by $\lambda \otimes 1$.

If V is a closed G-invariant subspace of $\ell^2(G) \otimes \mathcal{H}$, and P is the orthogonal projection onto V, we define the G-*dimension* of V as:

$$\dim_G V = \tau(P) \in [0, +\infty].$$

Example 2. $\dim_G \ell^2(G) = \tau(1) = 1$.

Properties:

D1: $\dim_G V = 0 \Leftrightarrow V = 0$.

D2: If V is G-isomorphic to a dense subspace of W, then $\dim_G V = \dim_G W$.

D3: (additivity) $\dim_G(V \oplus W) = \dim_G V + \dim_G W$.

D4: (continuity) If $(V_j)_{j>0}$ is a decreasing sequence of G-invariant subspaces, then

$$\dim_G(\cap_{j>0} V_j) = \lim_{j\to\infty} \dim_G V_j.$$

D5: (finite index subgroups) If $[G : H] = d$, and V is G-invariant: $\dim_H(V) = d \cdot \dim_G V$.

Example 3. *1. If G finite:* $\dim_G V = \frac{\dim V}{|G|}$.

2. *For $G = \mathbf{Z}$, by Fourier series for $n \in \mathbf{Z}$ the shift operator $\lambda(n)$ becomes multiplication by $e^{2\pi i n\theta}$ on $L^2(S^1, \mu)$ (μ the normalized Lebesgue measure). Invariant subspaces of this action are of the form*

$$\mathcal{H}_B = \{f \in L^2(S^1) : f \equiv 0 \text{ a.e. on } S^1 \setminus B\},$$

with B a Borel subset of S^1. The corresponding projection is multiplication by the characteristic function χ_B of B. Hence

$$\dim_{\mathbf{Z}} \mathcal{H}_B = \int_{S^1} \chi_B \, d\mu = \mu(B);$$

this takes all values in $[0, 1]$.

4 L^2-cohomology

Let X be a finite complex, $\tilde{X}$ its universal cover. Let $\tilde{X}^k$ denote the set of k-cells of $\tilde{X}$. Consider the complex of ℓ^2-cochains:

$$\ell^2(\tilde{X}^0) \xrightarrow{d_0} \ell^2(\tilde{X}^1) \xrightarrow{d_1} \ell^2(\tilde{X}^2) \to \dots$$

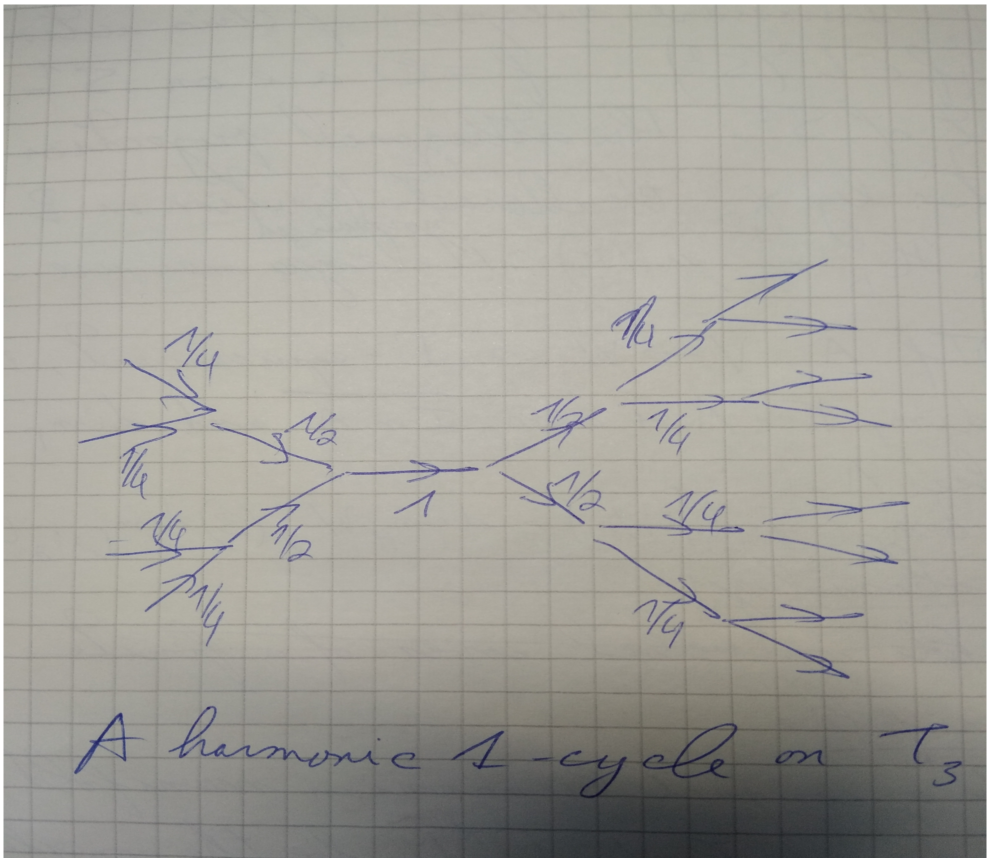
where $(d_k f)(\sigma) = \sum_{\tau\subset\sigma}(-1)^{\epsilon(\tau,\sigma)} f(\tau)$, for $\sigma \in \tilde{X}^{k+1}$.

Example 4. *$(d_0 f)(e) = f(e^+) - f(e^-)$ for $e \in \tilde{X}^1$.*

Definition 4.1. *The L^2-cohomology of $\tilde{X}$ is $H^k_{(2)}(\tilde{X}) = \ker\, d_k / \overline{Im\, d_{k-1}}$.*

We can realize the quotient as a subspace because we use Hilbert spaces. Let $\mathcal{H}^k_{(2)}(X) = \ker\, d_k \cap (Im\, d_{k-1})^\perp$ be the space of *harmonic k-cycles*: then we have a canonical identification $\mathcal{H}^k_{(2)}(\tilde{X}) = H^k_{(2)}(\tilde{X})$.

Example 5. $(d_0^* f)(x) = \sum_{e:e^+=x} f(e) - \sum_{e:e^-=x} f(e)$ *for* $x \in \tilde{X}^0$.



For $G = \pi_1(X)$, all $\ell^2(\tilde{X}^k)$'s become G-invariant subspaces of $\ell^2(G) \otimes \ell^2(\mathbf{N})$, so we may define:

Definition 4.2. *The k-th L^2-Betti number of X is* $b_k^{(2)}(X) = \dim_G H_{(2)}^k(\tilde{X})$.

Proposition 4.3. *(Atiyah 1976)* $\chi(X) = \sum_i (-1)^i b_i^{(2)}(X)$.

Proof: Set $\ell^2_{even/odd}(\tilde{X}) =: \oplus_{k\, even/odd} \ell^2(\tilde{X}^k)$. Consider

$$S = d + d^* : \ell^2_{even}(\tilde{X}) \to \ell^2_{odd}(\tilde{X}).$$

Then

$$\sum_k (-1)^k b_k^{(2)}(X) = \dim_G \ker\, S - \dim_G \ker\, S^*$$

$$= (\dim_G \ker\, S + \dim_G \overline{Im\, S}) - (\dim_G \overline{Im\, S} + \dim_G \ker\, S^*)$$
$$= \dim_G \ell^2_{even}(\tilde{X}) - \dim_G \ell^2_{odd}(\tilde{X}).$$

Choosing representatives for orbits of the free G-action on $\tilde{X}^k$, identify $\ell^2(\tilde{X}^k)$ with $\ell^2(G)^{|X^k|}$ in a G-equivariant way. So $\dim_G \ell^2(\tilde{X}^k) = |X^k|$ and $\sum_k (-1)^k b_k^{(2)}(X) = \sum_k (-1)^k |X^k| = \chi(X)$. ■

Theorem 4.4. *(Dodziuk 1977) $H^k_{(2)}(\tilde{X})$ is a homotopy invariant of X.* □

Also enjoyable:

Proposition 4.5. *(Poincaré duality) If X is a triangulation of a closed orientable manifold M^n, then $b_k^{(2)}(X) = b_{n-k}^{(2)}(X)$.*

5 Invariants of discrete groups

Suppose the countable group G has a finite classifying space BG (=Eilenberg-McLane space $K(G,1)$). Define:

$$b_k^{(2)}(G) = b_k^{(2)}(BG).$$

(well-defined by Dodziuk's theorem).

Example 6. 1. *Construct BG from the presentation 2-complex of G. So the 1-skeleton of $\tilde{X}$ is a Cayley graph of G, hence $\tilde{X}^0 = G$ and $H_0^{(2)}(\tilde{X})$ is the space of square-integrable constant functions on G. So*

$$b_0^{(2)}(G) = \begin{cases} 0 & if \quad G\, infinite \\ 1/|G| & if \quad G\, finite \end{cases}$$

Moreover $b_1^{(2)}(G) \leq dim_G \ell^2(\tilde{X}^1)$, hence $b_1^{(2)}(G) \leq d(G)$, the minimal number of generators of G.

2. *$G = \mathbf{Z}^n$. Then $BG = \mathbf{T}^n$, which is a double cover of itself. So $b_k^{(2)}(\mathbf{Z}^n) = b_k^{(2)}(\mathbf{T}^n) = 2b_k^{(2)}(\mathbf{T}^n)$, hence $b_k^{(2)}(\mathbf{Z}^n) = 0$.*

3. *$G = \mathbf{F}_n$. Then we may take for $X = BG$ a bouquet of n circles, so*

$$\chi(X) = 1 - n = b_0^{(2)}(\mathbf{F}_n) - b_1^{(2)}(\mathbf{F}_n) = -b_1^{(2)}(\mathbf{F}_n)$$

hence $b_1^{(2)}(\mathbf{F}_n) = n - 1$.

4. $G = \pi_1(\Sigma_g)$, *with* Σ_g *a closed Riemann surface of genus* $g > 0$. *Take* $BG = \Sigma_g$, *so by Poincaré duality:*

$$\chi(\Sigma_g) = 2 - 2g = b_0^{(2)}(G) - b_1^{(2)}(G) + b_2^{(2)}(G) = -b_1^{(2)}(G)$$

hence $b_1^{(2)}(G) = 2g - 2$.

Theorem 5.1. *The property* $b_k^{(2)}(G) = 0$ *is invariant under:*

- *quasi-isometry (Soardi for* $k = 1$, *Pansu for* k *arbitrary);*
- *measure equivalence (Gaboriau)* □

6 Methods of computation

Theorem 6.1. *(Cheeger-Gromov 1986) There is a Künneth formula:*

$$b_k^{(2)}(G_1 \times G_2) = \sum_{i=0}^{k} b_i^{(2)}(G_1) b_{k-i}^{(2)}(G_2).$$

□

Theorem 6.2. *(Cheeger-Gromov 1986; Paschke 1992) For* $G = A *_C B$*:*

$$b_1^{(2)}(G)-b_0^{(2)}(G) = (b_1^{(2)}(A)-b_0^{(2)}(A))+(b_1^{(2)}(B)-b_0^{(2)}(B))-(b_1^{(2)}(C)-b_0^{(2)}(C)).$$

$\square$

Example 7. *For* $G = SL_2(\mathbf{Z}) = (\mathbf{Z}/6) *_{\mathbf{Z}/2} (\mathbf{Z}/4)$*:*

$$b_1^{(2)}(G) = (0 - \frac{1}{6}) + (0 - \frac{1}{4}) - (0 - \frac{1}{2}) = \frac{1}{12}.$$

$Infinite\ G$	$b_1^{(2)}(G)$	$b_k^{(2)}(G)\ (k \geq 2)$	$Who?$
$\mathbf{F}_n$	$n-1$	0	
$SL_2(\mathbf{Z})$	$\frac{1}{12}$	0	
$\pi_1(\Sigma_g)$	$2g-2$	0	
$Amenable$	0	0	$Cheeger-Gromov$ 198
$Thompson's\ F$	0	0	$Lueck$ 1994
$Prop.\ (T)$	0	$*$	$Bekka-V.$ 1996
$Lattice\ in\ SL_n(\mathbf{R}),\ n \geq 3$	0	0	$Borel$ 1985
$Lattice\ in\ SO(2n+1,1)$	0	0	$Borel$ 1985
$G \rtimes \mathbf{Z}\ (G\ fin.\ gen.)$	0	0	$Lueck$1995
$H \wr G$	0	$*$	$Martin-V.$ 2003
$1-relator\ on\ g\ gen.$	$\max\{g-2,0\}$	0	$Dicks-Linnell$ 2006
$Higman's\ group\ H_4$	0	$b_2^{(2)}=1$ $b_k^{(2)}=0\ (k>2)$	$Fernos-V.$ 2016

Theorem 6.3. *(Borel 1985) Let G be a lattice in a rank 1 simple Lie group S:*

$S = SO(2n,1)$	$b_n^{(2)}(G) > 0$	$Others: \ 0$
$S = SU(n,1)$	$b_n^{(2)}(G) > 0$	$Others: \ 0$
$S = Sp(n,1)$	$b_{2n}^{(2)}(G) > 0$	$Others: \ 0$

□

THANK YOU FOR YOUR ATTENTION!

Following a question: D. Osin (2008) constructed finitely generated torsion groups with $b_1^{(2)}(G) > 0$, where non-amenability follows from non-vanishing of $b_1^{(2)}$.