On delocalization in the six-vertex model

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Outline

1. The *six-vertex model* and its height function on a finite torus
2. Two results:
    1. *Existence* and *ergodicity* of the infinite volume limit for $c \in [\sqrt{3}, 2]$
    2. *Delocalization* of the height function for $c \in (\sqrt{2+\sqrt{2}}, 2]$
3. Ingredients of proofs:
    - Baxter–Kelland–Wu '73 correspondence with the *critical random cluster model* with $q \in [1, 4]$
    - *continuity of phase transition* in the random cluster model with $q \in [1, 4]$ (Duminil–Copin & Sidoravicius & Tassion '15)
    - *spin representation* of the six-vertex model (Rys '63)
    - *FK-type representation* of the spin model (Glazman & Peled '18, Ray & Spinka '19, L. '19)

The six-vertex model

A *six-vertex* (or *arrow*) *configuration* on a 4-regular graph is an assignment of an arrow to each edge which yields a *conservative flow*, i.e., such that there are two incoming and two outgoing arrows at every vertex

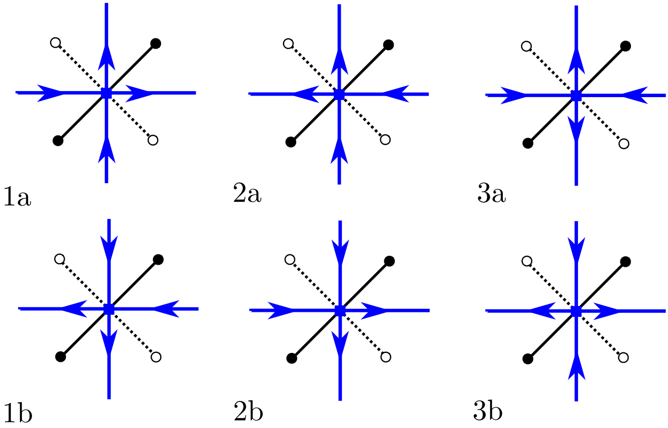
For $\mathbf{n} = (n_1, n_2) \in \mathbb{N}^2$, let $\mathbb{T}_\mathbf{n} = (\mathbb{Z}/2n_1\mathbb{Z}) \times (\mathbb{Z}/2n_2\mathbb{Z})$, and let $\mathcal{O}_\mathbf{n}$ and $\mathcal{O}$ be the set of arrow configurations on $\mathbb{T}_\mathbf{n}$ and $\mathbb{Z}^2$ respectively

We consider the *six-vertex model* (or more precisely the *F-model*) on $\mathbb{T}_\mathbf{n}$ with parameter $c > 0$. This is a probability measure on $\mathcal{O}_\mathbf{n}$ given by

$$\mu_\mathbf{n}(\alpha) \propto c^{N(\alpha)}, \qquad \alpha \in \mathcal{O}_\mathbf{n},$$

where $N(\alpha)$ is the number of vertices of type $3a$ or $3b$ in α

The six-vertex model



Existence of the infinite-volume limit

Theorem 1. (Dumnil-Copin et al. '20, L. '20)

For $c \in [\sqrt{3}, 2]$,

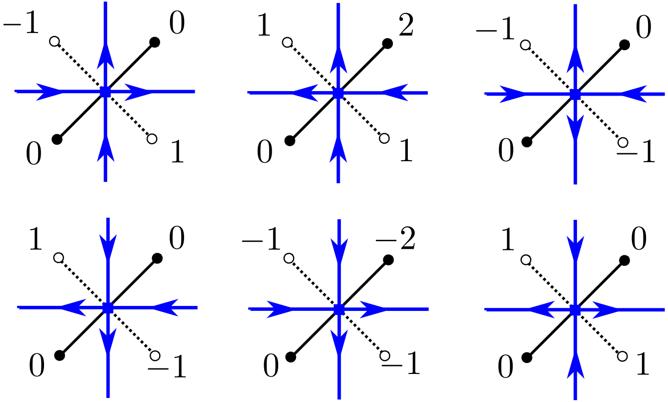
(i) there exists a translation invariant probability measure μ on $\mathcal{O}$ such that

$$\mu_{\mathbf{n}} \to \mu \quad \text{as} \quad |\mathbf{n}| \to \infty$$

(ii) the limiting measure μ is ergodic with respect to translations by the even sublattice of $\mathbb{Z}^2$

$c \in [\sqrt{3}, 2]$ corresponds to $q \in [1, 4]$ in the BKW representation

The height function



$h(u) = \#_{\leftarrow} - \#_{\rightarrow}$ on a path from u_0 to u

Behaviour of the height function

Question

What is the behaviour of

$$\mathbf{Var}_\mu[h(u)] \quad \text{as} \quad |u| \to \infty$$

where u is a face of $\mathbb{Z}^2$?

- variance bounded $\leftrightarrow$ *localization*
- variance unbounded $\leftrightarrow$ *delocalizatoin*

So far

- localization was proved for $c > 2$ (Duminil-Copin et al. '16, Glazman & Peled '18)
- delocalization for $c = 2$ (Duminil-Copin & Sidoravicius & Tassion '15, Glazman & Peled '18), $c = \sqrt{2}$ (Kenyon '99) and its small neighbourhood (Giuliani & Mastropietro & Toninelli '14), and $c = 1$ (Chandgotia et al. '18, Duminil-Copin et al. '19)

Delocalization of the height function

Theorem 2. (Dumnil-Copin et al. ’20, L. ’20)

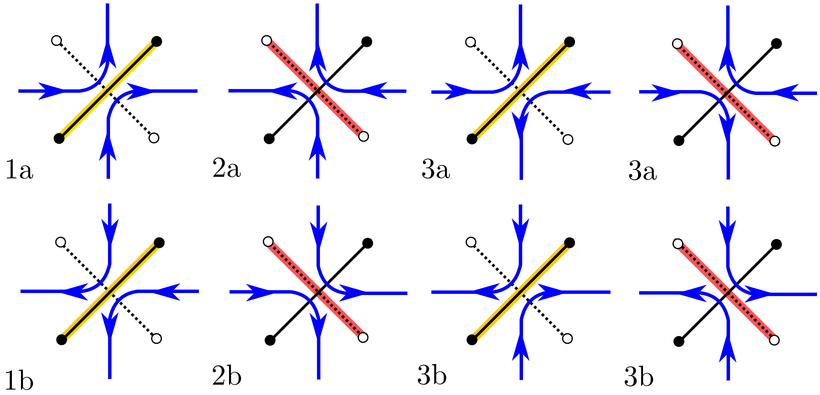
For $c \in (\sqrt{2+\sqrt{2}}, 2]$,

$$\mathbf{Var}_\mu[h(u)] \to \infty \quad \text{as} \quad |u| \to \infty$$

The result of Dumnil-Copin & Karrila & Manolescu & Oulamara uses different techniques, works for all $c \in [1,2]$ and gives logarithmic divergence of the variance

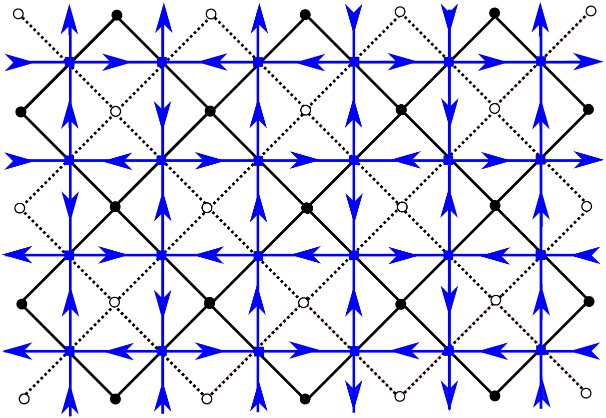
$c \in (\sqrt{2+\sqrt{2}}, 2]$ corresponds to $q \in (2,4]$ in the BKW representation

BKW representation



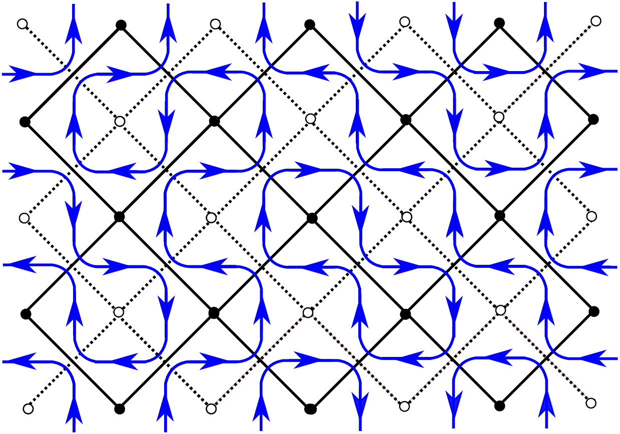
$$e^{\frac{i\lambda}{4}}e^{-\frac{i\lambda}{4}} = 1 \qquad e^{\frac{i\lambda}{2}} + e^{-\frac{i\lambda}{2}} = c$$

BKW representation



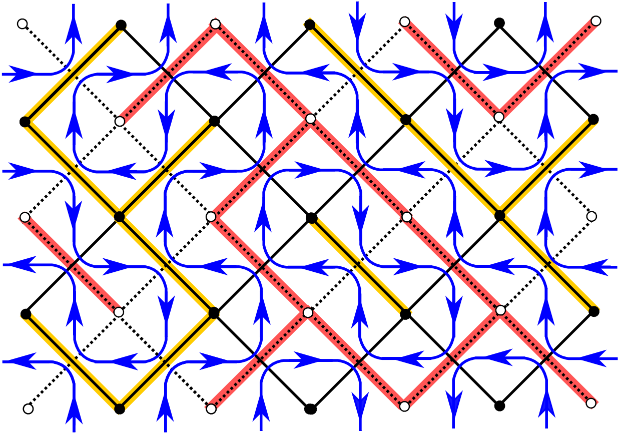
$$w(\alpha) = c^{N_3(\alpha)}$$

BKW representation



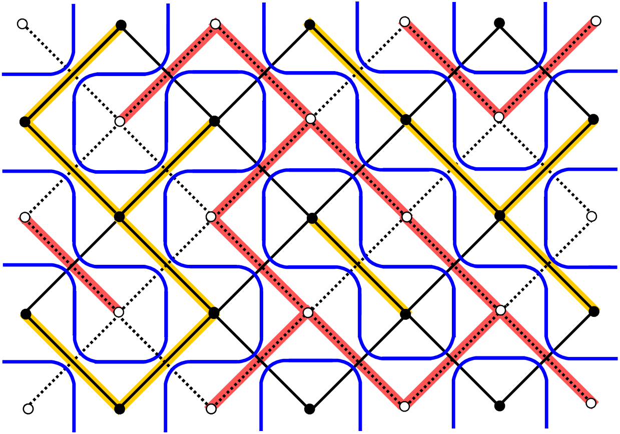
$$w(\vec{L}) = e^{\frac{i\lambda}{4}(\mathrm{left}(\vec{L}) - \mathrm{right}(\vec{L}))} = \prod_{\vec{\ell} \in \vec{L}} e^{\frac{i\lambda}{4}(\mathrm{left}(\vec{\ell}) - \mathrm{right}(\vec{\ell}))} = \prod_{\vec{\ell} \in \vec{L}} e^{i\lambda \mathrm{wind}(\vec{\ell})}$$

BKW representation



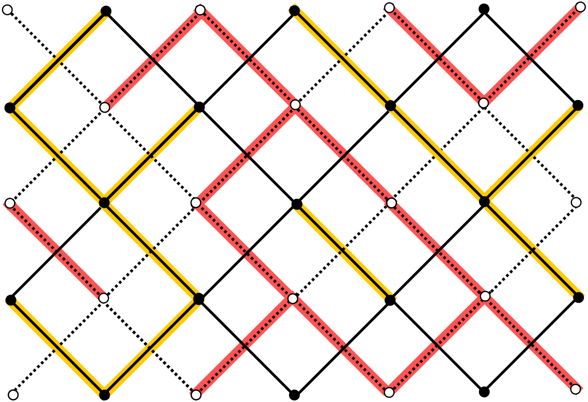
$$w(\vec{L}) = \prod_{\vec{\ell} \in \vec{L}} e^{i\lambda \text{wind}(\vec{\ell})}$$

BKW representation



$$\phi_{\mathbf{n}}(L) \propto \prod_{\ell \in L} \left(e^{i\lambda \mathrm{wind}(\vec{\ell})} + e^{-i\lambda \mathrm{wind}(\vec{\ell})}\right) \propto \sqrt{q}^{|L|} \left(\tfrac{2}{\sqrt{q}}\right)^{|L_{\mathrm{nctr}}|}$$

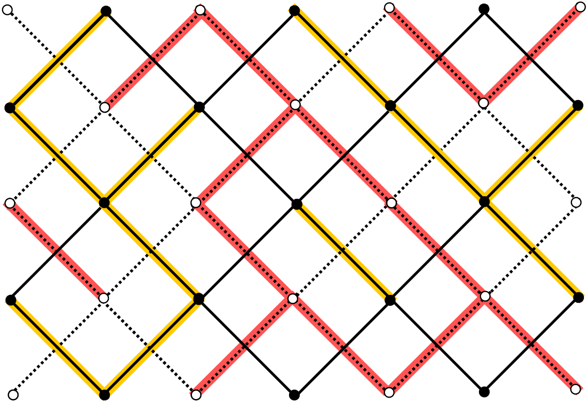
BKW representation



$$\phi_{\mathbf{n}}(\xi) \propto \sqrt{q}^{|L(\xi)|} \left(\tfrac{2}{\sqrt{q}}\right)^{|L_{\mathrm{nctr}}(\xi)|}$$

(“almost” FK(q) measure)

BKW representation



$\phi_{\mathbf{n}}^{\mathrm{FK}}(\xi) \propto \sqrt{q}^{|L(\xi)|} q^{s(\xi)}$ (FK(q) measure)

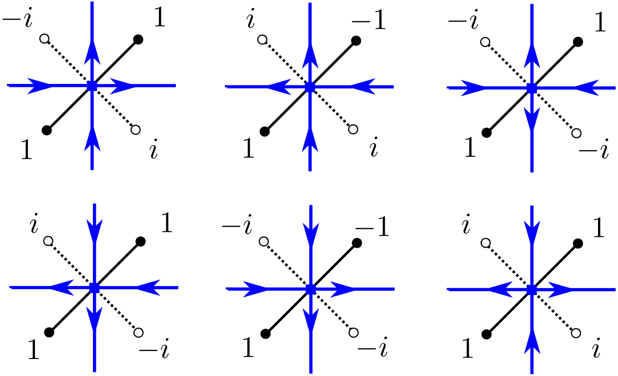
Convergence of $\phi_{\mathbf{n}}$ to ϕ

- All subsequential limits of $\phi_{\mathbf{n}}$ satisfy the DLR conditions of a critical random cluster measure
- By uniqueness of the critical random cluster measure ϕ, we get convergence of $\phi_{\mathbf{n}}$ to ϕ

Question

How to infer convergence of $\mu_{\mathbf{n}}$ without a probabilistic coupling between $\mu_{\mathbf{n}}$ and $\phi_{\mathbf{n}}$?

The spin model



$$\sigma(u) = i^{h(u)}$$

Answer

It is enough to prove convergence of spin correlations!

Outline of proof of Theorem 1

(i) use the BKW representation to get

$$\mathbf{E}_{\mu_{\mathbf{n}}}[\sigma(u_1)\cdots\sigma(u_{2m})] = \mathbf{E}_{\phi_{\mathbf{n}}}\Big[\prod_{\ell\in\mathcal{L}}\rho(\ell)\Big]$$

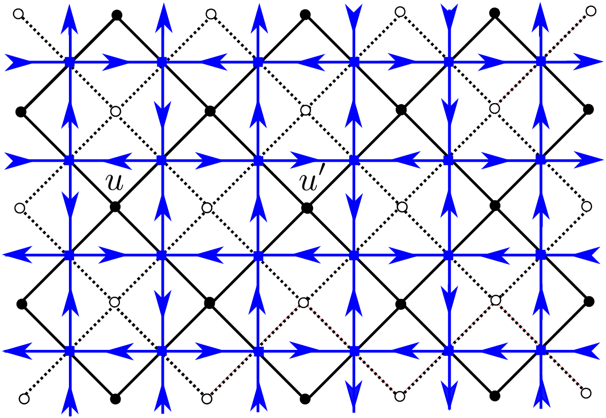
(ii) use *percolation* properties of the critical random cluster measure ϕ obtained by Duminil-Copin, Sidoravicius & Tassion '15, to get convergence

$$\mathbf{E}_{\phi_{\mathbf{n}}}\Big[\prod_{\ell\in\mathcal{L}}\rho(\ell)\Big] \to \mathbf{E}_{\phi}\Big[\prod_{\ell\in\mathcal{L}}\rho(\ell)\Big] \quad \text{as} \quad \mathbf{n}\to\infty$$

(iii) conclude convergence in distribution of $\mu_{\mathbf{n}}$ to an infinite volume measure μ

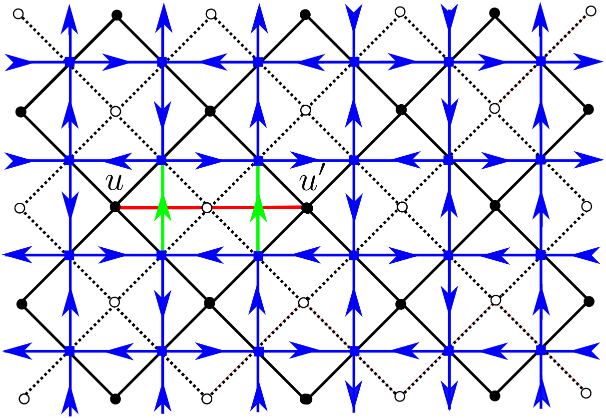
(iv) use *mixing* of ϕ to get ergodicity of μ

BKW representation of spin correlations



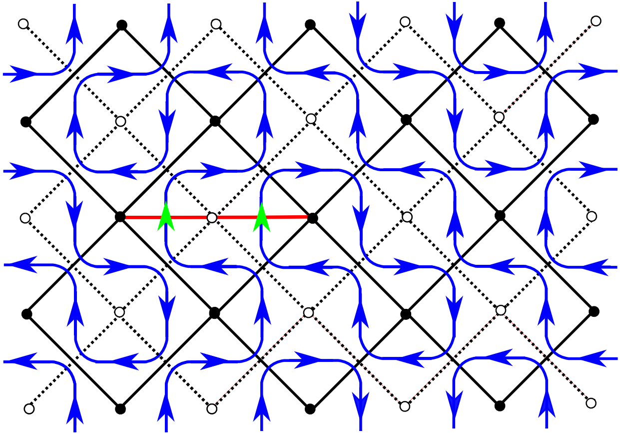
$$\mathbf{E}_{\mu_\mathbf{n}}[\sigma(u)\sigma(u')] = \mathbf{E}_{\mu_\mathbf{n}}[i^{h(u)+h(u')}] = \mathbf{E}_{\mu_\mathbf{n}}[i^{h(u)-h(u')}]$$

BKW representation of spin correlations



$$\mathbf{E}_{\mu_{\mathbf{n}}}[\sigma(u)\sigma(u')] = \mathbf{E}_{\mu_{\mathbf{n}}}[i^{|\Gamma\cap\alpha|}(-i)^{|\Gamma\cap(-\alpha)|}] =: \mathbf{E}_{\mu_{\mathbf{n}}}[\epsilon(\alpha)]$$

BKW representation of spin correlations



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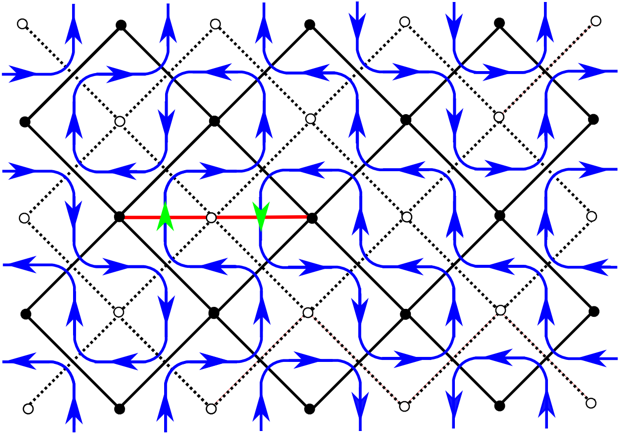
BKW representation of spin correlations



$$\mathbf{E}_{\mu_{\mathbf{n}}}[\sigma(u)\sigma(u')] = \mathbf{E}_{\mu_{\mathbf{n}}}[i^{|\Gamma\cap\alpha|}(-i)^{|\Gamma\cap(-\alpha)|}] =: \mathbf{E}_{\mu_{\mathbf{n}}}[\epsilon(\alpha)]$$

BKW representation of spin correlations

$$\begin{aligned}
\mathbf{E}_{\mu_{\mathbf{n}}}[\sigma(u)\sigma(u')] &= \mathbf{E}_{\mu_{\mathbf{n}}}[\epsilon(\alpha)] \\
&= \frac{1}{Z_{\mathbf{n}}} \sum_{\vec{L} \in \vec{\mathcal{L}}} \epsilon(\vec{L}) w(\vec{L}) \\
&= \frac{1}{Z_{\mathbf{n}}} \sum_{\vec{L} \in \vec{\mathcal{L}}} \prod_{\vec{\ell} \in \vec{L}} e^{i\lambda \mathrm{wind}(\vec{\ell})} \epsilon(\vec{\ell}) \\
&= \frac{1}{Z_{\mathbf{n}}} \sum_{L \in \mathcal{L}} \Big(\prod_{\ell \in L} \rho(\ell) \Big) \sqrt{q}^{|L|} \Big(\tfrac{2}{\sqrt{q}} \Big)^{|L_{\mathrm{nctr}}|} \\
&= \mathbf{E}_{\phi_{\mathbf{n}}} \Big[\prod_{\ell \in L} \rho(\ell) \Big]
\end{aligned}$$

where

$$\rho(\ell) = \frac{e^{i\lambda \mathrm{wind}(\vec{\ell})} \epsilon(\vec{\ell}) + e^{i\lambda \mathrm{wind}(-\vec{\ell})} \epsilon(-\vec{\ell})}{e^{i\lambda \mathrm{wind}(\vec{\ell})} + e^{i\lambda \mathrm{wind}(-\vec{\ell})}}$$

BKW representation of spin correlations

Let $u_1, \dots, u_{2m}$ be black faces. We call a face *source* if one of the fixed paths starts at this face, and otherwise the face is a *sink*. For a contractible loop ℓ, let $\delta(\ell)$ be the number of sources minus the number o sinks enclosed by the loop. Then

$$\rho(\ell) = \begin{cases} 1 & \text{if } \delta(\ell) = 0 \bmod 4, \\ -\tan\lambda & \text{if } \delta(\ell) = 1 \bmod 4, \\ -1 & \text{if } \delta(\ell) = 2 \bmod 4, \\ \tan\lambda & \text{if } \delta(\ell) = 3 \bmod 4 \end{cases}$$

Proposition [L. '20]

$$\mathbf{E}_{\mu_{\mathbf{n}}}[\sigma(u_1)\cdots\sigma(u_{2m})] = \mathbf{E}_{\phi_{\mathbf{n}}}\Big[\prod_{\ell\in\mathcal{L}}\rho(\ell)\Big]$$

End of proof of Theorem 1

- By *"quasilocality"* of the loop observables and non-percolation of ϕ, we have

$$\mathbf{E}_{\phi_{\mathbf{n}}}\Big[\prod_{\ell\in\mathcal{L}}\rho(\ell)\Big] \to \mathbf{E}_{\phi}\Big[\prod_{\ell\in\mathcal{L}}\rho(\ell)\Big] \quad \text{as} \quad \mathbf{n}\to\infty \qquad \square$$

Outline of proof of Theorem 2

(i) The fact that there is no infinite cluster under ϕ, implies that

$$\mathbf{E}_{\mu}[\sigma(u)\sigma(u')] \to 0 \qquad \text{as} \quad |u - u'| \to \infty$$

for $c \in (\sqrt{2+\sqrt{2}}, 2]$.

(ii) Decorrelation of spins implies no infinite cluster in the ***FK-type representation*** ω of σ

(iii) No percolation of ω implies delocalization of the height function (L. '19).

Decorrelation of spins

For two black faces u, u', we get

$$\mathbf{E}_\mu[\sigma(u)\sigma(u')] = \mathbf{E}_\phi\left[\rho^{2N(u,u')}(-1)^{N(u,u',\infty)}\right],$$

where

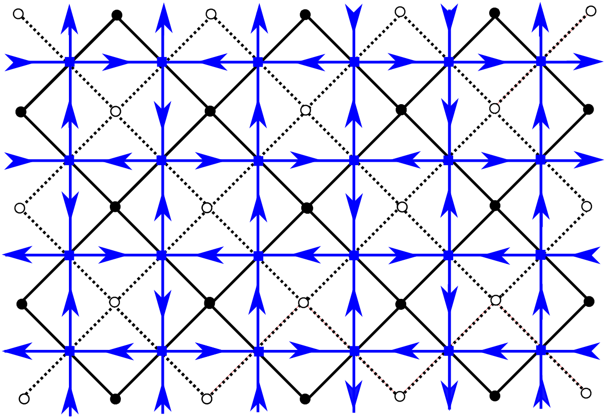
$$\rho = \tan\lambda \qquad \text{with} \qquad \sqrt{q} = 2\cos\lambda$$

Here

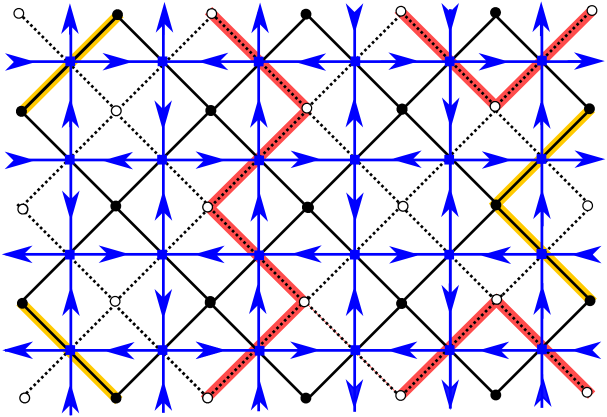
- $N(u, u')$ is the number of clusters in the random cluster model on $\mathbb{Z}^2_\circ$ that *disconnect* u from u'
- $N(u, u', \infty)$ is the number of clusters that *disconnect* all three points u, u', and ∞ from each other

$$\rho < 1 \quad \Leftrightarrow \quad c \in \left(\sqrt{2+\sqrt{2}}, 2\right]$$

FK representation of σ

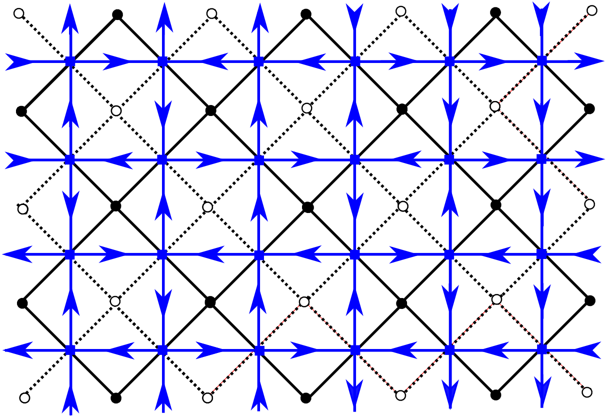


FK representation of σ



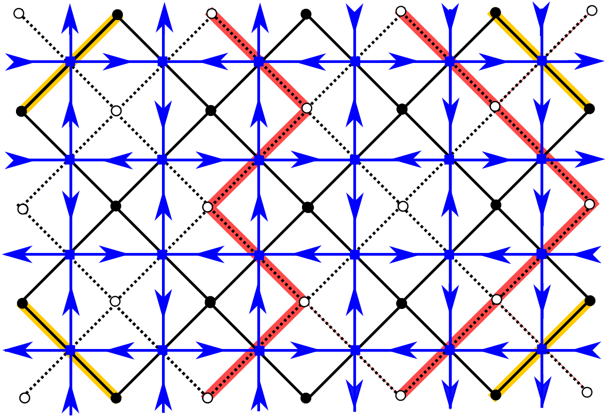
Draw primal and dual contours between spins of different value

FK representation of σ



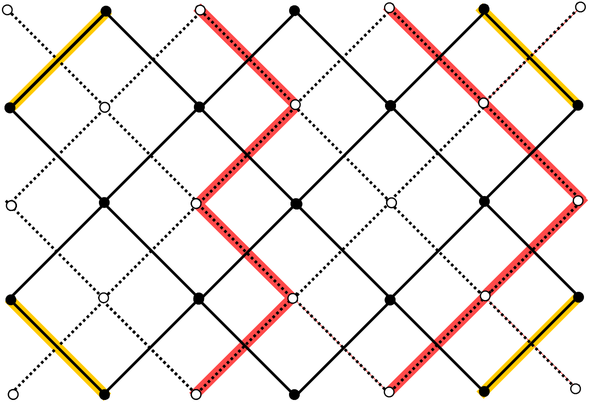
Condition on $\mathcal{O}^0_{\mathbf{n}}$ so that σ is globally well-defined

FK representation of σ

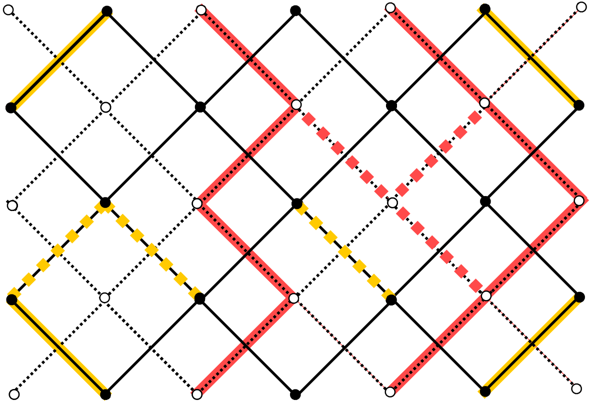


Condition on $\mathcal{O}^0_{\mathbf{n}}$ so that σ is globally well-defined

FK representation of σ



FK representation of σ



Open primal and dual edges with probability $1 - c^{-1}$

Call the resulting yellow configuration ω and its law $\mathbf{P_n}$

Properties of the coupling (σ, ω)

We have the following *Edwards–Sokal property*

Proposition [Glazman & Peled '18, L. '19]

Under $\mathbf{P_n}$, conditionally on ω, the spins σ are distributed like an *independent* uniform assignment of a ± 1 spin to each connected component of ω.

One can show that conditioning on $\mathcal{O}^0_{\mathbf{n}}$ does not change the limit distribution. This implies that $\mathbf{P_n}$ converges to an ergodic infinite-volume limit $\mathbf{P}$

In particular,

$$\mathbf{E}_\mu[\sigma(u)\sigma(u')] = \mathbf{P}(u \overset{\omega}{\leftrightarrow} u'),$$

where $\{u \overset{\omega}{\leftrightarrow} u'\}$ is the event that u and u' are in the same cluster of ω.

Hence if spins decorrelate, then ω does not percolate!

Properties of the coupling (σ, ω)

For two black faces u, u', let $N(u, u')$ be the number of clusters of ω *disconnecting* u from u'

Proposition [L. '19]

$$\mathbf{Var}_{\mu}[h(u) - h(u')] \asymp \mathbf{E_P}\left[N(u, u')\right]$$

Properties of the coupling (σ, ω)

Theorem [L. '19]

If

$$\mathbf{P}(\omega \text{ percolates}) = 0,$$

then

$$\mathbf{P}(\text{infinitely many clusters of } \omega \text{ surround the origin}) = 1$$

and

$$\mathbf{Var}_\mu[h(u)] \to \infty \quad \text{as} \quad |\mathbf{n}| \to \infty$$

What next?

0. $c = \sqrt{2+\sqrt{2}}$
1. $\sqrt{3} \le c < \sqrt{2+\sqrt{2}}$
2. Polynomial decay of correlations
3. Other boundary conditions
4. Scaling limit at $c = \sqrt{2+\sqrt{2}}$? Then $q = (c^2-2)^2 = 2$
5. Connection to Gaussian imaginary chaos

Thank you for your attention!