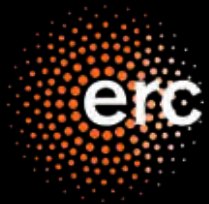


CRITICAL PARKING on RANDOM TREES



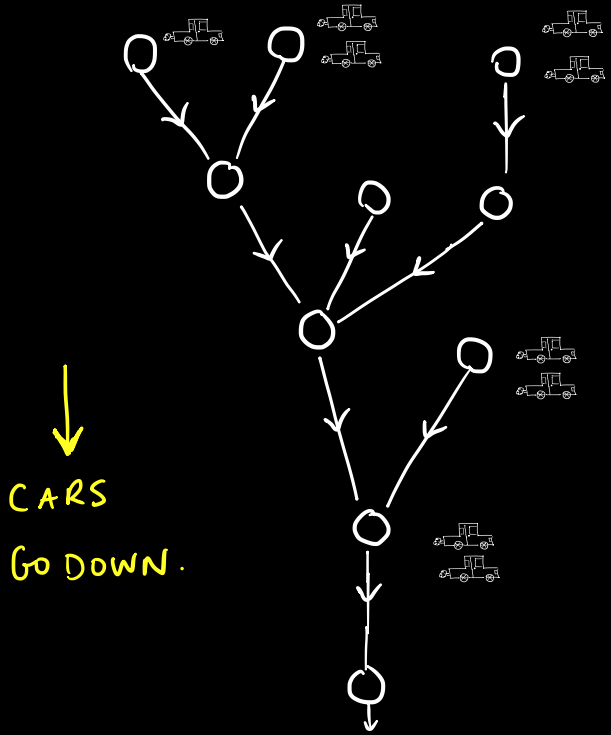
Work in progress
with

Olivier HÉNARD.



Nicolas CURIEN
INRIA & Université Paris-Saclay

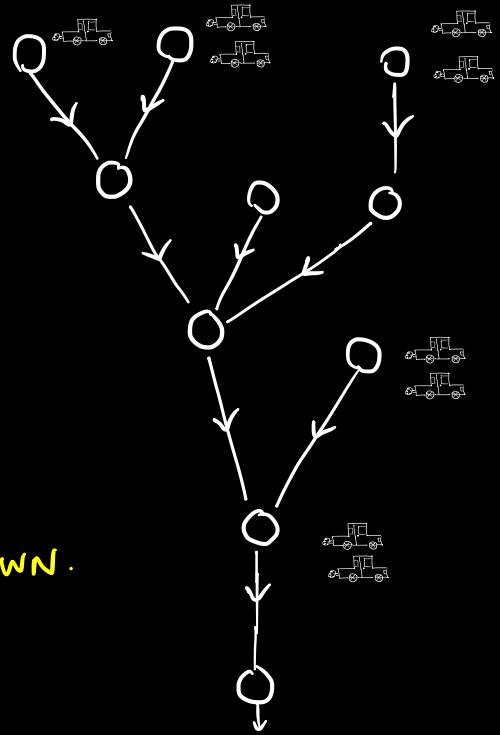
PARKING ON TREES



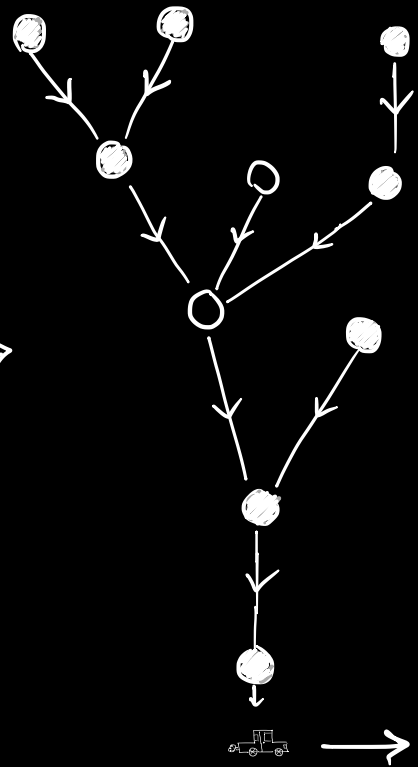
PARKING ON TREES

- spot occupied
- free spot.

↓
CARS
Go DOWN.



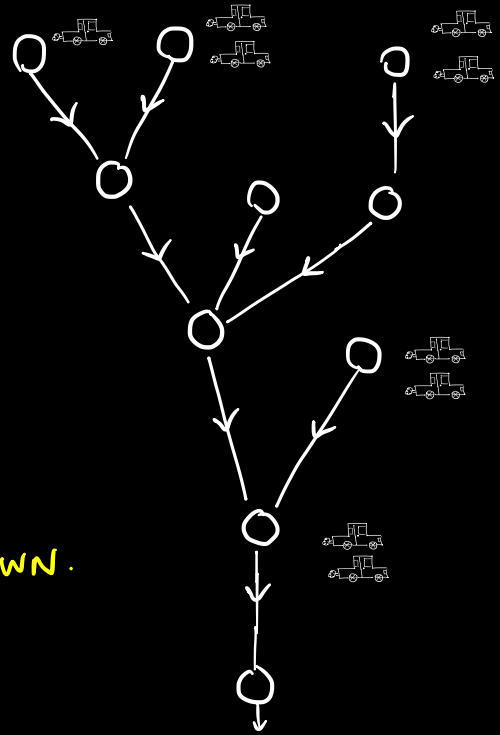
→



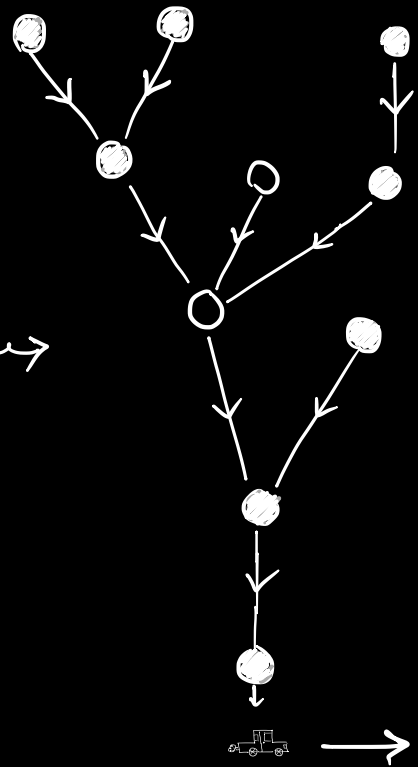
PARKING ON TREES

- spot occupied
- free spot.

↓
CARS
Go DOWN.



← llll →



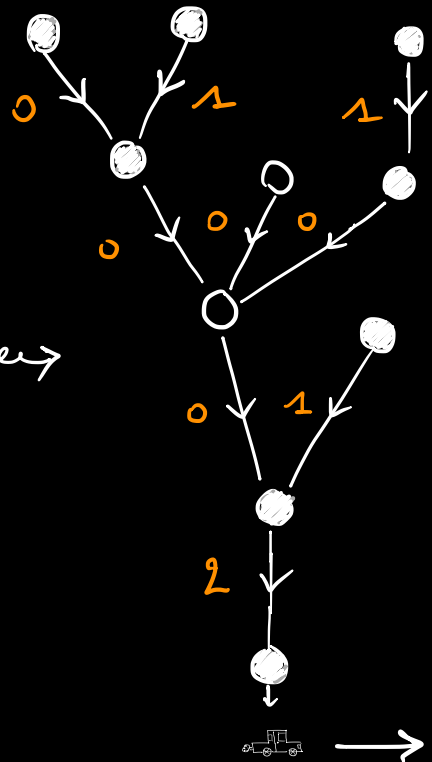
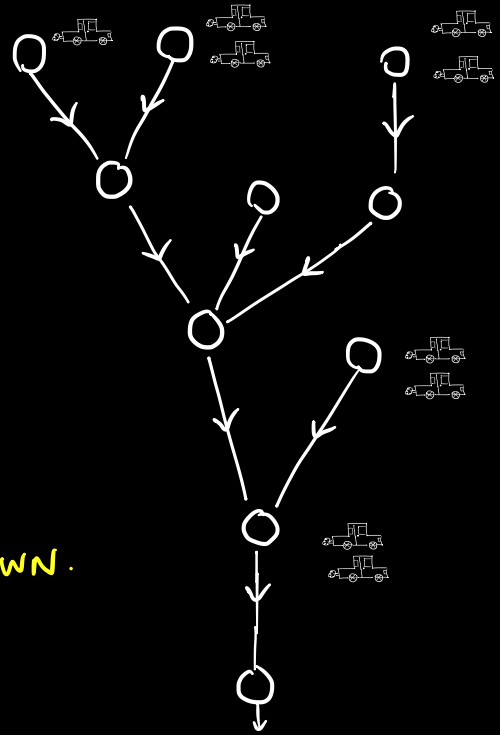
ABELIAN PROPERTY.



PARKING ON TREES

- spot occupied
- free spot.

↓
CARS
Go DOWN.



→ 1

ABELIAN PROPERTY.



MOTIVATIONS

- Parking on the line / hash tables.
- Hydrological modelling.
- Complex processes on trees.
- Abelian Sandpile / Avalanches
- Truth: Because it produces nice maths.



Model:

TREE [$T_n \sim$ some random tree on n edges
e.g: uniform, Cayley, conditioned critical GW.

CARS [i.i.d. arrivals on each vertices of T_n , with
a parameter p controlling the mean number
of cars on each vertex (increasing).
ex: Poisson(p); Geom(p); Binary(p)



Model:

TREE $\left[\begin{array}{l} T_n \sim \text{some random tree on } n \text{ edges} \\ \text{e.g.: uniform, Cayley, conditioned critical GW.} \end{array} \right.$

CARS $\left[\begin{array}{l} \text{iid arrivals on each vertex of } T, \text{ with} \\ \text{a parameter } p \text{ controlling the mean number} \\ \text{of cars on each vertex (increasing).} \\ \text{ex: Poisson}(p); \text{Geom}(p); \text{Binary}(p) \end{array} \right.$

$$\mathbb{P}_n^p = \int d\mathbb{P}_{T_n}(t) \int d\mathbb{P}_t^p(\text{cars}).$$

Expect: A phase transition!



number of
cars not parked

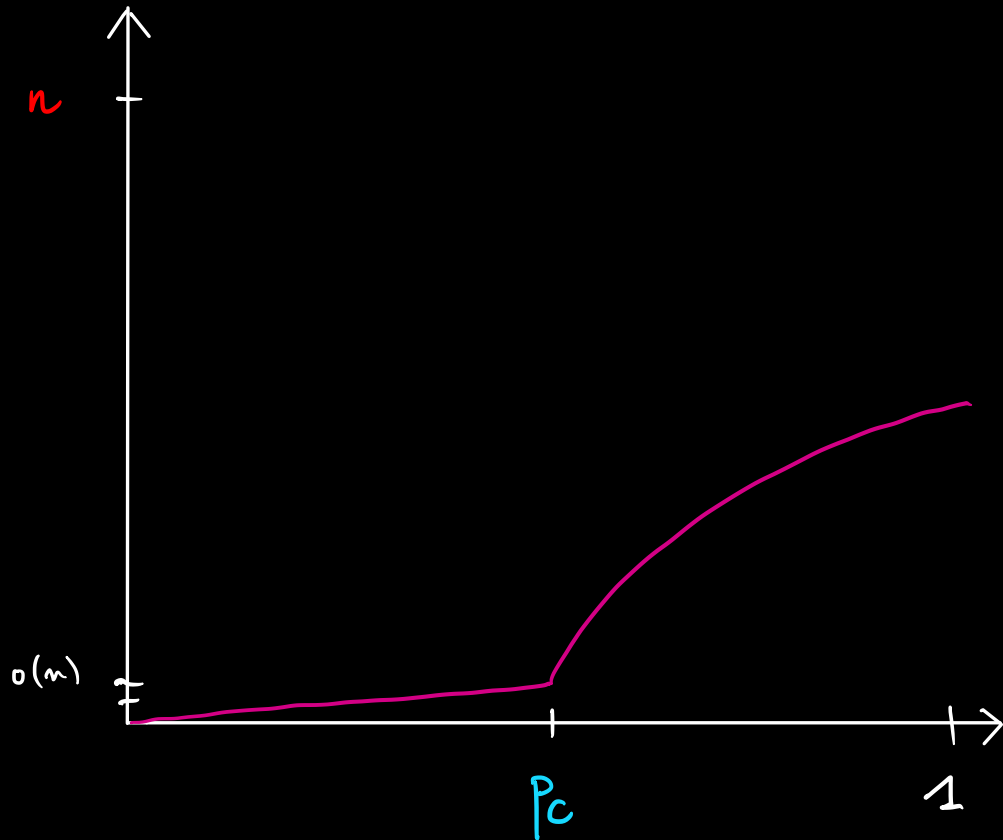
n

density of
cars.

1



number of
cars not parked



density of
cars.



number of
cars not parked

n

$o(n)$

- The value $p_c \in (0,1)$ depends on the model
- Universality of scaling limit at criticality.
- Fractal properties -
- Critical exp.

p_c

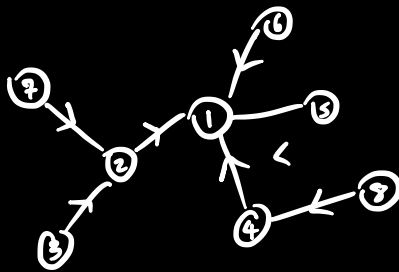
1

density of
cars.



KNOWN RESULTS

- * Uniform labeled trees
CAYLEY trees



Poisson (λ) cars iid
on the vertices.

2015

- Lackner & Panholzer



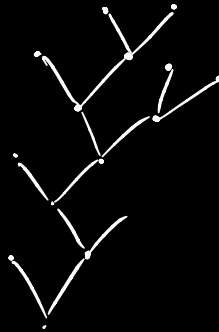
$\exists$ phase transition
 $p_c = 1/2$.

2016

- Goldschmidt & Przykucki



- * Binary plane trees
BINARY trees



0/2 cars on each
vertex prob $1-p / p$.

2018

- O.D. Jones.



$\exists$ phase transition
 $p_c = \frac{3-\sqrt{5}}{2}$.

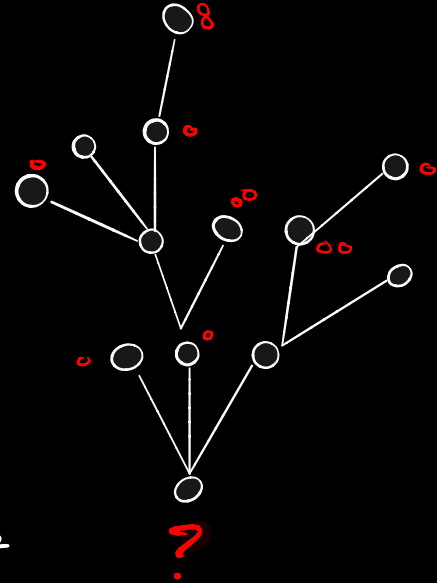
CRITICAL EXPONENTS



$$P_n^{p_c}(\text{all cars park}) \sim n^{-1/6}$$

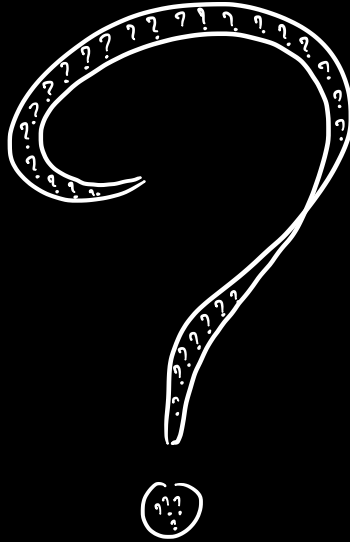


$$P_{\text{GW}}^{p_c}(k \text{ cars not parked}) \sim \text{cte } k^{-5/2}$$



not much geometry---





???

What do		You	want to	prove	???
		we			



PLAN OF (THE REST OF) THE TALK.

1) Description of our model (uniform plane trees).

2) Decomposition of parked trees into clusters

2a) Study of clusters of parked cars.

link with enumeration of planar maps.

2b) Decomposition into clusters.

multi-type GW trees, stable trees, looptrees.

3) Our current results

4) Our quest (future results?).

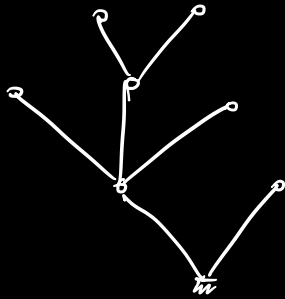
Growth-fragmentation trees, subordination of CRT,

$3/2$ -stable processes.

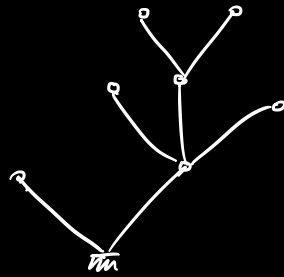


OUR MODEL

We focus on uniform plane trees and undistinguishable caus.



$\neq$



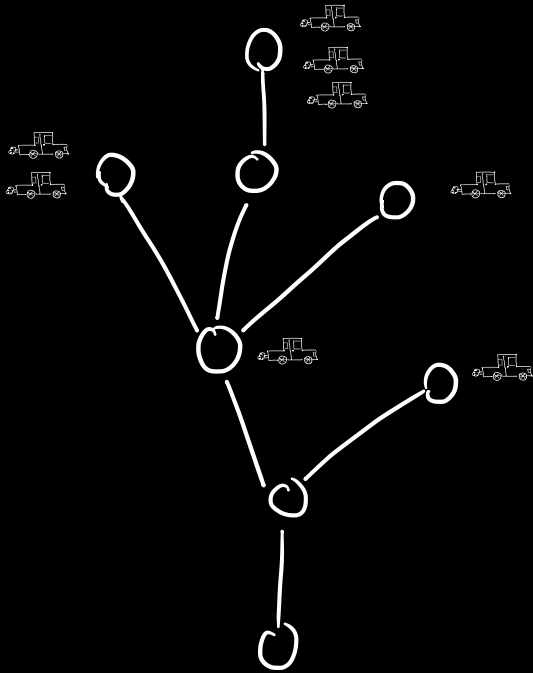
+ multiset of caus.

$T_n \sim$ Uniform plane tree with n edges. mean.
Caus $\sim$ iid on each vertex geometric (p) -

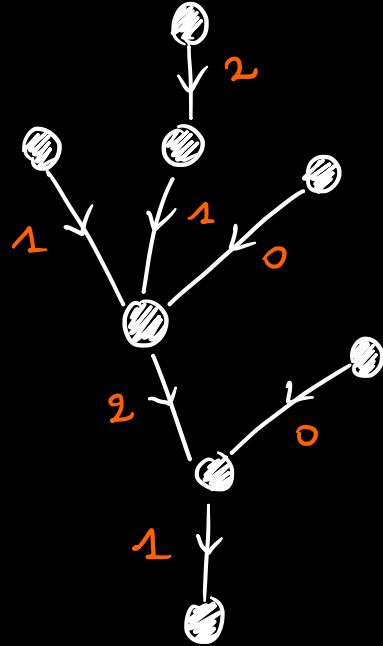


FULLY PARKED TREES (with no outgoing flux)

"Cluster of Cars".



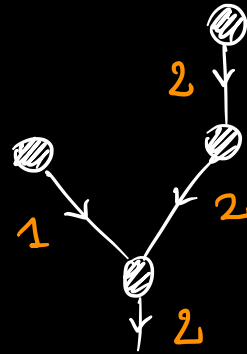
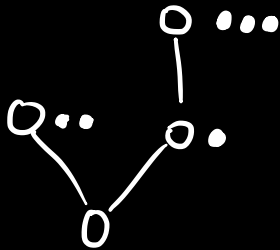
↔



ENUMERATION

Want to enumerate configurations of fully packed trees

→ Harder is simpler: enumerate FPT with possible flux at the origin.



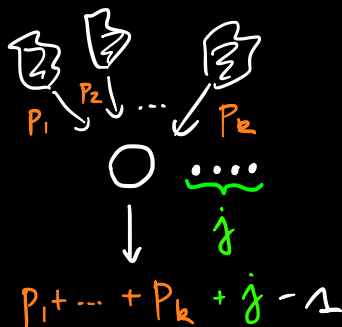
$$C_n^{(p)} = \# \left\{ \begin{array}{l} \text{configurations of} \\ \text{tree with } n \text{ vertices} \\ (n+p) \text{ cars s.t.} \\ p \text{ cars do not park} \end{array} \right. \equiv \left. \begin{array}{l} \text{FPT with } n \text{ vertices} \\ \text{and} \\ \text{outgoing flux } p \end{array} \right\}$$

Analytic combinatorics

$$F(x, y) = \sum_{n \geq 1} \sum_{p \geq 0} C_n^{(p)} x^n y^p \quad \text{satisfies}$$

$$F(x, y) = x \frac{1}{1-y} + x \frac{1}{1-y} \frac{1}{y} \frac{F}{1-F}$$

$$\begin{aligned} O^\bullet &= \text{rooted node with 0 children} \\ O^{\bullet\bullet} &= \text{rooted node with 1 child} \\ O^{\bullet\bullet\bullet} &= \text{rooted node with 2 children} \\ O^{\bullet\bullet\bullet\bullet} &= \text{rooted node with 3 children} \end{aligned}$$



not true if $p_1 + \dots + p_k = 0 \dots$



$$F(x, y) = \sum_{n \geq 1} \sum_{p \geq 0} C_n^{(p)} x^n y^p \quad \text{satisfies}$$

$$F(x, y) = \frac{x}{1-y} + \frac{x}{1-y} \left[\frac{1}{y} \left(\frac{F}{1-F} - [y'] \frac{F}{1-F} \right) + [y'] \frac{F}{1-F} \right]$$

$$\begin{aligned} 0^{\bullet} &= \text{diagram with one node and one arrow} \\ 0^{\bullet\bullet} &= \text{diagram with one node and two arrows} \\ 0^{\bullet\bullet\bullet} &= \text{diagram with one node and three arrows} \\ 0^{\bullet\bullet\bullet\bullet} &= \text{diagram with one node and four arrows} \end{aligned}$$

$$\begin{aligned} &\text{diagram with } k \text{ boxes labeled } P_1, P_2, \dots, P_k \text{ pointing to a central node} \\ &\text{diagram with } j \text{ boxes labeled } \dots \text{ pointing to a central node} \\ &P_1 + \dots + P_k + j - 1 \end{aligned}$$

$$\begin{aligned} &\text{diagram with } k \text{ boxes labeled } P_1, P_2, \dots, P_k \text{ pointing to a central node} \\ &\text{diagram with } j \text{ boxes labeled } \dots \text{ pointing to a central node} \\ &j - 1 \end{aligned}$$

$$\text{Problem: } [y'] F(x, y) = \sum_{n \geq 1} C_n^{(0)} x^n$$

is precisely what we want!



ON the SHOULDERS of Tutte, Brown, Bonsquet-Mélon & Jehanne...

This is very similar to equations encountered using Tutte's decomposition to enumerate planar maps, see also Cori & Schaeffer's "Description trees" and more recently Duché & Guerrini & Rinaldi & Schaeffer's "Fighting fish".

There are techniques to solve those equations (quadratic method, kernel method, one-cut assumption) and eventually: $F(x, 0)$ is algebraic and $F(x, y)$ is tractable.



In our case

$$F(x, 0) = 1 \cdot x + 2x^2 + 8x^3 + 41x^4 + 240x^5 + \dots$$

OEIS A294084 "Number of indecomposable intervals
in the Tamari lattices"



In our case

$$F(x, 0) = 1 \cdot x + 2x^2 + 8x^3 + 41x^4 + 240x^5 + \dots$$

OEIS A294084 "Number of indecomposable intervals
in the Tamari lattices"

$$= 1 - \frac{1}{T(x)}$$

with $T(x)$. G.S of
3-connected triangulations
of the sphere.

see Wenjie FANG (16)



In our case

$$F(x, 0) = 1 \cdot x + 2x^2 + 8x^3 + 41x^4 + 240x^5 + \dots$$

OEIS A294084 "Number of indecomposable intervals
in the Tamari lattices"

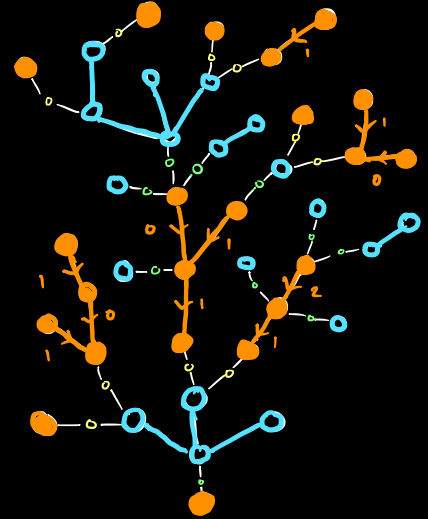
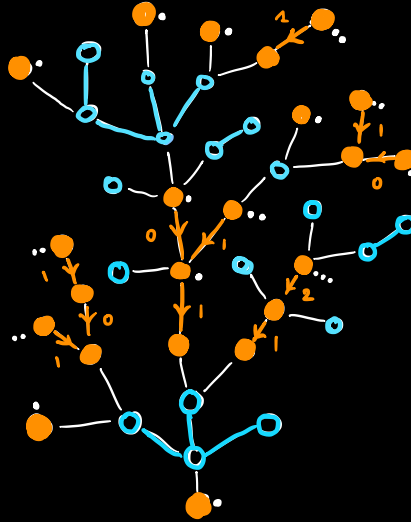
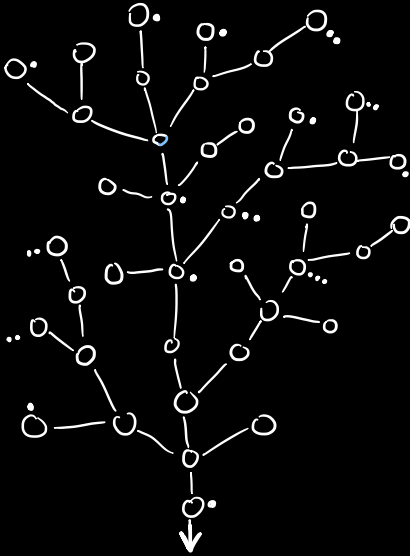
$$\# C_n^{(p)} \simeq \text{cte} \left(\frac{256}{27} \right)^n 4^p \sqrt{p} n^{-5/2} \exp \left(-\frac{p^2}{3n} \right)$$

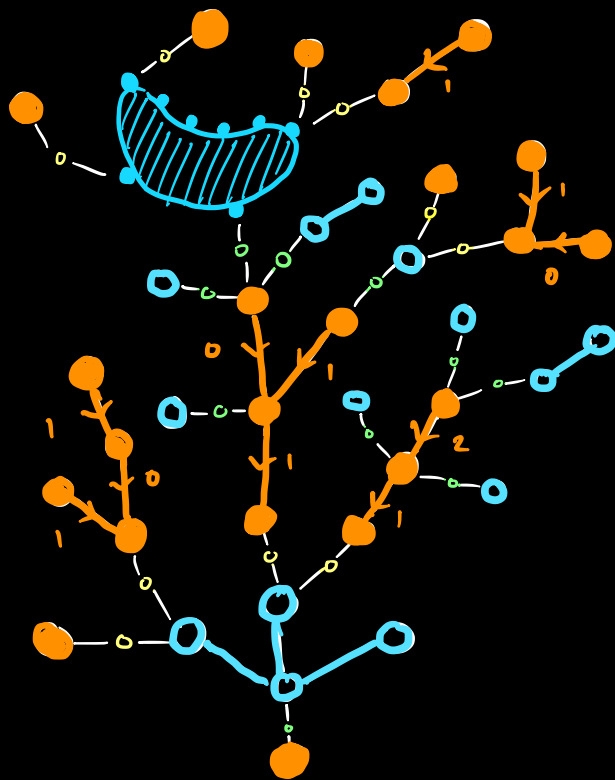
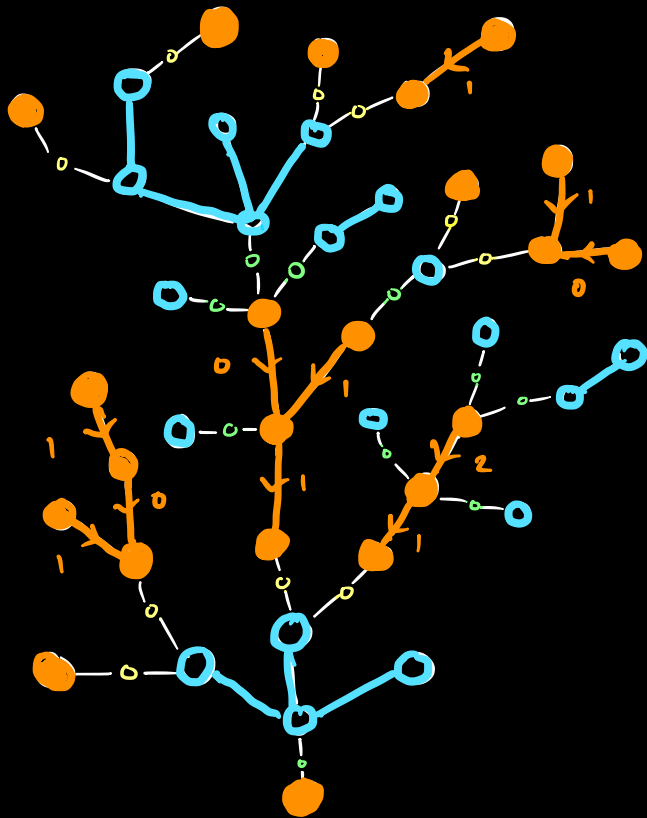
$$\# C_n^{(10)} \simeq \text{cte} \left(\frac{256}{27} \right)^n n^{-5/2}$$

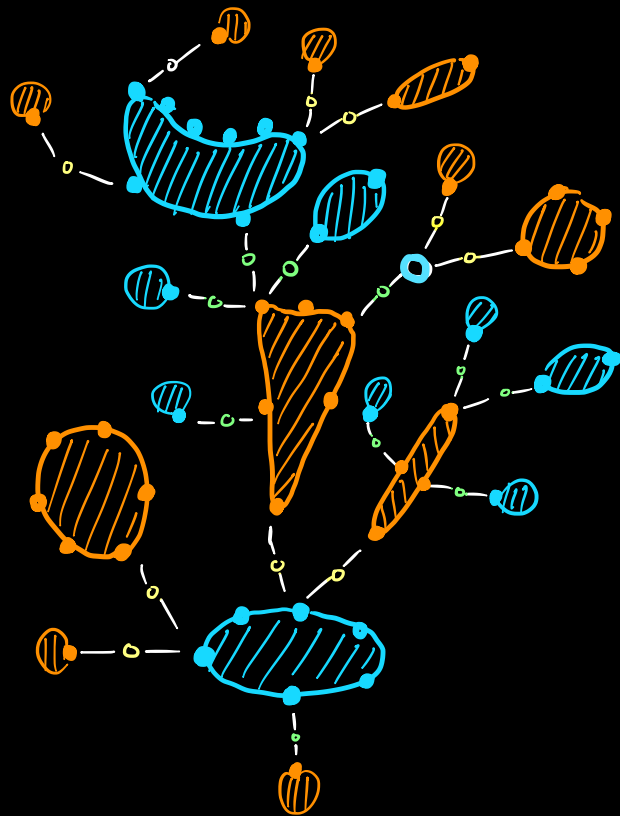
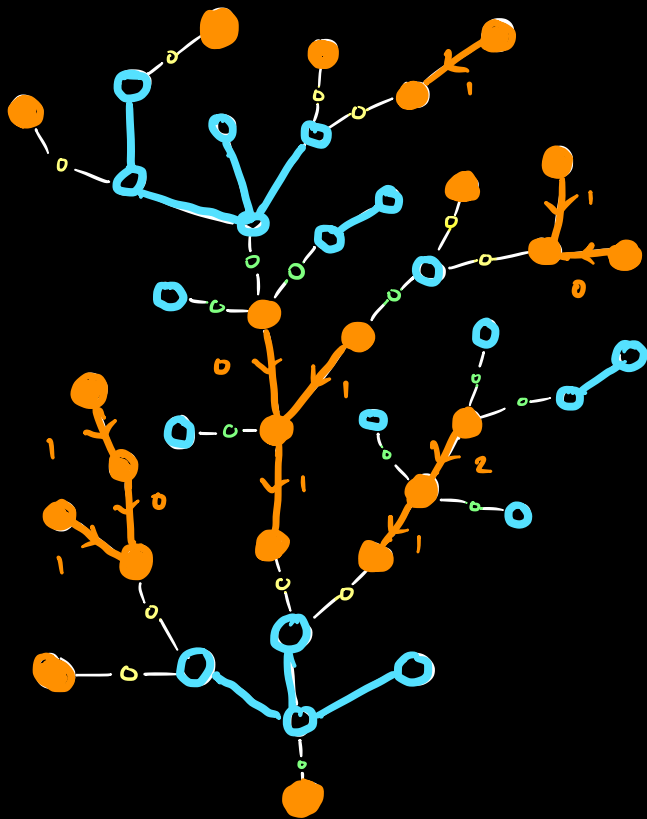
$\leadsto 3/2$ -stable laws.



DECOMPOSITION BY CLUSTERS







4-type tree.



4-TYPES TREE & GALTON-WATSON

Prop: If $(T_n \text{ Cars})$ is sampled according to $P_n^P(\cdot | \text{parking})$
the last 4-type tree is a Galton-Watson tree
whose offspring distributions $\mu_{\bullet}^P$ $\mu_{\circ}^P$ $\mu_{\bullet}^P$ $\mu_{\bullet}^P$
are explicit (thank to knowledge of $F_{\dots}$) -
conditionned to have "size n " -

In particular: $1 = \mathbb{E}[\mu_{\bullet}^{P_c}] \mathbb{E}[\mu_{\circ}^{P_c}] \mathbb{E}[\mu_{\bullet}^{P_c}] \mathbb{E}[\mu_{\bullet}^{P_c}]$

$$\mu_{\circ}^{P_c}(k) \sim c k^{-5/2} \quad \text{"}\frac{3}{2}\text{-stable"}$$

~~~~~ Phase transition at

$$P_c = \frac{1}{3}$$



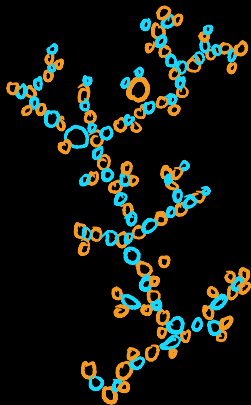
# 4-TYPES TREE & GALTON-WATSON

Prop: If  $(T_n, \text{Caus})$  is sampled according to  $P_n^P(\cdot | \text{parking})$  the last 4-type tree is a Galton-Watson tree whose offspring distributions  $\mu_{\bullet}^P, \mu_{\bullet}^P, \mu_{\bullet}^P, \mu_{\bullet}^P$  are explicit (thank to knowledge of  $F_{\dots}$ ) - conditioned to have "size  $n$ ".

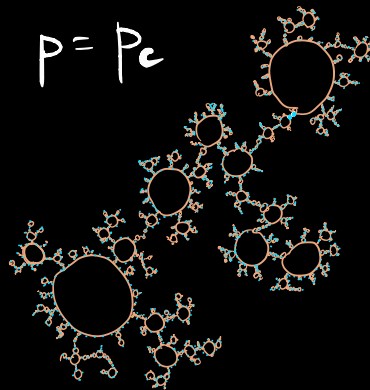
In particular:  $1 = \mathbb{E}[\mu_{\bullet}^P] \mathbb{E}[\mu_{\bullet}^P] \mathbb{E}[\mu_{\bullet}^P] \mathbb{E}[\mu_{\bullet}^P]$   
 $\mu_{\bullet}^P(k) \sim c k^{-5/2}$  "  $\frac{3}{2}$ -stable".

Phase transition at  $p_c = \frac{1}{3}$

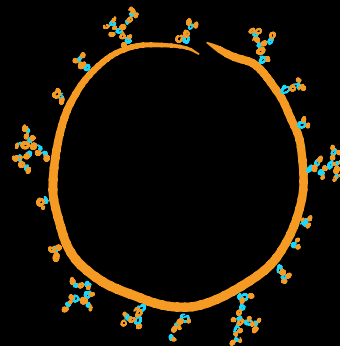
$p < p_c$



$p = p_c$



$p > p_c$



$c, c' > 0$

$$P_n^P(\text{parking}) \sim \text{cte.}$$

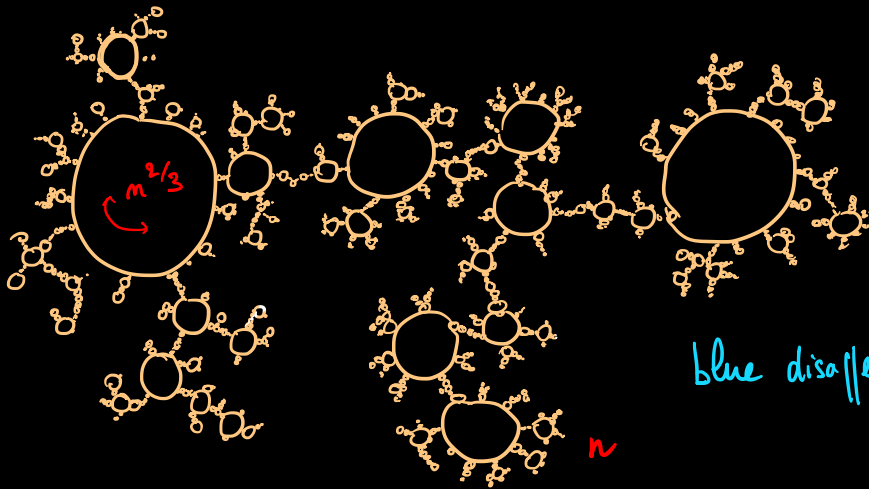
$$P_n^P(\text{parking}) \sim \frac{c}{n} e^{-c'n}$$



MORE :

Explains Lackner & Panholzer  $n^{-1/6}$  ...

Under  $\mathbb{P}_n^{P_c}(\cdot | \text{parking})$ , the 4-type tree converges after renormalisation towards the  $3/2$ -stable tree (Duquesne; Le Gall).

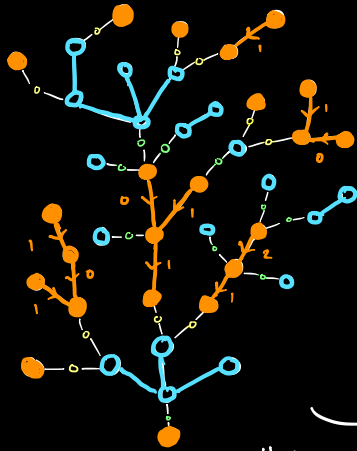


A  $3/2$ -stable tree  
in its "dual" form  
as a looptree -

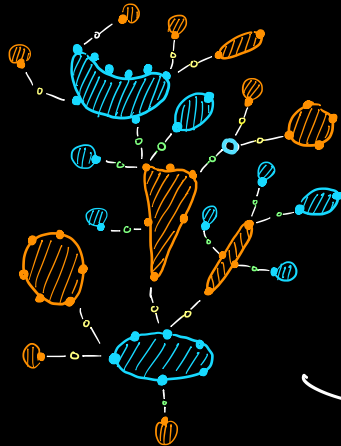
blue disappears in the limit -



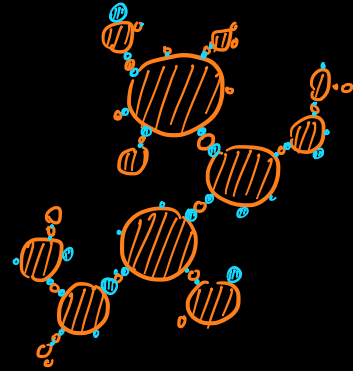
# Summary:



→  
"tensorisation"



→  
 $n$  large  
critical.



Conditionally on the  
structure, iid  
Fully packed trees in  
each orange blob.

It remains to understand  
Scaling limits of uniform fully packed  
trees with  $n$  vertices.



# SCALING LIMIT of FULLY PARKED TREES

Take  $\mathcal{T}_n = (\mathcal{T}_n; (\phi_e)_{e \in \mathcal{T}_n})$  a fully parked tree drawn uniformly at random among all FPT with  $n$  vertices.

Conjecture: We have the convergence in distribution GH + Sko.

(Work in progress)

$$(n^{-3/4} \mathcal{T}_n; n^{-1/2} (\phi_e)_{e \in \mathcal{T}_n})$$

(d)  
 $\xrightarrow{n \rightarrow +\infty}$   $\left( \begin{array}{l} \text{Growth-Fragmentation tree} \\ \text{associated to } 3/2\text{-stable Lévy} \\ \text{process conditioned on having} \\ \text{mass 1 and started from 0} \end{array} \right)$



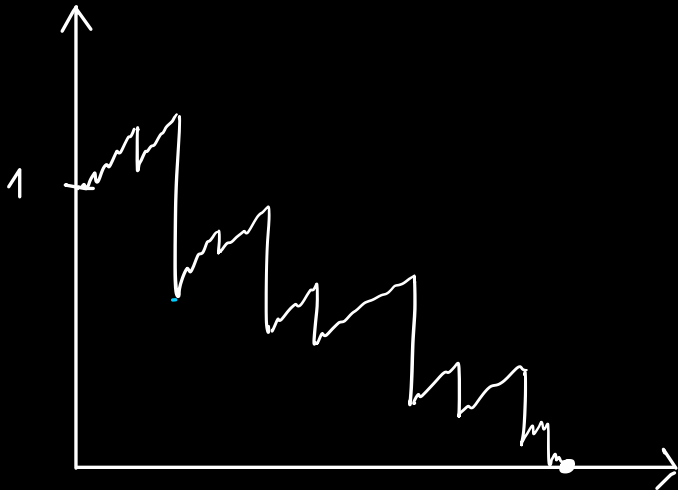
## GROWTH-FRAGMENTATION TREES (BERTOIN).

Let  $(X_t)_{t \geq 0}$  be a positive self similar Markov process started from  $x \geq 0$  under  $\mathbb{P}_x$  and killed when it touches 0. We can create a random tree  $\mathcal{C}$  together with a labeling  $(\phi_u)_{u \in \mathcal{C}}$  as follows:



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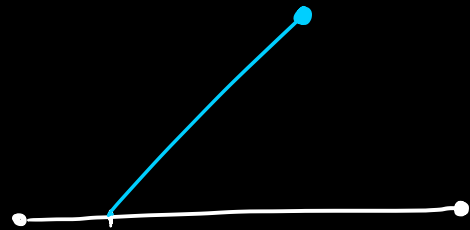
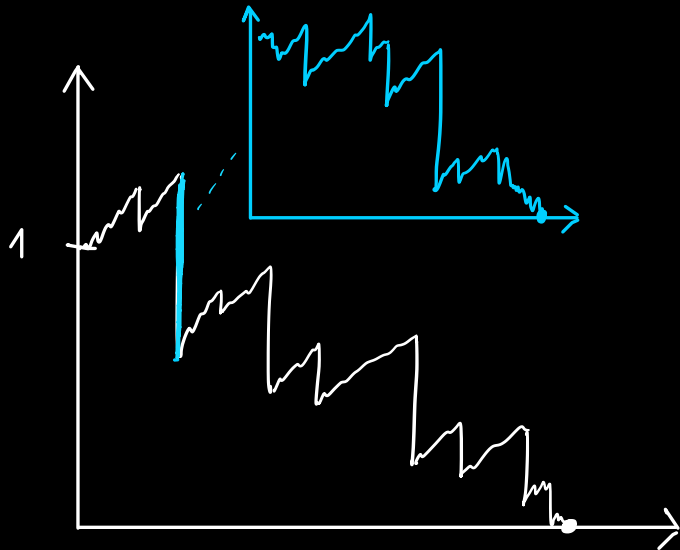
↑  
labels given  
by the process  
of the corresponding  
color -





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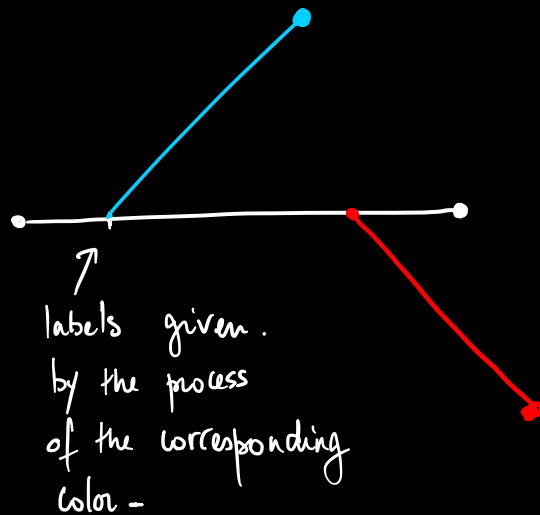
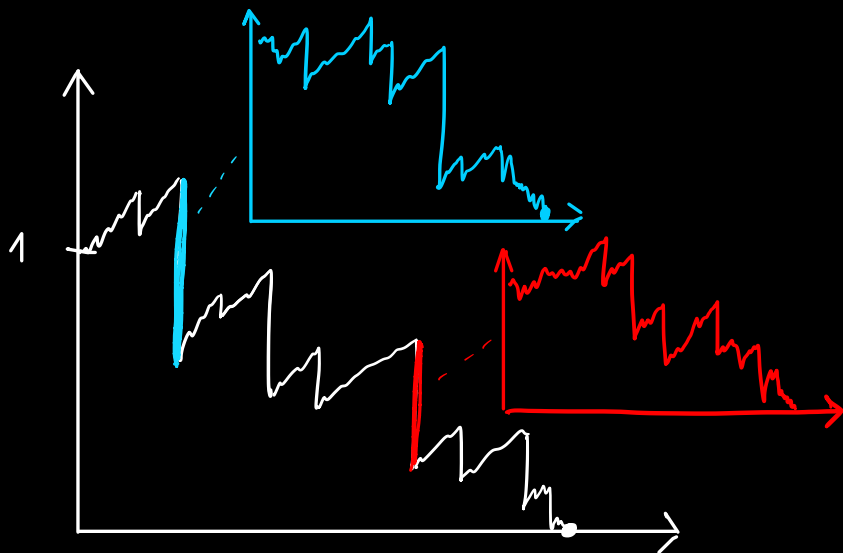


↑  
labels given  
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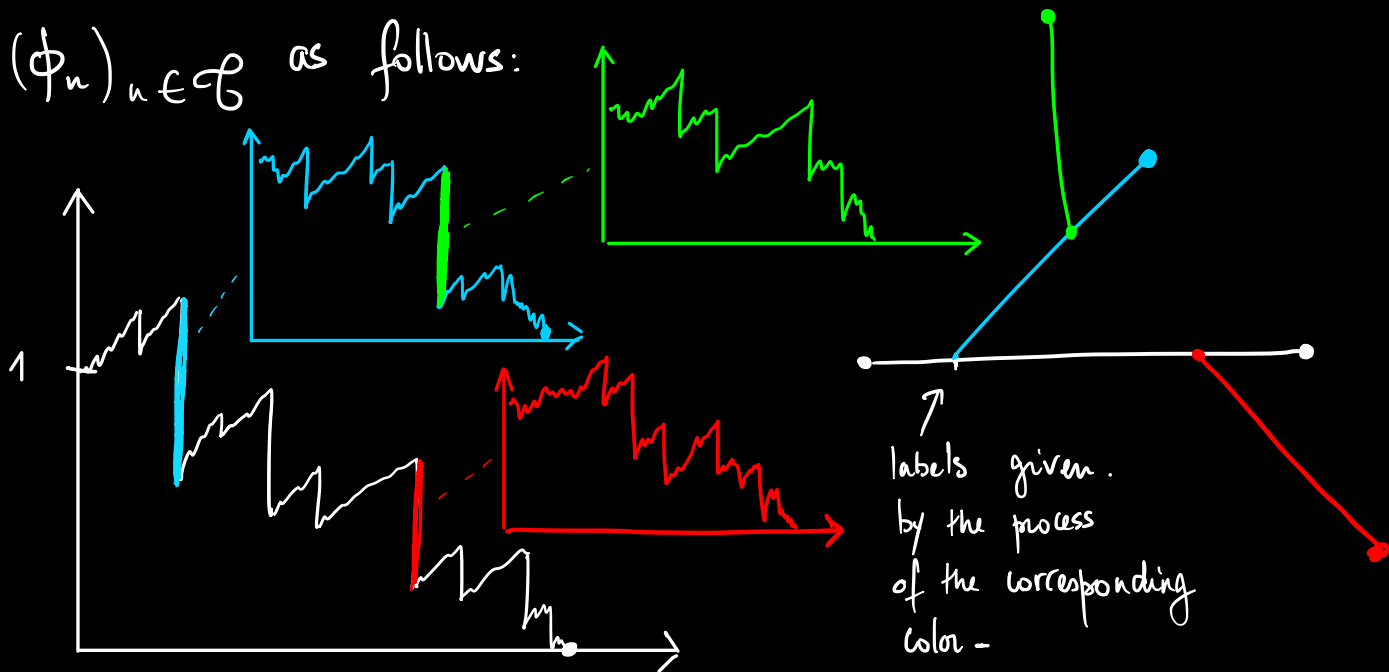
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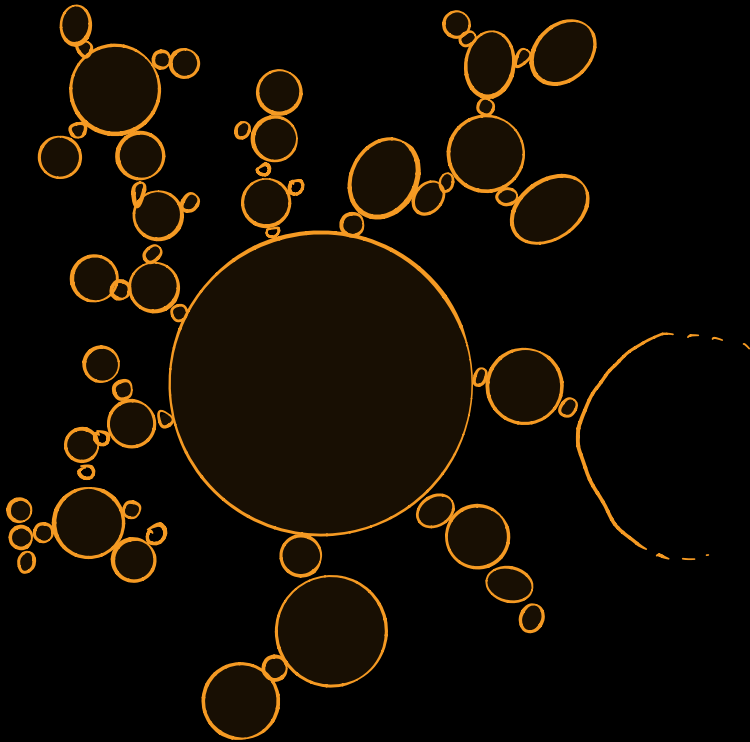


- There is a law of pssmp which produces a GF tree such that along a "typical" branch the process of labels evolve as a  $\frac{3}{2}$ -stable process with no positive jumps -
- Although the tree  $\mathcal{G}$  is composed of as many branches we can give a sense to its "mass"  $\in (0, \infty)$ .
- Needs to be done: Make sense of  $(\mathcal{G}, \phi)$  on the singular event of starting from 0 and Conditioned on having mass = 1.  
(see Bertoin; Budd; C.; Kortchemski)



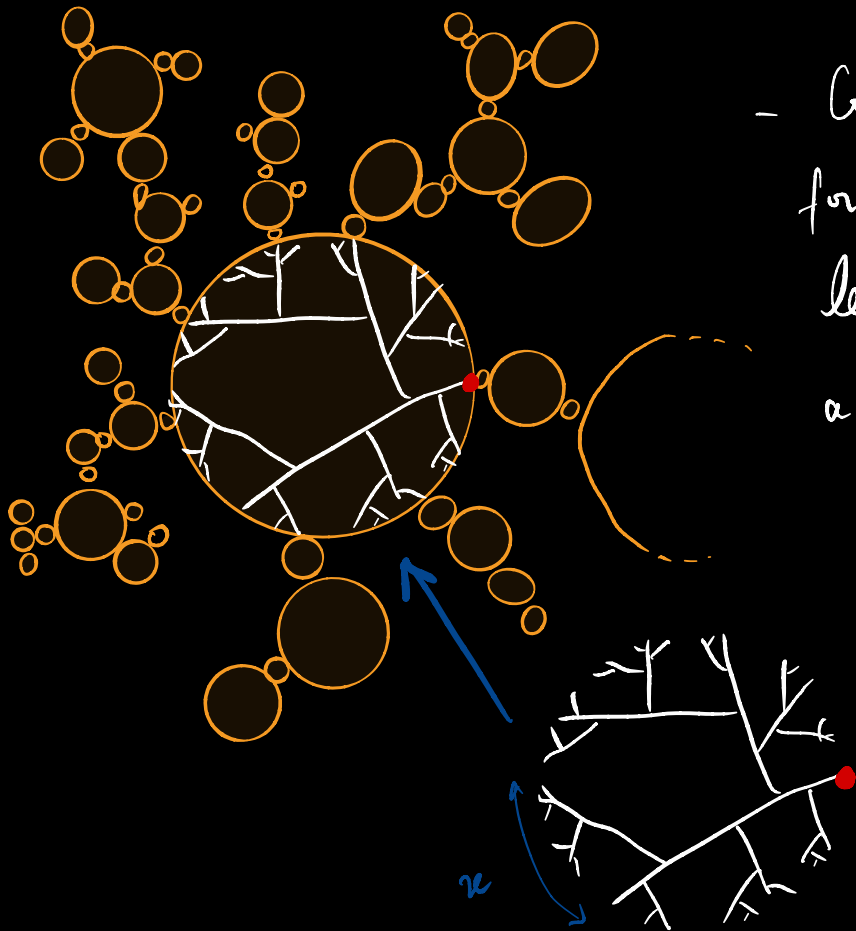
# THE BIG PICTURE (2020+)

- Sample a  $\frac{3}{2}$ -stable  
looptree -

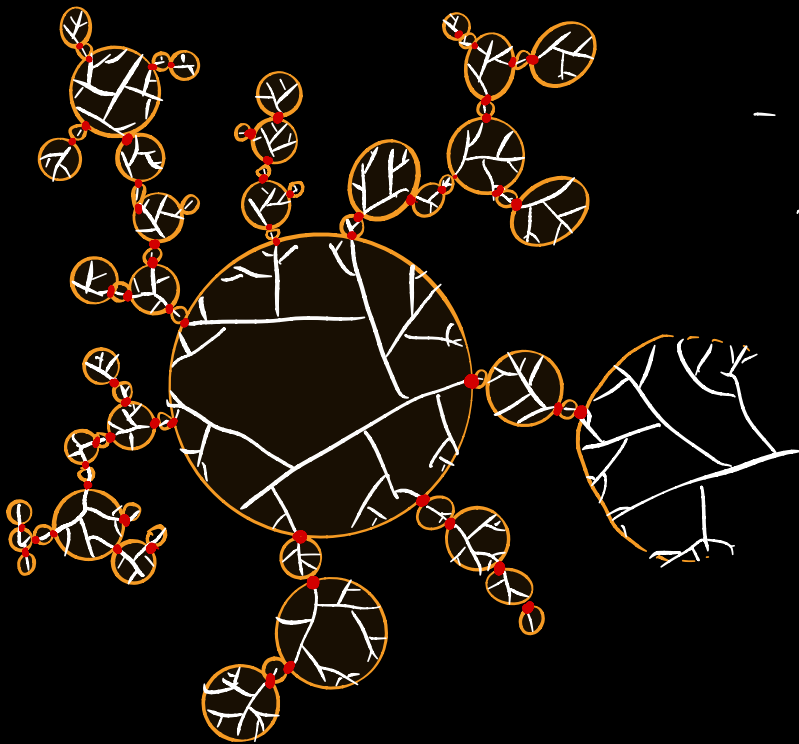


# THE BIG PICTURE (2020+)

- Sample a  $\frac{3}{2}$ -stable looptree -
- Conditionally on it, for each loop of length  $x$ , sample a  $GF_0^x$ -tree



# THE BIG PICTURE (2020+)



- Sample a  $\frac{3}{2}$ -stable looptree -
- Conditionally on it, for each loop of length  $x$ , sample a  $GF_0^x$ -tree
- Glue



*That's all Folks!*



THANKS!





LONG TERM GOAL: WHAT WOULD BE A SCALING LIMIT.

Ideal Theorem: Fix a "nice" model (tree + distrib of cars) -

Then there is a critical density of cars  $\rho_c \in (0,1)$   
and we have under  $\mathbb{P}_\rho^{\rho_c}$

$$\left( n^{-1/2} \cdot T_\rho; n^{-?} \cdot (\phi_e)_{e \in T} \right) \quad \leftarrow \begin{array}{l} \text{flux of cars} \\ \text{along edges.} \end{array}$$

$$\xrightarrow[n \rightarrow +\infty]{(d)} \left( c_1 \cdot \mathcal{G}; c_2 \cdot (Z_u)_{u \in \mathcal{G}} \right)$$

Brownian CRT

"Flux of" cars.

"universality". + more geom. properties.



# Questions:

- Dynamical version?
- Noise sensitivity?
- Near critical regime?
- Other universality classes?
- Continuum "cutoff" on CRT?
- Applications to random maps?

