Elliptic dimer models and genus 1 Harnack curves

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work in progress with

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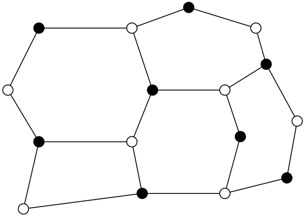
Outline

- Dimer model
- Dimer model and Harnack curves
- Minimal isoradial immersions
- Elliptic dimer model
- Results

Dimer model: definition

- Planar, bipartite graph $\mathsf{G} = (\mathsf{V} = \mathsf{B} \cup \mathsf{W}, \mathsf{E})$.



- Dimer configuration M: subset of edges s.t. each vertex is incident to exactly one edge of $\mathsf{M} \rightsquigarrow \mathcal{M}(\mathsf{G})$.
- Positive weight function on edges: $\nu = (\nu_{\mathsf{e}})_{\mathsf{e} \in \mathsf{E}}$.
- Dimer Boltzmann measure (G finite):

$$\forall\, \mathsf{M} \in \mathcal{M}(\mathsf{G}), \quad \mathbb{P}_{\mathrm{dimer}}(\mathsf{M}) = \frac{\prod_{\mathsf{e} \in \mathsf{M}} \nu_{\mathsf{e}}}{Z_{\mathrm{dimer}}(\mathsf{G}, \nu)}.$$

where $Z_{\mathrm{dimer}}(\mathsf{G}, \nu)$ is the dimer partition function.

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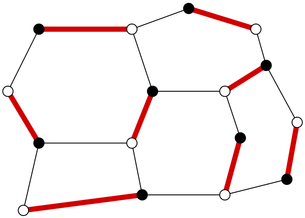

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Dimer model: Kasteleyn matrix

- Kasteleyn matrix (Percus-Kuperberg version)
  - Edge $wb \rightsquigarrow$ angle ϕ_{wb} s.t. for every face $w_1, b_1, \ldots, w_k, b_k$:

$$\sum_{j=1}^{k} (\phi_{w_j b_j} - \phi_{w_{j+1} b_j}) \equiv (k-1)\pi \text{ mod } 2\pi.$$

  - K is the corresponding twisted adjacency matrix.

$$\mathsf{K}_{w,b} = \begin{cases} \nu_{wb} e^{i\phi_{wb}} & \text{if } w \sim b \\ 0 & \text{otherwise.} \end{cases}$$

Dimer model: founding results

- Assume G finite.

Theorem ([Kasteleyn'61] [Kuperberg'98])

$$Z_{\text{dimer}}(\mathsf{G},\nu) = |\det(\mathsf{K})|.$$

Theorem (Kenyon'97)

Let $\mathcal{E} = \{\mathsf{e}_1 = w_1 b_1, \dots, \mathsf{e}_n = w_n b_n\}$ be a subset of edges of G, then:

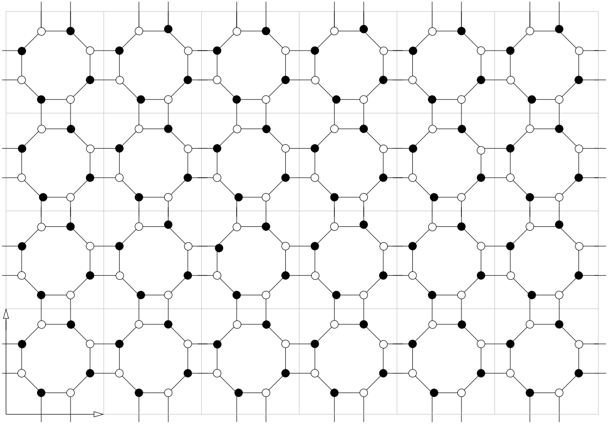
$$\mathbb{P}_{\text{dimer}}(\mathsf{e}_1,\dots,\mathsf{e}_n) = \left|\left(\prod_{j=1}^{n} \mathsf{K}_{w_j,b_j}\right)\det(\mathsf{K}^{-1})_{\mathcal{E}}\right|,$$

where $(\mathsf{K}^{-1})_{\mathcal{E}}$ is the sub-matrix of K^{-1} whose rows/columns are indexed by black/white vertices of $\mathcal{E}$.

- G infinite: Boltzmann measure $\rightsquigarrow$ Gibbs measure
  - Periodic case [Cohn-Kenyon-Propp'01], [Ke.-Ok.-Sh.'06]
  - Non-periodic [dT'07], [Boutillier-dT'10], [B-dT-Raschel'19]

Dimer model: periodic case

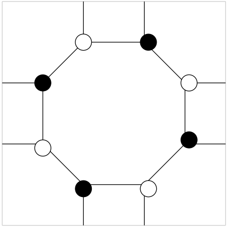
- Assume G is bipartite, infinite, $\mathbb{Z}^2$-periodic.



- Exhaustion of G by toroidal graphs: $(\mathsf{G}_n) = (\mathsf{G}/n\mathbb{Z}^2)$.

Dimer model: periodic case

- Fundamental domain: G_1



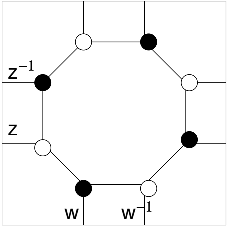
- Let K_1 be the Kasteleyn matrix of fundamental domain G_1.
- Multiply edge-weights by $\mathsf{z}, \mathsf{z}^{-1}, \mathsf{w}, \mathsf{w}^{-1} \to \mathsf{K}_1(\mathsf{z},\mathsf{w})$.
- The characteristic polynomial is:

$$P(\mathsf{z},\mathsf{w}) = \det \mathsf{K}_1(\mathsf{z},\mathsf{w}).$$

Example: weight function $\nu \equiv 1$, $P(\mathsf{z},\mathsf{w}) = 5 - \mathsf{z} - \frac{1}{\mathsf{z}} - \mathsf{w} - \frac{1}{\mathsf{w}}$.

Dimer model: periodic case

- Fundamental domain: G_1



- Let K_1 be the Kasteleyn matrix of fundamental domain G_1.
- Multiply edge-weights by $z, z^{-1}, w, w^{-1} \to K_1(z, w)$.
- The characteristic polynomial is:

$$P(z, w) = \det K_1(z, w).$$

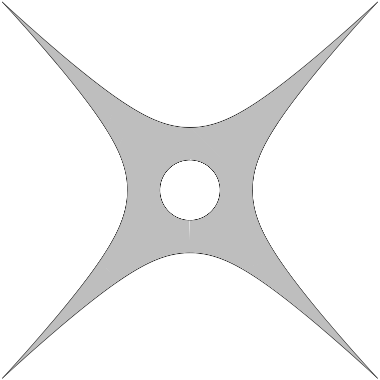
Example: weight function $\nu \equiv 1$, $P(z, w) = 5 - z - \frac{1}{z} - w - \frac{1}{w}$.

Dimer model: spectral curve

- The spectral curve:

$$\mathcal{C} = \{(\mathsf{z}, \mathsf{w}) \in (\mathbb{C}^*)^2 : \; P(\mathsf{z}, \mathsf{w}) = 0\}.$$

- Amoeba: image of $\mathcal{C}$ through the map $(\mathsf{z}, \mathsf{w}) \mapsto (\log|\mathsf{z}|, \log|\mathsf{w}|)$.



Amoeba of the square-octagon graph

Dimer model and Harnack curves

Theorems

- *Spectral curves of bipartite dimers* $\overset{\text{[Ke.-Ok.-Sh.'06] [Ke.-Ok.'06]}}{\longleftrightarrow}$ *Harnack curves with points on ovals.*
- *Spectral curves of isoradial, bipartite dimer models with critical weights [Kenyon '02]* $\overset{\text{[Kenyon-Okounkov'06]}}{\longleftrightarrow}$ *Harnack curves of genus 0.*
- *Spectral curves of minimal, bipartite dimers* $\overset{\text{[Goncharov-Kenyon '13]}}{\longleftrightarrow}$ *Harnack curves with points on ovals.*

  Explicit ($\longrightarrow$) map
- *[Fock'15] Explicit ($\longleftarrow$) map for all algebraic curves. (no check on positivity).*

Dimer model and Harnack curves of genus 1

Theorem ([Boutillier-dT-Cimasoni'20+])

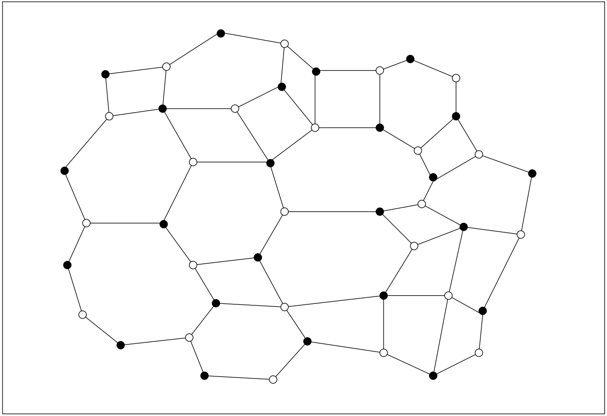
Spectral curves of minimal, bipartite dimer models with Fock's weights

$\longleftrightarrow$

Harnack curves of genus 1 with a point on the oval.

Quad-graph, train-tracks

- Infinite, planar, embedded graph G; embedded dual graph G^*.
- Corresponding quad-graph $\mathsf{G}^\diamond$, train-tracks.



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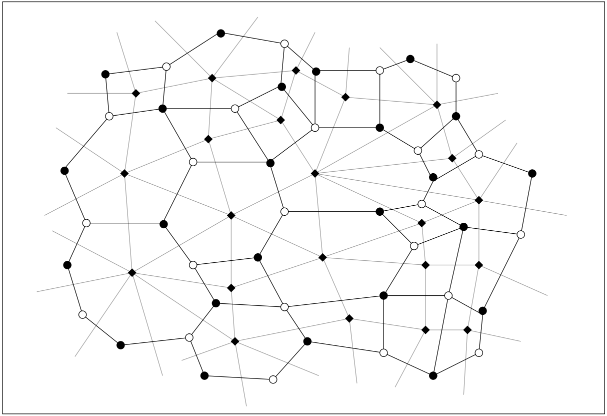

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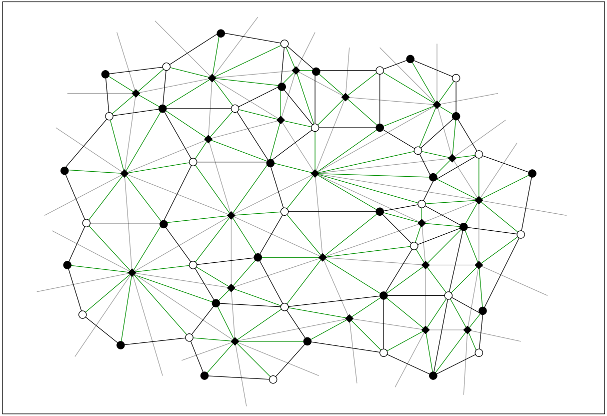

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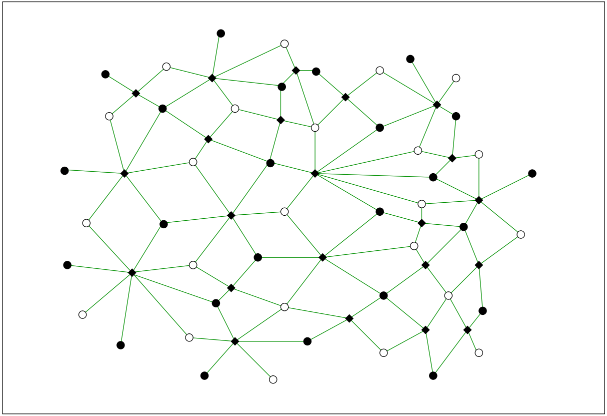

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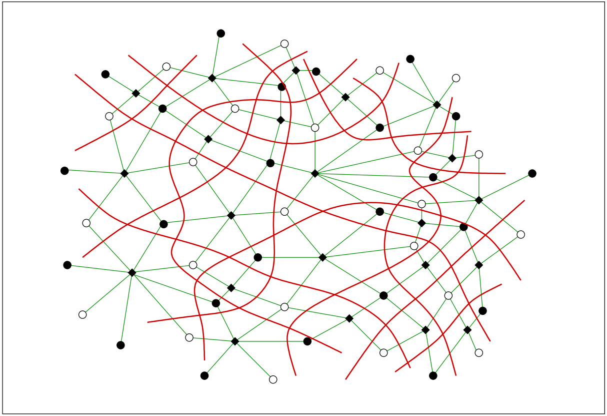

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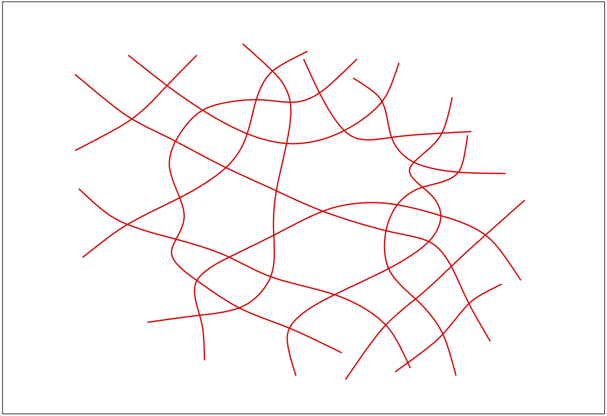
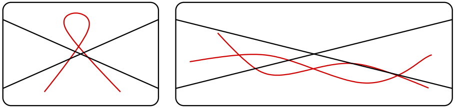


Isoradial graphs

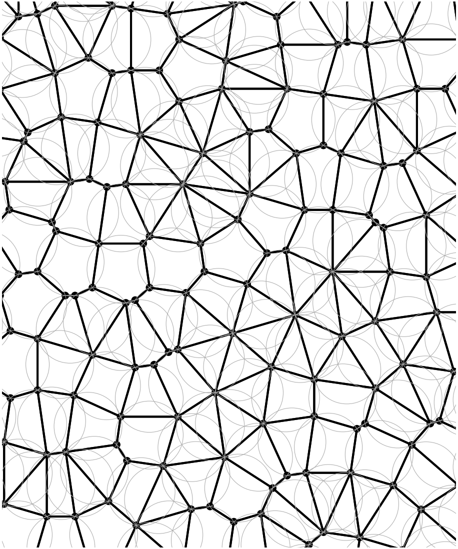
- An isoradial embedding of an infinite, planar graph G is an embedding such that all of its faces are inscribed in a circle of radius 1, and such that the center of the circles are in the interior of the faces [Duffin] [Mercat] [Kenyon].
- Equivalent to: the quad-graph $\mathsf{G}^\diamond$ is embedded so that of all its faces are rhombi.

Theorem (Kenyon-Schlencker'04)

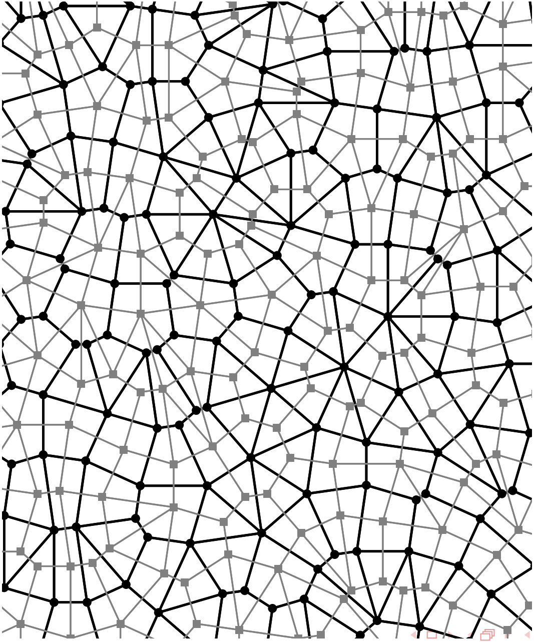
An infinite planar graph G *has an isoradial embedding iff*



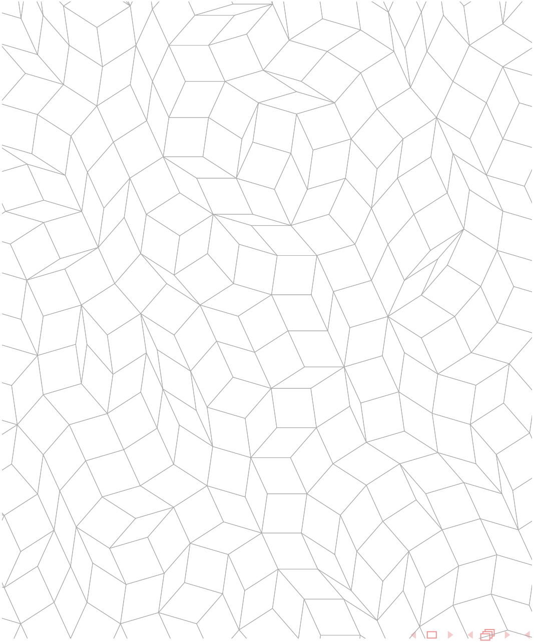
Isoradial embeddings



Isoradial embeddings

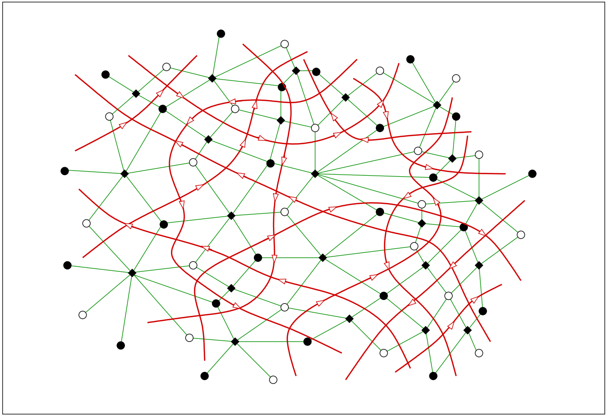


Isoradial embeddings



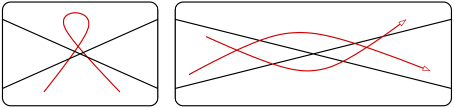
Minimal graphs

- If the graph G is bipartite, train-tracks are naturally oriented (white vertex of the left, black on the right).



Minimal graphs

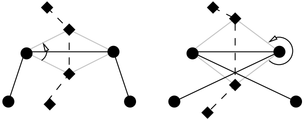
- If the graph G is bipartite, train-tracks are naturally oriented (white vertex of the left, black on the right).
- A bipartite, planar graph G is minimal if



[Thurston'04] [Gulotta'08] [Ishii-Ueda'11] [Goncharov-Kenyon'13]

Immersions of minimal graphs

- A minimal isoradial immersion of an infinite planar graph G is an immersion of the quadgraph $\mathsf{G}^\diamond$ such that:
  - all of the faces are rhombi (flat or reversed)



  - the immersion is flat: the sum of the rhombus angles around every vertex and every face is equal to 2π.

Proposition (Boutillier-dT-Cimasoni'19)
The flatness condition is equivalent to :

- *around every vertex there is at most one reversed rhombus*
- *around every face, the cyclic order of the vertices differ by at most disjoint transpositions in the embedding and in the immersion.*

Theorem (Boutillier-dT-Cimasoni'19)
An infinite, planar, bipartite graph G has a minimal isoradial immersion iff it is minimal.

Dimer version of Fock's weights

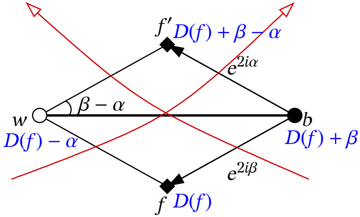
- **Tool 1.** Jacobi's (first) theta function.
  - Parameter $q = e^{i\pi\tau}$, $\Im(\tau) > 0$, $\Lambda(q) = \pi\mathbb{Z} + \pi\tau\mathbb{Z}$, $\mathbb{T}(q) = \mathbb{C}/\Lambda$.

$$\theta(z) = 2q^{\frac{1}{4}} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)} \sin(2n+1)z.$$

  - Allows to represent $\Lambda(q)$-periodic meromorphic functions.
  - $\theta(z) \sim 2q^{\frac{1}{4}} \sin(z)$ as $q \to 0$.
- **Tool 2.** Isoradially immersed, bipartite, minimal graph G.
  - each train-track T is assigned direction $e^{i2\alpha_T}$.
  - each edge $e = wb$ is assigned train-track directions $e^{2i\alpha}, e^{2i\beta}$, and a half-angle $\beta - \alpha \in [0, \pi)$.

Dimer version of Fock's adjacency matrix

- Tool 3. Discrete Abel map [Fock] $D \in (\mathbb{R}/\pi\mathbb{Z})^{\mathsf{V}(\mathsf{G}^\diamond)}$
  - Fix face f_0 and set $D(f_0) = 0$,
  - ○: degree -1, ●: degree 1, faces: degree 0,
  - when crossing T: increase/decrease D by α_T accordingly.



- Point $t \in \frac{\pi}{2}\tau + \mathbb{R}$.
- Fock's adjacency matrix

$$\mathsf{K}^{(t)}_{w,b} = \begin{cases} \dfrac{\theta(\beta-\alpha)}{\theta(t+D(b)-\beta)\theta(t+D(w)-\alpha)} & \text{if } w \sim b \\ 0 & \text{otherwise.} \end{cases}$$

Dimer version of Fock's adjacency matrix

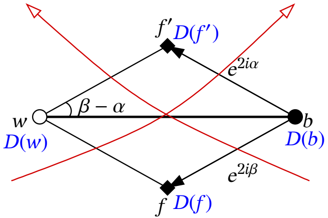
Lemma ([Boutillier-dT-Cimasoni'20+])

Under the above assumptions, the matrix $\mathsf{K}^{(t)}$ *is a Kasteleyn matrix for a dimer model (positive weights) on* G.

FUNCTIONS IN THE KERNEL OF $\mathsf{K}^{(t)}$

- Define $g^{(t)} : \mathsf{V}^\diamond \times \mathsf{V}^\diamond \times \mathbb{C} \to \mathbb{C}$
  - $g^{(t)}_{x,x}(u) = 1$,
  - If $f \sim w$, $g^{(t)}_{f,w}(u) = g^{(t)}_{w,f}(u)^{-1} = \dfrac{\theta(u + t + D(w))}{\theta(u - \alpha)}$,
  - If $f \sim b$, $g^{(t)}_{b,f}(u) = g^{(t)}_{f,b}(u)^{-1} = \dfrac{\theta(u - t - D(b))}{\theta(u - \alpha)}$,

    where $e^{2i\alpha}$ is the direction of the tt crossing the edge.
  - If distance ≥ 2, take product along path in $\mathsf{G}^\diamond$.



Property of the function $g^{(t)}$

Lemma ([Fock'15] [Boutillier-dT-Cimasoni'20+])

- *The function $g^{(t)}$ is well defined.*
- *The function $g^{(t)}$ is in the kernel of $\mathsf{K}^{(t)}$:*

$$\forall w \in \mathsf{W}, x \in \mathsf{V}^{\diamond}, \qquad \sum_{b: b \sim w} \mathsf{K}^{(t)}_{w,b}\, g^{(t)}_{b,x}(u) = 0.$$

Proof.

Weierstrass identity: $s, t \in \mathbb{T}(q),\ \ a, b, c \in \mathbb{C}$,

$$\frac{\theta(b-a)}{\theta(s-a)\theta(s-b)}\frac{\theta(u+s-a-b)}{\theta(u-a)\theta(u-b)} + \frac{\theta(c-b)}{\theta(s-b)\theta(s-c)}\frac{\theta(u+s-b-c)}{\theta(u-b)\theta(u-c)} + \frac{\theta(a-c)}{\theta(s-c)\theta(s-a)}\frac{\theta(u+s-c-a)}{\theta(u-c)\theta(u-a)} = 0.$$

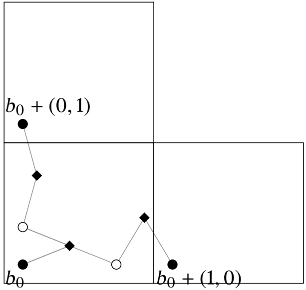
$\square$

Explicit parameterization of the spectral curve

- Assume G is $\mathbb{Z}^2$-periodic. Define the map ψ,

$$\begin{aligned}\psi : \mathbb{T}(q) &\to \mathbb{C}^2\\ u &\mapsto \psi(u) = (\mathsf{z}(u), \mathsf{w}(u))\end{aligned}$$

where $\mathsf{z}(u) = g^{(t)}_{b_0, b_0+(1,0)}(u)$, $\mathsf{w}(u) = g^{(t)}_{b_0, b_0+(0,1)}(u)$.



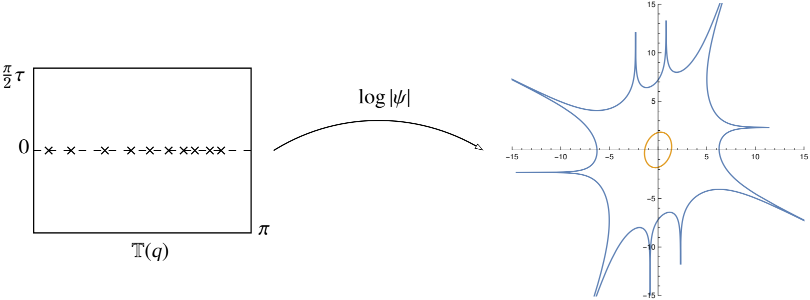
Explicit parameterization of the spectral curve

Proposition ([B-dT-C'20+])

The map ψ is an explicit birational parameterization of the spectral curve $\mathcal{C}$ of the dimer model with Kasteleyn matrix $\mathsf{K}^{(t)}$.

The real locus of $\mathcal{C}$ is the image under ψ of the set $\mathbb{R}/\pi\mathbb{Z} \times \{0, \frac{\pi}{2}\tau\}$, where the connected component with ordinate $\frac{\pi}{2}\tau$ is bounded and the other is not.

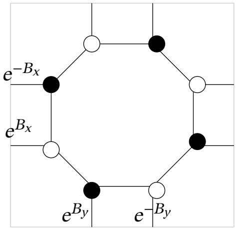
(The spectral curve is independent of t).

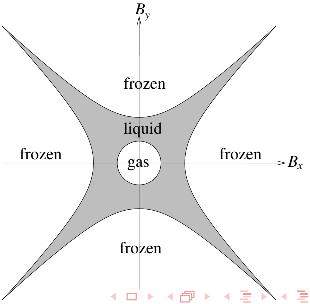


Gibbs measures for bipartite dimer models

Theorems (Kenyon-Okounkov-Sheffield'06)

- *The dimer model on a $\mathbb{Z}^2$-periodic, bipartite graph has a two-parameter family of ergodic Gibbs measures indexed by the slope (h, v), i.e., by the average horizontal/vertical height change.*
- *The latter are obtained as weak limits of Boltzmann measures with magnetic field coordinates (B_x, B_y).*
- *The phase diagram is given by the amoeba of the spectral curve $\mathcal{C}$.*





Local expression for Gibbs measures, genus 1

Suppose t fixed. Omit it from the notation.

Theorem (Boutillier-dT-Cimasoni'20+)

The 2-parameter set of EGM of the dimer model with Kasteleyn matrix K *is* $(\mathbb{P}^{u_0})_{u_0\in D}$, *where* $\forall$ *subset of edges* $\mathcal{E} = \{e_1 = w_1 b_1, \dots, e_n = w_n b_n\}$,

$$\mathbb{P}^{u_0}(e_1,\dots,e_n) = \Big(\prod_{j=1}^{n} \mathsf{K}_{w_j,b_j}\Big)\det(\mathrm{A}^{u_0})_{\mathcal{E}},$$

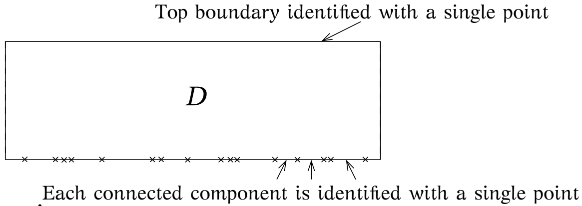
where $\forall b \in \mathsf{B},\ w \in \mathsf{W}, \quad \mathrm{A}^{u_0}_{b,w} = \frac{i\theta'(0)}{2\pi}\int_{\mathsf{C}^{u_0}_{b,w}} g_{b,w}(u)du.$

Moreover, when u_0

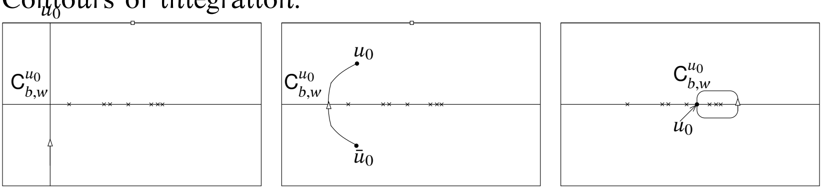
- *is the unique point corresponding to the top boundary of* D*, the dimer model is gaseous,*
- *is in the interior of* D*, the dimer model is liquid,*
- *is a point corresponding to a cc of the lower boundary, the model is solid.*

Local expressions for ergodic Gibbs measures, genus 1

- Domain D.



- Contours of integration.



Corollary

The slope of the Gibbs measure $\mathbb{P}^{u_0}$ is:

$$h^{u_0} = \frac{1}{2\pi i}\int_{\mathsf{C}^{u_0}} \frac{d}{du}(\log \mathsf{w}(u))du, \quad v^{u_0} = \frac{1}{2\pi i}\int_{\mathsf{C}^{u_0}} \frac{d}{du}(\log \mathsf{z}(u))du.$$

Idea of the proof

- Proof 1. Using [C-K-P], [K-O-S] the Gibbs measure $\mathbb{P}^B$ with magnetic field coordinates $B = (B_x, B_y)$ has the following expression on cylinder sets:

$$\mathbb{P}^{(B_x,B_y)}(e_1, \ldots, e_k) = \Big(\prod_{j=1}^{k} \mathsf{K}_{w_j,b_j}\Big) \det(\mathrm{A}^B)_{\mathcal{E}},$$

where

$$\mathrm{A}^B_{b+(m,n),w} = \int_{\mathbb{T}_B} \frac{Q(\mathsf{z},\mathsf{w})_{b,w}}{P(\mathsf{z},\mathsf{w})} \mathsf{z}^{-m}\mathsf{w}^{-n} \frac{d\mathsf{w}}{2i\pi\mathsf{w}} \frac{d\mathsf{z}}{2i\pi\mathsf{z}},$$

  - Perform one integral by residues.
  - Do the change of variable $u \mapsto \psi(u) = (\mathsf{z}(u), \mathsf{w}(u))$.
  - Non-trivial cancellation !

Idea of the proof

- Proof 2. Show that for every u_0, A^{u_0} is an inverse of K.
  - Use Weierstrass identity.
  - Show that the contours of integration are such that one has 1 on the diagonal.

  Use uniqueness statements of inverse operators.

Consequences

- Suitable for asymptotics.
- Explicit local expressions for edge probabilites.

Connection to previous work

- Genus 0. [Kenyon'02].
- Genus 1. Two specific cases were handled before:
  - the bipartite graph arising from the Ising model [Boutillier-dT-Raschel'20]
  - the $Z^{(t)}$-Dirac operator [dT'18] $\rightsquigarrow$ massive discrete holomorphic functions.

Perspectives

- 2-parameter family of Gibbs measures for non-periodic graphs.

  Missing: every finite, simply connected subgraph of an isoradial immersion can be embedded in a bipartite, $\mathbb{Z}^2$-periodic isoradial immersion.
- Extension to genus $g > 1$.
  - [Fock] gives a candidate for the dimer model.
  - Weierstrass identity $\rightsquigarrow$ Fay's trisecant identity.