Nodal sets of random spherical harmonics

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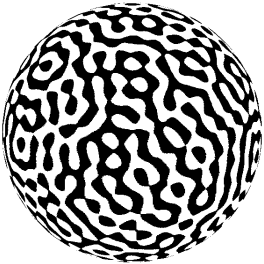

Workshop on Mathematical Physics

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Reasons to study topology of zero sets of smooth random functions:

mathematical physics:

Bogomolny-Schmit percolation model for nodal domains of random plane waves, i.e. random Gaussian solutions to $\Delta F + F = 0$ in $\mathbb{R}^2$

Berry's conjecture: the high energy Laplace eigenfuctions on negatively curved surfaces behave like random plane waves

algebraic topology: a statistical version of (the first part of) Hilbert's 16th problem

analysis/probability theory: natural and intriguing questions with lack of ideas and almost no progress

In this talk, all random functions are Gaussian, defined on $\mathbb{S}^2$ or on $\mathbb{R}^2$, and have distribution invariant w.r.t. to isometries.

We are interested in the topology of the zero set, primarily, with the number of its connected components.

Other intriguing questions: size of connected components, level sets, ...

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Random spherical harmonics

$\mathcal{H}_n$ real Hilbert space of 2D spherical harmonics equipped with the $L^2(\mathbb{S}^2)$-norm, $\dim \mathcal{H}_n = 2n+1$, (Y_k) orthonormal basis in $\mathcal{H}_n$

(ξ_k) Gaussian IIDs, $\mathbb{E}|\xi_k|^2 = \frac{1}{2n+1}$

$f_n = \sum_{k=-n}^{n} \xi_k Y_k$ random spherical harmonic of degree n

The distribution of f_n

is independent of the choice of the ONB in $\mathcal{H}_n$

is invariant w.r.t. isometries of the sphere $\mathbb{S}^2$

$Z(f_n) = f^{-1}\{0\}$ the zero set of f_n

$N(f_n)$ the number of connected components of $Z(f_n)$, $n \gg 1$

Major difficulties:

Slow off-diagonal decay (and *sign changes*) of the covariance $\mathbb{E}[f_n(x)f_n(y)] = P_n(\cos\Theta(x,y))$

P_n Legendre polynomial of degree n, $\Theta(x,y)$ angle between $x, y \in \mathbb{S}^2$.

Scaled covariance: $P_n\big(cos\frac{z}{n}\big) \sim J_0(z)$ $(n \to \infty)$

It is natural to think of f_n as defined on the sphere $n\mathbb{S}^2$ of radius n and of area $\simeq n^2$. In this scale the covariance decays as dist$^{-1/2}$, recall that $J_0(z) \sim \sqrt{\frac{2}{\pi z}}\cos\big(z - \frac{\pi}{4}\big)$ as $z \to \infty$.

Scaling limit $n \to \infty$: Random Plane Wave - the $2D$ Fourier transform of the (hermitean symmetric) white noise on $\mathbb{S}^1 \subset \mathbb{R}^2$ - a Gaussian solution to the Helmholtz equation $\Delta F + F = 0$ with the covariance $J_0(|X - Y|)$

Another difficulty: "non-locality" (contrary to the length or the Euler characteristics).

Bogomolny and Schmit percolation model

In 2001, Bogomolny and Schmit proposed a remarkable random loop model for description of the topology of the zero set $Z(F)$ of the RPW F.

Their model completely ignores slow decaying correlations and is very far from being rigorous.

Attempts to digest their work stimulated much of the progress recently achieved in this area.

What is known

LLN + Exponential concentration:

THEOREM 1 (F.Nazarov, M.S., arXiv 2007) There exists $\nu > 0$ s.t.

$$\mathbb{P}\Big[\big|N(f_n) - \nu n^2\big| > \varepsilon n^2\Big] < Ce^{-c(\varepsilon)n}$$

with $c(\varepsilon) \gtrsim \varepsilon^{15}$.

The proof (based on the Gaussian isoperimetry) gives

$$\nu = \lim_{n\to\infty} \mathbb{E}\Big[\frac{1}{\text{area}(G_n)}\Big],$$

where G_n is a nodal domain of f_n on $n\mathbb{S}^2$ that contains a marked point x.

Later, we have shown (using the ergodic theorem and some functional analysis) that the Law of Large Numbers with a positive limit (but without the exponential concentration) holds for rather general classes of smooth Gaussian fields on $\mathbb{R}^d$ and of smooth Gaussian ensembles on manifolds (arXiv, 2015).

Related works and extensions:

- "derandomization" on the torus: Bourgain, Buckley – Wigman, Ingremeau;
- other topological observables: Gayet – Welschinger, Lerario – Lundberg (upper and lower bounds for mean values), Sarnak – Wigman, Canzani – Sarnak (the Law of Large Numbers);
- fields and ensembles with positive correlations: Malevich (1972, sic!), Alexander (1996), Beliaev – Muirhead – Wigman, Rivera – Vanneuville;
- level/excursion sets: Swerling (1963, sic!), Beliaev – McAuley – Muirhead;

by no means is this list complete.

A recent advance

Power low bound for the variance

THEOREM 2 (work in progress with Fedya Nazarov):

$$\mathrm{Var}[N(f_n)] \gtrsim n^{\sigma}$$

with some $\sigma > 0$.

REMARK The exponential concentration from Theorem 1

$$\mathbb{P}\Big[\big|N(f_n) - \nu n^2\big| > \varepsilon n^2\Big] < Ce^{-c\varepsilon^{15} n}$$

yields the upper bound: $\mathrm{Var}[N(f_n)] \lesssim n^{4-\frac{2}{15}}$.

The Bogomolny and Schmit prediction $\mathrm{Var}[N(f_n)] \sim n^2$ remains widely open.

REMARK Our lower bound holds for *any* non-degenerated isotropic smooth Gaussian fields on $n\,\mathbb{S}^2$ with decay of correlations $\gtrsim \mathrm{dist}^{-c}$ with some $c > 0$.

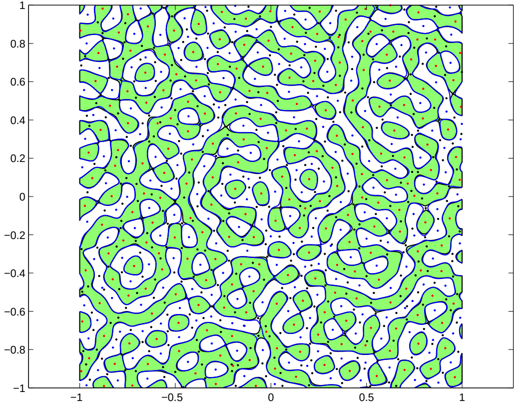
We will discuss main ideas from the proof of the lower bound. The proof can be viewed as the first (though, modest) step towards justification of the Bogomolny-Schmit heuristcs.

Saddle points with small critical values:

Heuristically, the fluctuations in the topology of $Z(f_n)$ are caused by saddle points of f_n with small critical values that yield so called "avoided crossings" of the zero set $Z(f_n)$.

I.e., switches in the topology of the zero set of f_n are caused by a point process that has a low intensity but strong long range dependence, as illustrated on the following simulation produced by Dima Beliaev.

Instead of random spherical harmonics Beliaev simulated the random plane wave (RPW) but one may safely ignore the difference.



Blue lines are zero lines of a RPW F_0, blue and red points are maxima and minima of F_0, and black points are saddle points of F_0. Black lines are zero lines of the sum $F_0 + \frac{1}{10}F_1$, where F_1 is another RPW, equidistributed with F_0 and independent of F_0, green domains are connected components of the set where this sum is positive.

Step 1: Low level critical points

$f = f_n$ random spherical harmonic of degree n on $n\mathbb{S}^2$, $\mathbb{E}|f|^2 = 1$

$\mathrm{Cr}(\alpha) = \{z \in n\mathbb{S}^2 \colon \nabla f(z) = 0, |f(z)| \leqslant \alpha\}$, $0 < \alpha \ll 1$

"With high probability" (w.h.p.) means except of an event of probability $O(n^{-c})$ with some $c > 0$.

LEMMA 1 Let $n^{-2+\varepsilon} \leqslant \alpha \leqslant n^{-2+2\varepsilon}$. Then, w.h.p., the set $\mathrm{Cr}(\alpha)$ is relatively large: $|\mathrm{Cr}(\alpha)| \gtrsim n^{c\varepsilon}$, and the points in this set are $n^{1-C\varepsilon}$-separated.

Step 2: Introducing a small perturbation

$f_\alpha = \sqrt{1-\alpha^2} f + \alpha g$, g is an independent copy of f. The random function f_α has the same distribution as f.

We condition on f.

LEMMA 2 Let $\alpha = n^{-2+\varepsilon}$ and $\alpha' = \alpha n^{\varepsilon} = n^{-2+2\varepsilon}$. Then, w.h.p., topology of $Z(f_\alpha)$ is determined by the collection of signs of $f_\alpha(z)$ at $z \in \mathrm{Cr}(\alpha')$.

This lemma allows us "to localize" the problem. Its proof needs a caricature of a quantitative Morse theory.

Step 3: Random loops model on planar graphs of degree 4

Recall: $\alpha = n^{-2+\varepsilon}$, $\alpha' = n^{-2+2\varepsilon}$, $f_\alpha = \sqrt{1-\alpha^2} f + \alpha g$, g is an independent copy of f

We replace g by its independent copy g_z (some linear algebra with estimates).

This step needs a good separation between the points of $\mathrm{Cr}(\alpha')$ provided Lemma 1.

Define a collection of independent random functions $\widetilde{f_\alpha} = \sqrt{1-\alpha^2} f + \alpha g_z$, $z \in \mathrm{Cr}(\alpha')$.

LEMMA 3 W.h.p., $\mathrm{sgn}(f_\alpha(z)) = \mathrm{sgn}(\widetilde{f_\alpha}(z))$, $z \in \mathrm{Cr}(\alpha')$.

This reduces the problem to the independent random loop model on a planar graph with the degree of each vertex either 4 (saddle points of f) or 0 (maxima and minima of f).

Discard the latter case and assume that the degree of each vertex is 4.

Step 4: Lower bound for the variance of the number of loops

$G = G(V, E)$ a graph embedded in $n\,\mathbb{S}^2$

The degree of each vertex $v \in V$ is 4.

In each vertex v, we independently replace the edges crossing by one of two possible avoided crossing configuration, $p(v), 1 - p(v)$ are the corresponding probabilities.

Γ random configuration of loops,

$N(\Gamma)$ the number of loops in Γ.

LEMMA 4 : For any $p_0 > 0$,

$$\mathrm{Var}[N(\Gamma)] \geqslant c(p_0)\,|\{v \in V : p_0 \leqslant p(v) \leqslant 1 - p_0\}|$$

This completes the proof of the lower bound for fluctuations of $N(f_n)$.

Questions that await new ideas

Do large nodal domains exist?

QUESTION 1 Show that a.s. the RPW has no infinite nodal domain.

Do large nodal domains exist? ("a spherical version")

G_n nodal domain of a random spherical harmonic f_n on $n\mathbb{S}^2$ that contains a marked point

The only thing we know about the distribution of area(G_n) is that, for some positive constants C, c,

$$\mathbb{P}\big[\text{area}(G_n) < C\big] \geqslant c$$

which yields positivity of the limiting constant $\nu = \lim_{n\to\infty} \mathbb{E}\Big[\frac{1}{\text{area}(G_n)}\Big]$ in Theorem 1.

QUESTION 2 Is it true that $\lim_{C\to\infty} \limsup_{n\to\infty} \mathbb{P}\big[\text{area}(G_n) \geqslant C\big] = 0$?

We do not know the answer to a much weaker question:

QUESTION 2a Show that for any $\delta > 0$, $\lim_{n\to\infty} \mathbb{P}\big[\text{area}(G_n) \geqslant \delta n^2\big] = 0$.

We know nothing about domains of a large diameter that contain a given point.

Level sets of random spherical harmonics:

Though the sets $\{f_n > \varepsilon\}$ and $\{f_n < \varepsilon\}$ have roughly the same areas, the former one should look as a collection of many small islands in ocean formed by the latter one.

Given $\varepsilon, \delta > 0$ consider that event $\mathcal{X}_n(\varepsilon, \delta)$ that the level set $\{f_n > \varepsilon\}$ has a connected component of diameter at least δn.

QUESTION 3 Show that for any $\varepsilon, \delta > 0$, $\lim_{n\to\infty} \mathbb{P}\big[\mathcal{X}_n(\varepsilon, \delta)\big] = 0$.

A similar question can be asked for the RPW.

Gaussian ensemble on $\mathbb{R}^2$ with correlations $e^{-\frac{1}{2}|X-Y|^2}$ ("Fock-Bargmann wave")

Due to positivity and very fast decay of correlations certain tools from the percolation theory become available and the situation becomes more tractable.

In this case, the answer to the questions raised above are mostly known: Alexander (1996), Beffara – Gayet, Beliaev – Muirhead – Wigman, Rivera – Vanneuville, Muirhead – Vanneuville ...

A version of the Sir Michael Berry prediction:

Consider high-energy Laplace eigenfunctions on the sphere endowed with a generic smooth Riemannian metric close to the constant one.

QUESTION 4 Do they (or at least some portion of them) behave similarly to random spherical harmonics?

Instead of perturbing the round metric on the sphere $\mathbb{S}^2$, one can add a small random potential to the Laplacian on the round sphere. The question remains just as hard.

We must know. We will know.

(David Hilbert)

The End