Stochastic vertex models and bijectivisation of Yang-Baxter equation

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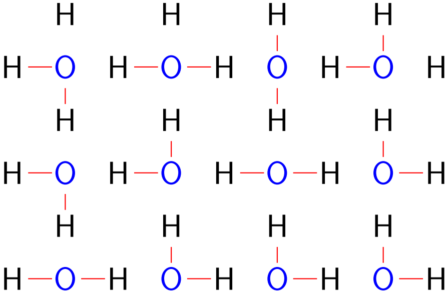
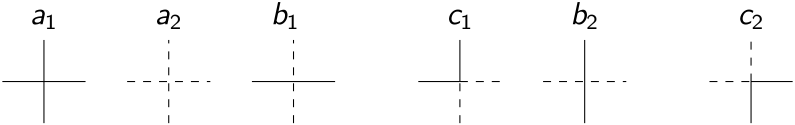
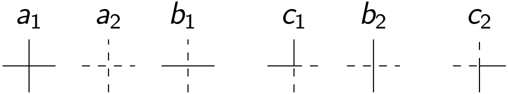
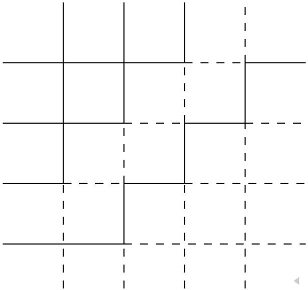
Six-vertex model



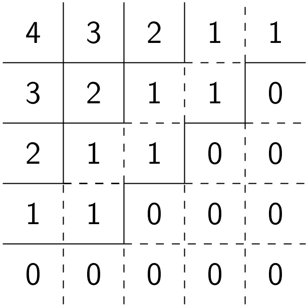




Weight of configuration is a product of weights of vertices.
Partition function: sum of weights over all possible configurations.
Consider random configuration: probability is proportional to the weight of configuration.



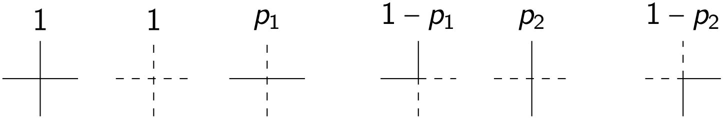
Height function:



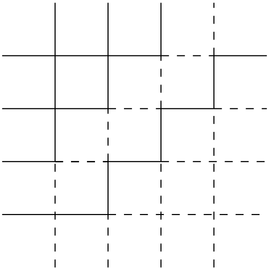
What is the asymptotic behavior of a height function of the (random) configuration of the six-vertex model ?

Very little is known rigorously outside of the free fermionic case.

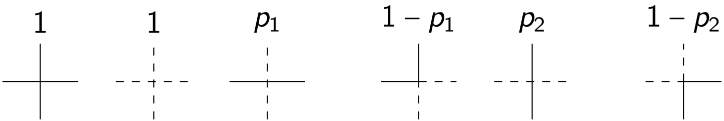
Consider one particular model: for $0 < t < 1$, $0 \le p_2 < p_1 \le 1$ let the weights have the form



Boundary conditions: quadrant, all paths enter from the left.



Consider one particular model: for $0 < t < 1$, $0 \leq p_2 < p_1 \leq 1$ let the weights have the form



Boundary conditions: quadrant, all paths enter from the left.

This is a *stochastic six vertex model* introduced by Gwa-Spohn'92.

It has a degeneration into ASEP (asymptotics of height function Tracy-Widom'07)

Borodin-Corwin-Gorin'14: law of large numbers and fluctuations for the height function at one point. Fluctuations are of order 1/3.

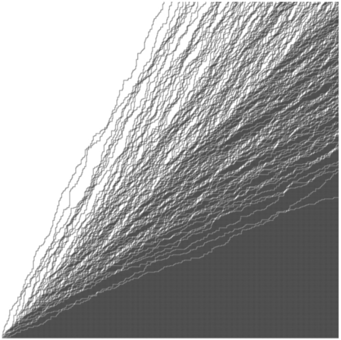
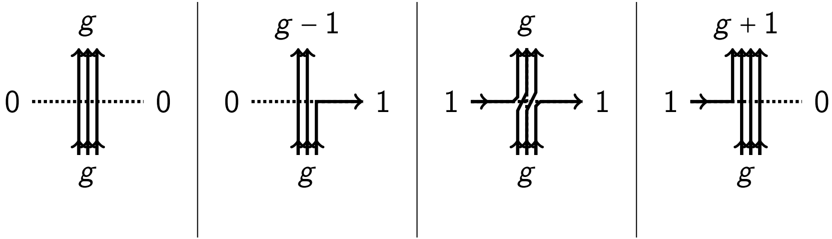


Figure by Leo Petrov.

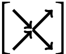
More general vertex models:

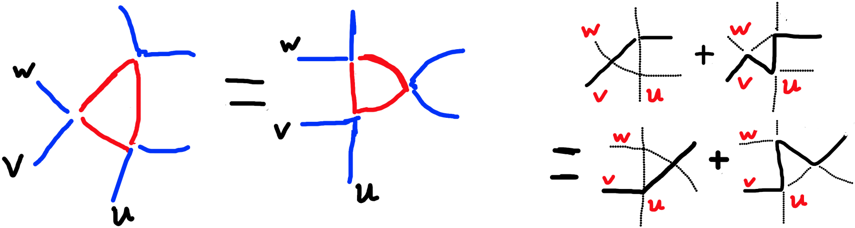
- higher spin vertex models: more than one arrow in horizontal and/or vertical directions.
- dynamical vertex models: the vertex weights might depend on the location of vertex and/or configuration parameters (such as height function at the vertex).



Yang-Baxter equation

[]$_{u,v} := 1,$ []$_{u,v} := \dfrac{u-v}{u-tv},$ []$_{u,v} := \dfrac{(1-t)v}{u-tv},$

[]$_{u,v} := 1,$ []$_{u,v} := \dfrac{t(u-v)}{u-tv},$ []$_{u,v} := \dfrac{(1-t)u}{u-tv}.$



Bijectivisation / coupling. General idea applied to equality $2+2 = 3+1$.

$$\begin{array}{ccc} & 3 & 1 \\ 2 & 2 & 0 \\ 2 & 1 & 1 \end{array} \qquad \begin{array}{ccc} & 3 & 1 \\ 2 & \frac{3}{2} & \frac{1}{2} \\ 2 & \frac{3}{2} & \frac{1}{2} \end{array}$$

Bufetov-Petrov'17:

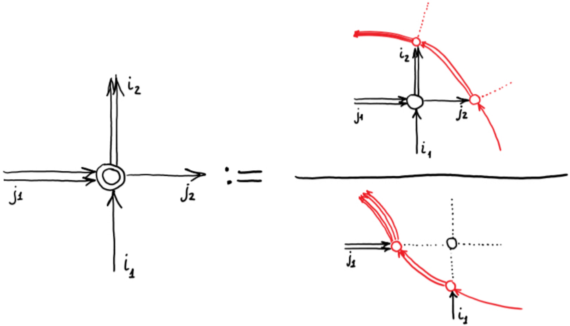
- Yang-Baxter equation can be viewed as a collection of equalities (one for any choice of boundary conditions). We "bijectivise" all of them.
- We obtain more structure on top of Yang-Baxter equation. One can try to use this structure...

Stochastization

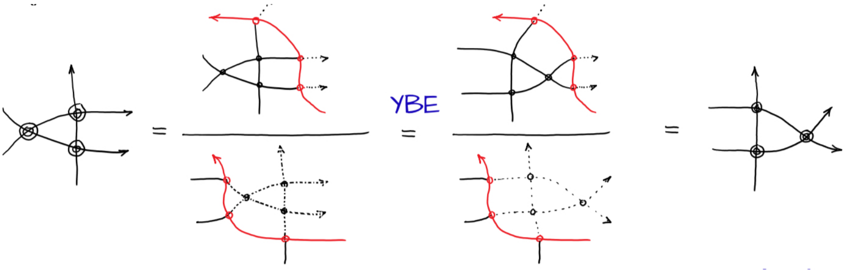
In certain cases there is a unique bijectivisation. This allows to construct a stochastic vertex from an arbitrary one.

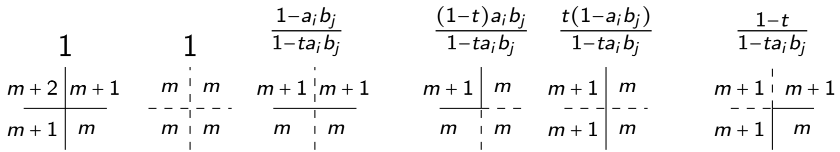
Aggarwal-Borodin-Bufetov'18

Define a stochastic vertex by equation:

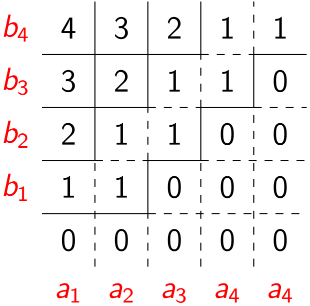


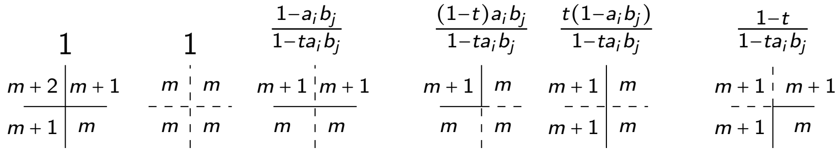
New stochastic vertices satisfy Yang-Baxter equation:



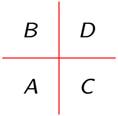


$0 < t < 1$, $0 < a_i b_j < 1$. In a homogeneous case $a_i b_j = z$, for all i and j, this is a *stochastic six vertex model*.





Let us try to find good observables



$$E(f(D)|A, B, C) = g(f(A), f(B), f(C))$$

In a stochastic six vertex model one obtains

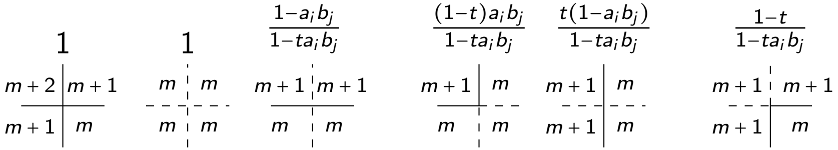
$$E(t^{nD}|A,B,C) = \alpha t^{nC} + \beta t^{(n-1)C+B} + \gamma t^{nA},$$

for some coefficients $\alpha(n), \beta(n), \gamma(n)$.

Borodin-Gorin'18, n=1,2; Bufetov'19+, general n.

This allows to write discrete non-linear equations for $E(t^{nh})$, solve them, and derive asymptotics of the height function.

Since $0 < t < 1$, the distribution of t^h is determined by its moments. Thus, we know all the information about the distribution of h (in principle).



$$F_{M,N}(z) := \prod_{1\leq i\leq N} \frac{tz - a_i}{z - a_i} \prod_{1\leq j\leq M} \frac{1 - zb_j}{1 - tzb_j}.$$

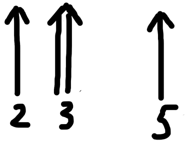
We have

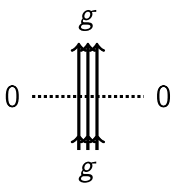
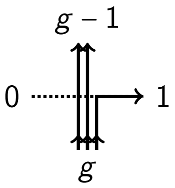
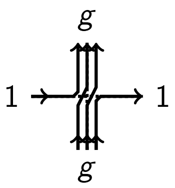
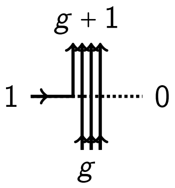
$$E\left(n\cdot h(M,N)\right) = \frac{1}{(2\pi i)^n} \oint \dots \oint \frac{dz_1 \dots dz_n}{z_1 z_2 \dots z_n} \prod_{1\leq i<j\leq n} \frac{z_i - z_j}{tz_i - z_j} \times \prod_{i=1}^{n} F_{M,N}(z_i)$$

where the contours are around a_i and 0 (and no other poles of the integrand).

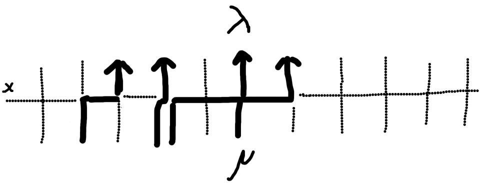
Young diagram: $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq \cdots \geq 0$.

Example: $\lambda = (5, 3, 3, 2)$.



			
1	x	x	$1 - t^{g+1}$

Define $Q_{\lambda/\mu}(x)$ as the weight of the following picture:



Define a linear operator on formal sums of Young diagrams via

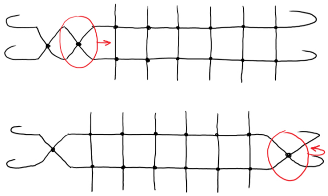
$Q(x) := \mu \mapsto \sum_\lambda Q_{\lambda/\mu}(x)\lambda$.

$Q(x_1)Q(x_2)\dots Q(x_n) =: \mu \mapsto \sum_\lambda Q_{\lambda/\mu}(x_1,\dots,x_n)\lambda$

Commuting operators:

$$Q(x_1)Q(x_2) = Q(x_2)Q(x_1)$$

Proof by Yang-Baxter equation.

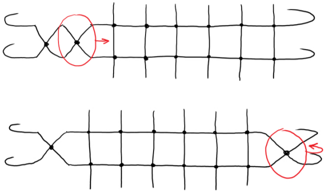


Therefore, $Q_\lambda(x_1, \ldots, x_N) := Q_{\lambda/\varnothing}(x_1, \ldots, x_N)$ is a symmetric polynomial (Hall-Littlewood polynomial).

Generalized Cauchy identity:

$$\frac{1-txy}{1-xy}\sum_{\mu} c(\lambda,\mu)Q_{\lambda/\mu}(x;t)Q_{\nu/\mu}(y;t) = \sum_{\rho} c(\rho,\nu)Q_{\rho/\nu}(x;t)Q_{\rho/\lambda}(y;t);$$

Also can be proved by Yang-Baxter equation.



Cauchy identity:

$$\sum_{\lambda\in\mathbb{Y}} c_\lambda Q_\lambda(x_1,\dots,x_N;t)Q_\lambda(y_1,\dots,y_N;t) = \prod_{i,j}\frac{1-tx_iy_j}{1-x_iy_j}$$

$a_i b_j < 1$, $a_i > 0, b_j > 0$. Schur measure ($t = 0$) on Young diagrams:

$$Prob(\lambda) = \prod_{i,j} (1 - a_i b_j)\, s_\lambda(a_1, \ldots, a_M) s_\lambda(b_1, \ldots, b_N).$$

Hall-Littlewood measure:

$$Prob(\lambda) = \prod_{i,j} \frac{1 - a_i b_j}{1 - t a_i b_j} c_\lambda Q_\lambda(a_1, \ldots, a_M; t) Q_\lambda(b_1, \ldots, b_N; t).$$

Okounkov'01: tools to analyze Schur measures.

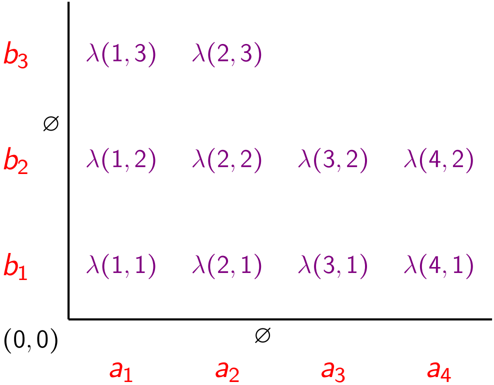
Borodin-Corwin'11: tools to analyze Macdonald / Hall-Littlewood measures.

Bijectivisation of generalized Cauchy identity: Multiple application of a bijectivisation of Yang-Baxter equation.

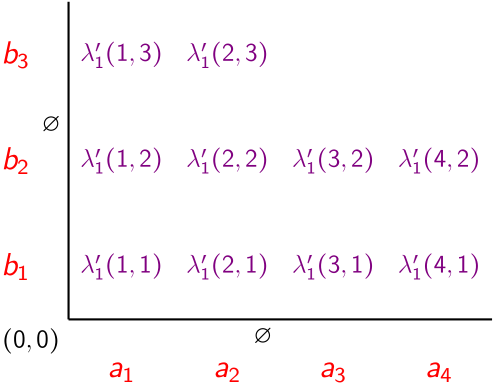
We can find coefficients $U(\mu;\lambda,\nu \to \rho)$ and $\hat{U}(\rho;\lambda,\nu \to \mu)$ such that

$$\frac{1-tab}{1-ab} c(\lambda,\mu) Q_{\nu/\mu}(a) Q_{\lambda/\mu}(b) U(\mu;\lambda,\nu \to \rho) = c(\rho,\nu) Q_{\rho/\lambda}(a) Q_{\rho/\nu}(b) \hat{U}(\rho;\lambda,\nu \to \mu),$$

This allows to construct two-dimensional arrays of random Young diagrams with the use of $U(\mu;\lambda,\nu \to \rho)$.



We have $P(\lambda(M,N)=\lambda) \sim c_\lambda Q_\lambda(a_1,\dots,a_M)Q_\lambda(b_1,\dots,b_N)$.
This is Hall-Littlewood measure.

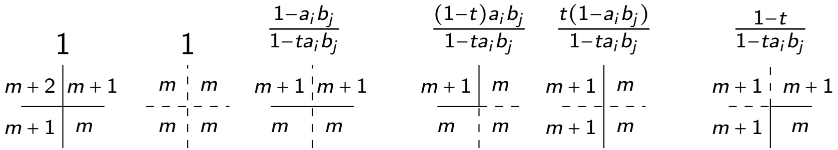


b_3 $\lambda_1'(1,3)$ $\lambda_1'(2,3)$
$\varnothing$
b_2 $\lambda_1'(1,2)$ $\lambda_1'(2,2)$ $\lambda_1'(3,2)$ $\lambda_1'(4,2)$
b_1 $\lambda_1'(1,1)$ $\lambda_1'(2,1)$ $\lambda_1'(3,1)$ $\lambda_1'(4,1)$
$(0,0)$
$\varnothing$
a_1 a_2 a_3 a_4

λ_1' is length of the first column of the Young diagram (= number of strictly positive integers).

Let us use $n - \lambda_1'(m, n)$.

b_4	4	3	2	1	1
b_3	3	2	1	1	0
b_2	2	1	1	0	0
b_1	1	1	0	0	0
	0	0	0	0	0
	a_1	a_2	a_3	a_4	a_4



Borodin-Bufetov-Wheeler'16, Bufetov-Petrov'17 the height function $H(M,N)$ for a stochastic six vertex model with weights above is distributed as $N-\lambda_1'(M,N)$, where λ is distributed as Hall-Littlewood measure with parameters $a_1,\dots,a_M$, $b_1,\dots,b_N$.

Borodin-Bufetov-Wheeler'16, Bufetov-Petrov'17 More generally, for $M_1 \geq \dots \geq M_k$ and $N_1 \leq \dots \leq N_k$ the height functions $\{H(M_i,N_i)\}$ is distributed as first columns of diagrams from Hall-Littlewood process.