Uniqueness of the limiting profile for monotone Lipschitz random surfaces

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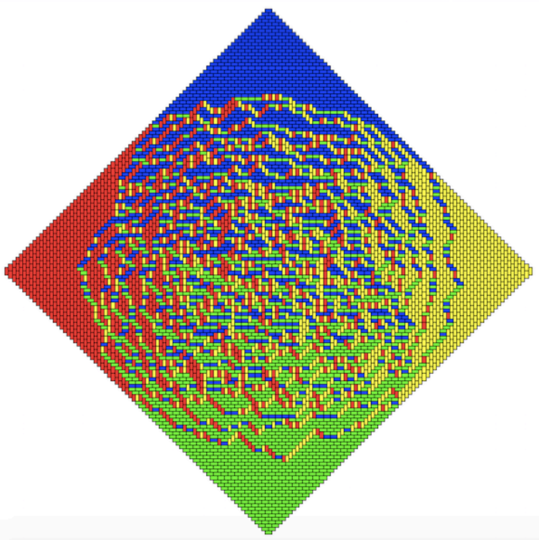
14 February 2020

The variational principle for domino tilings



Uniformly random domino tiling of an aztec diamond

The variational principle for domino tilings

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Theorem (Cohn, Kenyon, Propp, 2000)

Consider R an open region, ξ a continuous function on ∂R.

Assume: $\frac{1}{n}\Lambda_n \to R$ *and* $\frac{1}{n}\,\mathrm{Graph}(\omega_n|_{\partial\Lambda_n}) \to \mathrm{Graph}(\xi)$,

Then: $\frac{1}{n^2} Z_{\Lambda_n}^{\omega_n} \to \inf_{f:\ f|_{\partial R}=\xi} \int_R \sigma(\nabla f)\, d\lambda$,

where $\sigma(s)$ encodes the specific free energy of a gradient Gibbs measure of slope s.

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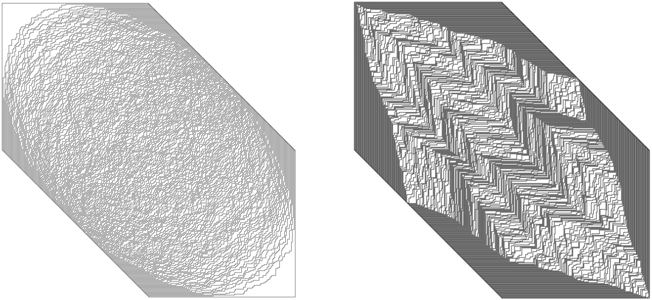
1. Uniqueness of the asymptotic profile,
2. The LDP with speed n^2 and rate function

$$I(f) := \int_R \sigma(\nabla f)d\lambda$$

   has a unique minimizer.

What if σ is not strictly convex?

May have “chaos” when the surface tension σ has affine regions:



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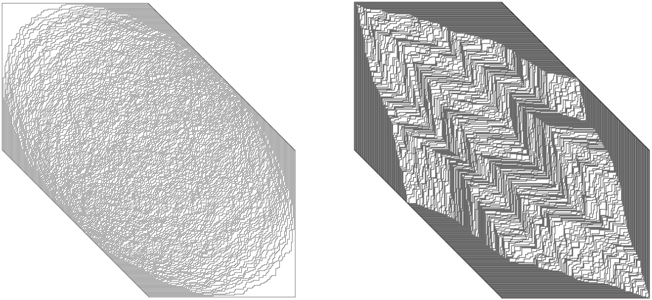

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1. Understand natural conditions that imply strict convexity,

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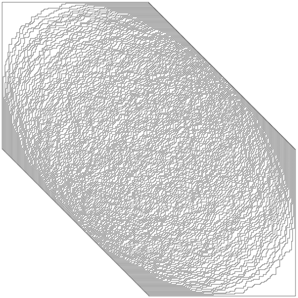

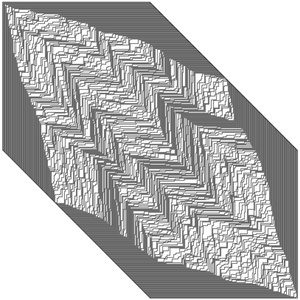

Goals:

1. Understand natural conditions that imply strict convexity,
2. Prove strict convexity for a large class of models.

What is known

Definition (Notation)

- Sample space $\Omega = E^{\mathbb{Z}^d}$, where $E \in \{\mathbb{Z}, \mathbb{R}\}$,

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- ϕ is generated by a potential Φ.

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It contains:

1. Dimer models,
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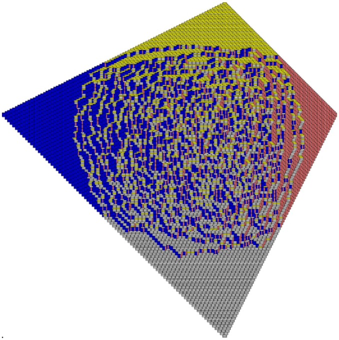
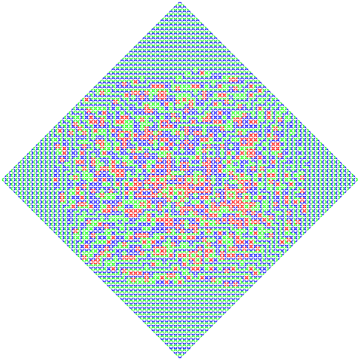
3. Uniformly random k-Lipschitz functions,
4. Discrete Gaussian free field.

Can we go further?

Limit shapes appear for models with non-local interactions.



tiling by 3×1 bars



tree-valued function

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- Get around cluster swapping techniques which use edge energy and thus only works for two-points interactions,
- Understand what convexity of the potential is really bringing to the table.

The answer to these problems is **stochastic monotonicity**.

Definition of stochastic monotonicity

Definition

A specification ϕ is **stochastically monotone** if

$$\phi_\Lambda^{\omega_1} \preccurlyeq \phi_\Lambda^{\omega_2} \qquad \text{whenever} \qquad \omega_1 \leq \omega_2,$$

where ω_1, ω_2 are height functions, $\Lambda \subset\subset \mathbb{Z}^d$.

Definition of Lipschitz

Definition

A specification ϕ is **Lipschitz** if there is a $K < \infty$ such that ϕ_Λ^ω is supported on K-Lipschitz functions for ω K-Lipschitz.

Statement of the result

Theorem (L, Tassy)

Let ϕ denote a shift-invariant gradient specification which is:

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2. *Lipschitz,*
3. *generated by some potential $\Phi \in \mathcal{C}$.*

Write U for the interior of $\{\sigma < \infty\} \subset (\mathbb{R}^d)^*$: the set of allowable slopes. The surface tension $\sigma : U \to \mathbb{R}$ is strictly convex if:

1. $E = \mathbb{R}$
2. $E = \mathbb{Z}$ *and σ is affine on ∂U, but not U.*

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$$|H_{\Lambda_n} - H^0_{\Lambda_n}| = o(n^d)$$

as $n \to \infty$ where $\Lambda_n = \{0, \ldots, n-1\}^d \subset \mathbb{Z}^d$.

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$*$ By which we mean: 1-Lipschitz w.r.t. some quasimetric q, in order to be as general as possible.

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- Is it possible to reformulate the variational principle in terms of specifications only?
- Can we find models non-monotone models for which there exists a $c < \infty$ such that $\omega_1 \leq \omega_2$ implies

$$\phi_\Lambda^{\omega_1} \preceq \phi_\Lambda^{\omega_2+c}?$$

  If it is the case the Moat Lemma still works and σ is strictly convex.