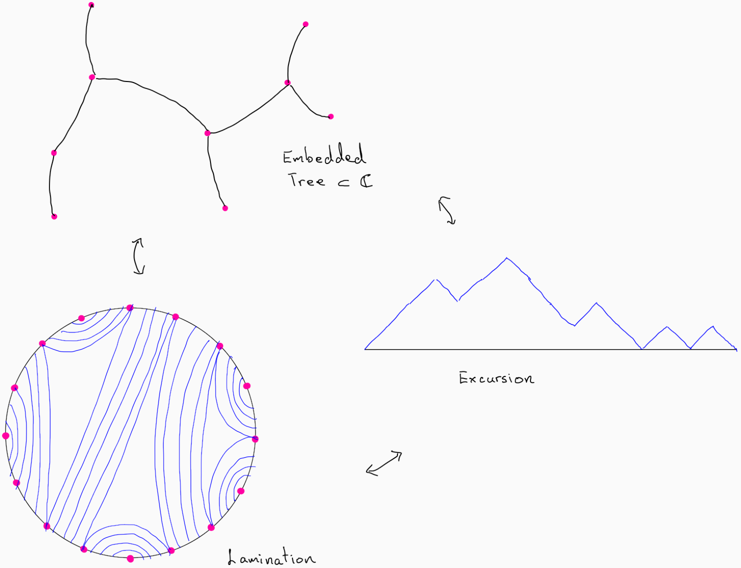
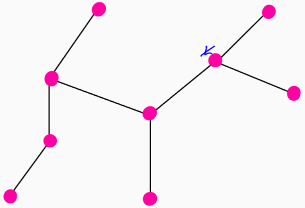
3 pictures to keep in mind



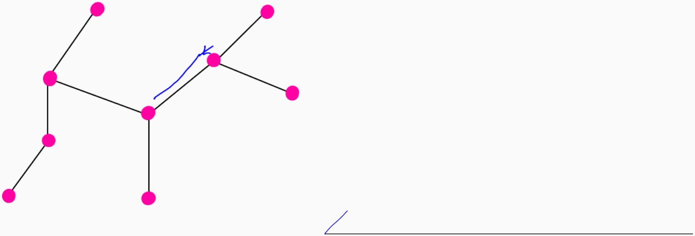
Tree to Excursion



Trace around the tree starting from root.

Go up if seeing new edge, go down if seeing edge that's already been seen.

Tree to Excursion



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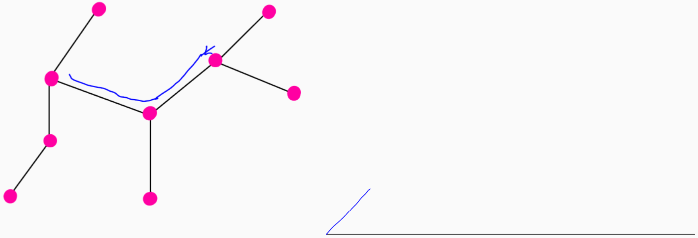

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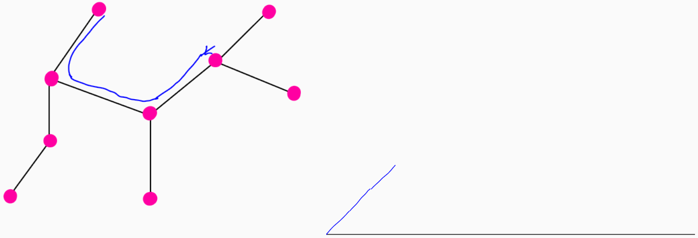

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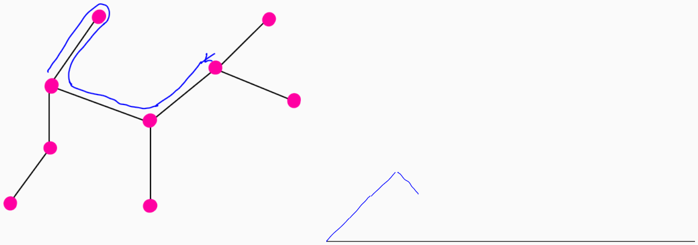

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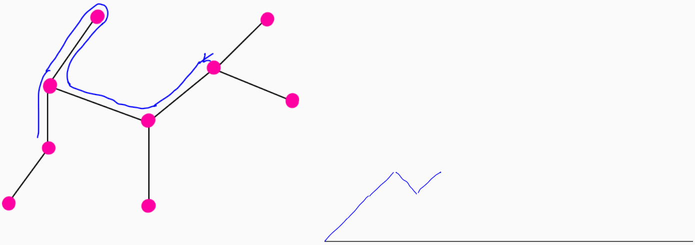

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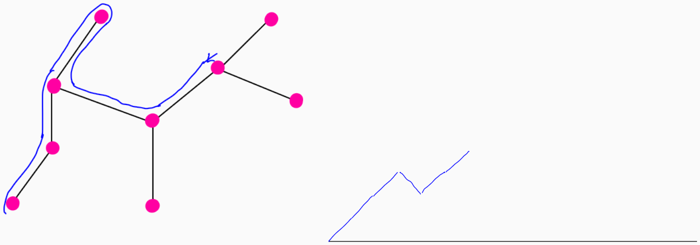

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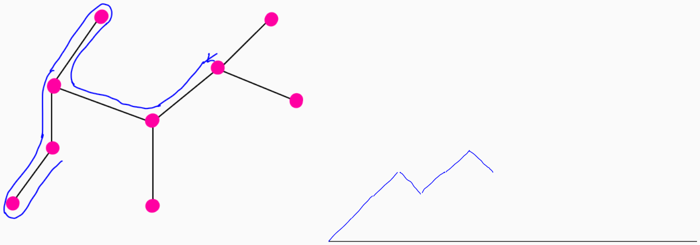

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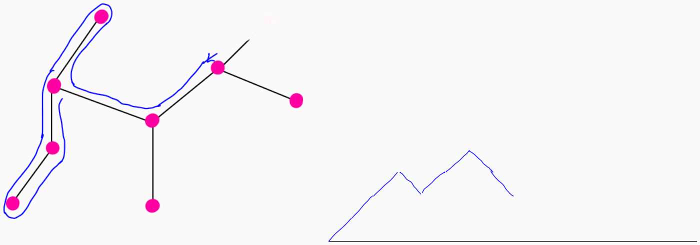

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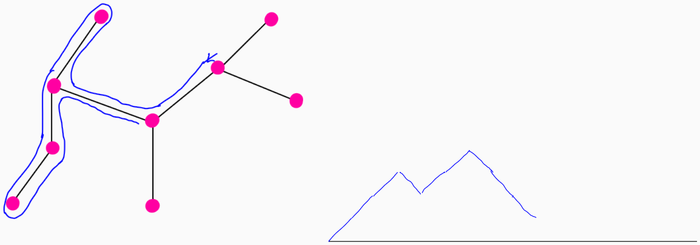

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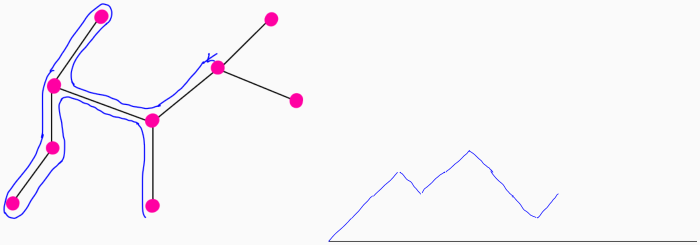

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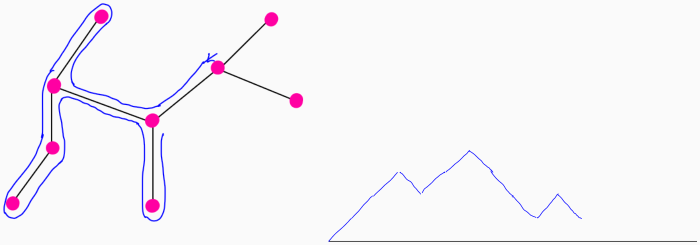

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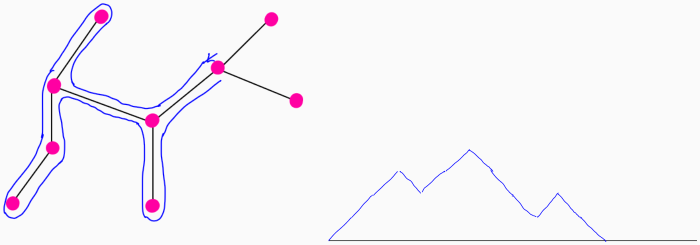

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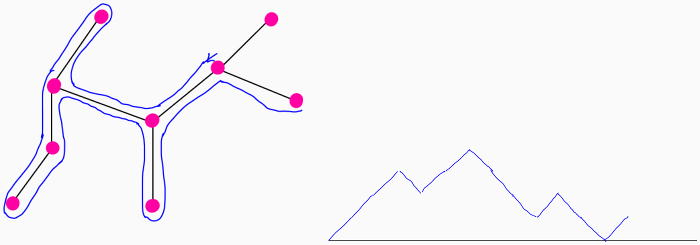

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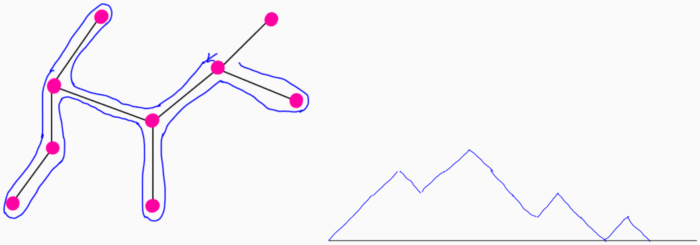

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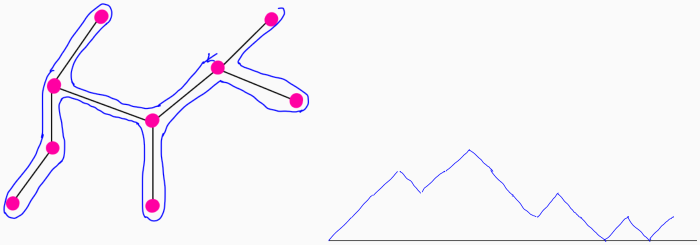

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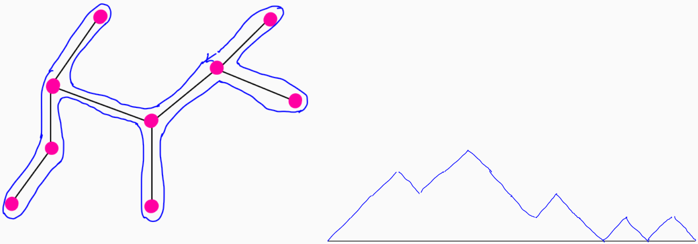

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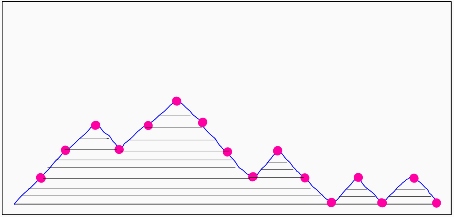
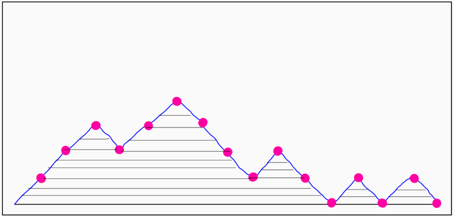
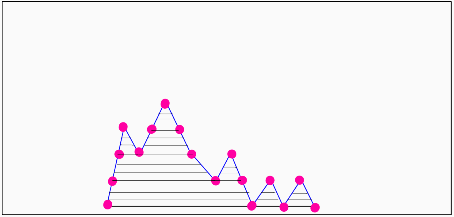
Excursion to Tree



Excursion to Tree





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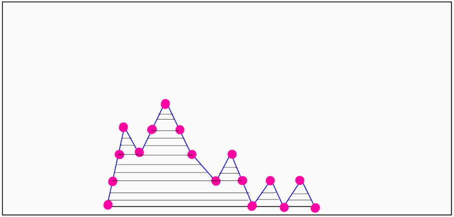

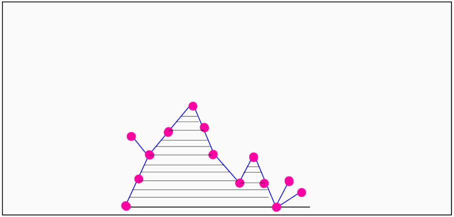

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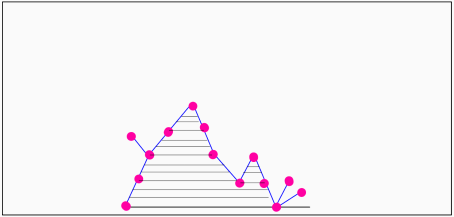

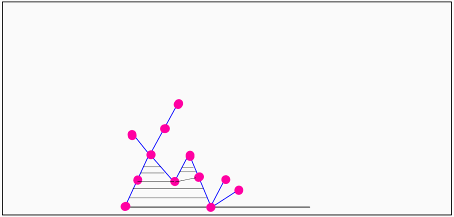

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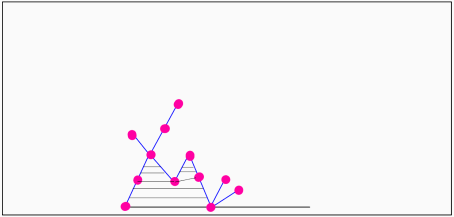
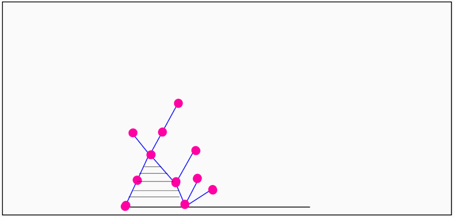
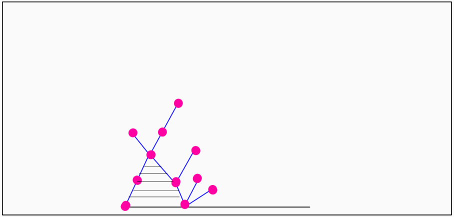
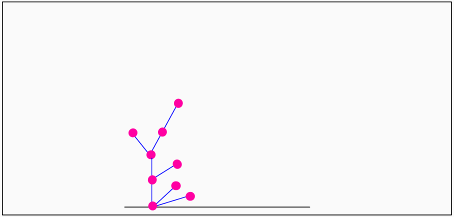


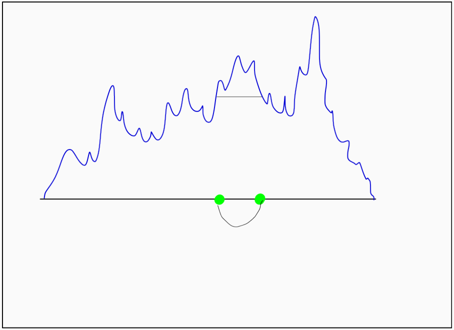


Excursion to Tree



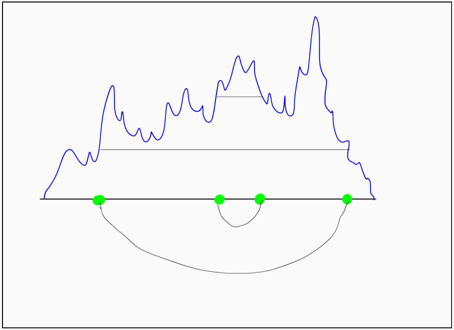


Excursion to Lamination



$$x \sim y \iff \inf_{[x,y]} \mathbb{e} = \mathbb{e}(x) = \mathbb{e}(y)$$

Excursion to Lamination



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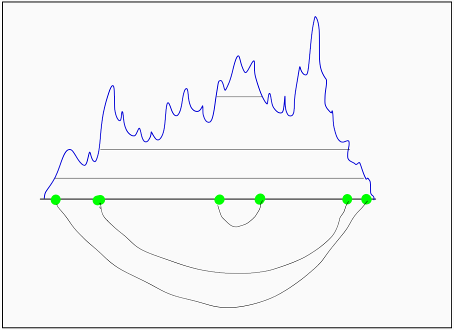
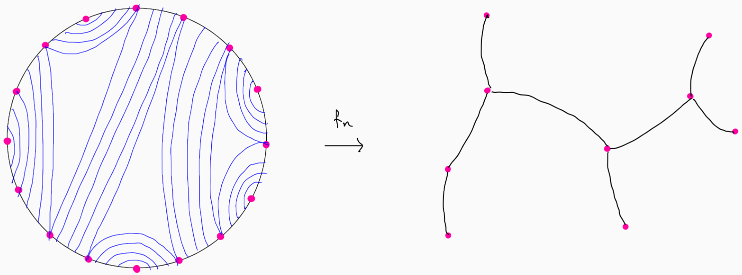


$$x \sim y \iff \inf_{[x,y]} \mathbb{e} = \mathbb{e}(x) = \mathbb{e}(y)$$

Theorem Statement

Let $f_n : \overline{\mathbb{D}^*} \to \mathbb{C}$ be the solution to the welding problem for a uniformly random plane tree. Then as $n \to \infty$, f_n converges in distribution (w.r.t uniform convergence on $\partial\mathbb{D}$) to a random map f.



Proof Sketch

Convergence

$\leftarrow$Equicontinuity/Tightness

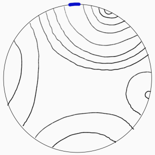
$\leftarrow$Estimate diam(f(arc))

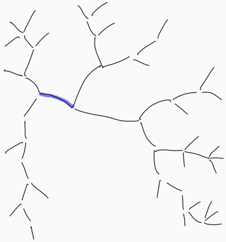
$\approx$Show that each edge is small

$\leftarrow$Find **lots** of **thick** annuli seperating edge from infinity

Proof Sketch

Need to find lots of thick annuli to bound diamater of f(arc).





Create annuli by joining points by geodesics (red).

Proof Sketch

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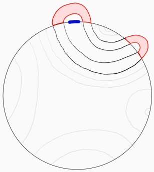

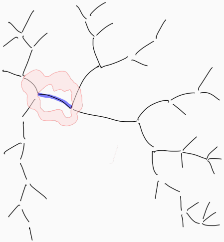

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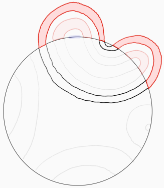
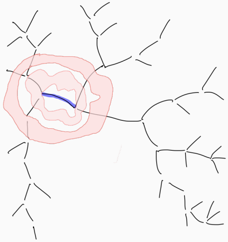
Need to find lots of thick annuli to bound diamater of $f(\text{arc})$.





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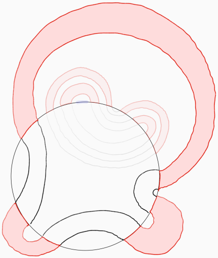
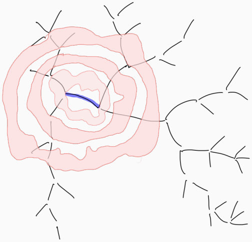
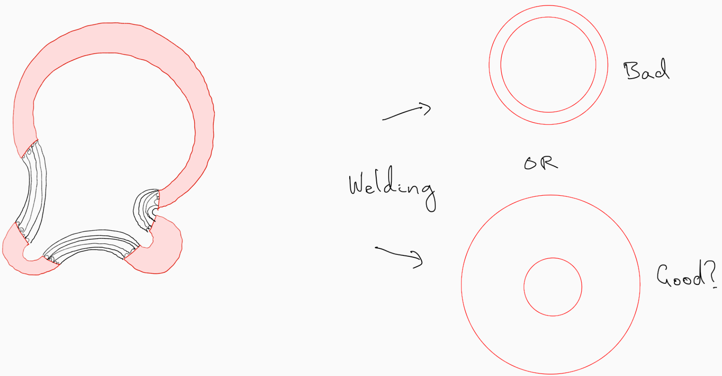




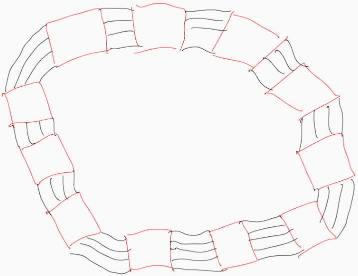
Create annuli by joining points by geodesics (red).

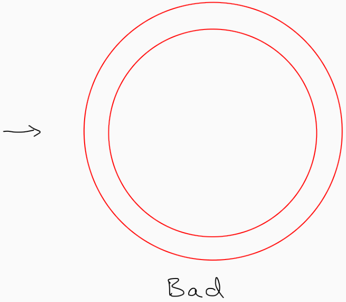
Wishlist to make thick annuli



Conditions to ensure annulus is thick after welding?

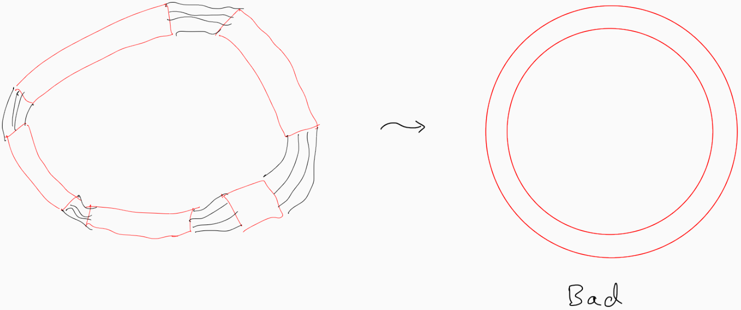
Wishlist to make thick annuli





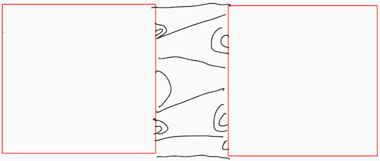
1. Need bounded number of rectangles.

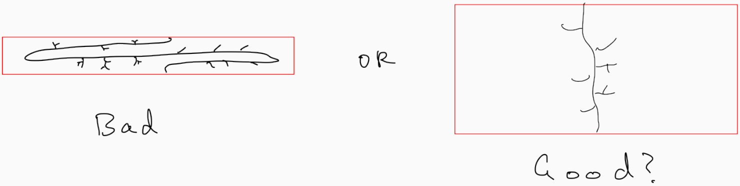
Wishlist to make thick annuli



2. Need bounded geometry for rectangles.

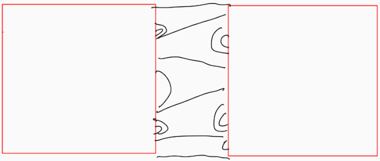
Wishlist to make thick annuli

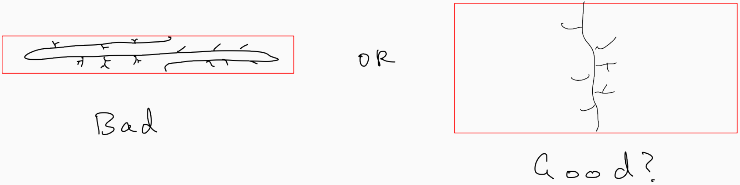




3. Need to know *something* about the welding map.

How to ensure that rectangle is still thick after welding

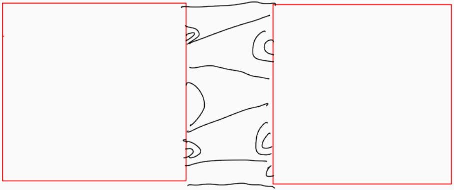




3. Need to know *something* about the equivalence relation.

If we can control the *quality* of the equivalence relation on a *large* subset of the edge then we get control on the modulus.

How to ensure that rectangle is still thick after welding

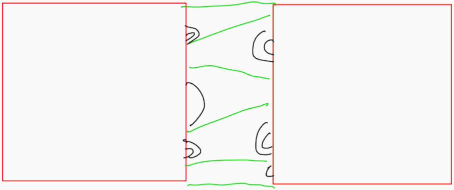


If we can control the *quality* of the equivalence relation on a *large* subset of the edge then we get control on the modulus.

It's not enough that the sets on each side are large.

The equivalence relation should also behave nicely.

How to ensure that rectangle is still thick after welding



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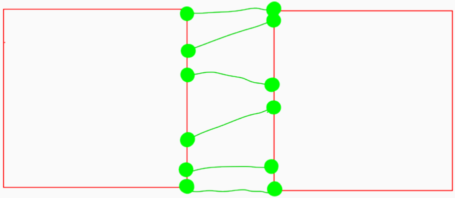

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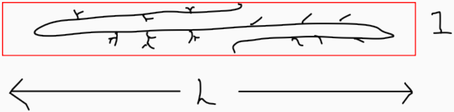
Modulus and Gluing of rectangles



Lemma

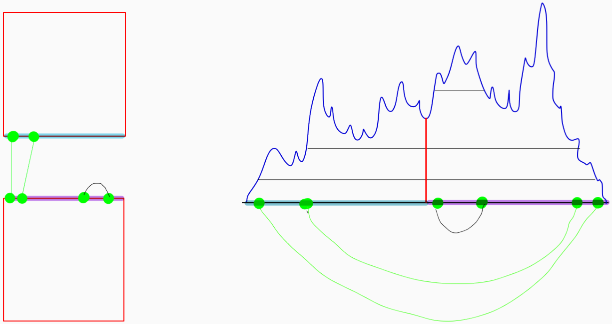
We have

$$L \lesssim \inf_{\mu^-,\mu^+} \mathcal{E}(\mu^-) + \mathcal{E}(\mu^+).$$

- μ^- ranges over probability measures on $I^- \cap \operatorname{supp}(\sim)$
- μ^+ ranges over probability measures on $I^+ \cap \operatorname{supp}(\sim)$
- μ^- and μ^+ must be be *equivalent* with respect to $\sim$.
- Energy:

$$\mathcal{E}(\mu) = \iint \log \frac{1}{|x-y|} d\mu(x) d\mu(y).$$

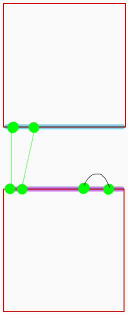
How to use this lemma for our problem:

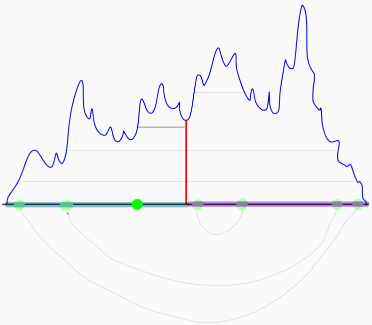


Consider a toy model where we glue two squares together using the equivalence relation from a random excursion.

What should we take for μ^- and μ^+?

How to use this lemma for our problem:





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Notice that the support of $\sim$ is exactly the images of the left sided and right sided inverse map of the excursion $\mathbb{e}$ on $[0, \mathbb{e}(1/2)]$.

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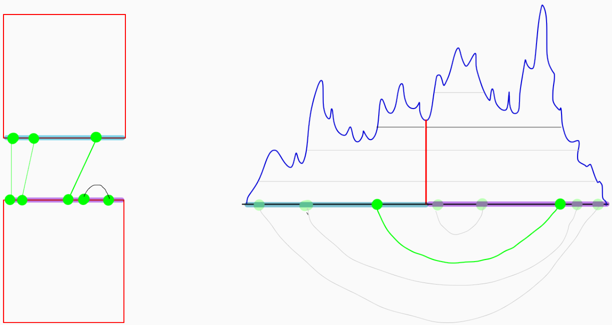

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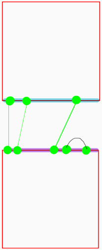
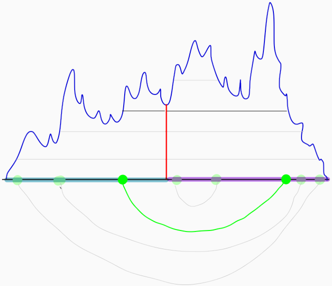
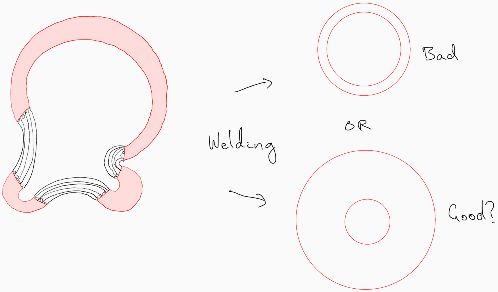




Thus we should take μ^-, μ^+ to be pullback of Lebesgue measure on $[0, \mathbb{e}(1/2)]$ via $\mathbb{e}$.

This demonstrates that the modulus L is controlled by the Hölder regularity of $\mathbb{e}$.

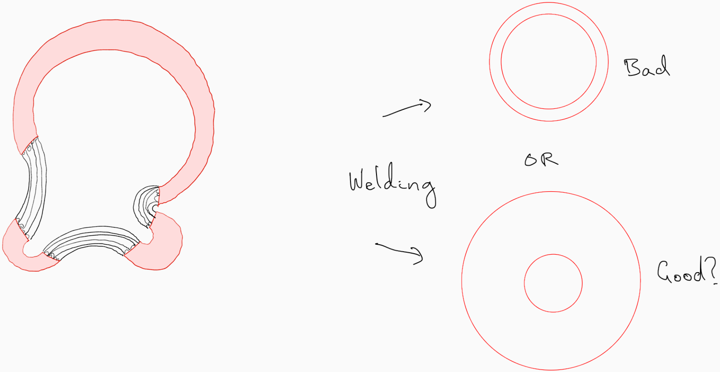
Wishlist to make thick annuli



To get thick annuli, want

1. Bounded number of rectangles
2. Bounded geometry rectangles
3. Control over $\sim$ on interface ($=$ regularity of excursion)

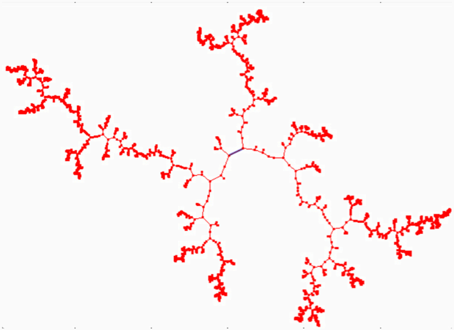
Finding lots of thick annuli



Now want to find lots of configurations that satisfy the conditions of our list.

How to construct the annuli

Fix $\lambda > 1$.



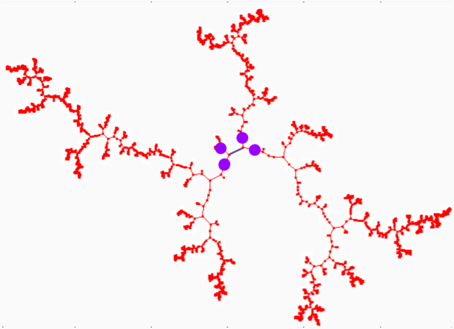
Start with large finite tree $\mathcal{T}$ with the graph distance.

Let X_k be the points on $\mathcal{T}$ which are distance λ^k from the root.

Join these points with geodesics in the complement of the tree.

How to construct the annuli

Fix $\lambda > 1$.



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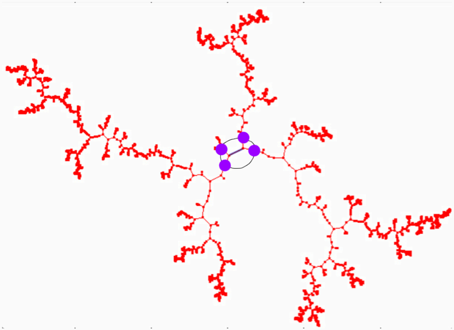

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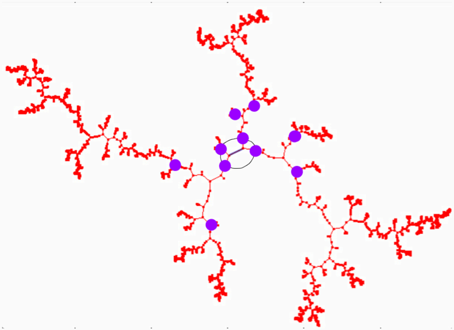

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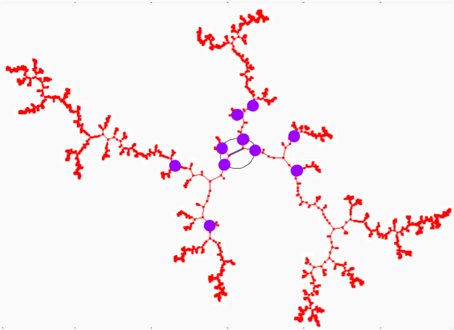

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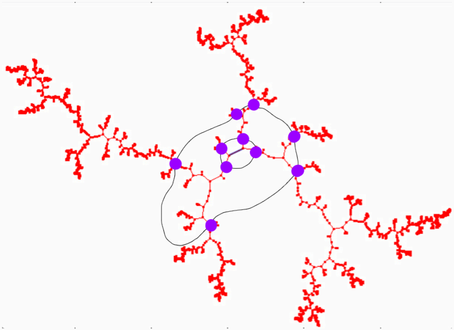

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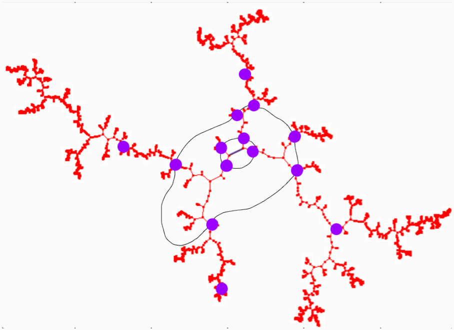

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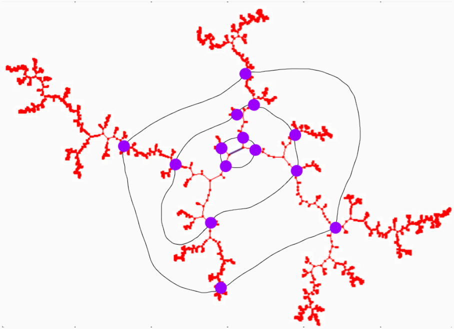

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Rest of the proof

- Analyze this exploration process via excursion picture, show that w.h.p. on *most* scales k the list of conditions is satisfied.
- Borel-Cantelli + Union bound $\implies$ Hölder equicontinuity of welding map
- Uniqueness of limit from Jones-Smirnov removability theorem.

Thank you!