Imaginary chaos

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Les Diablerets, February 10th 2020

joint work with

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Outline

1. Introduction
2. Moments
3. XOR-Ising model
4. Regularity, densities and monofractality

Introduction

Log-correlated Gaussian fields

Let us fix a bounded simply connected domain $D \subset \mathbb{R}^d$.

Heuristic definition

A Gaussian field $X : D \to \mathbb{R}$ is called log-correlated if

$$\mathbb{E}X(x)X(y) = C(x, y) := \log \frac{1}{|x - y|} + g(x, y)$$

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Caveat

Such fields cannot be defined pointwise and must instead be understood as distributions (generalized functions). This means that for all $\varphi, \psi \in C_c^\infty(\mathbb{R}^d)$ we have

$$\mathbb{E}X(\varphi)X(\psi) = \int \varphi(x)\psi(y)C(x, y)\, dx\, dy.$$

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  - $g \in L^1(D \times D) \cap C(D \times D)$
  - g is bounded from above
- These properties are enough to ensure that X exists.

  (Assuming that the kernel C is positive definite.)

Example: The 2D GFF

Definition

The 0-boundary GFF Γ in the domain D is a Gaussian field with the covariance

$$\mathbb{E}\Gamma(x)\Gamma(y) = G_D(x, y)$$

where G_D is the Green's function of the Dirichlet Laplacian in D.

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- **universality:**
  - appears in the scaling limit of various height function models, random matrices, QFT, ...
  - a recent characterisation: the only random field with conformally invariant law and domain Markov property (+some moment condition) [BPR19]

Example: The GFF in the unit square

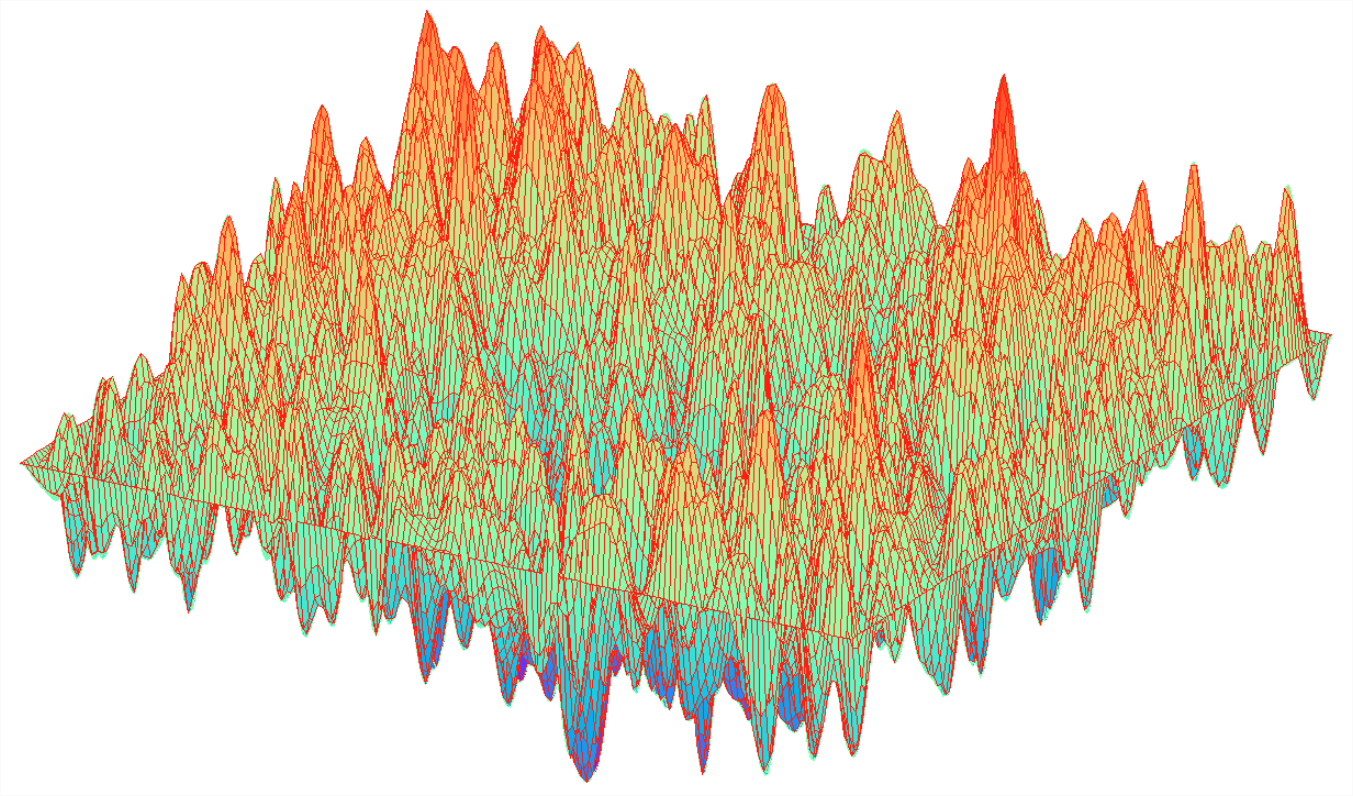

Figure 1: An approximation of the GFF in the unit square.

Gaussian Multiplicative Chaos

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Theorem/Definition ([Kah85; RV10; Sha16; Ber17])

For any given $\gamma \in (0, \sqrt{2d})$ the functions $\mu_n(x) := e^{\gamma X_n(x) - \frac{\gamma^2}{2}\mathbb{E}X_n(x)^2}$ converge to a random measure μ. We say that $\mu = \mu^\gamma$ is a GMC measure associated to X.

Existence of GMC when $\gamma \in (0, \sqrt{d})$

A simple L^2-computation

For any $f \in C_c^\infty(D)$ we have

$$\begin{aligned}
\mathbb{E}|\mu_n(f)|^2 &= \int_{D^2} f(x)f(y)\mathbb{E}e^{\gamma X_n(x)+\gamma X_n(y)-\frac{\gamma^2}{2}\mathbb{E}X_n(x)^2-\frac{\gamma^2}{2}\mathbb{E}X_n(y)^2}\,dx\,dy \\
&= \int_{D^2} f(x)f(y)e^{\gamma^2\mathbb{E}X_n(x)X_n(y)}\,dx\,dy \\
&\lesssim \|f\|_\infty^2 \int_{D^2} e^{\gamma^2 \log\frac{1}{|x-y|}}\,dx\,dy = \int_{D^2} \frac{dx\,dy}{|x-y|^{\gamma^2}} < \infty.
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- Otherwise one can do a similar computation to show that the sequence is Cauchy in $L^2(\Omega)$.

Complex values of γ

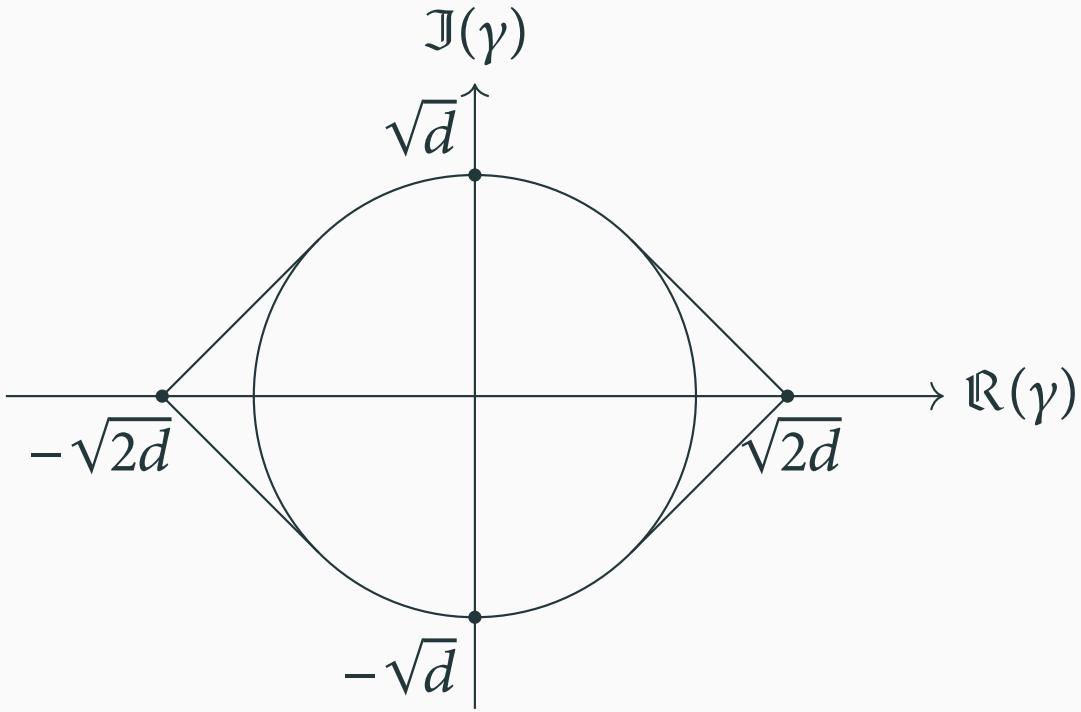

Figure 2: The subcritical regime A for γ in the complex plane.

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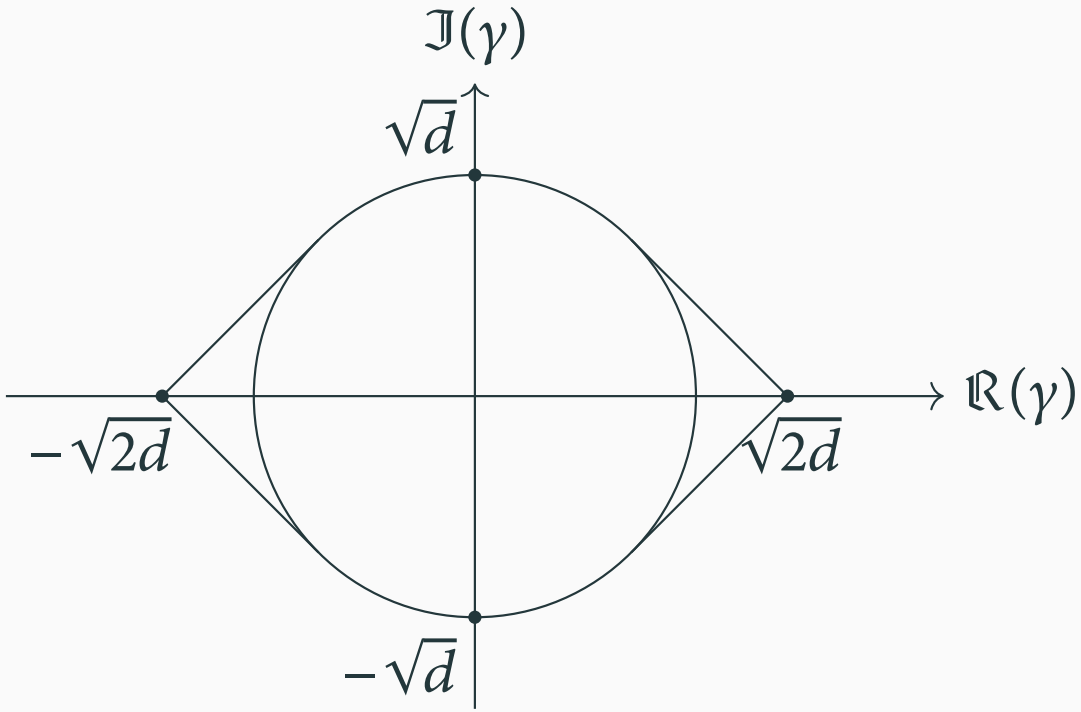

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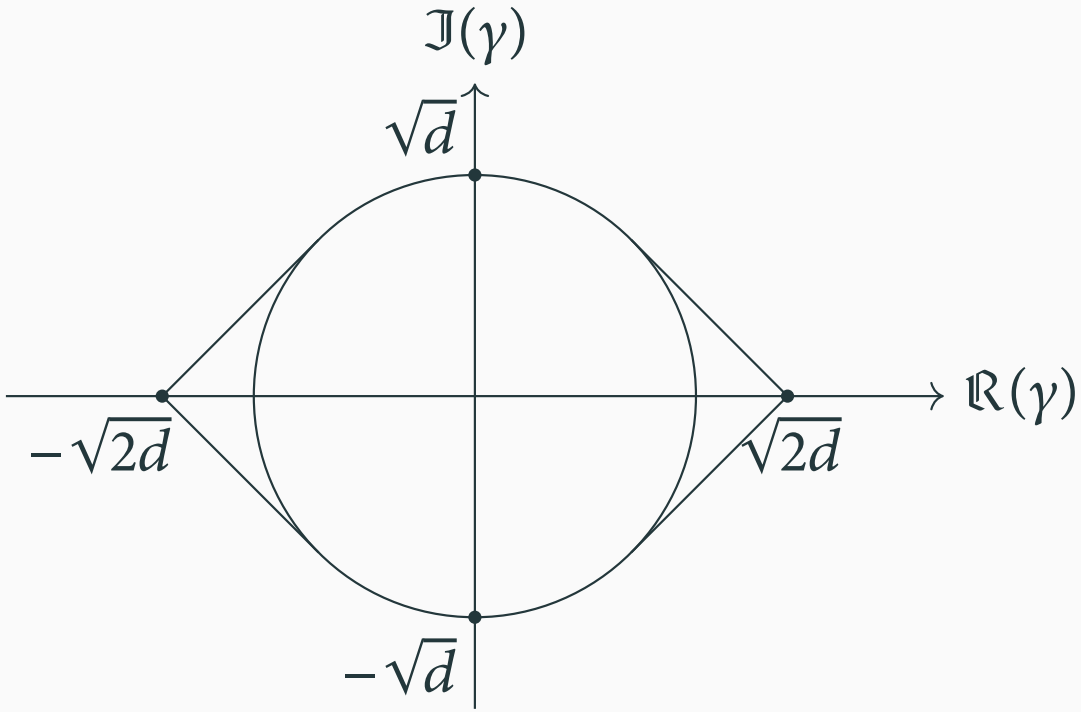

Figure 2: The subcritical regime A for γ in the complex plane.

- In fact, $\gamma \mapsto \mu^{\gamma}(f)$ is an analytic function on A [JSW19].
- The circle corresponds to the L^2-phase – in particular it contains the whole subcritical part of the imaginary axis.

Imaginary multiplicative chaos

Theorem/Definition ([JSW18; LRV15])

Let $\beta \in (0, \sqrt{d})$. Then the random functions

$$\mu_n(x) := e^{i\beta X_n(x) + \frac{\beta^2}{2}\mathbb{E}X_n(x)^2}$$

converge in probability in $H^{-d/2-\varepsilon}(\mathbb{R}^d)$ to a random distribution μ.

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- **Applications:** XOR-Ising model [JSW18], two-valued sets of the GFF [SSV19] and certain random fields constructed using the Brownian loop soup [CGPR19].

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There exists $C > 0$ such that for any $f \in C_c^\infty(D)$ and $N \geq 1$ we have

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Theorem ([JSW18])

Let $f \in C_c^\infty(D)$ be non-negative and non-zero, then there exists $C > 0$ such that

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Bounding moments – the naive way

- $\mathbb{E}|\mu(1)|^{2N}$ is (formally) given by

$$\int_{D^{2N}} \mathbb{E} \prod_{j=1}^{N} e^{i\beta X(x_j)+\frac{\beta^2}{2}\mathbb{E}X(x_j)^2} e^{-i\beta X(y_j)+\frac{\beta^2}{2}\mathbb{E}X(y_j)^2} dx_j dy_j =$$

$$\int_{D^{2N}} e^{\beta^2 \sum_{1\le j,k\le N} C(x_j,y_k) - \beta^2 \sum_{1\le j<k\le N}(C(x_j,x_k)+C(y_j,y_k))} dx_1 \dots dx_N dy_1 \dots dy_N,$$

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  where $C(x, y)$ is the covariance kernel of X.
- In the case $C(x, y) = \log \frac{1}{|x-y|}$ this is simply the partition function of Coulomb gas with N positive and N negative charges. Estimating this was done in [GP77] by using an electrostatic inequality due to Onsager [Ons39].

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- In the general case $C(x, y) = \log \frac{1}{|x-y|} + g(x, y)$ with g bounded one could simply estimate each $C(x, y)$ in the sums by $\log \frac{1}{|x-y|} \pm \|g\|_\infty$, but this would incur an error of order $O(N^2)$.

General Onsager inequalities

Theorem ([JSW18; JSW19])

Assume that either of the following conditions hold:

- $g \in H^{d+\varepsilon}_{loc}(D \times D)$ for some ε, or
- $d = 2$ and $g \in C^2(D \times D)$,

Then around any $z \in D$ there exists a neighbourhood $U \subset D$ and $C > 0$ such that for any $z_1, \dots, z_N \in U$ and $q_1, \dots, q_N \in \{-1, 1\}$ we have

$$-\sum_{1 \le j < k \le N} q_j q_k C(z_j, z_k) \le \frac{1}{2} \sum_{j=1}^{N} \log \frac{1}{\frac{1}{2} \min_{j \neq k} |z_j - z_k|} + CN.$$

The rest of the argument

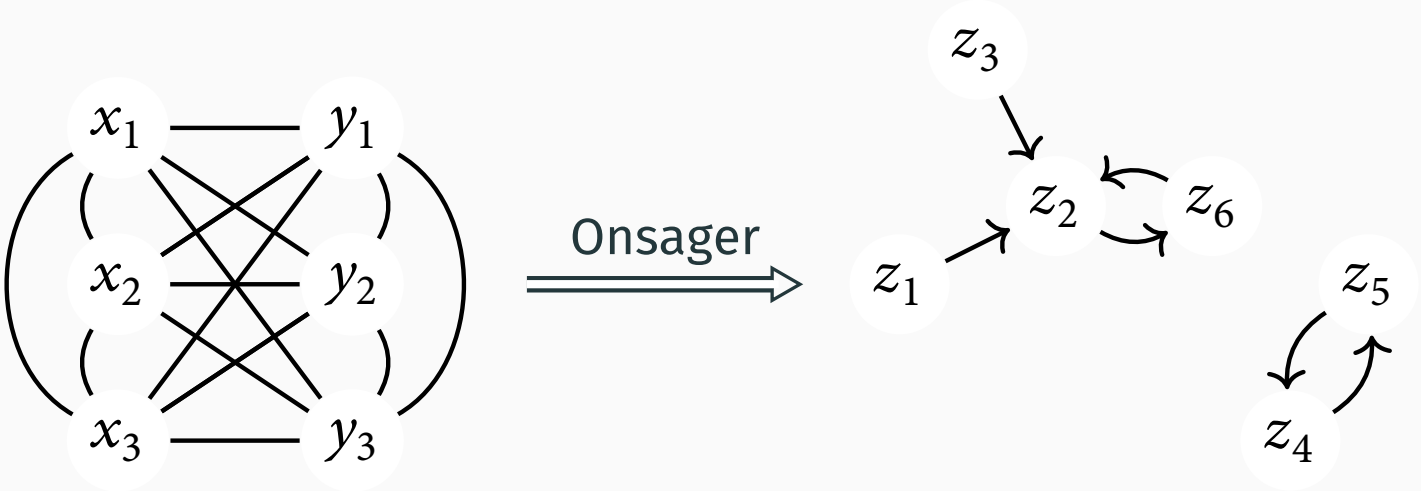

Figure 3: Dependencies between the variables in the integral.

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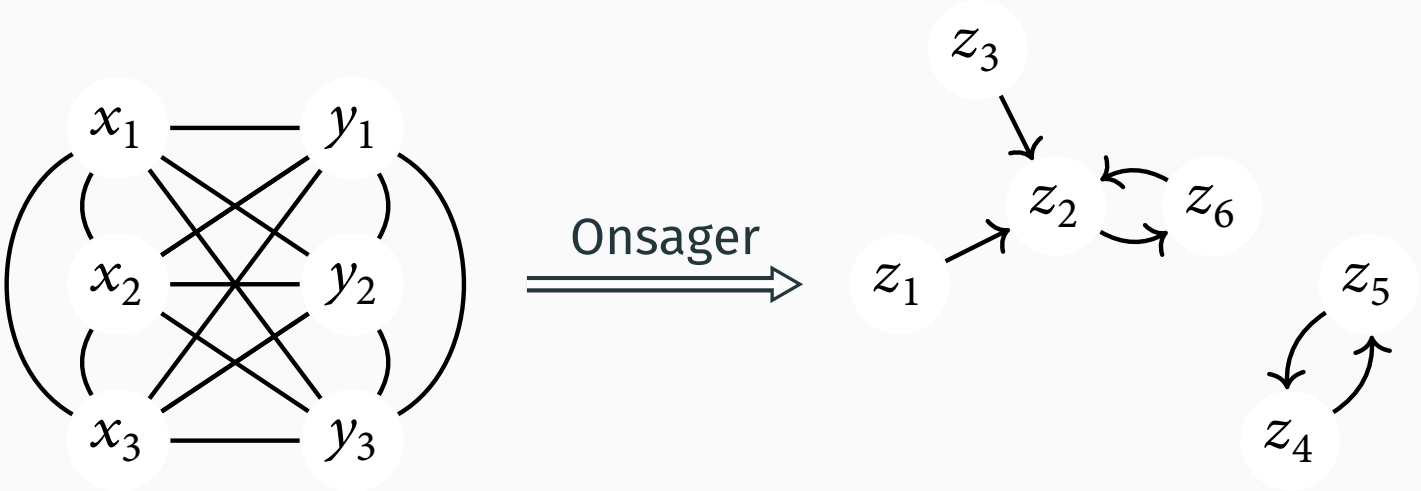

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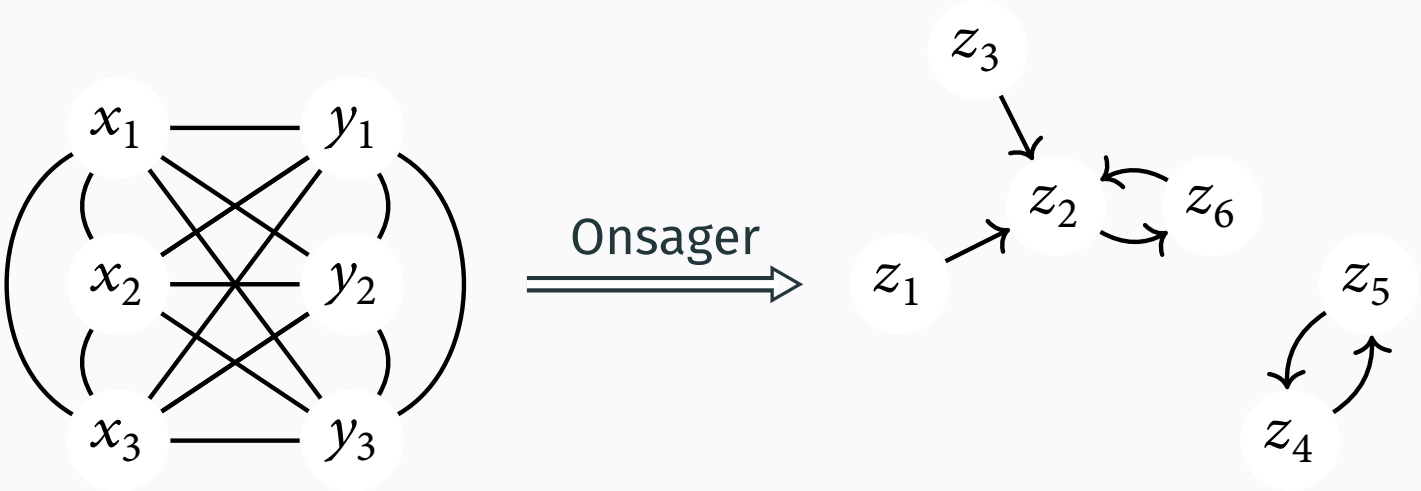

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- After applying Onsager the dependencies between the variables can be reduced to a set of 2-cycles with attached trees.
- The upper bound is now obtained by computing a uniform bound over all the graphs with a given number of components (integrate variables one by one starting from the leaves) and multiplying by the number of such graphs.

Proof of the Onsager inequality for nice fields

- Let A_j be centered Gaussians. From $\mathbb{E}\Big(\sum_{j=1}^N q_j A_j\Big)^2 \geq 0$ we get by expanding and rearranging the inequality

$$-\sum_{1\leq j<k\leq N}^{N} q_j q_k \mathbb{E} A_j A_k \leq \frac{1}{2}\sum_{j=1}^{N} \mathbb{E} A_j^2.$$

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- Assume that X has an approximation X_r with the following properties:
  - $X_r(x)$ is a martingale as $r \to 0$
  - $\mathbb{E} X_r(x)^2 \approx \log \frac{1}{r}$
  - $(X_u(x) - X_r(x)) \perp (X_v(y) - X_s(y))$ for all $u < r$ and $v < s$ if $r + s < |x - y|$

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- By choosing $A_j = X_{r_j}(z_j)$, where $r_j = \frac{1}{2}\min_{k\neq j} |z_j - z_k|$, we see that $\mathbb{E} A_j A_k = C(z_j, z_k)$ and the claim follows.

Generalizing to other fields

Theorem ([JSW19])

Assume that in the covariance $C(x, y) = \log \frac{1}{|x-y|} + g(x, y)$ the function g lies in $H^{d+\varepsilon}_{loc}(D \times D)$. Then around any point $x_0 \in D$ there exists a neighbourhood in which X can be decomposed as a sum of independent fields, $X = L + R$, where L is a nice log-correlated field (in particular it has the properties in the previous slide) and R is a regular field with Hölder continuous realisations.

XOR-Ising model

Ising model

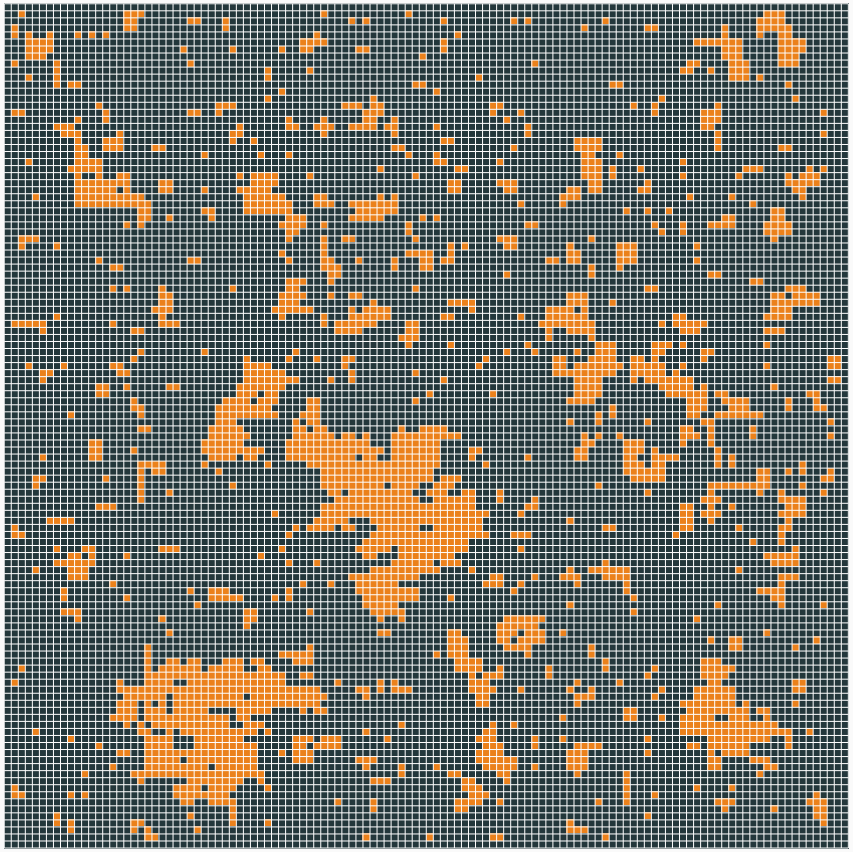

Figure 4: Critical Ising model

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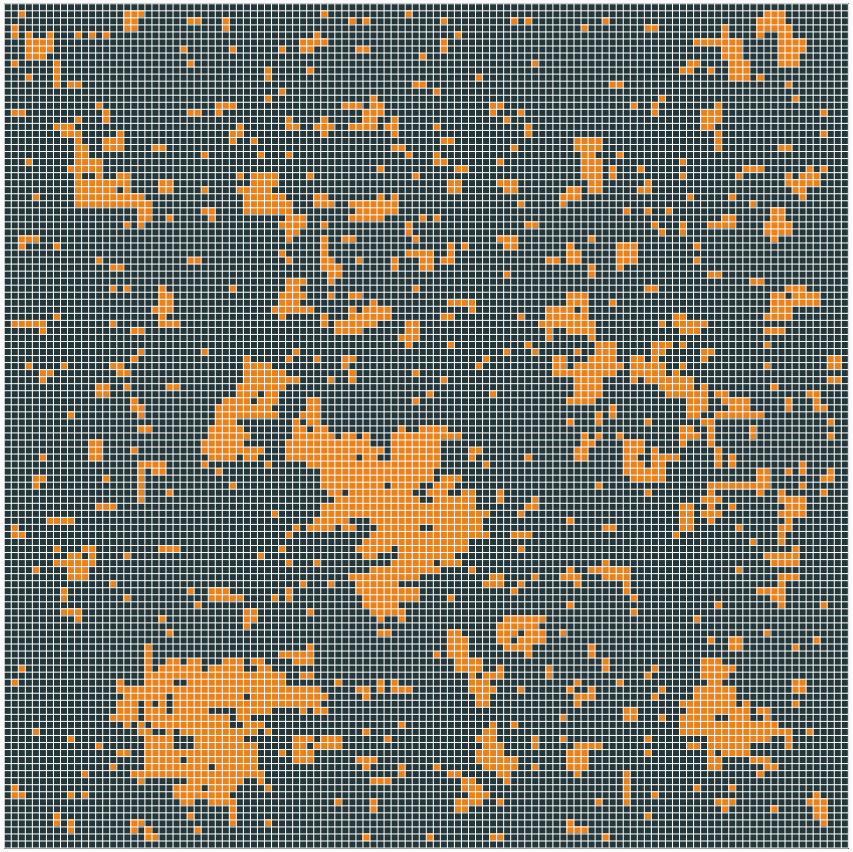

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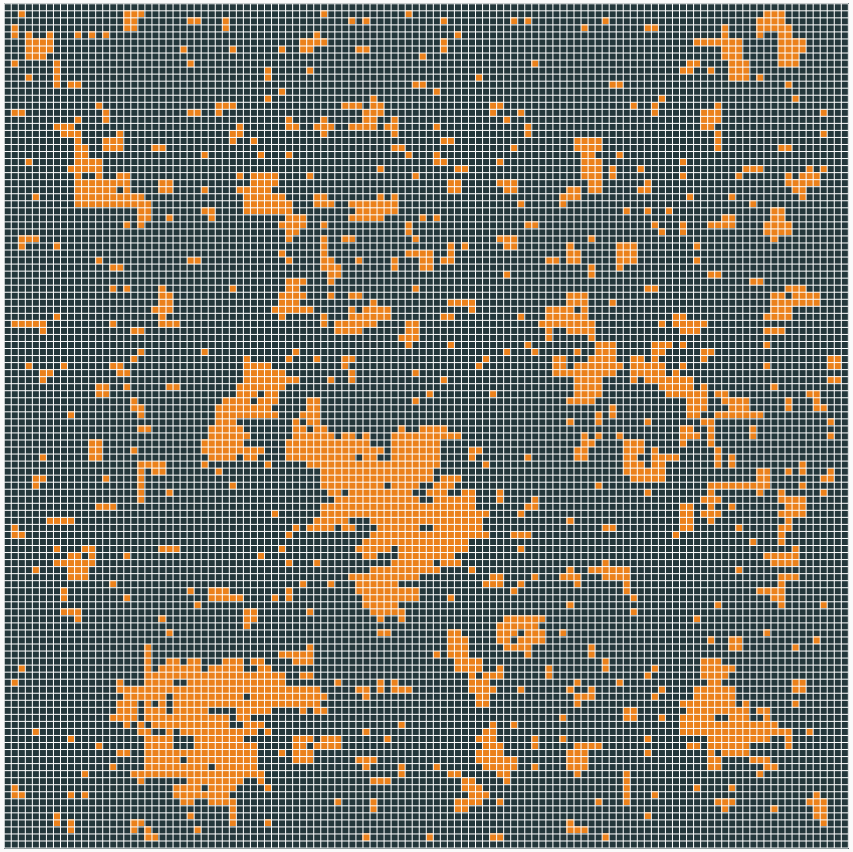

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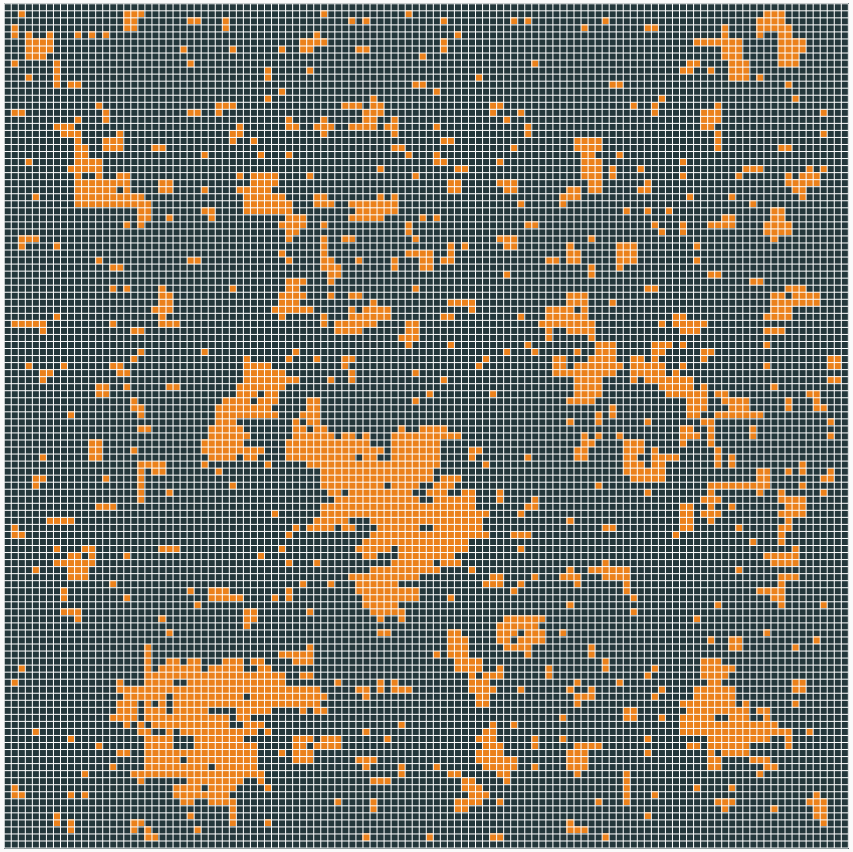

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- phase transition at $\beta = \beta_c = \log(1 + \sqrt{2})/2$.
- We denote $\sigma_\delta(x) = \sigma(f)$ for $x \in f$ and a given lattice length $\delta > 0$.

XOR-Ising model

- The XOR-Ising spin field is defined by $S_\delta(x) := \sigma_\delta(x)\tau_\delta(x)$, where σ and τ are two independent Ising spin fields.

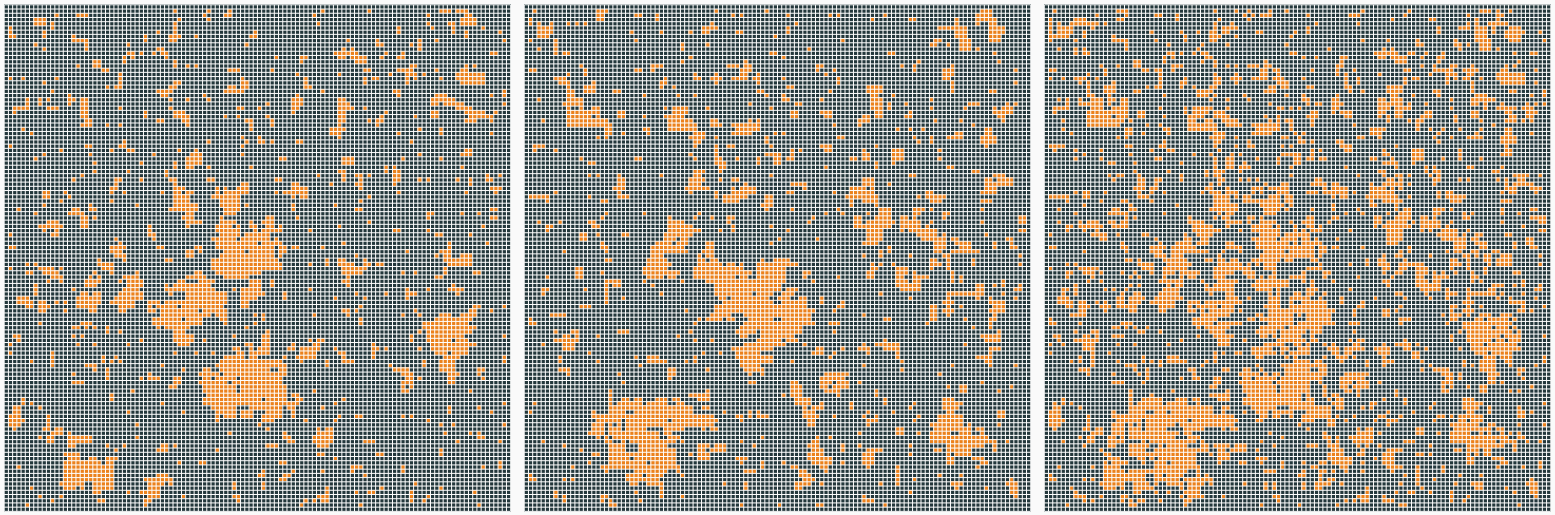

Figure 5: Ising, Ising, XOR-Ising

XOR-Ising and the real part of imaginary chaos

Theorem ([JSW18])

For any $f \in C_c^\infty(D)$ we have

$$\delta^{-1/4} \int_D f(x) S_\delta(x)\, dx \to C^2 \int_D f(x) \left(\frac{2|\varphi'(x)|}{\mathfrak{I}\varphi(x)} \right)^{1/4} \cos(2^{-1/2}\Gamma(x))\, dx$$

where $\cos(2^{-1/2}\Gamma(x))$ denotes the real part of the imaginary chaos distribution μ with parameter $\beta = 1/\sqrt{2}$ and $\varphi\colon D \to \mathbb{H}$ is a conformal bijection.

On the proof

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- A direct computation shows that the moments match formally.
- To justify dominated convergence, we prove an Onsager-type inequality for the Ising model:

$$\delta^{-n/8} \mathbb{E}\sigma_\delta(x_1) \dots \sigma_\delta(x_n) \le C^n \prod_{j=1}^{n} (\min_{k \ne j} |x_j - x_k|)^{-1/8}$$

Regularity, densities and monofractality

Besov spaces

The spaces $B^s_{p,q}(\mathbb{R}^d)$

- Banach spaces of distributions parametrised by smoothness parameter $s \in \mathbb{R}$ and two size parameters $p, q \in [1, \infty]$.
- Contain both Sobolev and Hölder spaces:
  - $B^s_{2,2}(\mathbb{R}^d) = H^s(\mathbb{R}^d)$ $(s \in \mathbb{R})$
  - $B^s_{\infty,\infty}(\mathbb{R}^d) = C^s(\mathbb{R}^d)$ $(s \in (0, \infty) \setminus \mathbb{N})$.

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- We say that $f \in B^s_{p,q,loc}(D)$ if and only if $\psi f \in B^s_{p,q}(\mathbb{R}^d)$ for all $\psi \in C^\infty_c(D)$.

Regularity of μ

Theorem ([JSW18])

We have for all $p, q \in [1, \infty]$ that

- $s < -\frac{\beta^2}{2} \Rightarrow \mu \in B^s_{p,q,loc}(D)$
- $s > -\frac{\beta^2}{2} \Rightarrow \mu \notin B^s_{p,q,loc}(D)$
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- One can get finiteness of Besov norms by computing moments.
- To show that μ is not a complex measure it suffices to show that $\mu(e^{-i\beta X_\delta}\psi) \to \infty$ as $\delta \to 0$ for some $\psi \in C_c^\infty(D)$.

Smooth and bounded densities

Theorem ([ABJJ20])

Assume that X is a GFF in some bounded domain D and let $f \in L^\infty(D)$ be a non-zero function. Then the random variable $\mu(f)$ has a smooth and bounded density in $\mathbb{C}$.

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- A rough first idea towards a proof: Look at $\int e^{i\beta \sum_{n=1}^\infty A_n \varphi_n(x) + \frac{\beta^2}{2} \sum_{n=1}^\infty \varphi_n(x)^2} \, dx$ and try to show that if one conditions for instance on A_1, A_2, then with a high probability the continuous map $(A_1, A_2) \mapsto \mu(f)$ sweeps a reasonable area in the complex plane for $|A_1|, |A_2| \leq 1$, say.

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- Central difficulty with this approach: How to rule out the rest of the chaos $e^{i\beta \sum_{n=3}^\infty A_n \varphi_n(x) + \frac{\beta^2}{2} \sum_{n=3}^\infty \varphi_n(x)^2}$ being close to 0?

Smooth and bounded densities

- In the case of real chaos on say the unit interval $[0,1]$ one heuristically has something like

$$\mathbb{P}[\mu([0,1]) \leq \varepsilon] \leq \mathbb{P}[\mu([0,\frac{1}{2}]) \leq \varepsilon, \mu([\frac{1}{2},1]) \leq \varepsilon] \approx \mathbb{P}[\mu([0,1]) \leq 2\varepsilon]^2.$$

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  Reasoning along these lines can indeed be made precise and yields the existence of all negative moments for $\mu([0,1])$.
- The crucial property here was non-negativity, which of course fails for imaginary chaos.
- In the end our proof goes through Malliavin calculus.

Monofractality

Theorem ([ABJJ20])

Almost surely for all $z \in D$ we have

$$\liminf_{r \to 0} \frac{\log |\mu(Q(z,r))|}{\log r} = 2 - \beta^2/2.$$

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- We refine this in two different ways:
  - A law of iterated logarithm -type result: For fixed x we have

$$\limsup_{r \to 0} \frac{|\mu(Q(x,r))|}{r^{2-\beta^2/2}(\log|\log r|)^{\beta^2/4}} = c_1(\beta)$$

  - Existence of exceptional (fast) points:

$$\sup_{x \in D} \limsup_{r \to 0} \frac{|\mu(Q(x,r))|}{r^{2-\beta^2/2}|\log r|^{\beta^2/4}} = c_2(\beta)$$

Thanks!

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