Gaussian Free Field in (self-adjoint) random matrices and random surfaces

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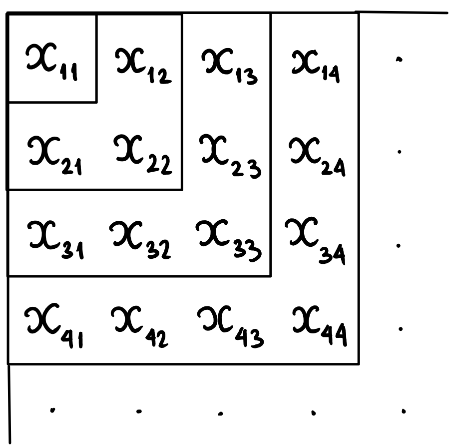
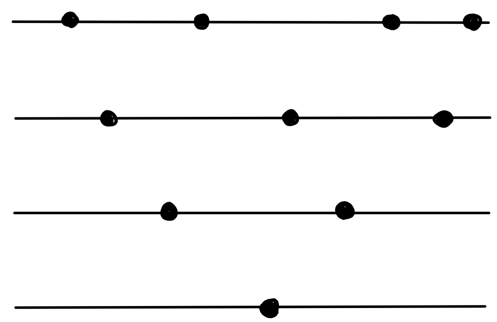
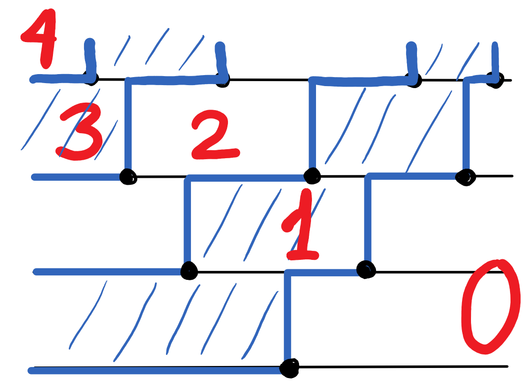
Corners of random matrices and GFF



spectra



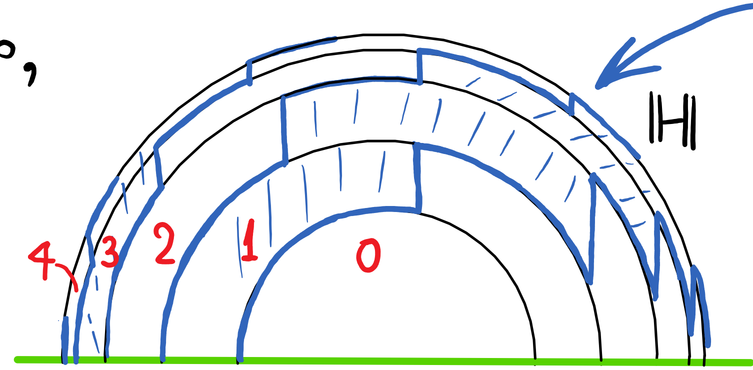
height function

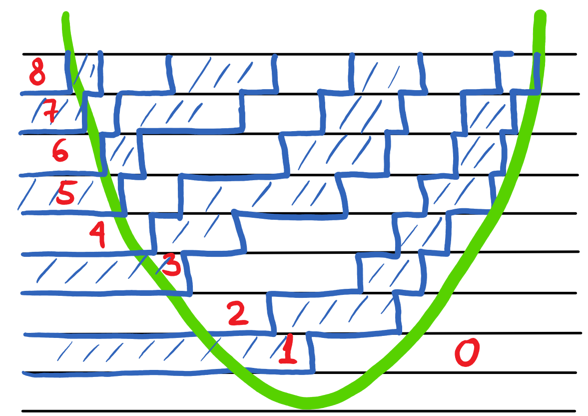


liquid region

$(x, y) = (2\mathrm{Re}(z), |z|^2)$

Theorem As $z \mapsto L^{-1} z$, $L \to \infty$,
Unscaled fluctuations $\Rightarrow$
Gaussian (massless)
Free Field on $\mathbb{H}$ with
zero boundary conditions





- Gaussianity follows from (more general) results of [Guionnet '02]
- GFF structure for GUE follows from [B-Ferrari, 2008], for GUE/GOE type Wigner matrices GFF fluctuations are proved in [B, 2010]

Why GFF?

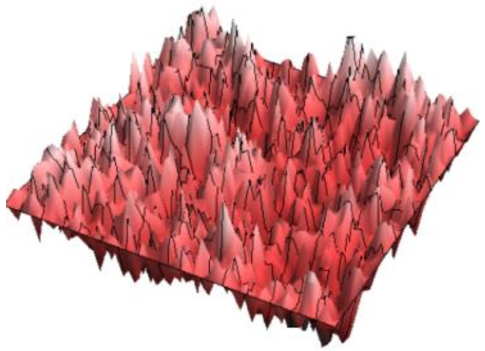
2d Gaussian Free Field

2d GFF (with zero boundary conditions) on a domain $\mathcal{D} \subset \mathbb{C}$ is a (conformally invariant) random generalized function:

$$\mathrm{GFF}(\Omega) = \sum_k \xi_k \cdot \frac{\varphi_k(\Omega)}{\sqrt{\lambda_k}}, \quad \Omega \in \mathcal{D},$$

1d analog: Brownian Bridge

where φ_k's are the eigenfunctions of $-\Delta$ on $\mathcal{D}$ with zero boundary conditions, λ_k is the corresp. eigenvalue, and ξ_k's are i.i.d. standard Gaussians.

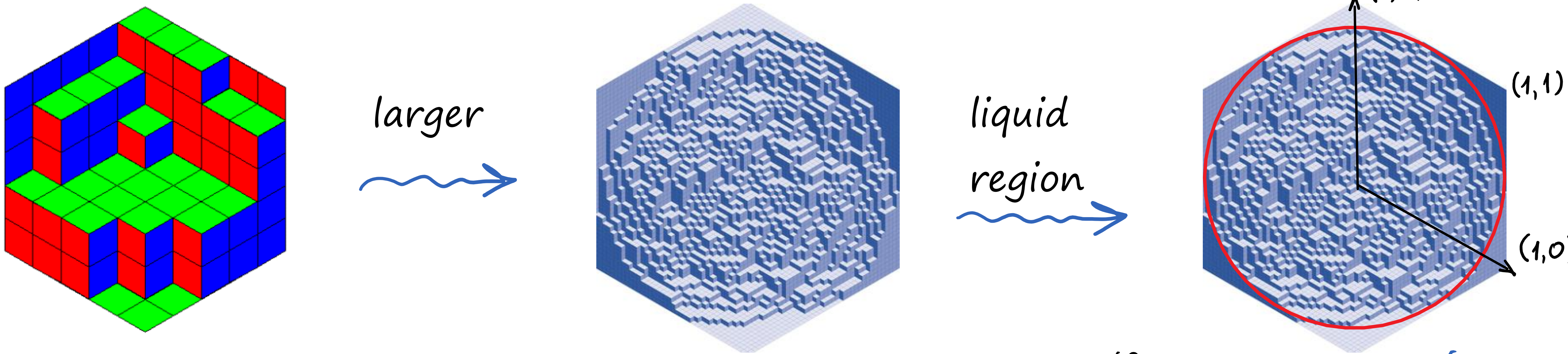


Other definitions:

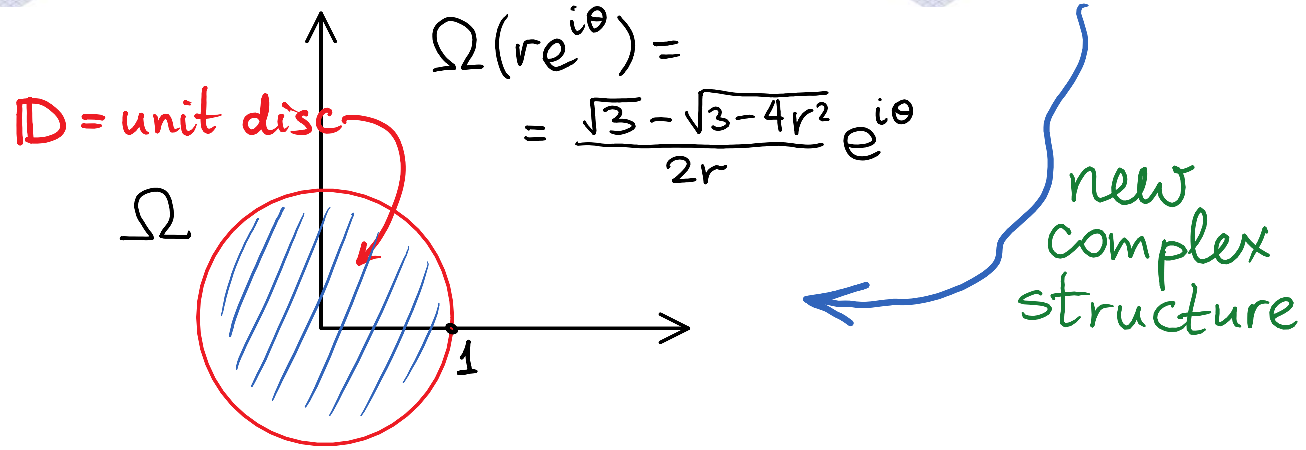
- GFF is a Gaussian process on $\mathcal{D}$ with Green's function of the Laplacian as the covariance kernel
- $\int_{\mathcal{D}} \varphi(\Omega)\,\mathrm{GFF}(\Omega)\,|d\Omega|^2$ are Gaussian (mean zero), $\mathbb{E}\left(\mathrm{GFF}(\Delta\psi_1), \mathrm{GFF}(\Delta\psi_2)\right) = \int_{\mathcal{D}} \nabla\varphi_1 \cdot \nabla\varphi_2$

  (where $\int_{\mathcal{D}} \varphi(\Omega)\,\mathrm{GFF}(\Omega)\,|d\Omega|^2 =: \mathrm{GFF}(\varphi)$)

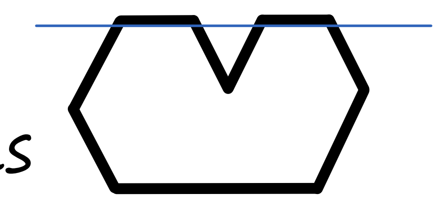
Uniform lozenge tilings and GFF



<u>Theorem</u> As mesh goes to zero, Fluctuations of height ⇒ Gaussian Free Field on $\mathbb{D}$ with zero boundary conditions.

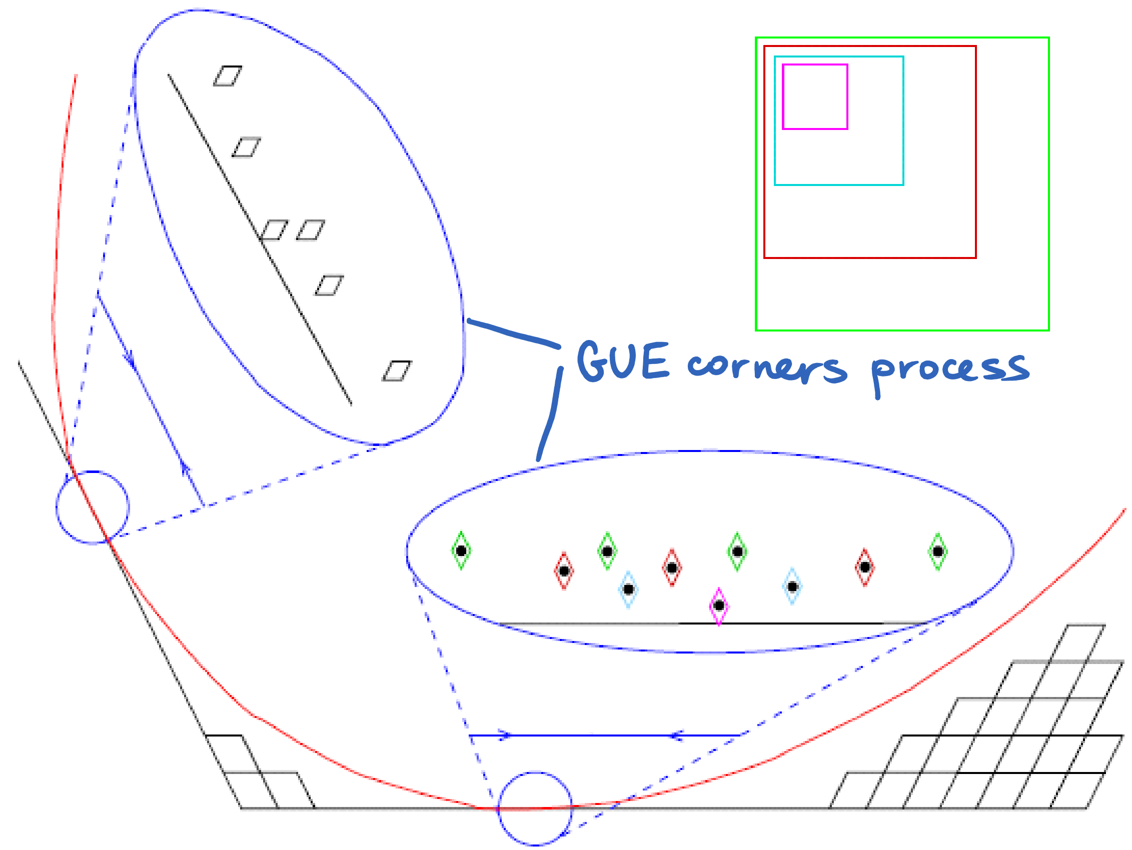


- [Kenyon '01+] conjectured for general lattices/domains, proved for lozenge tilings without facets in the limit shape.
- [Petrov '12], [Bufetov-Gorin '16-17]: certain polygons



From tilings to spectra

[Okounkov-Reshetikhin '06]:
GUE corners process *should* arise near every tangency point of the limit shape.



Explanation: The limit of the tiling measure must be Gibbs (uniform, given boundary conditions), [Olshanski-Vershik '96] classified all such, among them only GUE corners fit the bill.

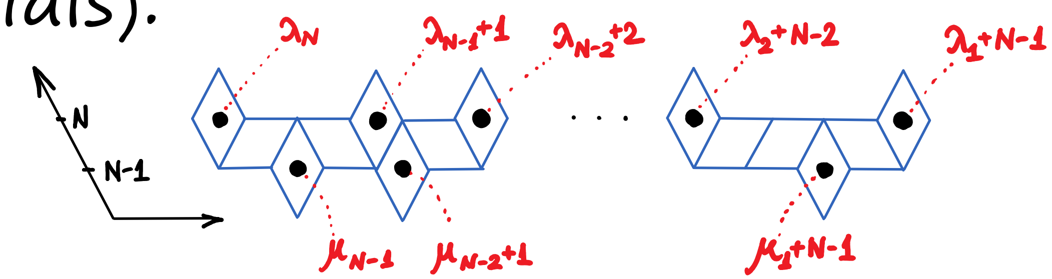
The Gibbs property was used by [Gorin '13], [Dimitrov '17] to prove convergence of the 6-vertex model to the GUE corners process.

Another explantion: semi-classical limit

Branching of irreps of unitary groups is encoded by lozenge tilings.

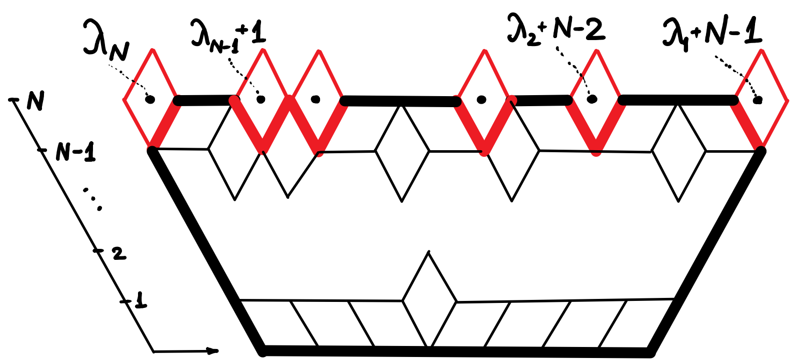
In terms of characters (Schur polynomials):

$$S_\lambda(z_1,\dots,z_{N-1},{\color{red}1}) = \sum_{\mu\prec\lambda} S_\mu(z_1,\dots,z_{N-1})$$

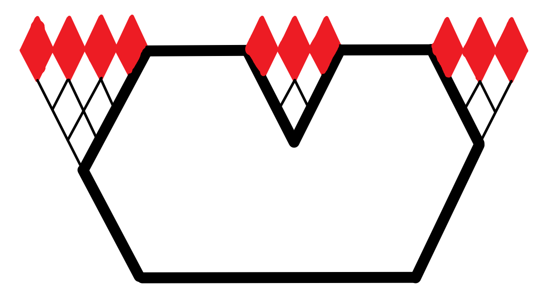


where μ interlaces λ : $\lambda_N \le \mu_{N-1} \le \lambda_{N-1} \le \dots \le \lambda_2 \le \mu_2 \le \lambda_1$.

Branching all the way down $U(N) \supset U(N-1) \supset \dots \supset U(2) \supset U(1)$



An example:



Large representations of Lie group behave as group-invariant measures on (dual to) the Lie algebra. Hence, tilings converge to random matrices.

Markov evolution of random matrices and GFF

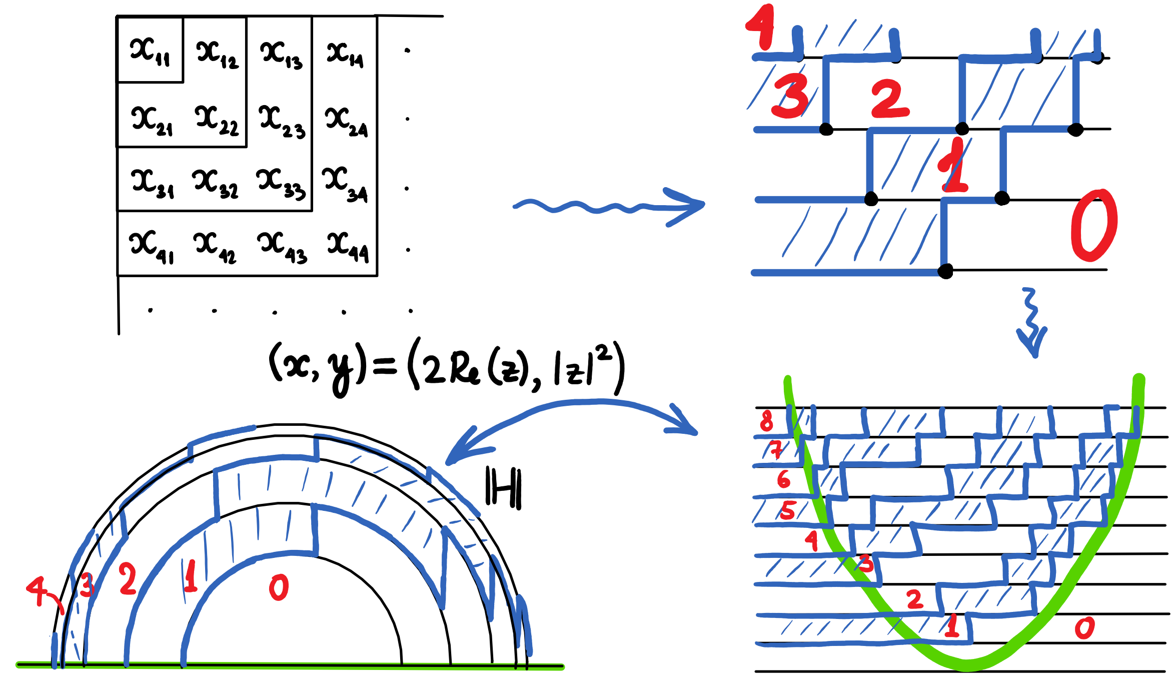
For GOE/GUE, consider

Dyson Brownian Motion:

Each matrix element x_{ij}

executes 1d stationary $\mathbb{R}$ or $\mathbb{C}$

Ornstein-Uhlenbeck process.



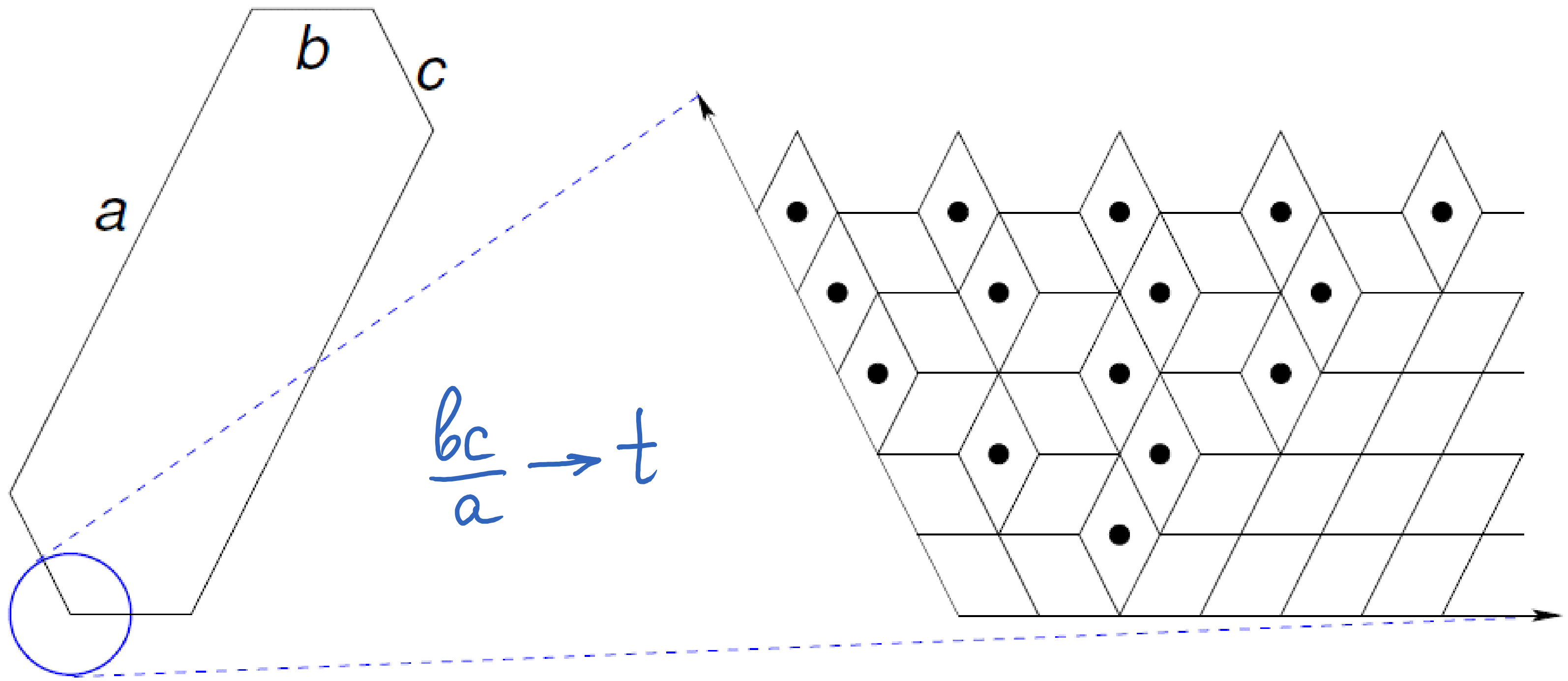
Theorem [B '10] Under the same identification of the liquid region with the upper half-plane at each time moment, the height fluctuations converge to a 3d generalized Gaussian process with the covariance kernel on $\mathbb{H}\times\mathbb{R}$

$$\mathrm{Cov}(z,s;w,t)=\begin{cases}-\dfrac{1}{2\pi}\log\left|\dfrac{e^{-|s-t|}z-w}{e^{-|s-t|}z-\overline{w}}\right|, & |z|\le w,\\[2ex] -\dfrac{1}{2\pi}\log\left|\dfrac{e^{-|s-t|}w-z}{e^{-|s-t|}w-\overline{z}}\right|, & |z|\ge w.\end{cases}$$

Conceptual meaning?

Markov evolution of random tilings and GFF

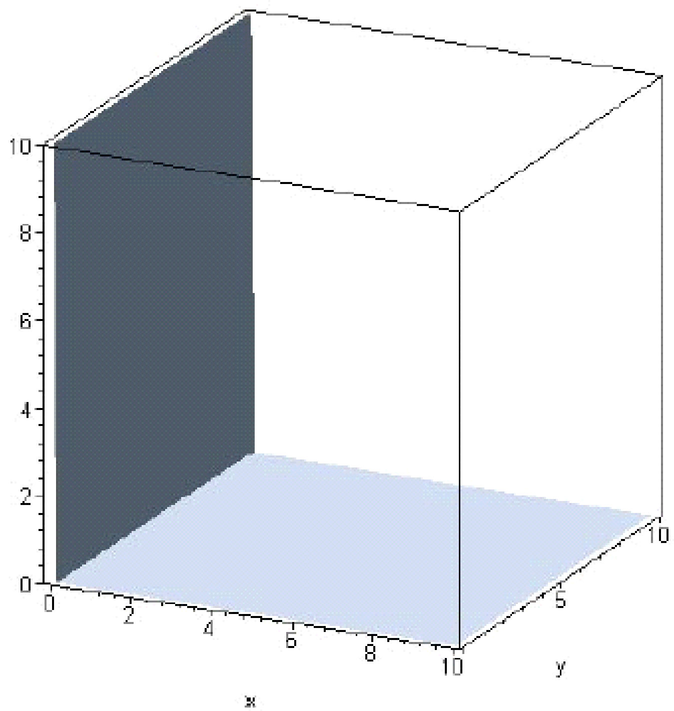
We focus on the simplest nontrivial example, which is a limit of uniform lozenge tilings of hexagons (can also be done for hexagons):

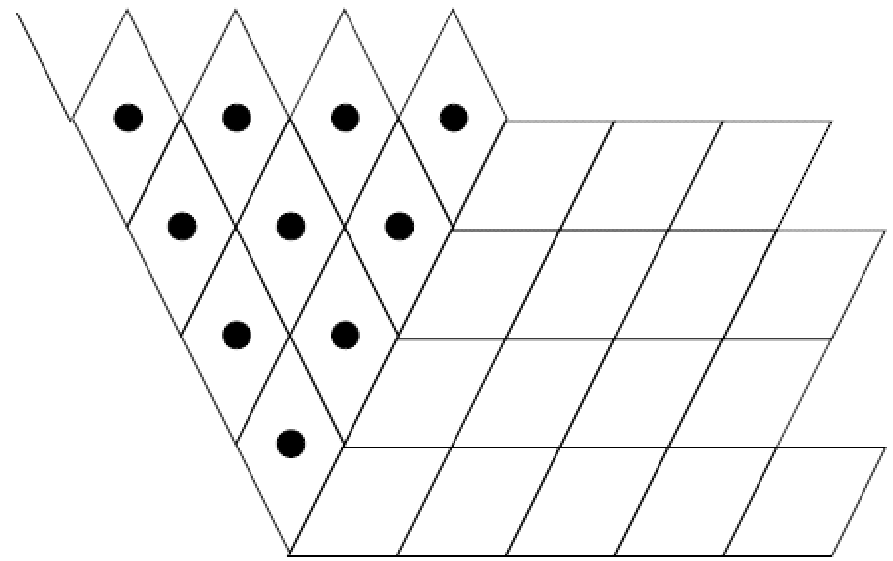


The resulting random tiling of a sector in the plane can be stochastically grown starting from a frozen configuration, with t serving as time.

An integrable random growth model [B-Ferrari '08]

Consider the `empty' initial condition

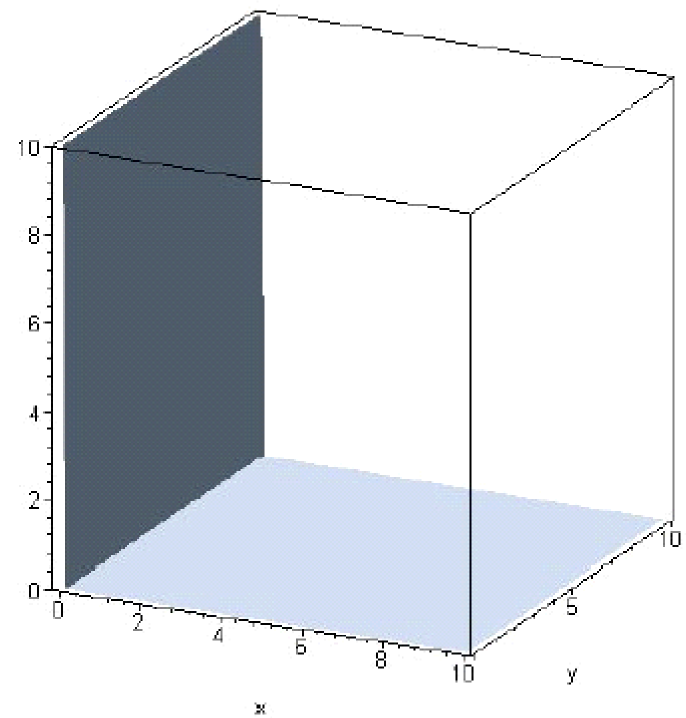


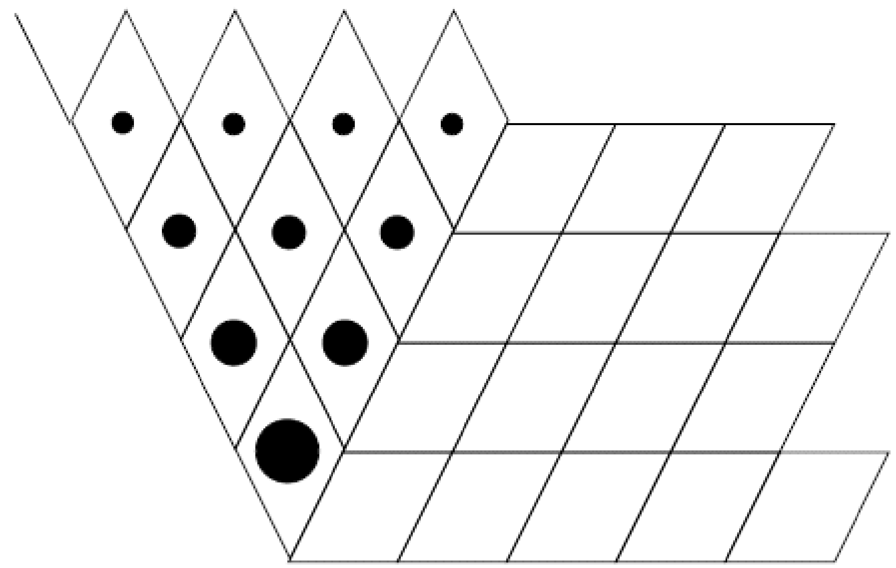


Place particles in centers of `vertical' lozenges.

An integrable random growth model

Consider the `empty' initial condition



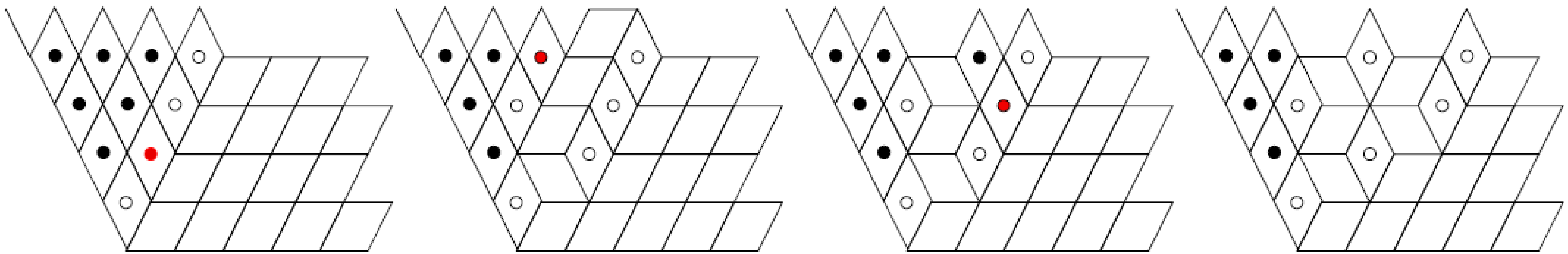


Imagine that particles have weights that decrease upwards.

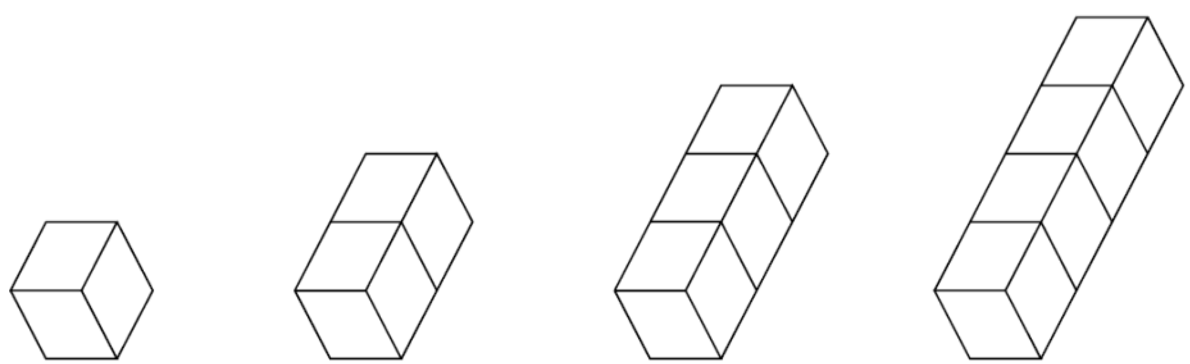
An integrable random growth model

Each particle jumps to the right independently with rate 1.

It is blocked by heavier particles and it pushes lighter particles.



In 3d, this can be viewed as adding directed columns



Column deposition - Animation

An integrable random growth model

Each particle jumps to the right independently with rate 1.
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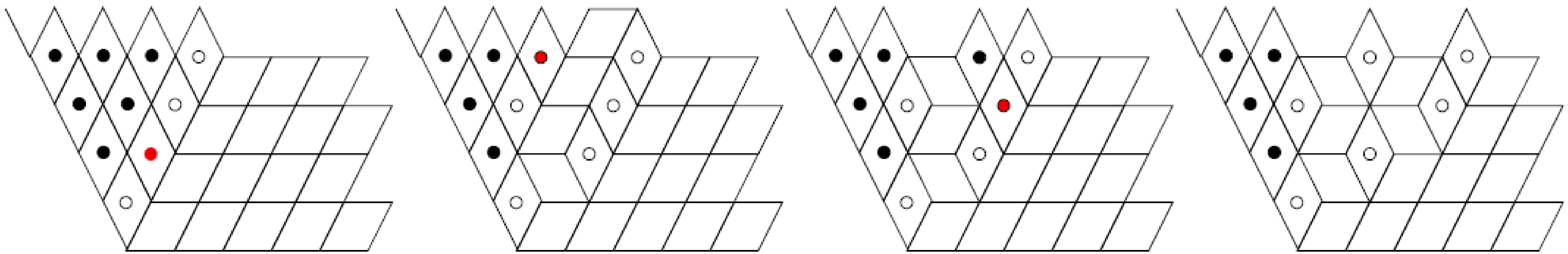
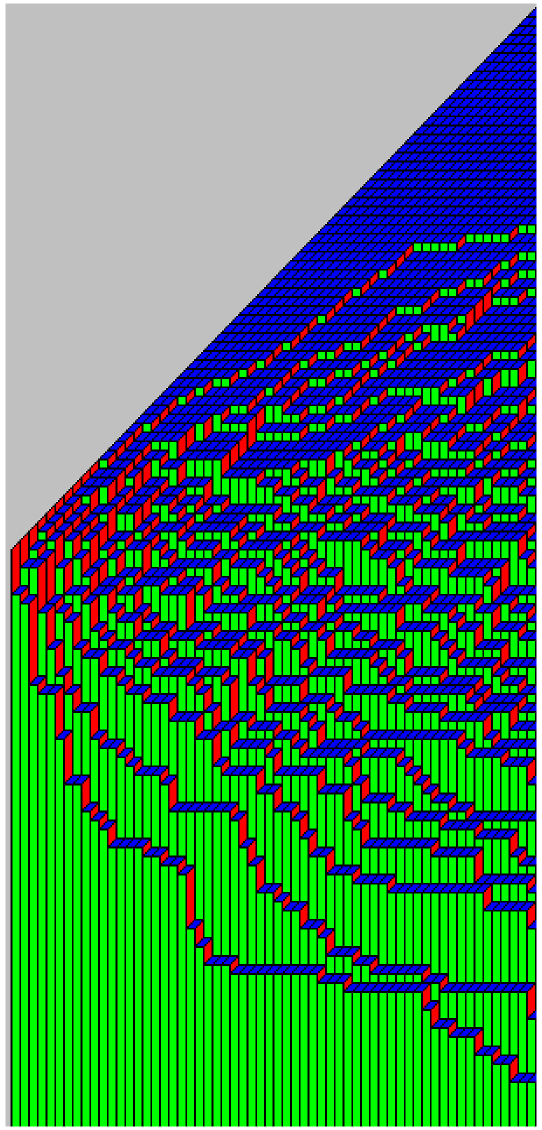


- Left-most particles form TASEP
- Right-most particles form PushTASEP
- Large time (diffusive) limit of the evolution of n particles on the n-th horizontal level is Dyson's Brownian motion for GUE

Large time behaviour

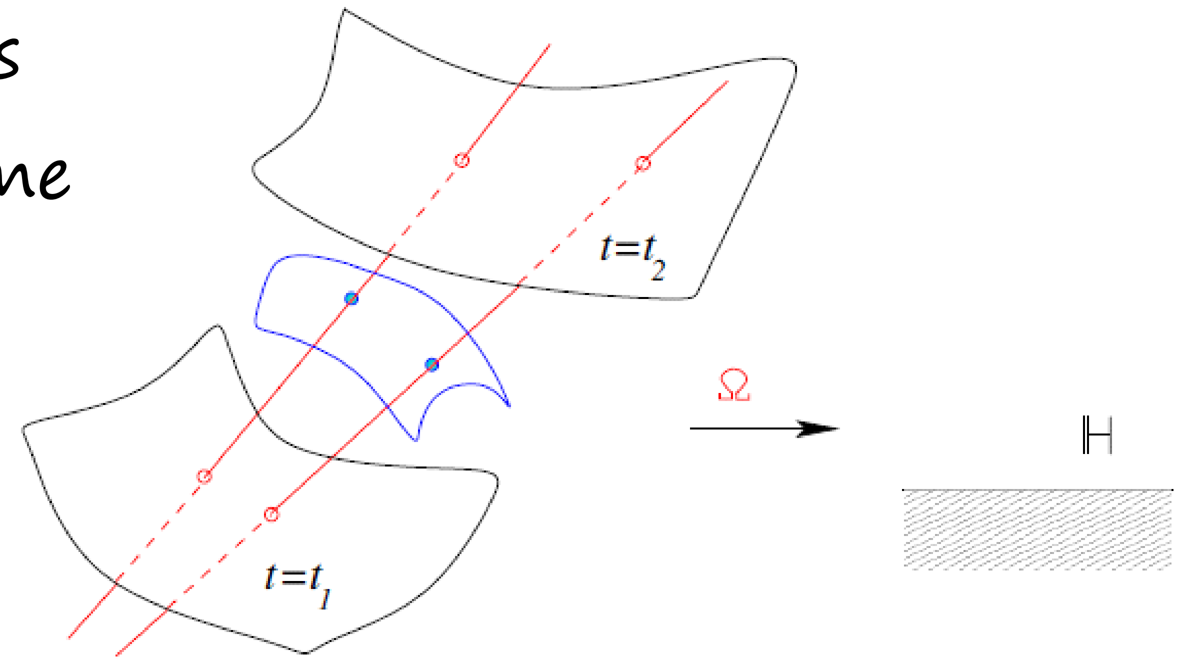
- In the hydrodynamic scaling, a deterministic limit shape arises. It is described by $\partial_t h = f(x, \nabla h)$.
- The models belong to the anisotropic KPZ universality class associated with the (formal) equation $\partial_t h = \Delta h + (\partial_x h)^2 - (\partial_y h)^2 + \text{white noise}$.
- One-point fluctuations in the bulk are Gaussian with log(t) variance (predicted in [Wolf '91])
- Unscaled multi-point fluctuations at fixed time are described by 2d GFF.



What about time dependent fluctuation structure?

Space-time fluctuations

To see fixed time GFF, one constructs the map Ω that sends 3d space-time to $\mathbb{H}$. Its level curves are the characteristics of the hydrodynamic equation $\partial_t h = f(x, \nabla h)$.



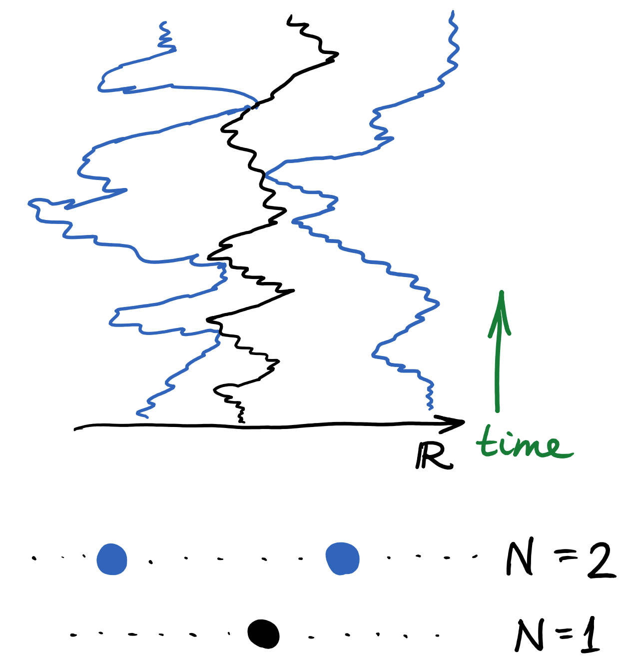
Slow decorrelation conjecture

claims that along characteristics fluctuations vary much slower. It agrees with established fluctuations on **space-like surfaces** [B-Ferrari '08]. It is also supported by numerics, results for (1+1)d KPZ models ([Ferrari '08] etc.), and Gaussian models [B-Corwin-Toninelli '16], [B-Corwin-Ferrari '17].

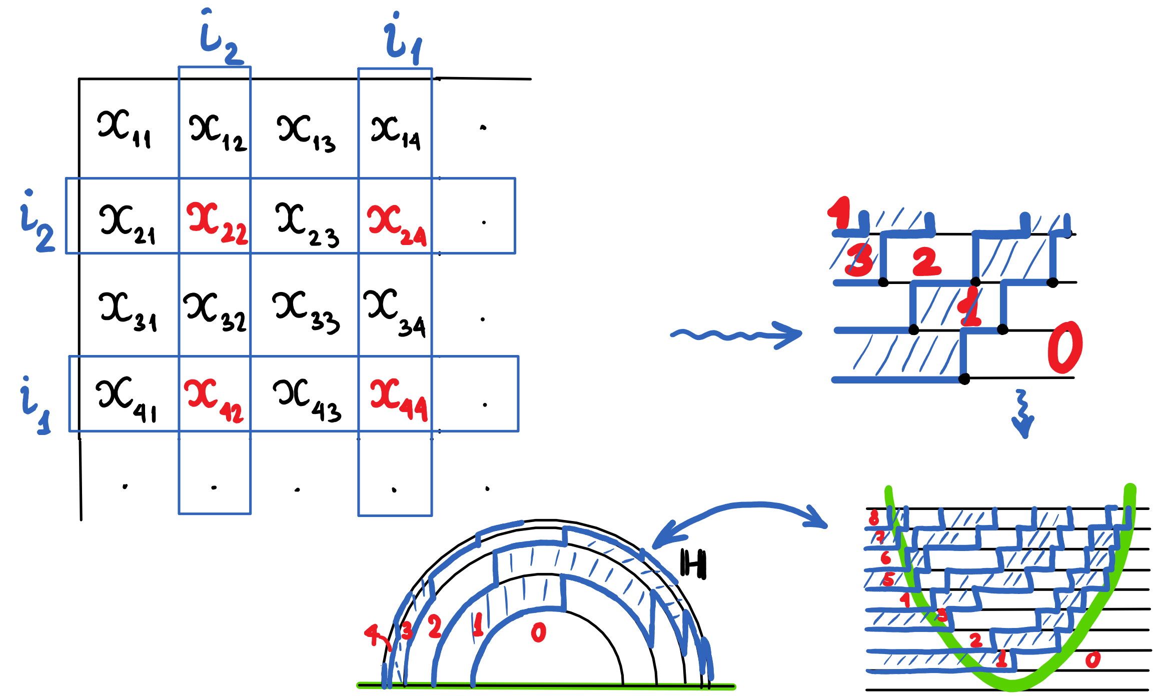
If true, it would imply that the fluctuations are different from Dyson Brownian Motion (despite agreeing on space-like surfaces).

The tiling analog of the Dyson Brownian Motion is a Quantum Random Walk on U(N) [Biane '90], that consists in tensoring with a fixed representation of a unitary group. Gaussian asymptotics was obtained in [Kuan '14,'16], [Bufetov-Gorin '16-17].

The random matrix limit of the Markov dynamics on tilings described above is Warren's process [Warren '05], proved by [Gorin-Shkolnikov '12]. It consists of a triangular array of 1d BMs with the level N ones reflecting off those on level (N-1). Its fluctuations should be as for tiling dynamics.



Submatrices of random matrices and GFF



Consider sequences $I = \{i_1, i_2, \dots\}$ of distinct natural numbers.

Define

$$\alpha(x, I; y, J) = \lim_{L\to\infty} \frac{\left|\{i_1, i_2, \dots, i_{xL}\} \cap \{j_1, j_2, \dots, j_{yL}\}\right|}{L}$$

Ex. $\alpha(x, \mathbb{N}; y, \mathbb{N}) = \min\{x, y\}$.

<u>Theorem</u> [B '10] Under the same map of spectra of GOE/GUE/Wigner submatrices to the height function on $\mathbb{H}$, its fluctuations converge to a generalized Gaussian process with the covariance kernel on $\mathbb{H} \times \{\text{sequences}\}$

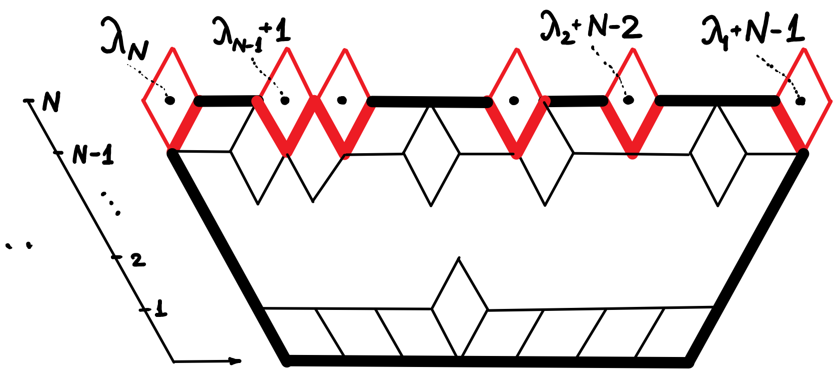
$$\mathrm{Cov}(z, I; w, J) = \frac{1}{2\pi} \log \left| \frac{\alpha(|z|^2, I; |w|^2, J) - zw}{\alpha(|z|^2, I; |w|^2, J) - z\bar{w}} \right|$$

Conceptual meaning?

The tiling analog is harder to see but it is very natural.

A commutative C^*-algebra with a state (positive linear functional) can be viewed as $C_0(\mathcal{X})$ for an abstract probability space $(\mathcal{X}, \mu)$. For representations of $U(N)$, Gelfand-Tsetlin subalgebra generated by centers of $\mathcal{U}(\mathfrak{gl}(k))$, $U(N) \supset U(N-1) \supset \ldots \supset U(2) \supset U(1)$, with trace is realized as poly functions on corresponding uniform tilings.

Given a sequence $I = \{i_1, i_2, \ldots\} \subset \{1, 2, \ldots, N\}$, take Gelfand-Tsetlin algebra of $U(e_{i_1}) \subset U(e_{i_1}, e_{i_2}) \subset \ldots$ For different sequences, they form a noncommutative probability space, but in the global scaling the limit is the same as for random matrices [B-Bufetov '12].



This can be viewed as a step towards fluctuation theory for representations.

General beta random matrices (log-gas) and GFF

We focus on the general beta Jacobi ensembles

$$P_N(x_1,\dots,x_N) \sim \prod_{1\le i<j\le N} |x_i - x_j|^{\beta} \prod_{j=1}^{N} x_j^{p}(1-x_j)^{q}, \qquad 0 < x_1,\dots x_N < 1.$$

The Laguerre/Wishart and Hermite/Gaussian cases can be obtained via straightforward limit transitions.

What is the 2d object (corners process)?

(Tridiagonal general beta matrix models do not help.)

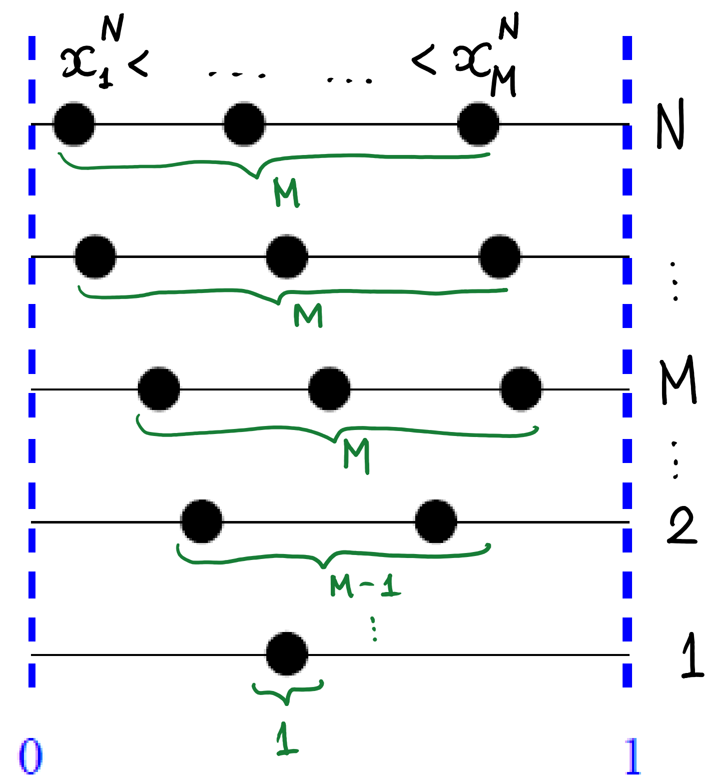
General beta Jacobi corners process

There exists a natural construction of 2d extension.

$$\text{j.p.d.} \sim \prod_{i<j} (x_j^N - x_i^N) \prod_j (x_j^N)^{\theta(\alpha+N-1)-1} (1-x_j^N)^{\theta(N-M+1)-1}$$

$$* \prod_{k=1}^{N-1} \left[\prod_j (x_j^k)^{-2\theta} \prod_{i<j} (x_j^k - x_i^k)^{2-2\theta} \prod_{a,b} |x_a^k - x_b^{k+1}|^{\theta-1} \right]$$

$\theta = \beta/2$

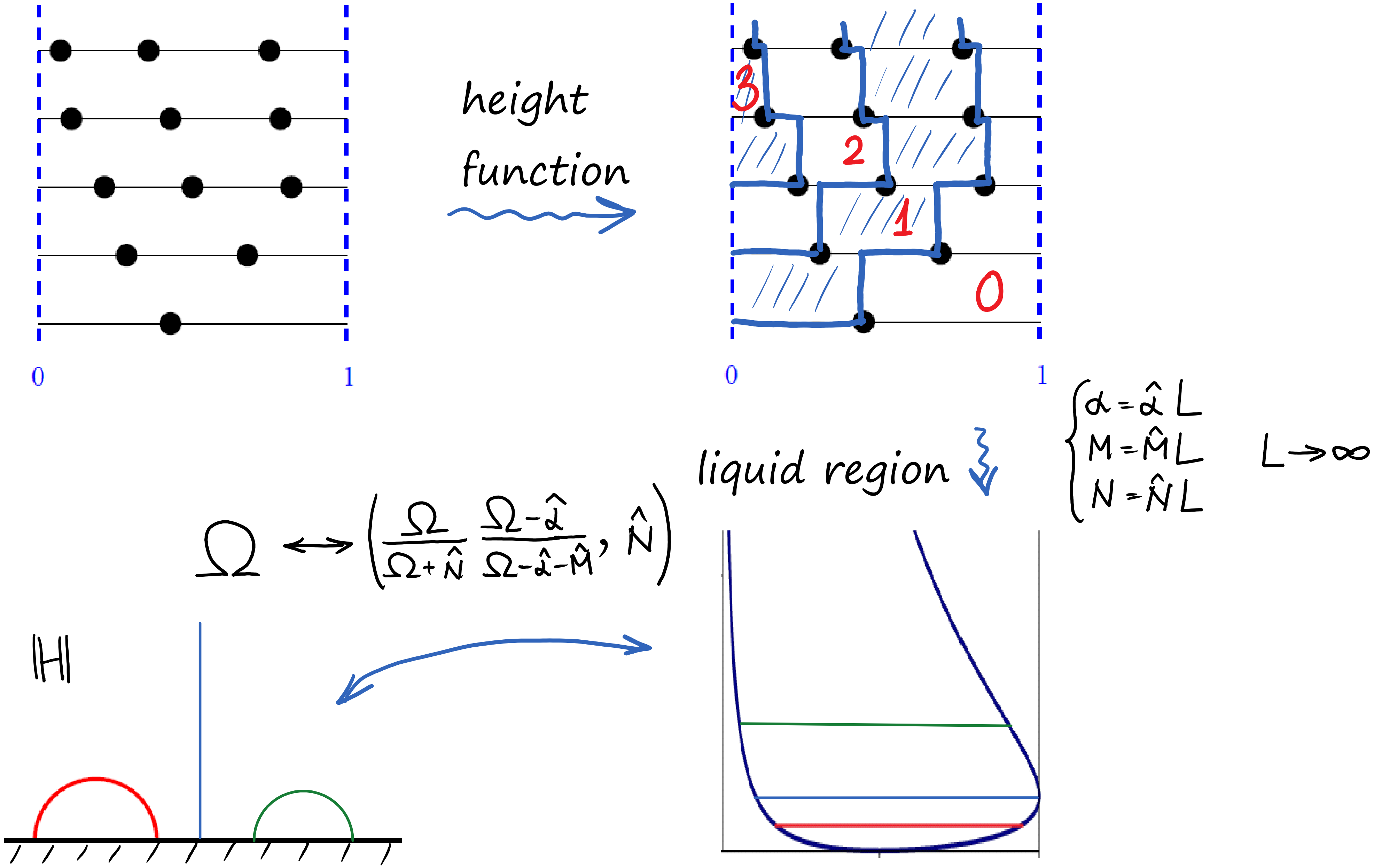


On level N:

$$\prod_{i<j} (x_j^N - x_i^N)^{2\theta} \prod_j (x_j^N)^{\theta\alpha-1} (1-x_j^N)^{\theta(|M-N|+1)-1}$$

Motivation: 1. Dixon–Anderson two-level Selberg type integrals.

2. Extrapolating off radial parts of Haar/Gaussian measures on symmetric spaces (e.g. eigenvalues of $XX^*/(XX^*+YY^*)$) [Sun '16]



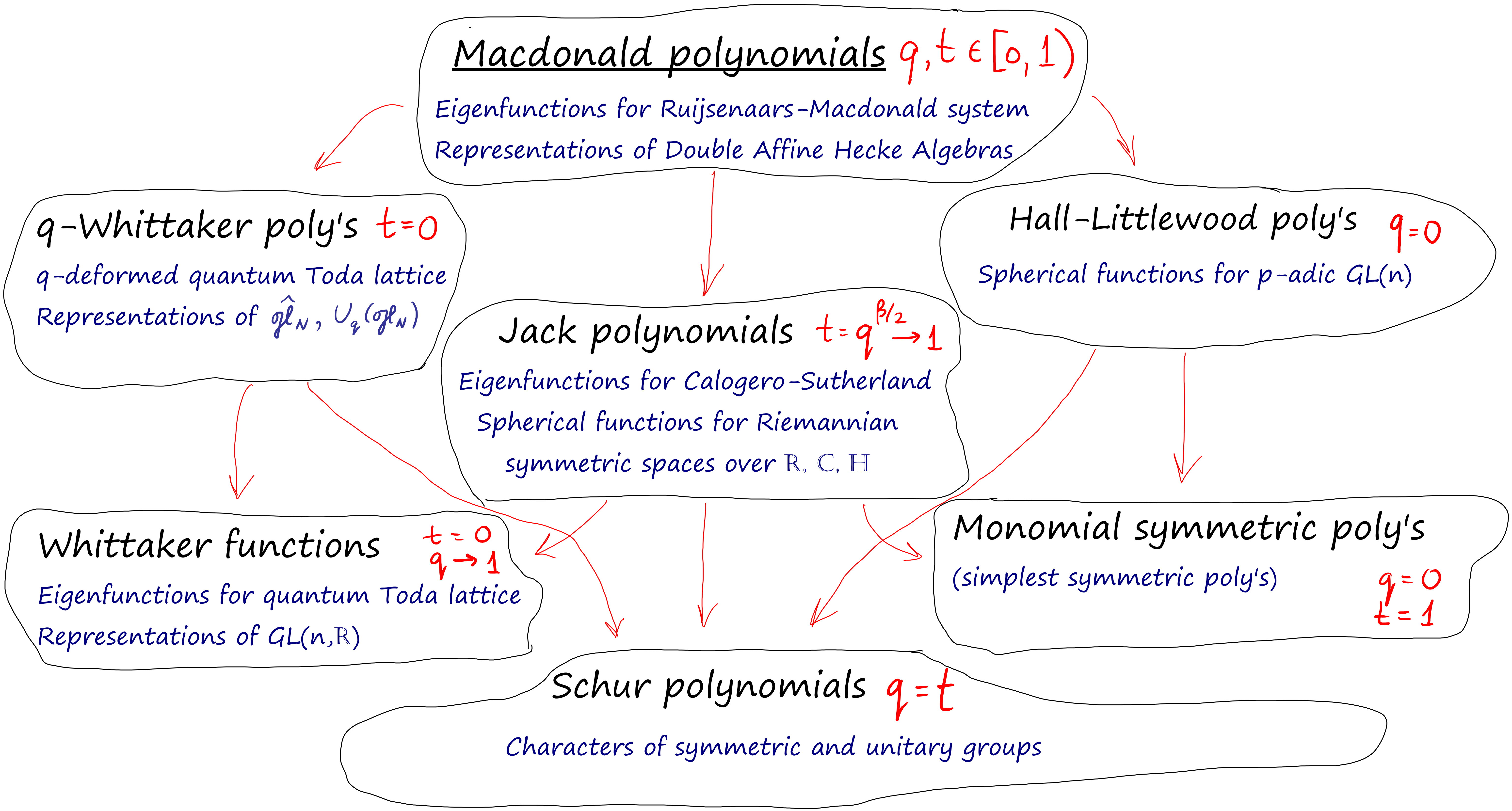
Theorem [B-Gorin '13] As $L \to \infty$, the fluctuations of the height function converge to the GFF on $\mathbb{H}$ with zero boundary conditions.

Related results for general beta ensembles

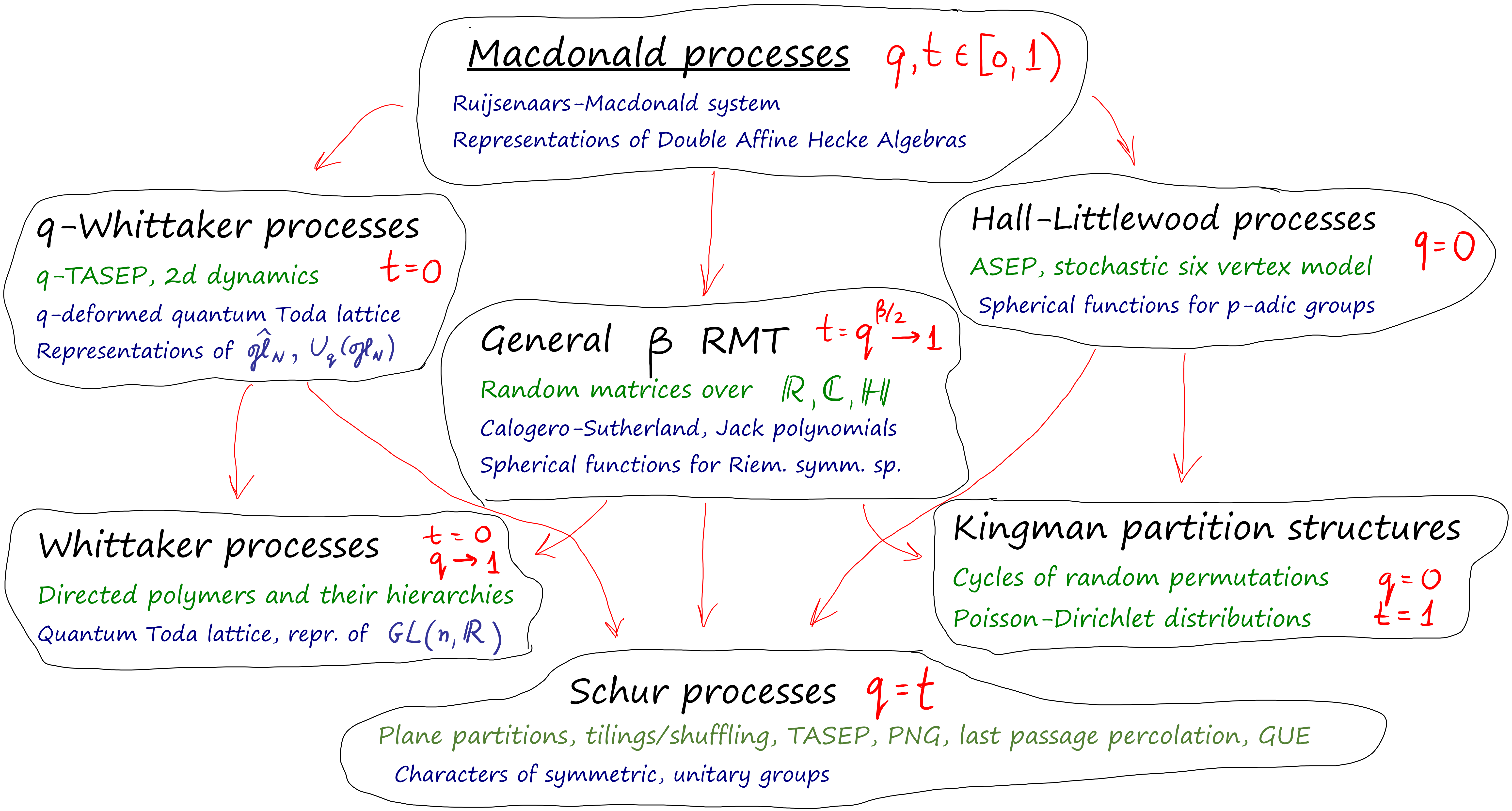
- [Johansson '98] proved single level CLT for the Hermite/Gaussian and much more general convex potentials.
- [Spohn '98] found GFF in the limit of the circular Dyson Brownian Motion.
- [Israelsson '01], [Bender '08], [Anderson-Guionnet-Zeitouni '10] proved multi-time CLT for DBM on the real line (GFF not ID'd).
- [Dumitriu-Paquette '12] proved single level CLT in our setting.
- [Edelman '13+] conjecturally has a (not tridiagonal) matrix model for our corner processes.

Further related results

- [Dumitriu-Paquette '14] showed that GFF arises in an analog of the corners process for Wishart matrices
- [Ganguly-Pal '14] found time-dependent GFF in an analog of the Brownian evolution of the corners process for random graphs
- [B-Gorin-Guionnet '15] proved CLT for discrete beta-ensembles with generic weights via Nekrasov's discrete loop equations
- [Bufetov-Knizel '17] showed GFF fluctuations for the height in a domino tiling model with Arctic curve and many parameters
- [Duits '15] proved GFF behavior for ensembles of nonintersecting paths
- [Huang '17] showed Gaussian behavior in a discrete space version of the general beta nonintersecting Poisson processes



Macdonald polynomials $q, t \in [0,1)$
Eigenfunctions for Ruijsenaars–Macdonald system
Representations of Double Affine Hecke Algebras
q-Whittaker poly's $t=0$
q-deformed quantum Toda lattice
Representations of $\widehat{\mathfrak{gl}}_N$, $U_q(\mathfrak{gl}_N)$
Hall–Littlewood poly's $q=0$
Spherical functions for p-adic GL(n)
Jack polynomials $t=q^{\beta/2}\to 1$
Eigenfunctions for Calogero–Sutherland
Spherical functions for Riemannian
symmetric spaces over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$
Whittaker functions $t=0$ $q\to 1$
Eigenfunctions for quantum Toda lattice
Representations of $GL(n,\mathbb{R})$
Monomial symmetric poly's
(simplest symmetric poly's)
$q=0$ $t=1$
Schur polynomials $q=t$
Characters of symmetric and unitary groups



Macdonald processes $q, t \in [0,1)$
Ruijsenaars-Macdonald system
Representations of Double Affine Hecke Algebras
q-Whittaker processes $t=0$
q-TASEP, 2d dynamics
q-deformed quantum Toda lattice
Representations of $\widehat{\mathfrak{gl}}_N$, $U_q(\mathfrak{gl}_N)$
Hall-Littlewood processes $q=0$
ASEP, stochastic six vertex model
Spherical functions for p-adic groups
General β RMT $t=q^{\beta/2}\to 1$
Random matrices over $\mathbb{R}, \mathbb{C}, \mathbb{H}$
Calogero-Sutherland, Jack polynomials
Spherical functions for Riem. symm. sp.
Whittaker processes $t=0$ $q\to 1$
Directed polymers and their hierarchies
Quantum Toda lattice, repr. of $GL(n,\mathbb{R})$
Kingman partition structures
Cycles of random permutations $q=0$
Poisson-Dirichlet distributions $t=1$
Schur processes $q=t$
Plane partitions, tilings/shuffling, TASEP, PNG, last passage percolation, GUE
Characters of symmetric, unitary groups

Summary

- The two-dimensional Gaussian Free Field appears to be a universal and unifying object for global fluctuations of spectra of random matrices and random tilings, `explaining' previously known single level 1d Gaussian processes.

- Natural probabilitistic extensions lead to Gaussian processes on larger spaces, with extra coordinates being time and/or different flags/commutative subalgebras. Those appear to be universal as well, but their conceptual understanding is missing.