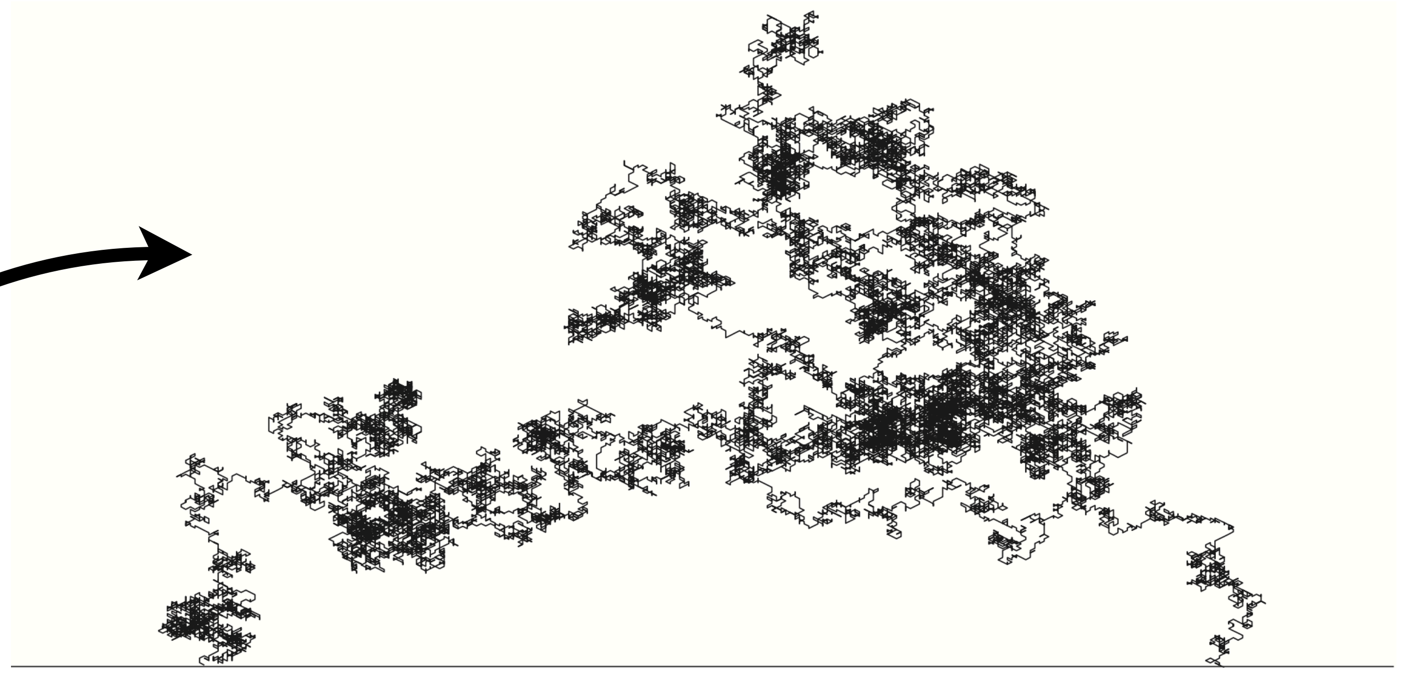
Growth Fragmentations, Brownian Motion and Random Geometry

Les Diablerets, February 2023

Ellen Powell, Durham University.
Based on joint work with Juhan Aru, Nina Holden, Xin Sun

Aim

Planar Brownian Excursions



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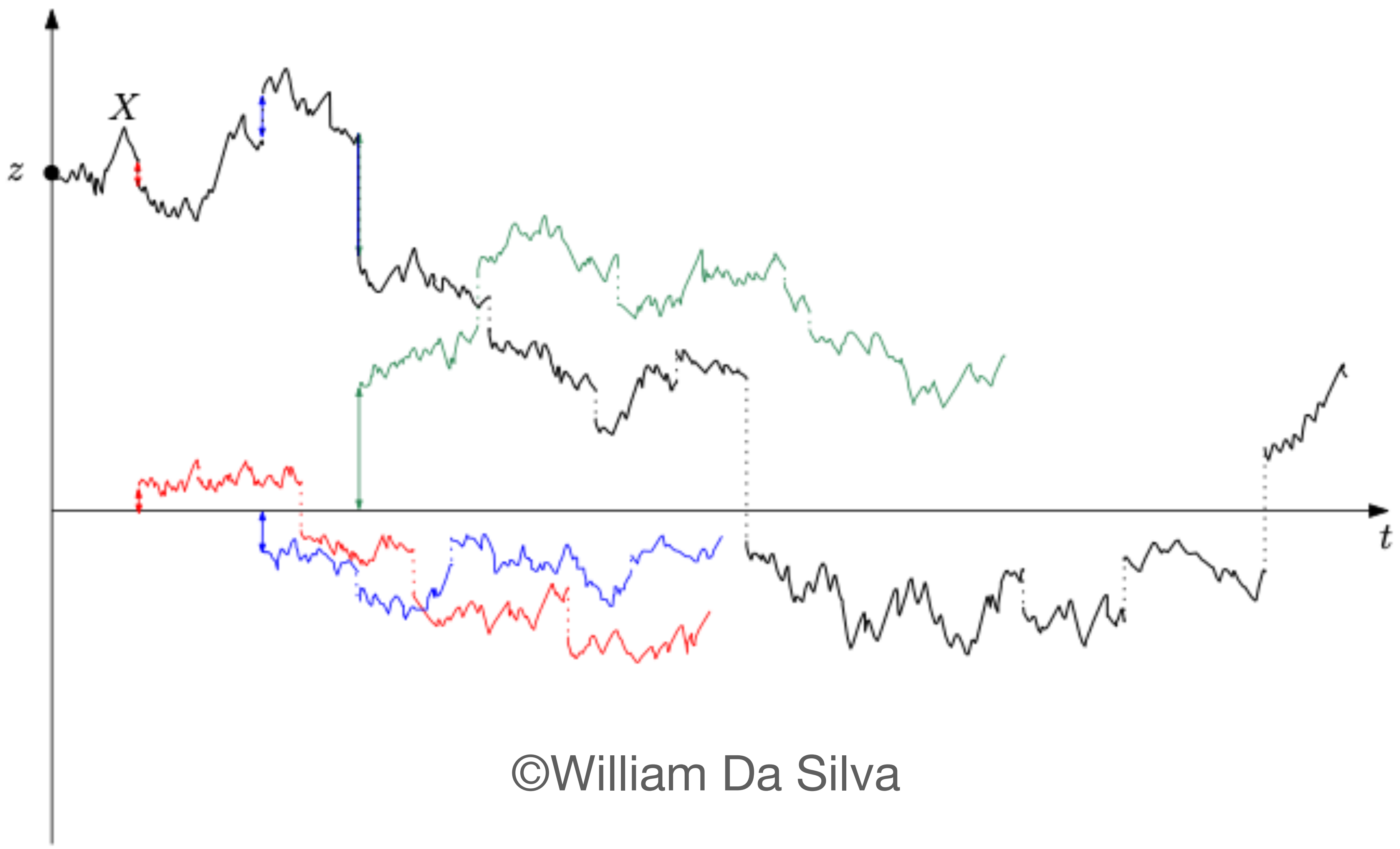
Growth Fragmentations



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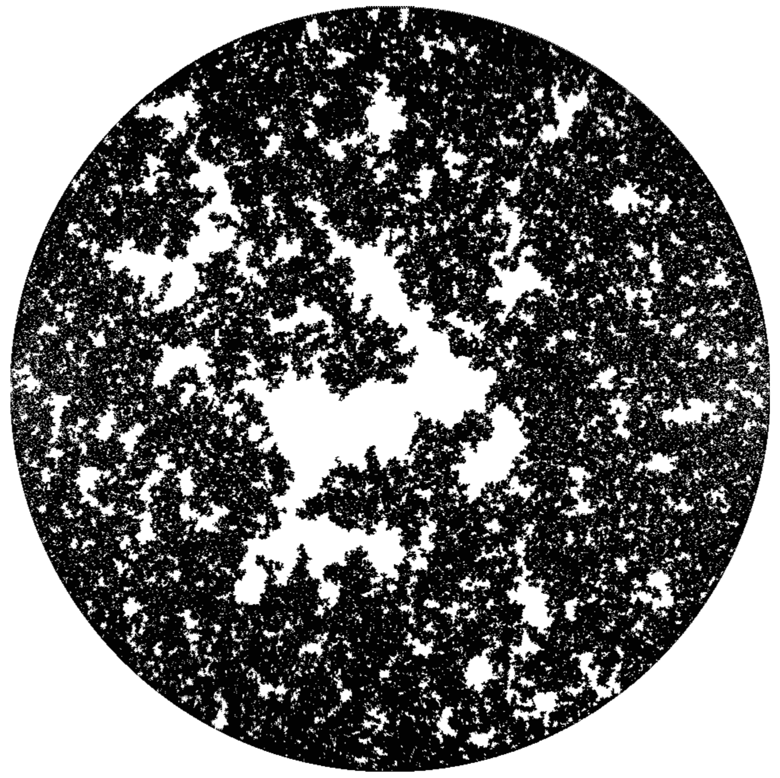
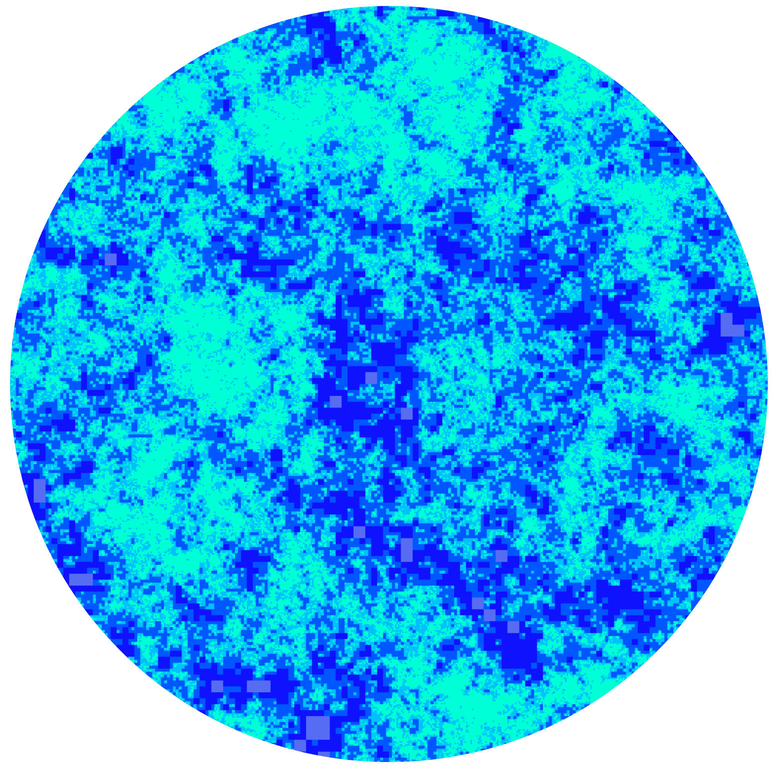
Gaussian Multiplicative Chaos & Conformal Loop Ensembles





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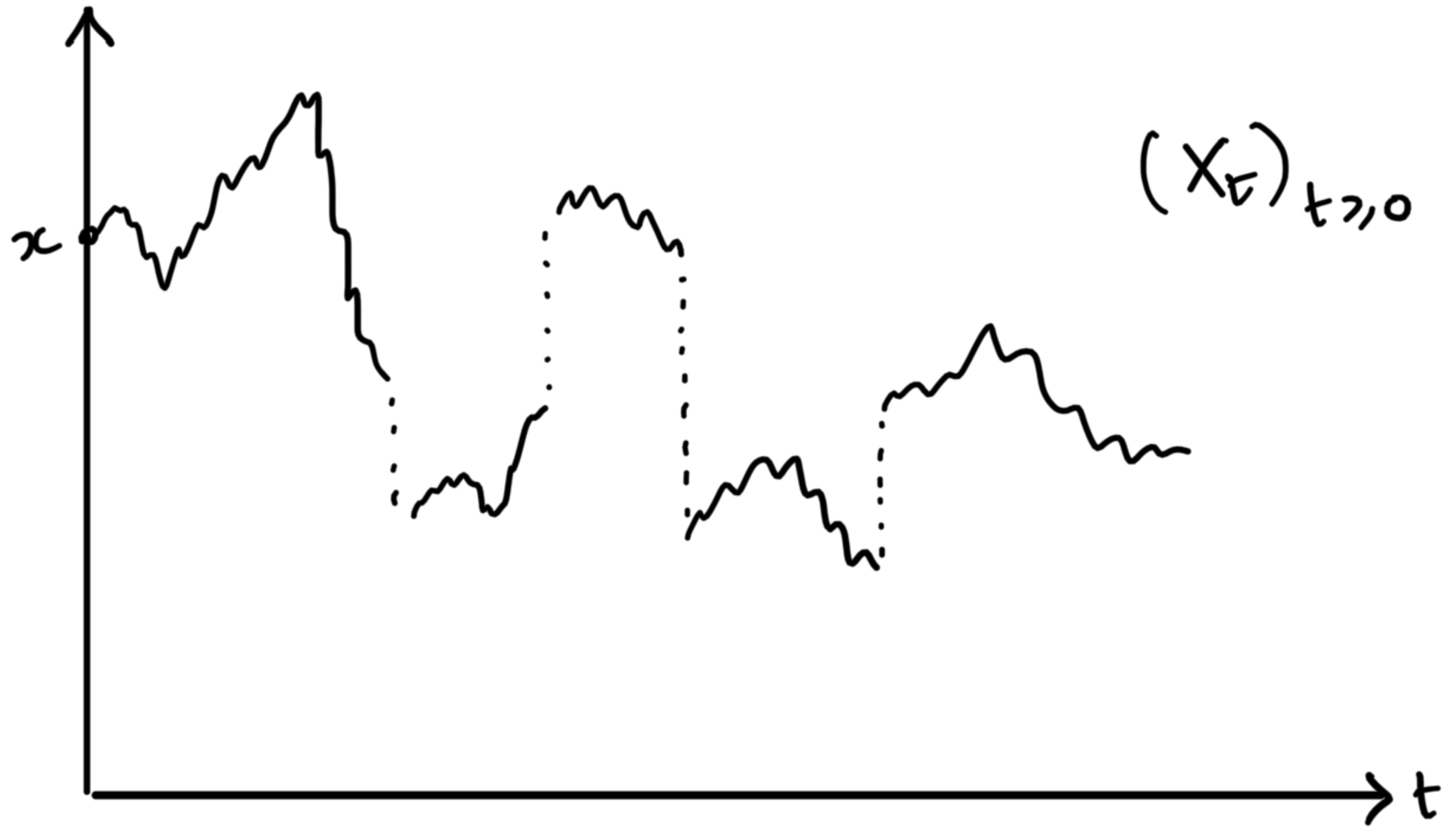
Growth fragmentations

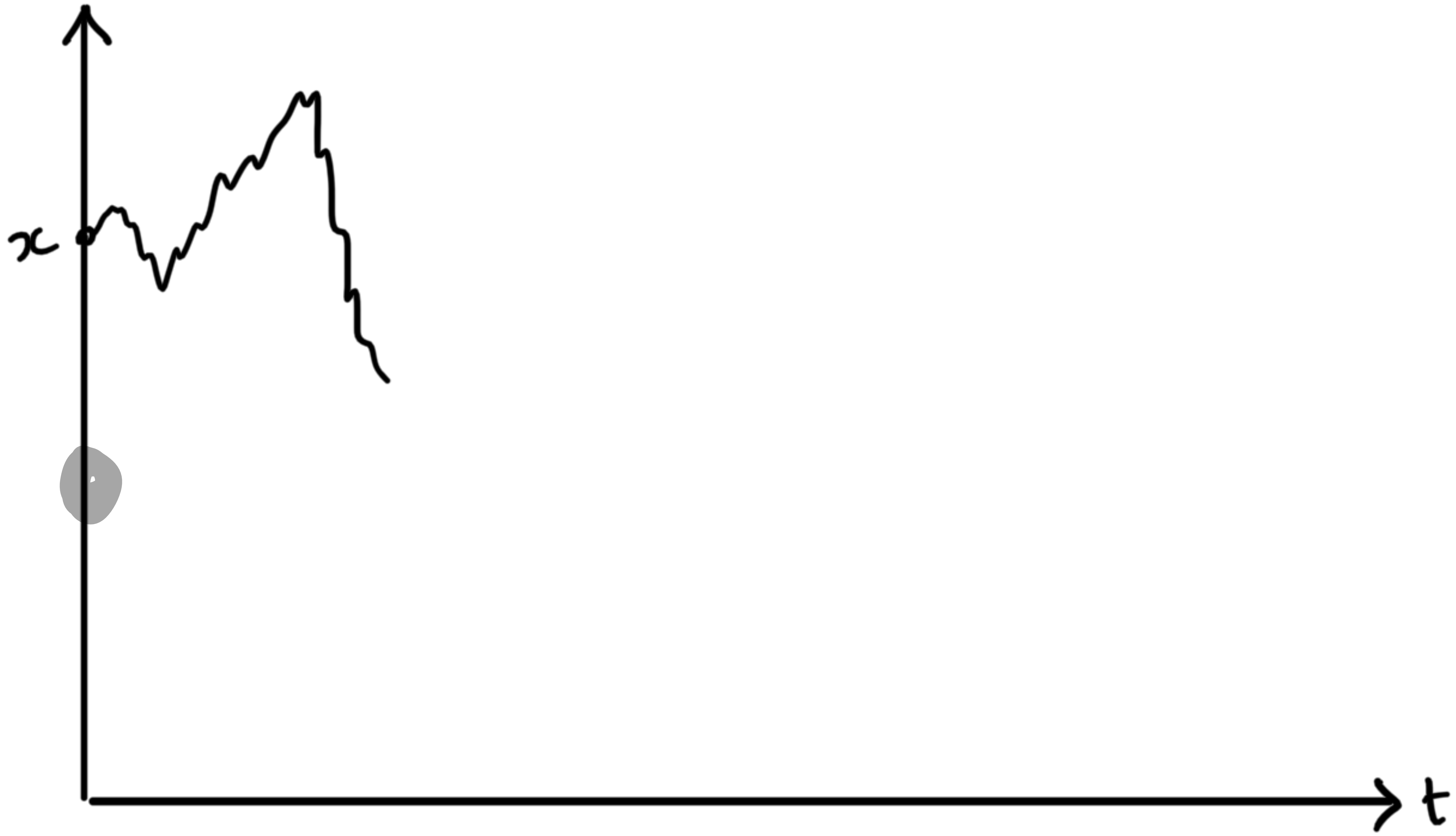
Growth Fragmentations

- $X =$ positive self-similar Markov process, some initial value x
- *E.g. Stable Lévy process conditioned to be die continuously at 0*



Growth Fragmentations

- $X =$ positive self-similar Markov process, some initial value x
- **Growth (or shrinking) of cells:** evolution of X



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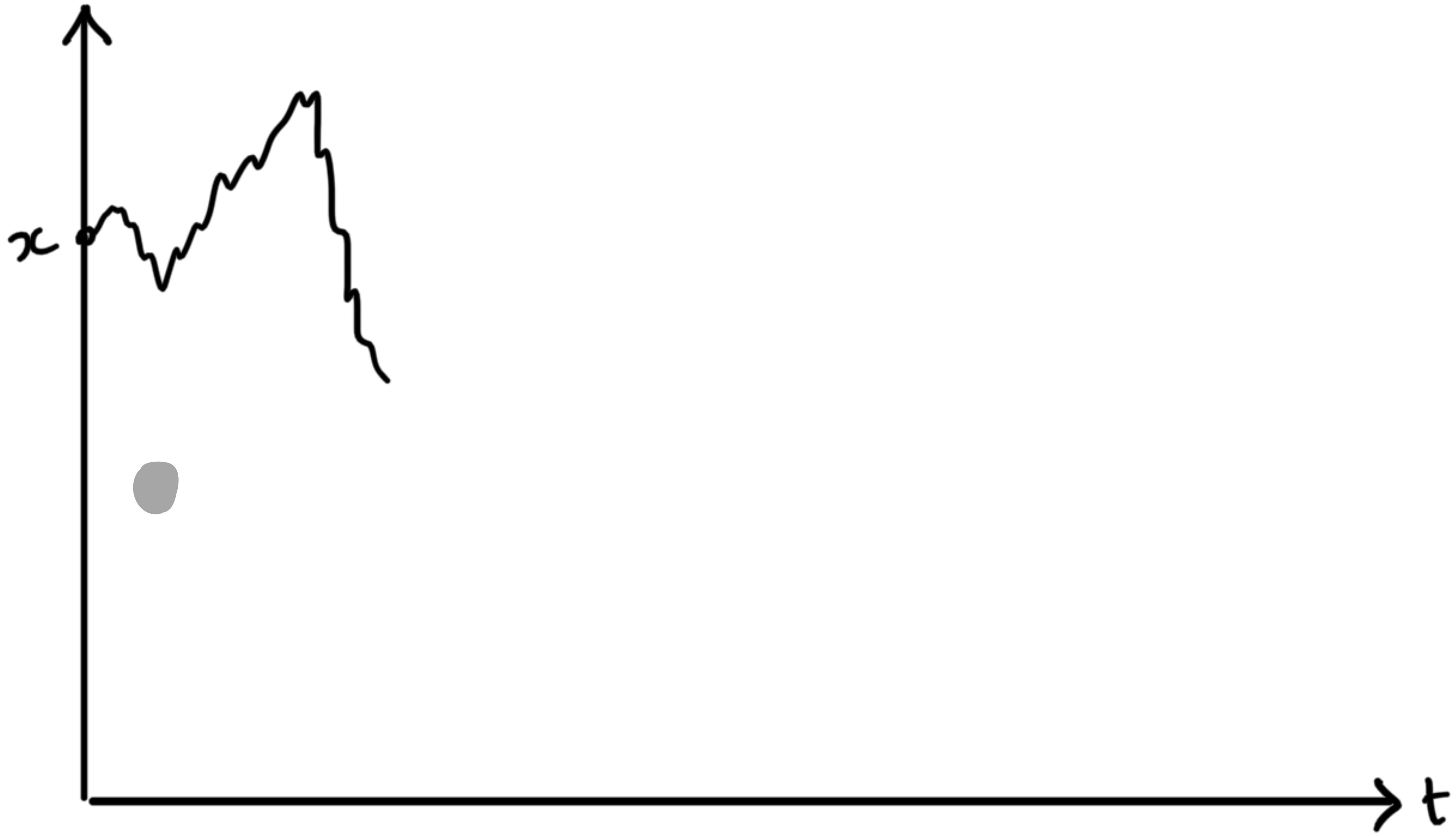

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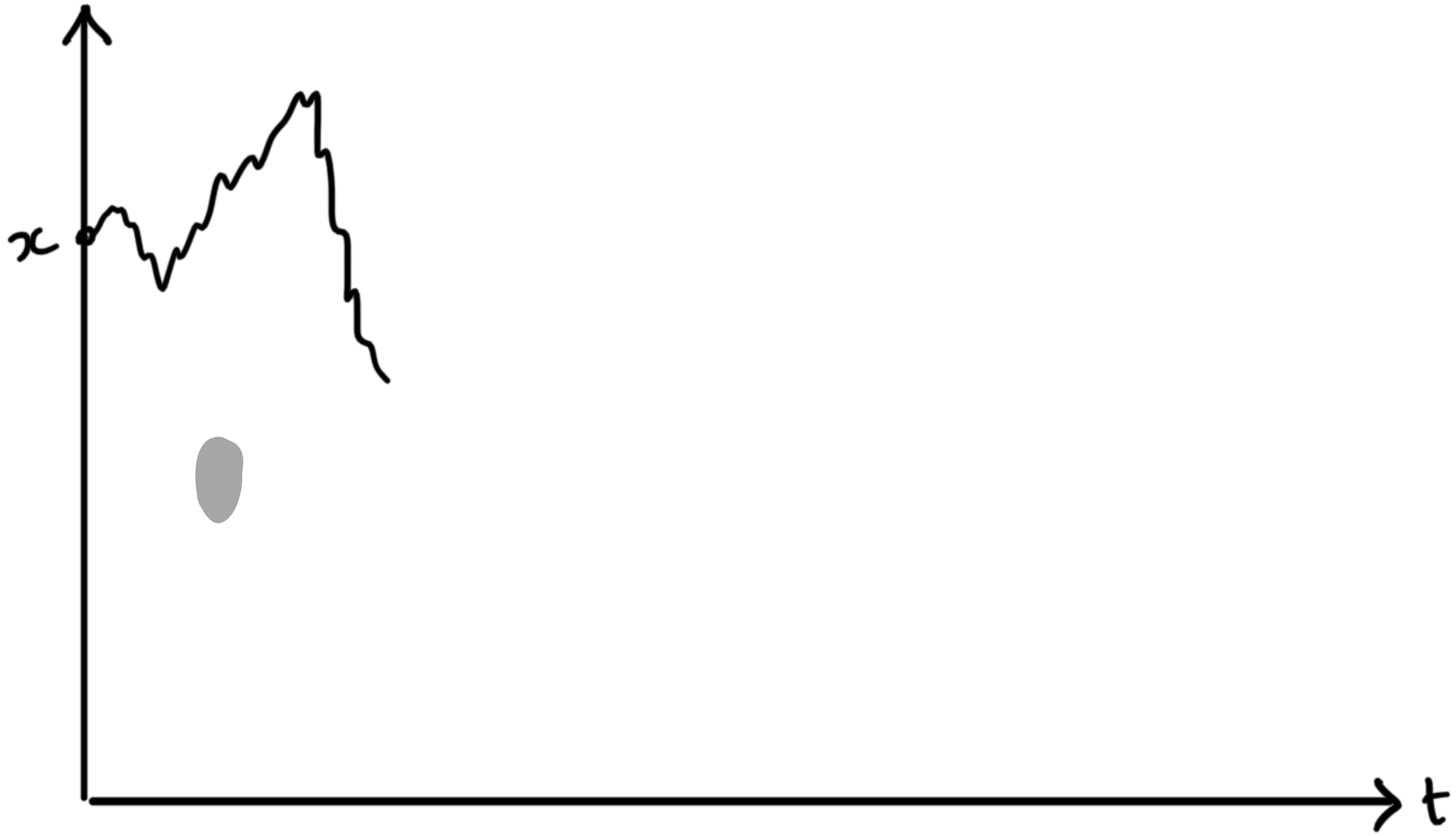

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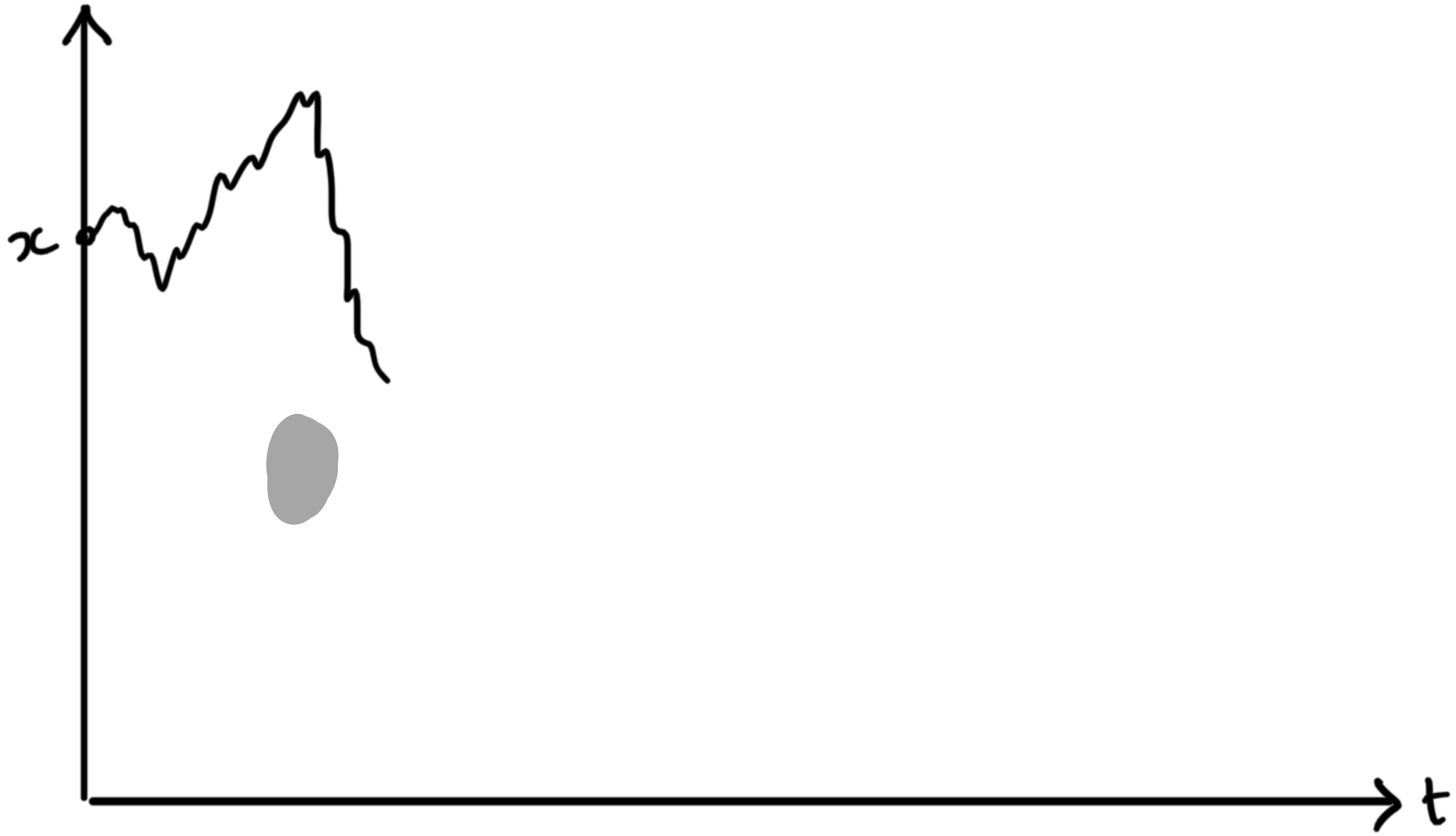

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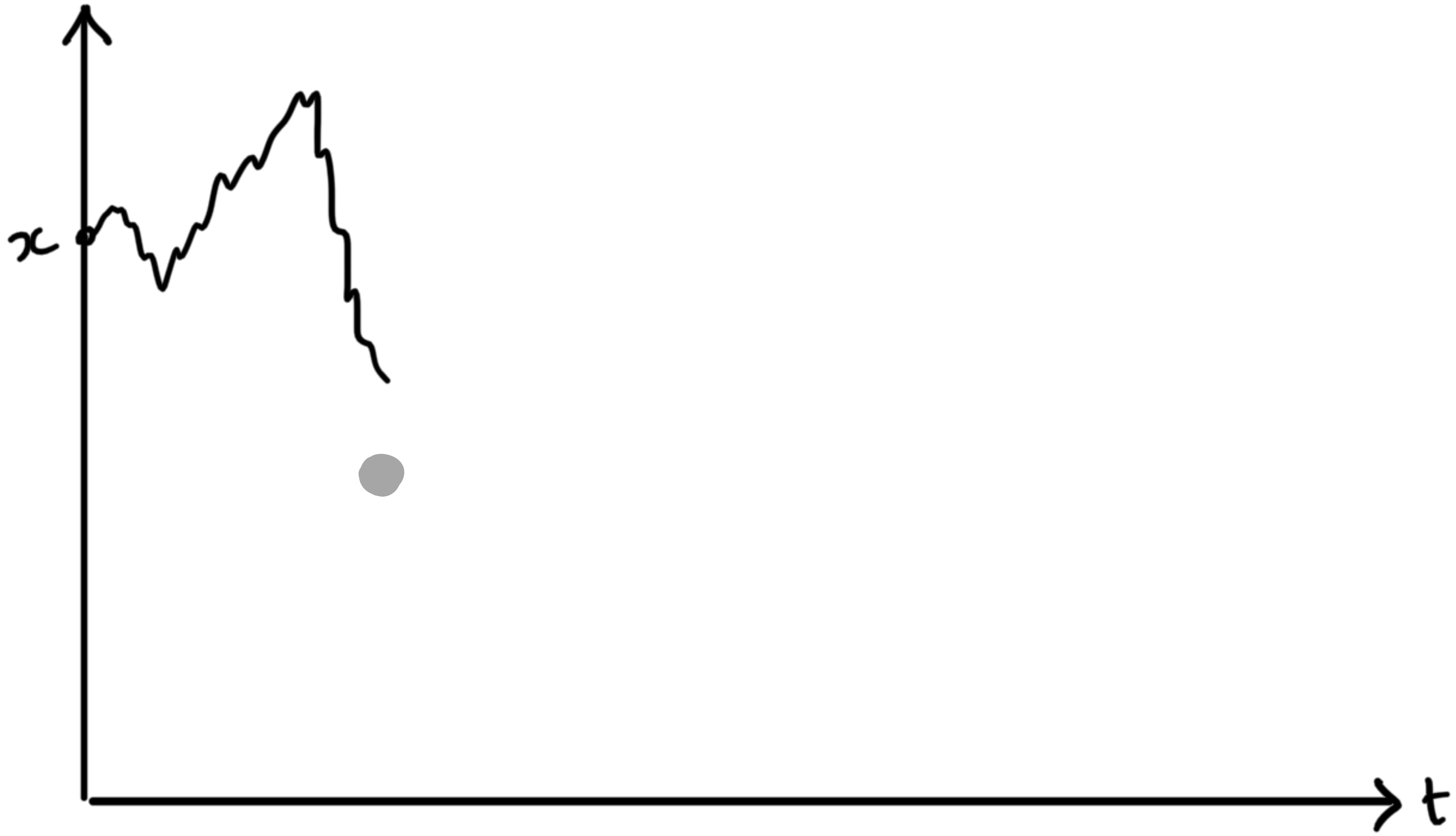
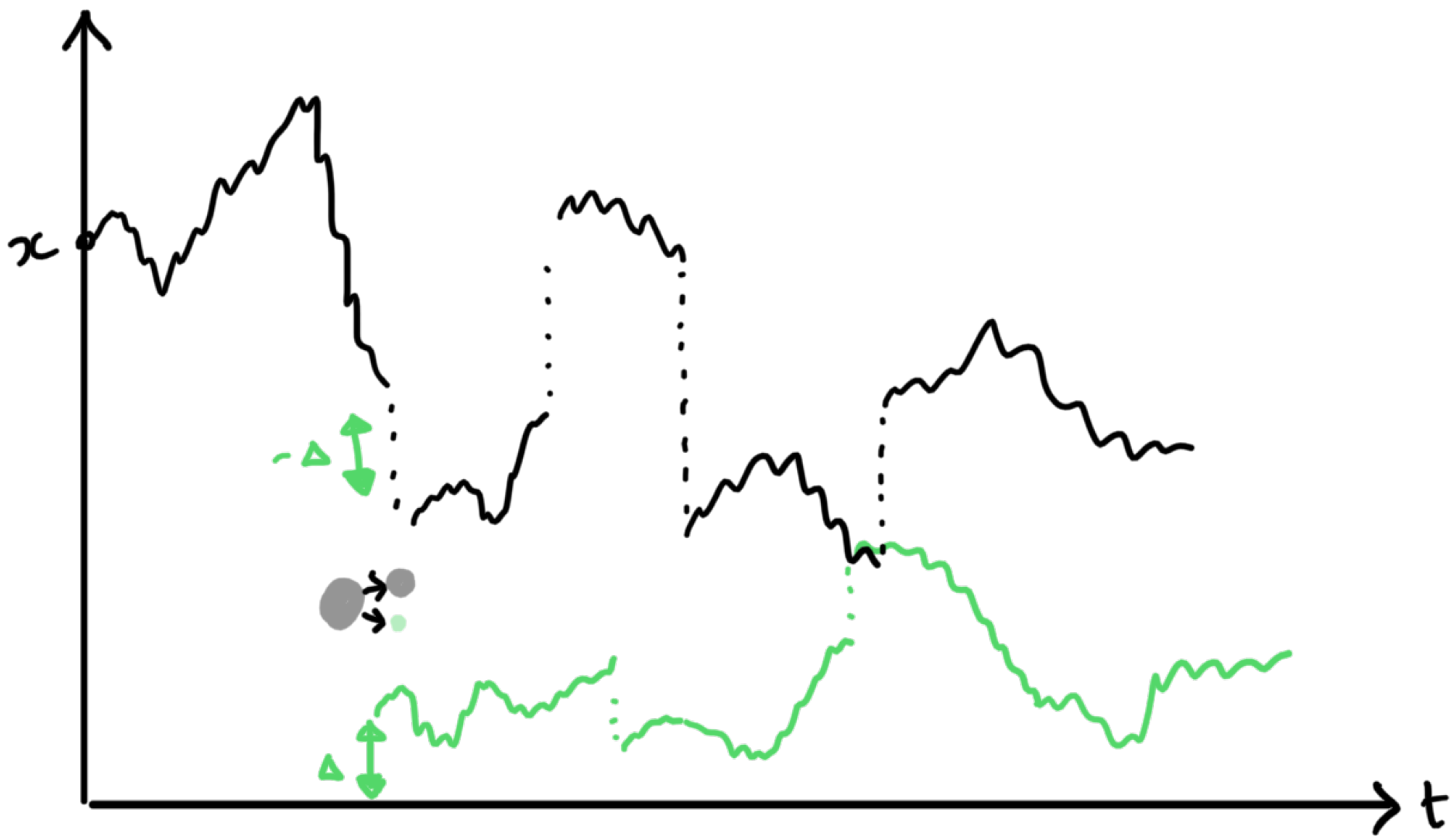


Growth Fragmentations

- $X =$ positive self-similar Markov process, some initial value x
- **Growth (or shrinking) of cells:** evolution of X
- **Fragmentation:** negative jump - Δ of $X \rightsquigarrow$ new particle with initial size Δ, then evolves independently under same law as X **(mass is conserved)**



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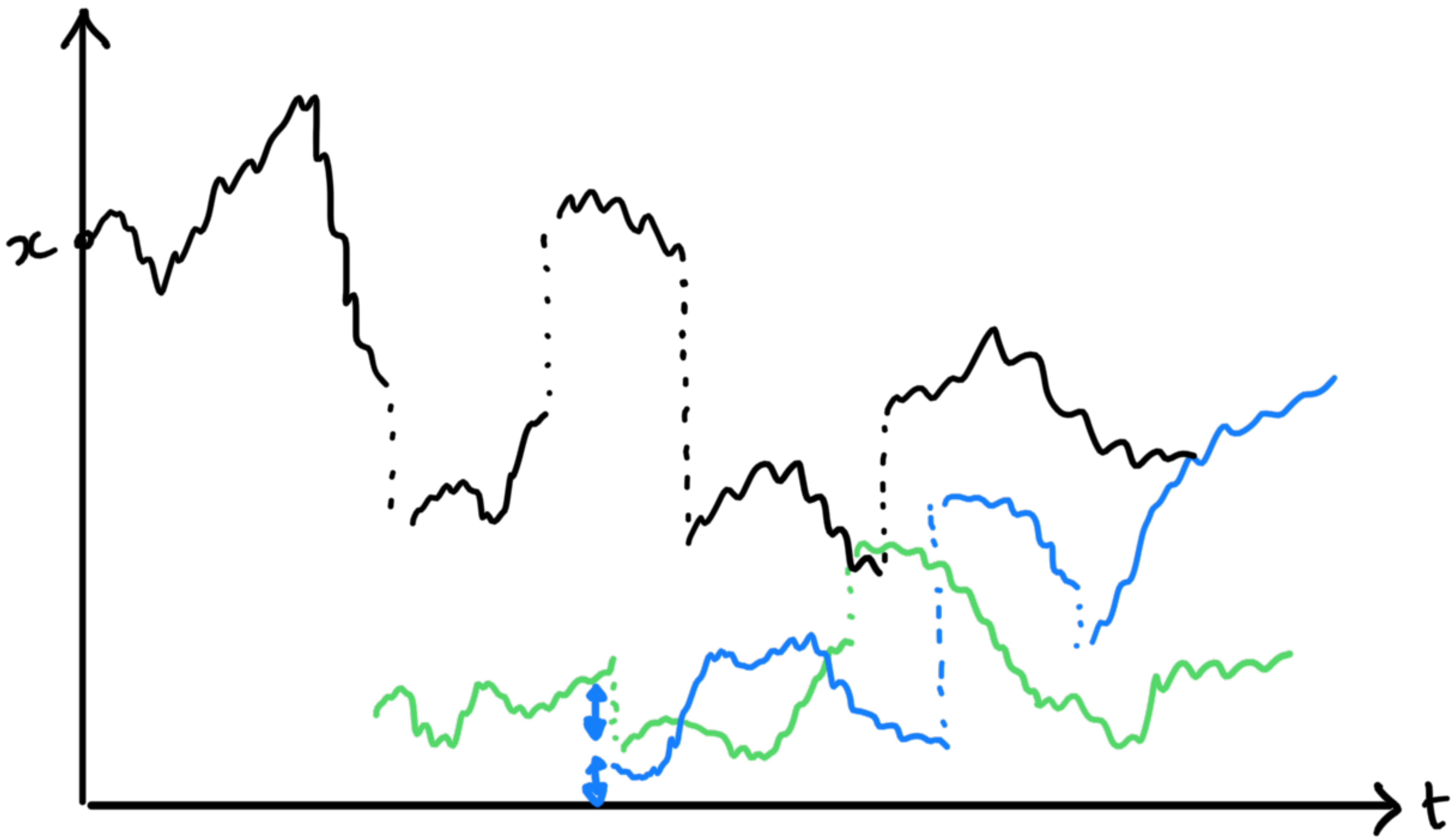
- **Iterates**



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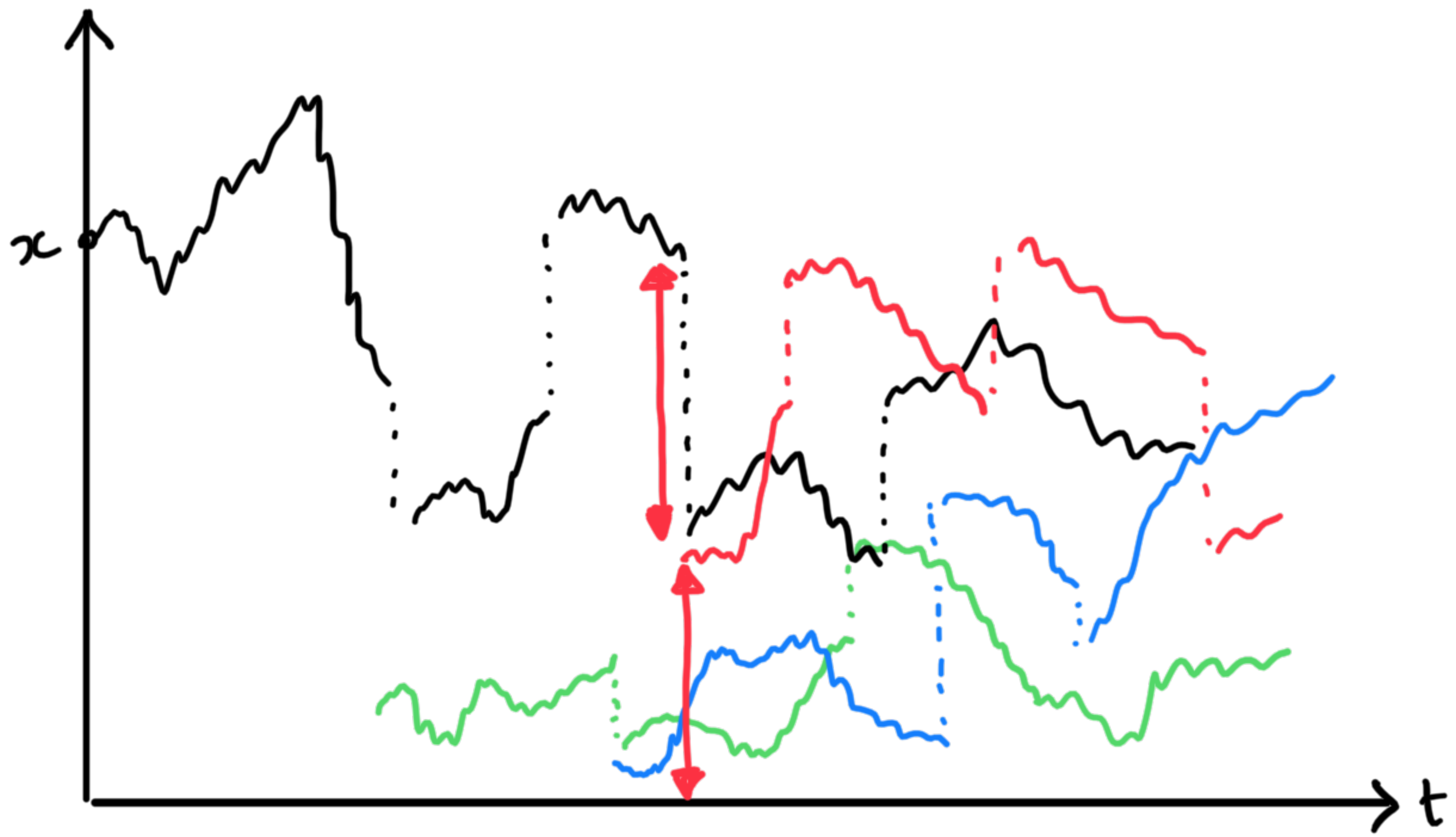
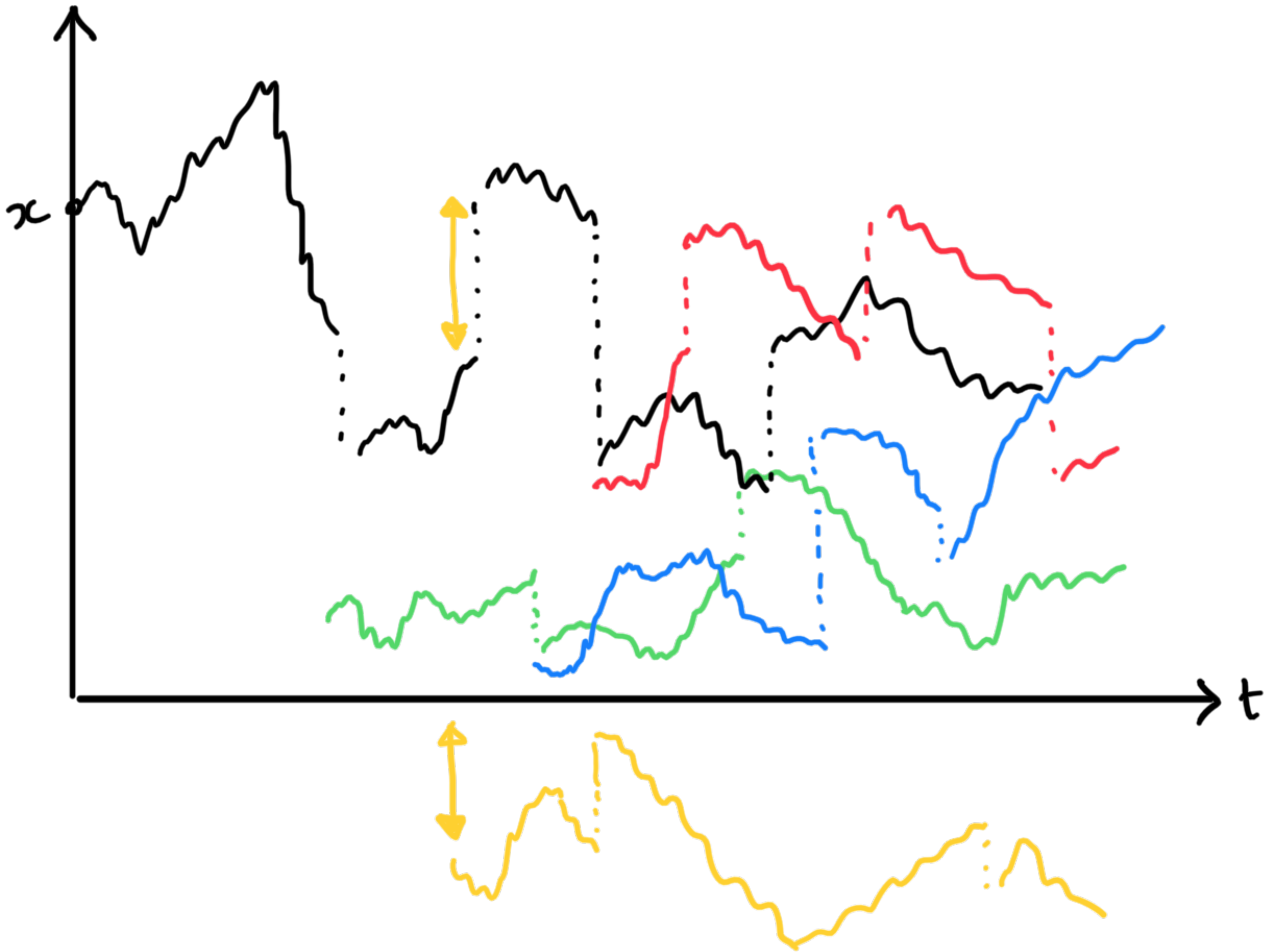


Growth Fragmentations

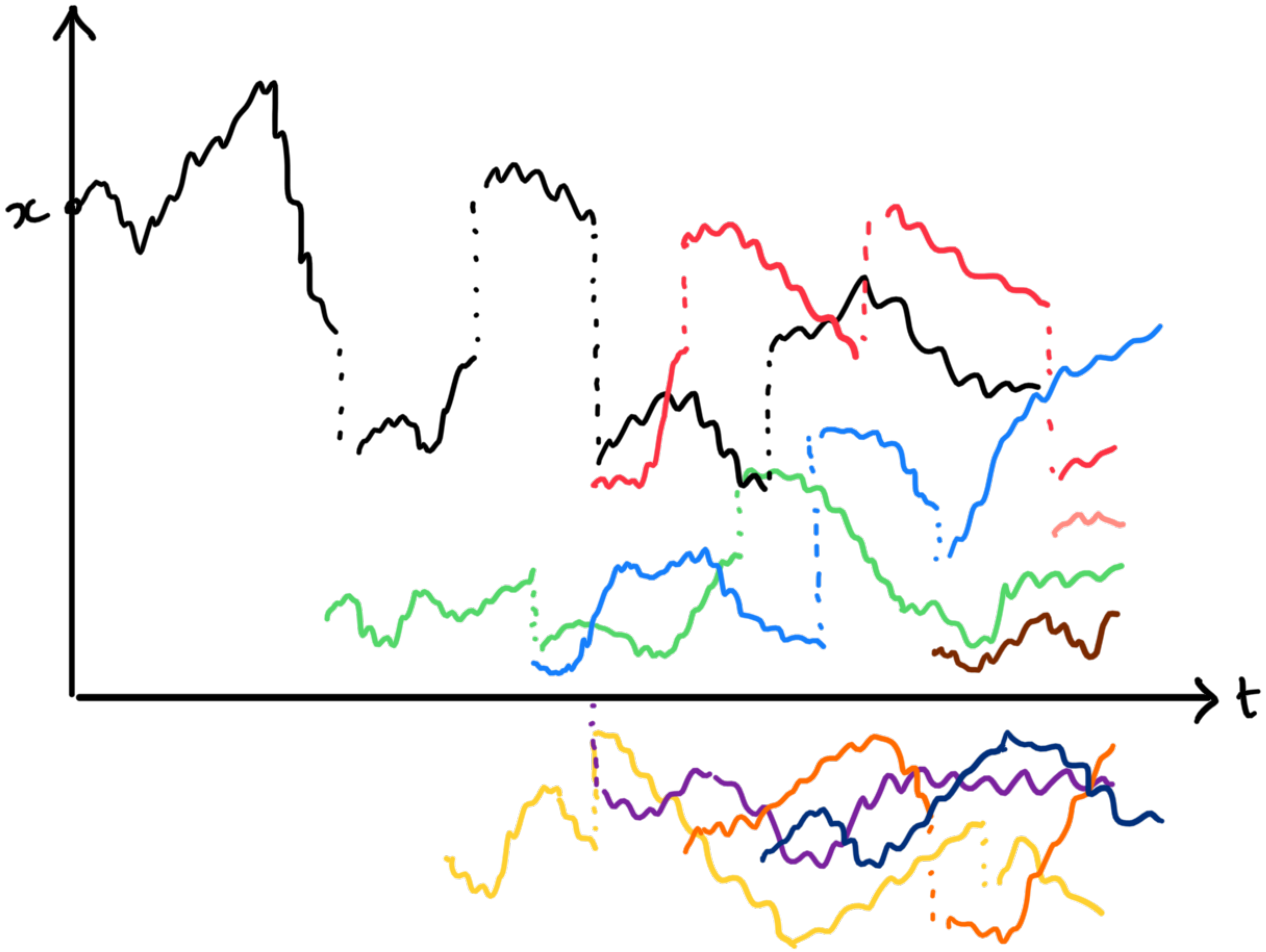
- $X =$ positive self-similar Markov process, some initial value x
- **Growth (or shrinking) of cells:** evolution of X
- **Fragmentation:** negative jump $-\Delta$ of $X \rightsquigarrow$ new particle with initial size Δ
- **Signed version:** positive jumps $\rightsquigarrow$ new particles of negative mass



Growth Fragmentations

Bertoin, Bertoin-Budd-Curien-Kortchemski, Aïdékon-Da Silva, Da Silva …

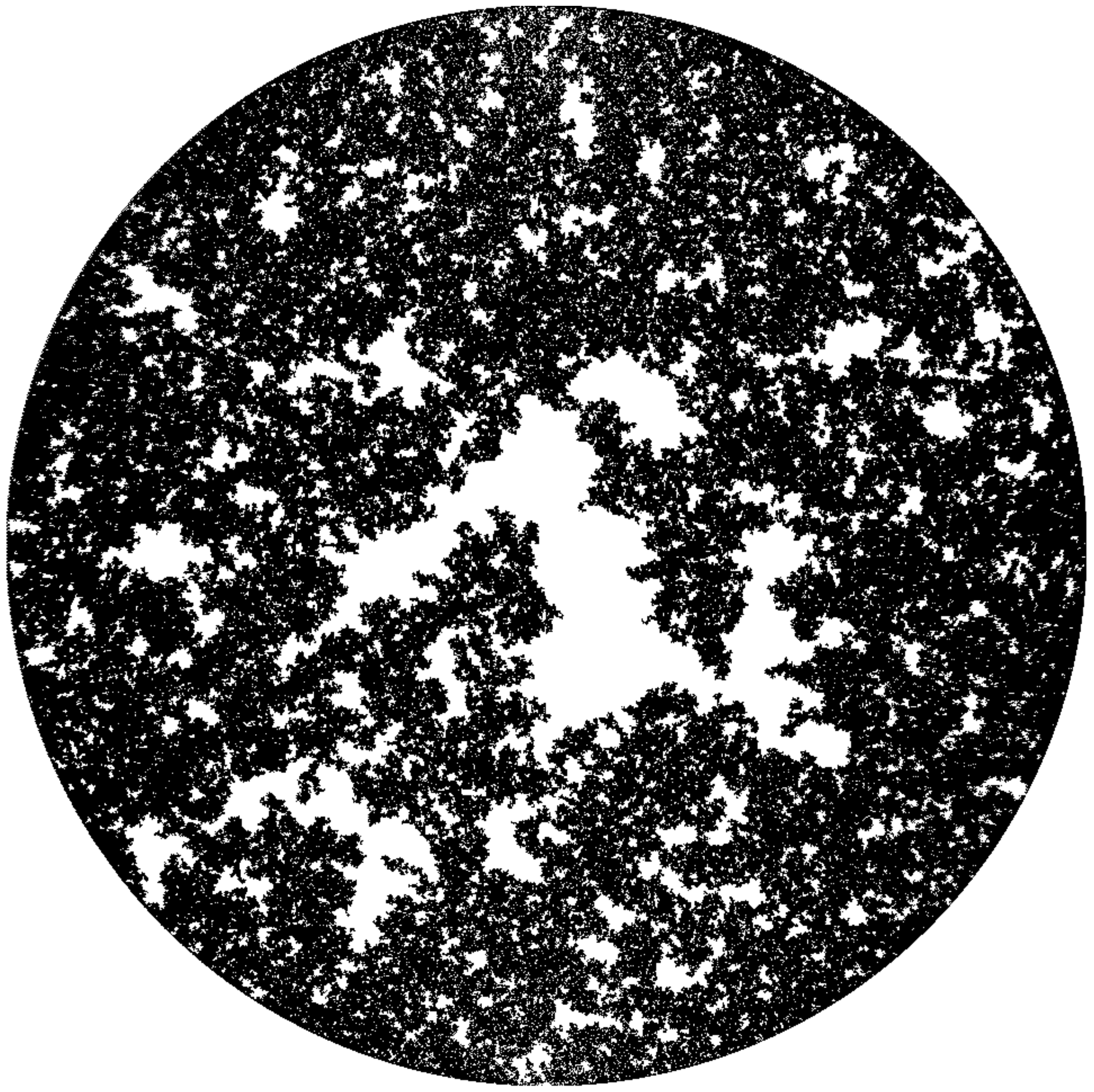
- $X =$ positive self-similar Markov process, some initial value x
- **Growth (or shrinking) of cells:** evolution of X
- **Fragmentation:** negative jump $-\Delta$ of $X \rightsquigarrow$ new particle with initial size Δ
- At time $t \geq 0$, system = collection of particles with (signed) masses



Conformal Loop Ensembles

Conformal Loop Ensembles $\kappa \in (8/3,4]$

- Simple CLE_κ = random collection of disjoint simple loops in a simply connected domain of $\mathbb{C}$, introduced by (Sheffield-Werner)



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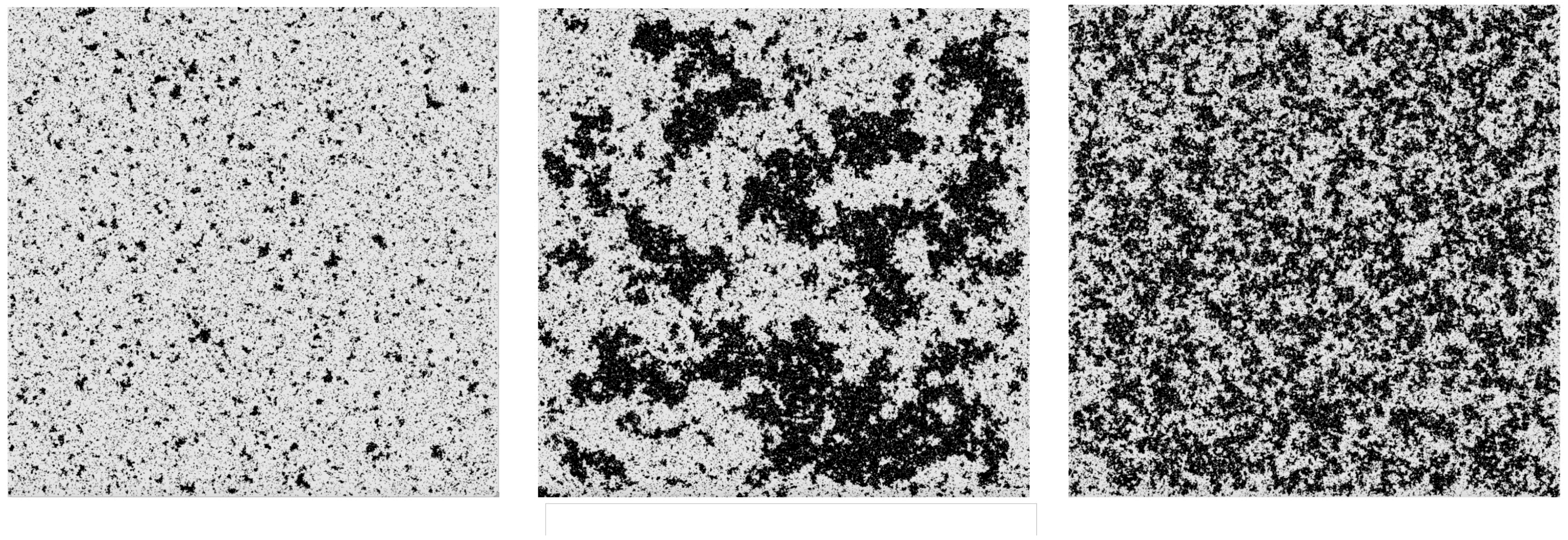
Conformal Loop Ensembles $\kappa \in (8/3, 4]$

- Simple CLE_κ = random collection of disjoint simple loops in a simply connected domain of $\mathbb{C}$, introduced by (Sheffield—Werner)

- (Conjectured) **scaling limit of interfaces** in discrete models

- CLE_3 (top, bottom left): Chelkak—Duminil-Copin—Hongler—Smirnov, Benoist—Hongler



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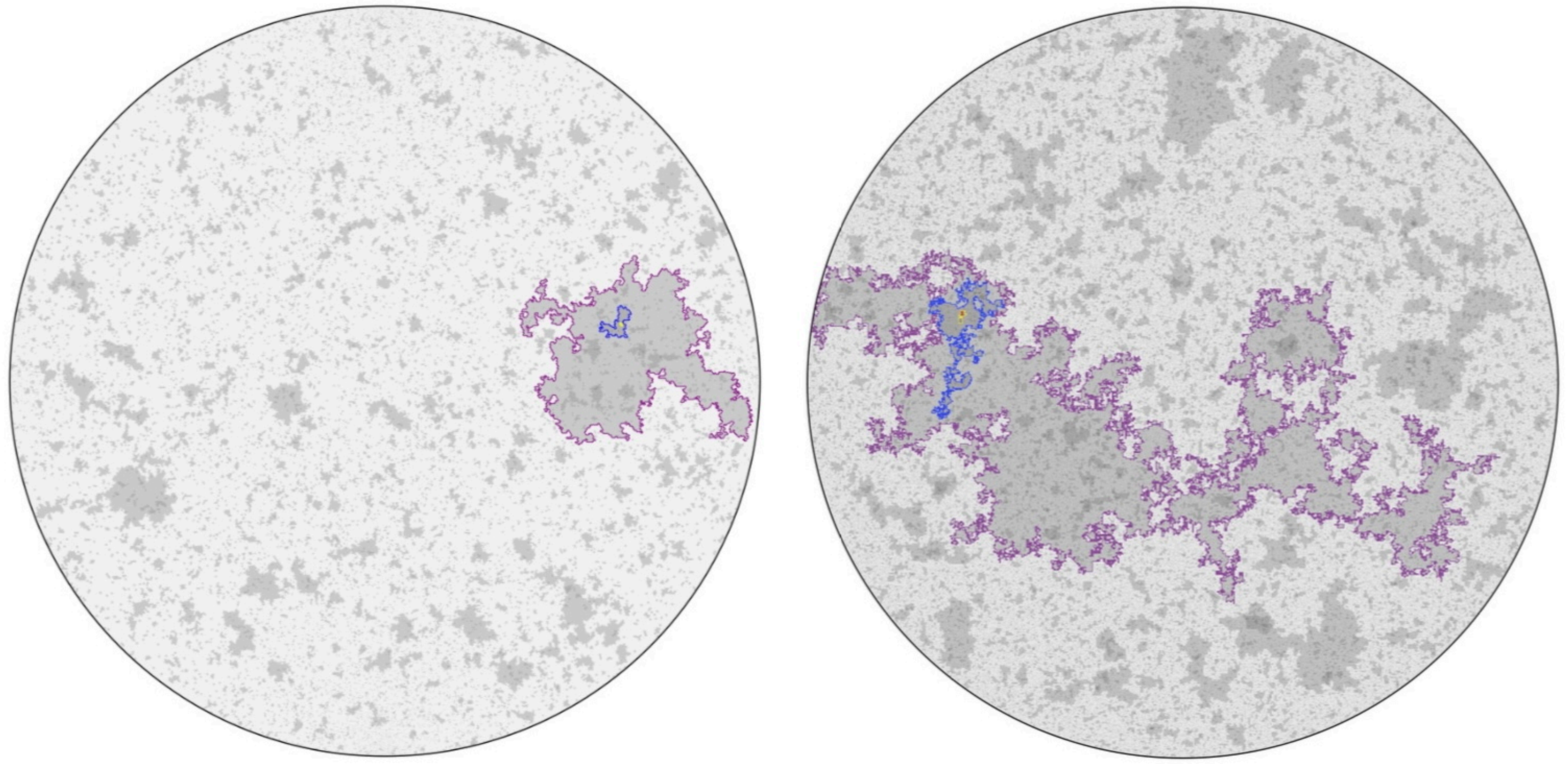

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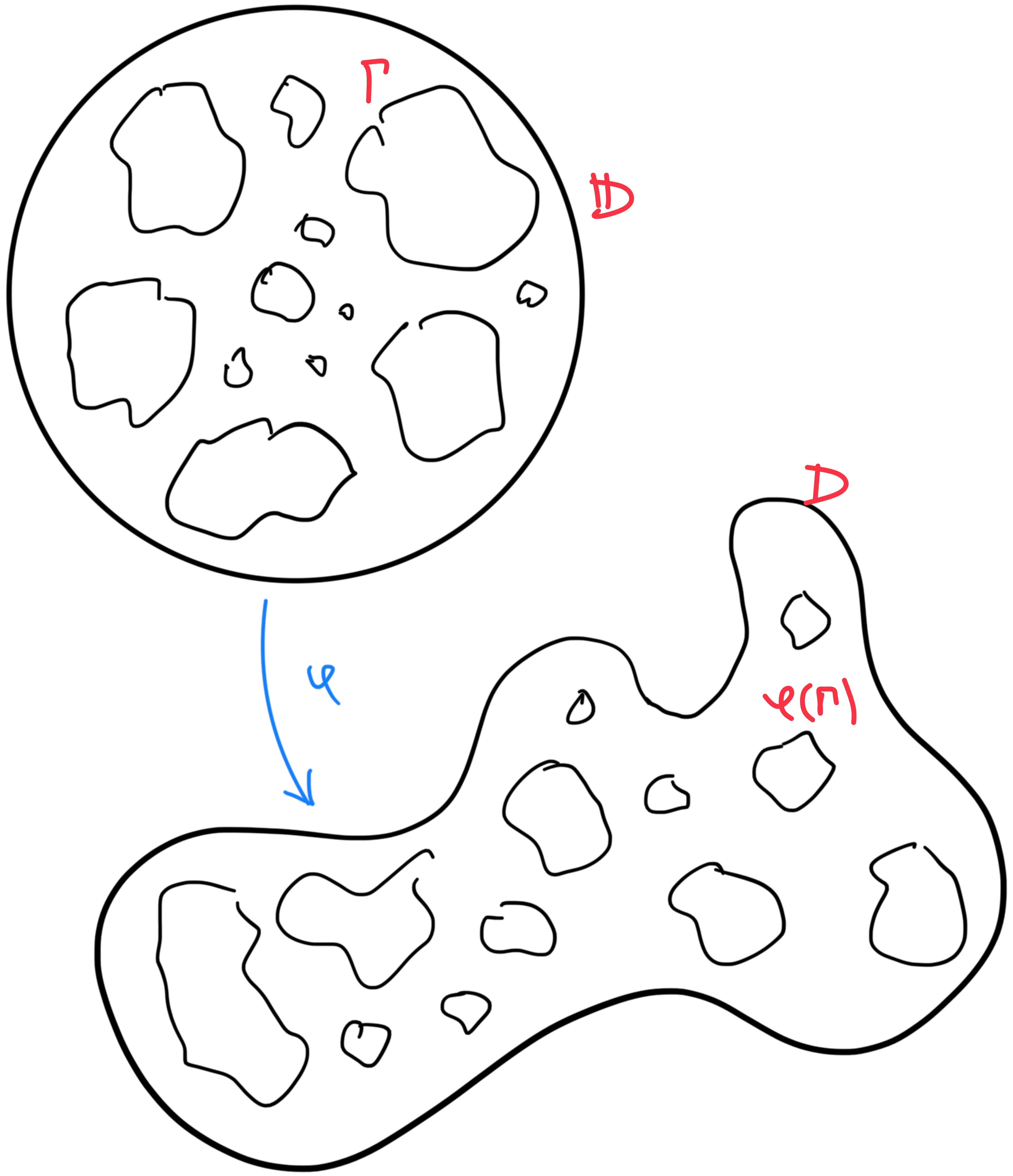
Conformal Loop Ensembles $\kappa \in (8/3,4]$

- Simple CLE_κ = random collection of disjoint simple loops in a simply connected domain of $\mathbb{C}$, introduced by (Sheffield-Werner)
- (Conjectured) **scaling limit of interfaces** in discrete models
- **Conformally invariant**
- $\Gamma \overset{(d)}{=} \text{CLE}_\kappa \text{ in } D \Rightarrow \varphi(\Gamma) \overset{(d)}{=} \text{CLE}_\kappa \text{ in } D'$



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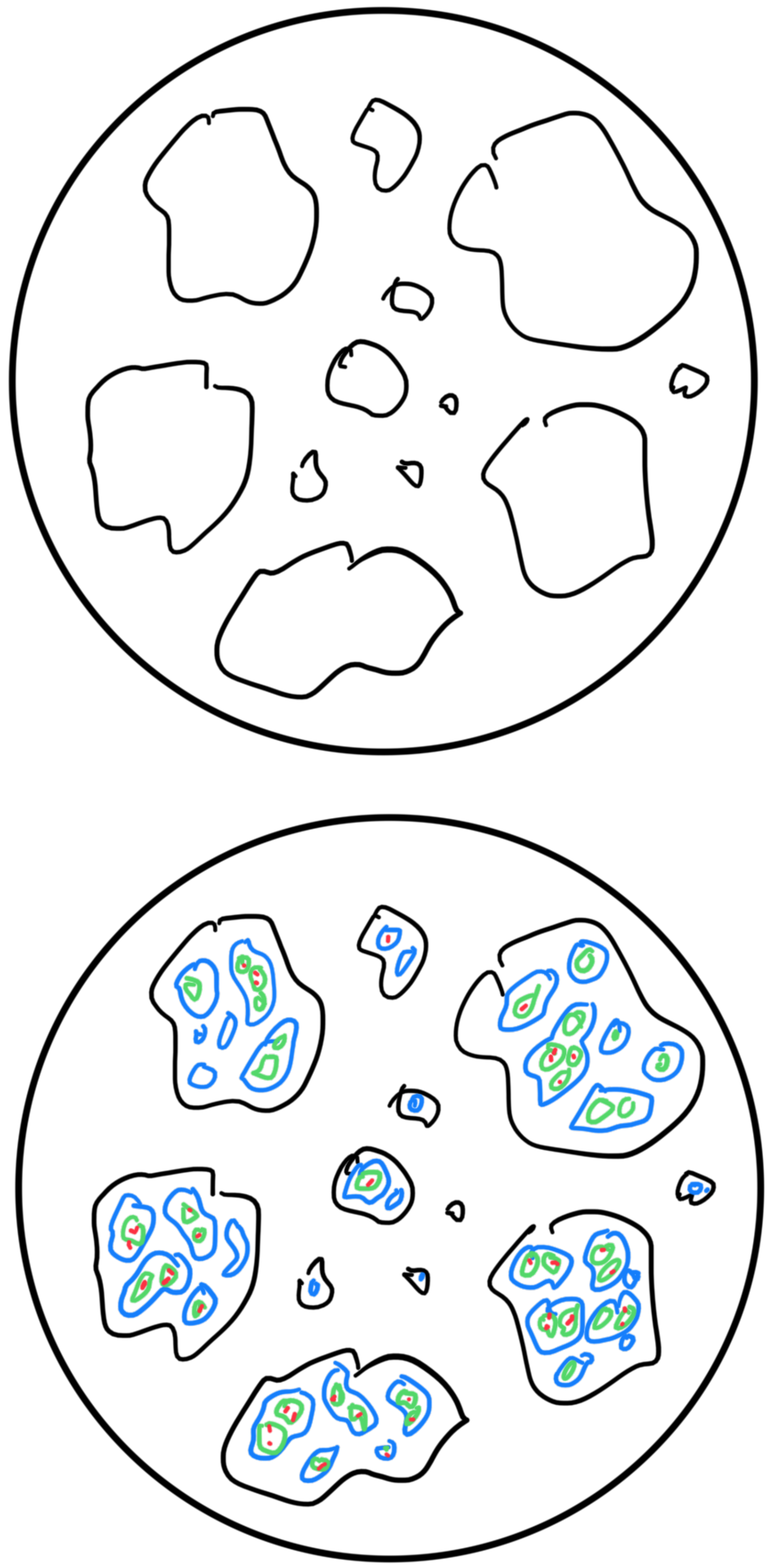
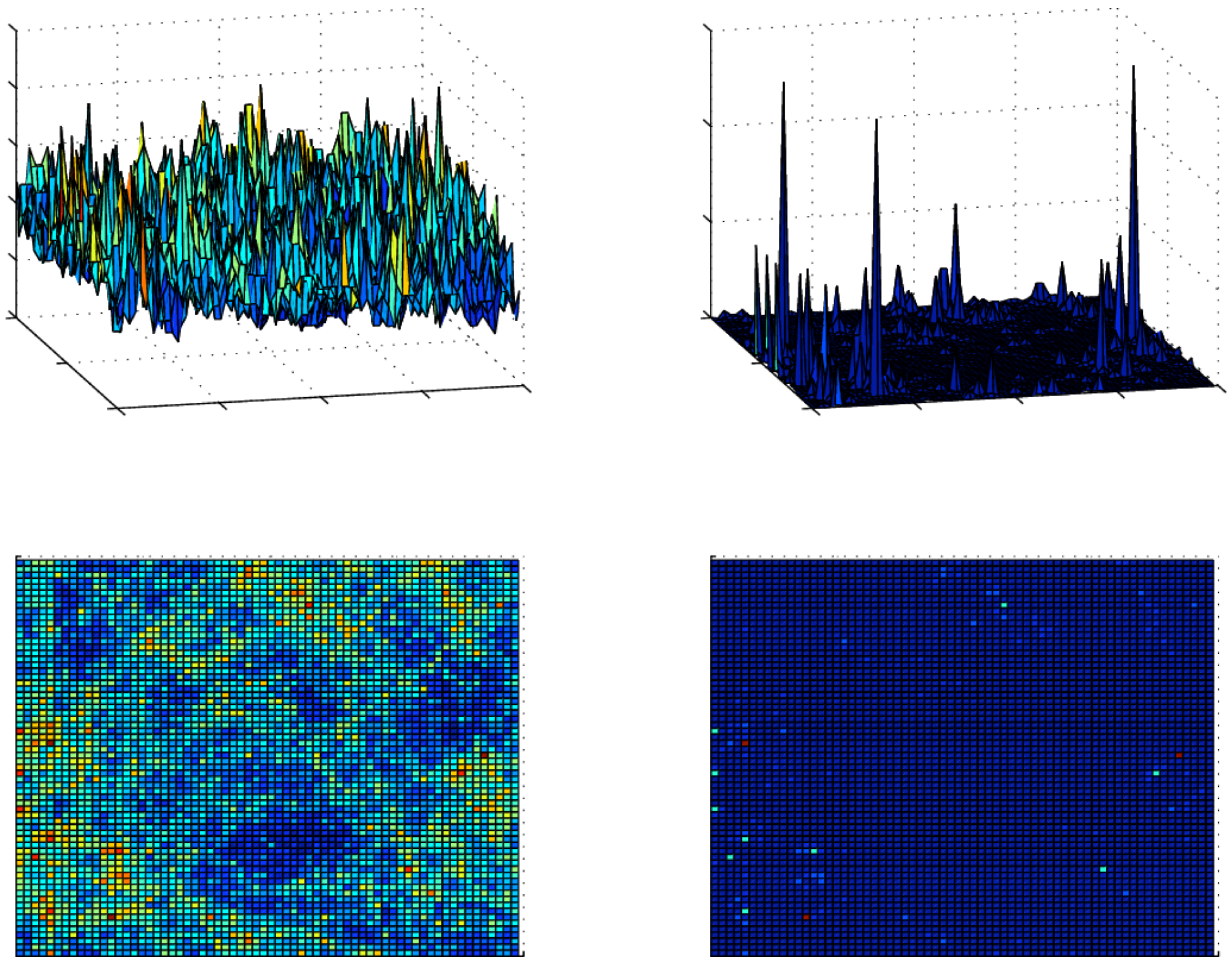
- (Conjectured) **scaling limit of interfaces** in discrete models
- **Conformally invariant**
- **Nested version** defined by iteration



Gaussian Multiplicative Chaos

Gaussian Multiplicative Chaos/ Liouville Quantum Gravity

- Family of measures on $D \subset \mathbb{R}^d$,
- Parameter $\gamma \in (0, \sqrt{2d})$
- $\mu_\gamma(dx)$ " = " $\exp(\gamma h(x))\, dx$, h a **Gaussian log-correlated field on** D



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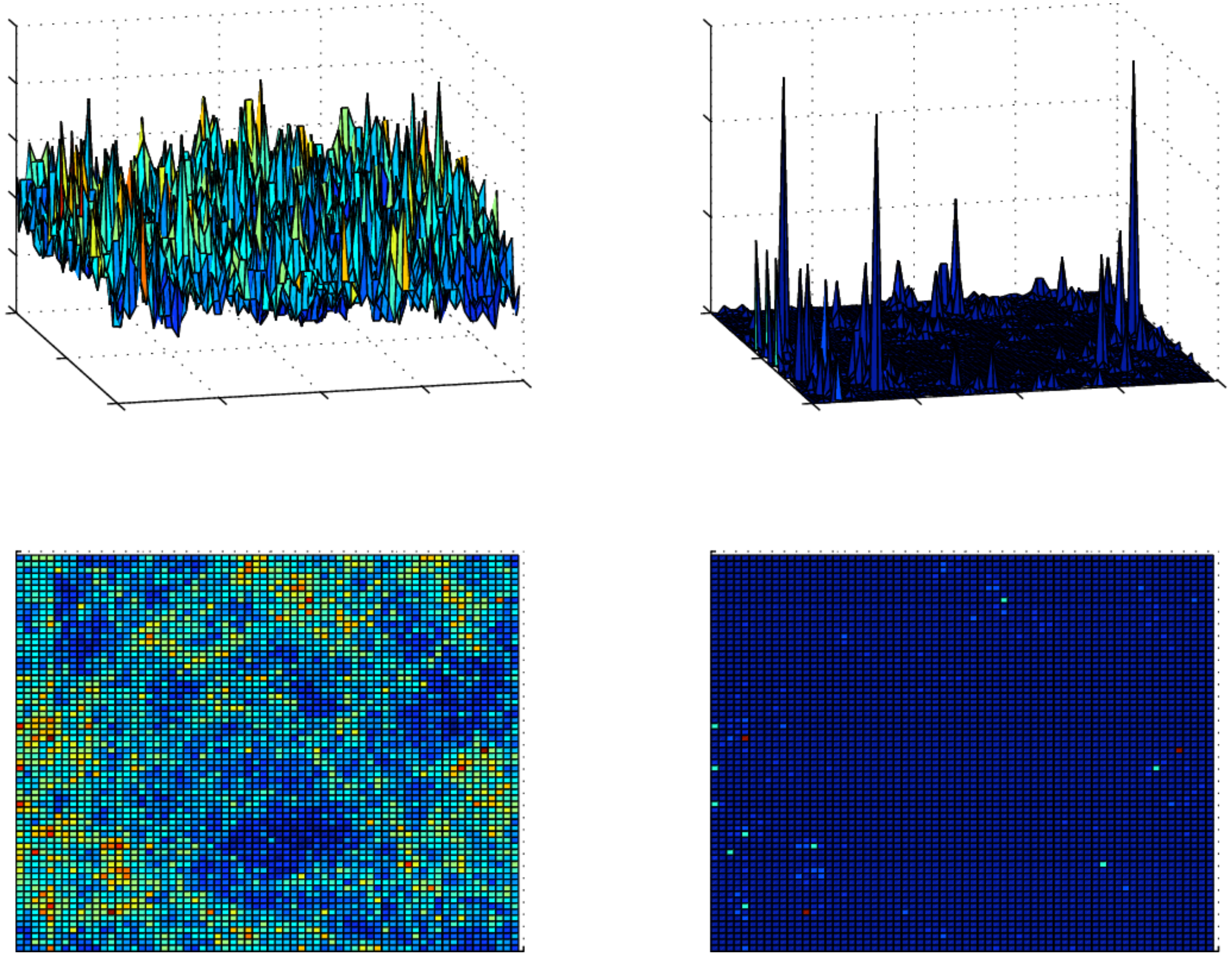
- Constructed by regularisation
- Defines **areas** of regions and **lengths** of (some) curves (Kahane, Duplantier-Sheffield, Robert-Vargas, Rhodes-Vargas, Berestycki, Shamov, Junnila-Saksman …)



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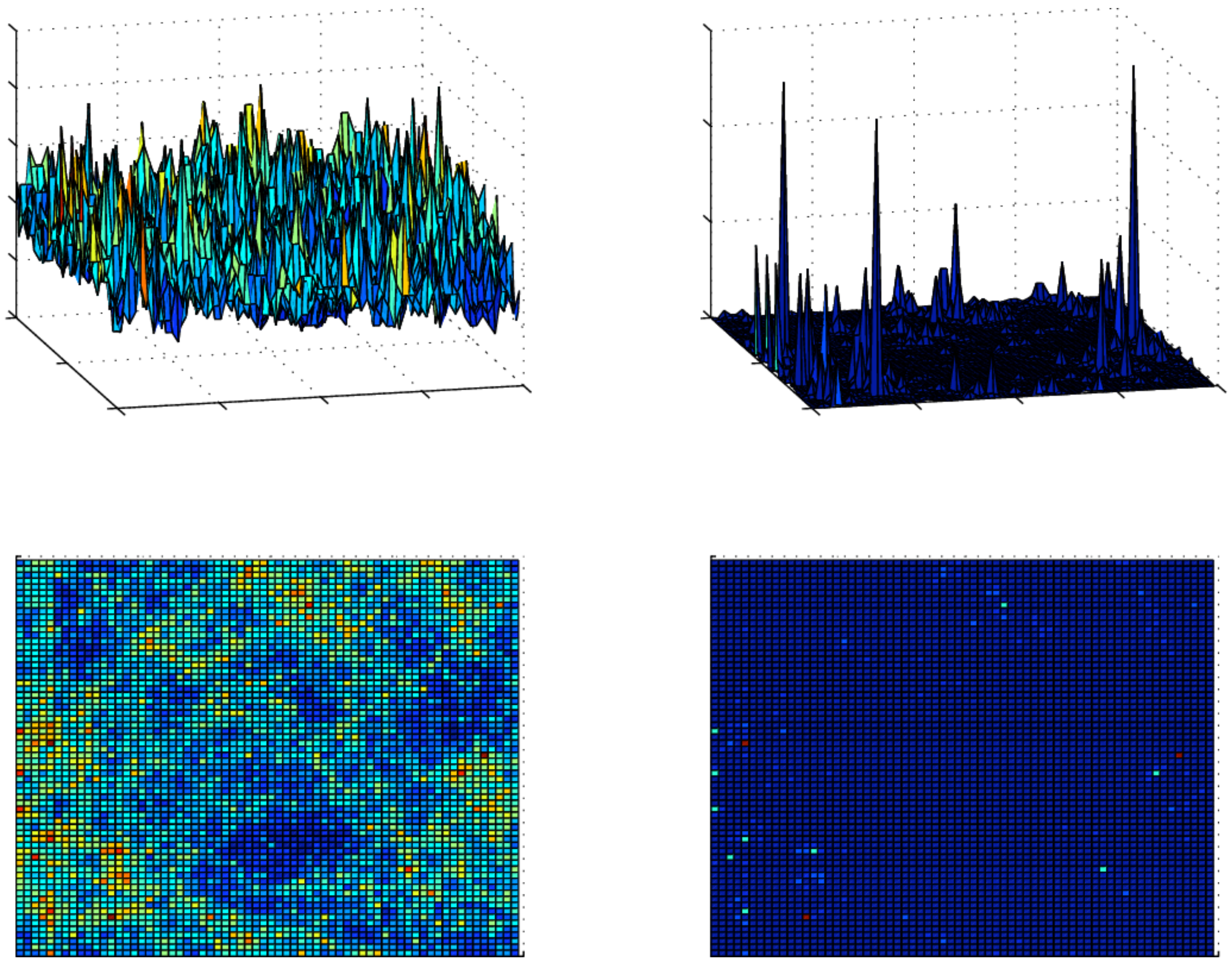
Gaussian Multiplicative Chaos/ Liouville Quantum Gravity

- Family of measures on $D \subset \mathbb{R}^2$,
- Parameter $\gamma \in (0,2)$
- $\mu_\gamma(dx)$ " $=$ " $\exp(\gamma h(x))\, dx$, with h a **Gaussian free field on** D
- Constructed by regularisation
- Defines **areas** of regions and **lengths** of (some) curves (Kahane, Duplantier-Sheffield, Robert-Vargas, Rhodes-Vargas, Berestycki, Shamov, Junnila-Saksman …)



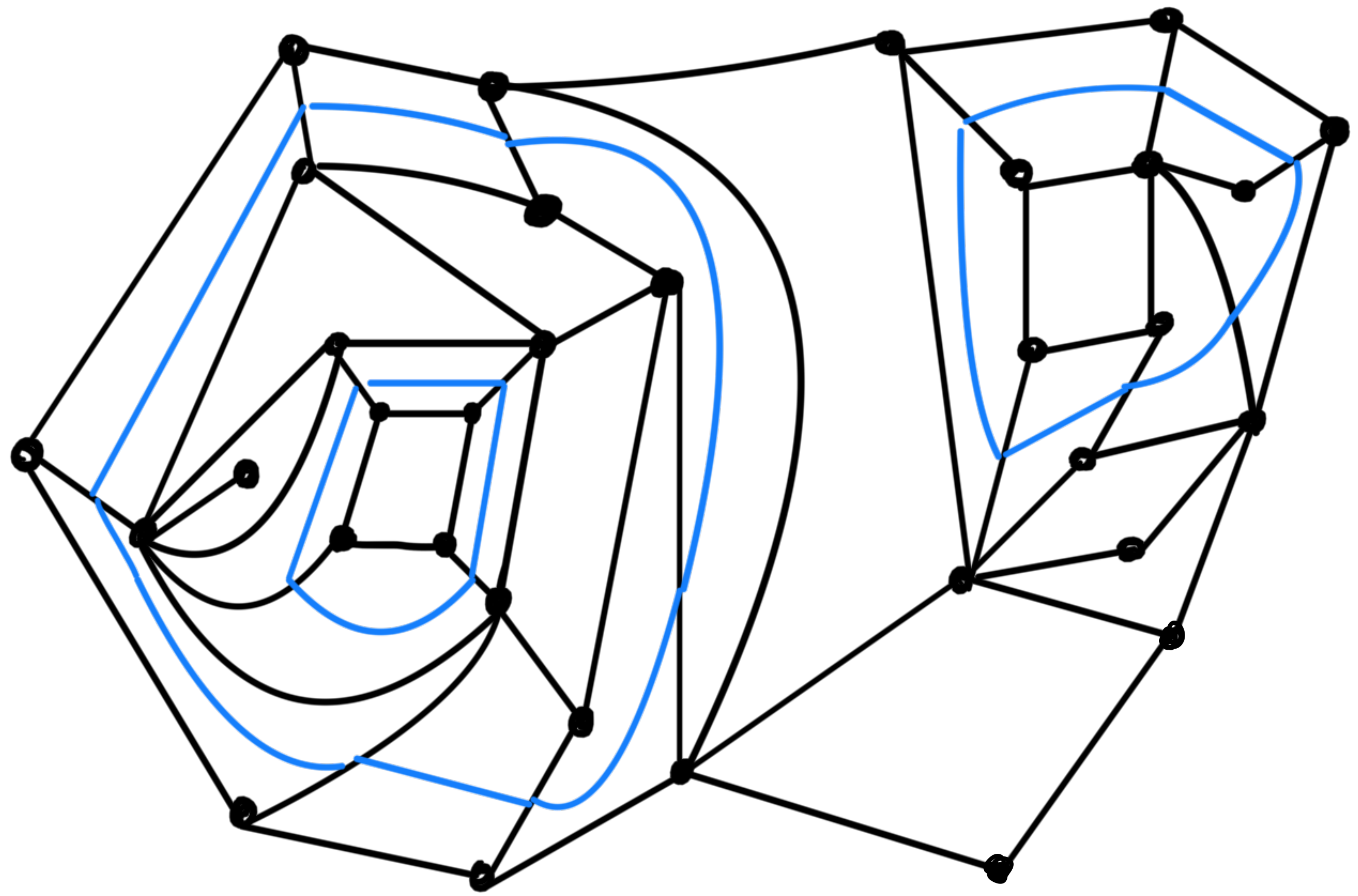
Loops and Chaos

Growth Fragmentations and Random Quadrangulations

- **Example:** O(n) model of random quadrangulation with fixed perimeter p plus loops

$$(q, l)$$

Borot-Bouttier-Guittier



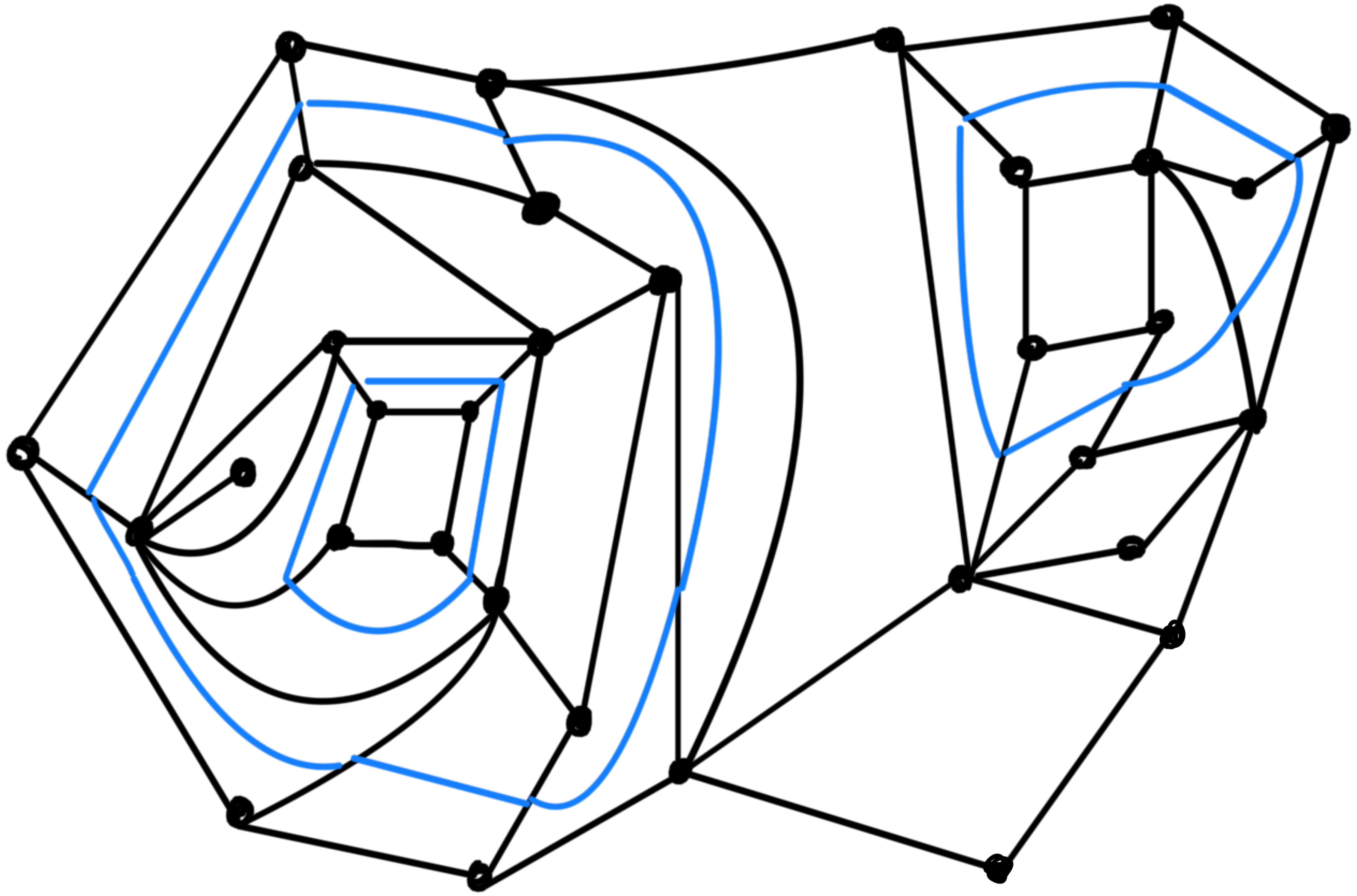
Growth Fragmentations and Random Quadrangulations

Borot-Bouttier-Guittier

- **Example:** O(n) model of random quadrangulation with fixed perimeter p plus loops

$$(q, l)$$

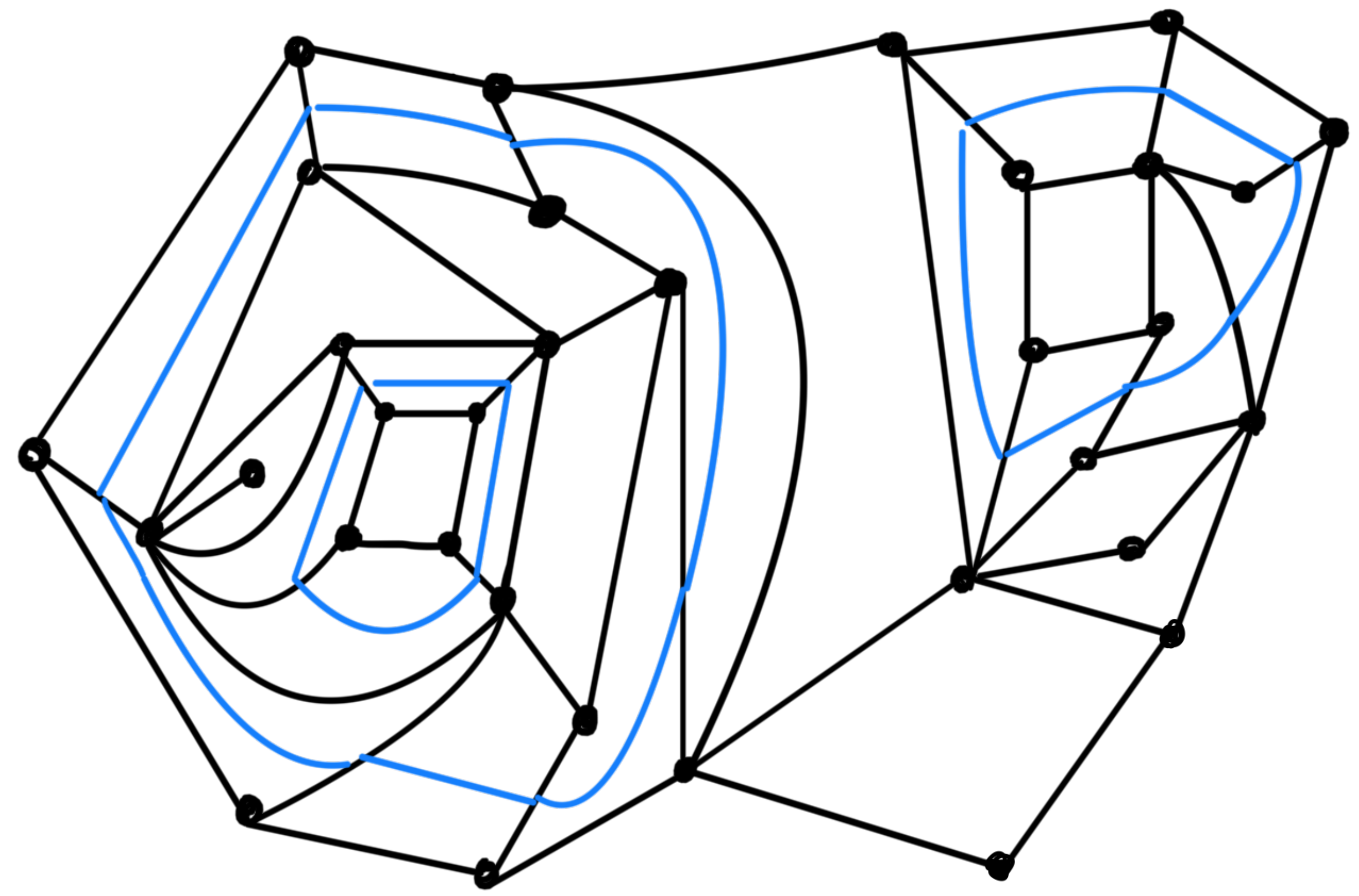
- $\mathbf{P}((q,l)) \propto g^{\#\text{faces}\, q}\, h^{\text{total length}\, l} n^{\# l}$



Loop decorated random planar maps

- **Example:** O(n) model of quadrangulation with fixed perimeter p plus loops (q, l)

- $\mathbf{P}_p((q,l)) \propto g^{\#\text{faces}\, q}\, h^{\text{total length}\, l} n^{\#l}$



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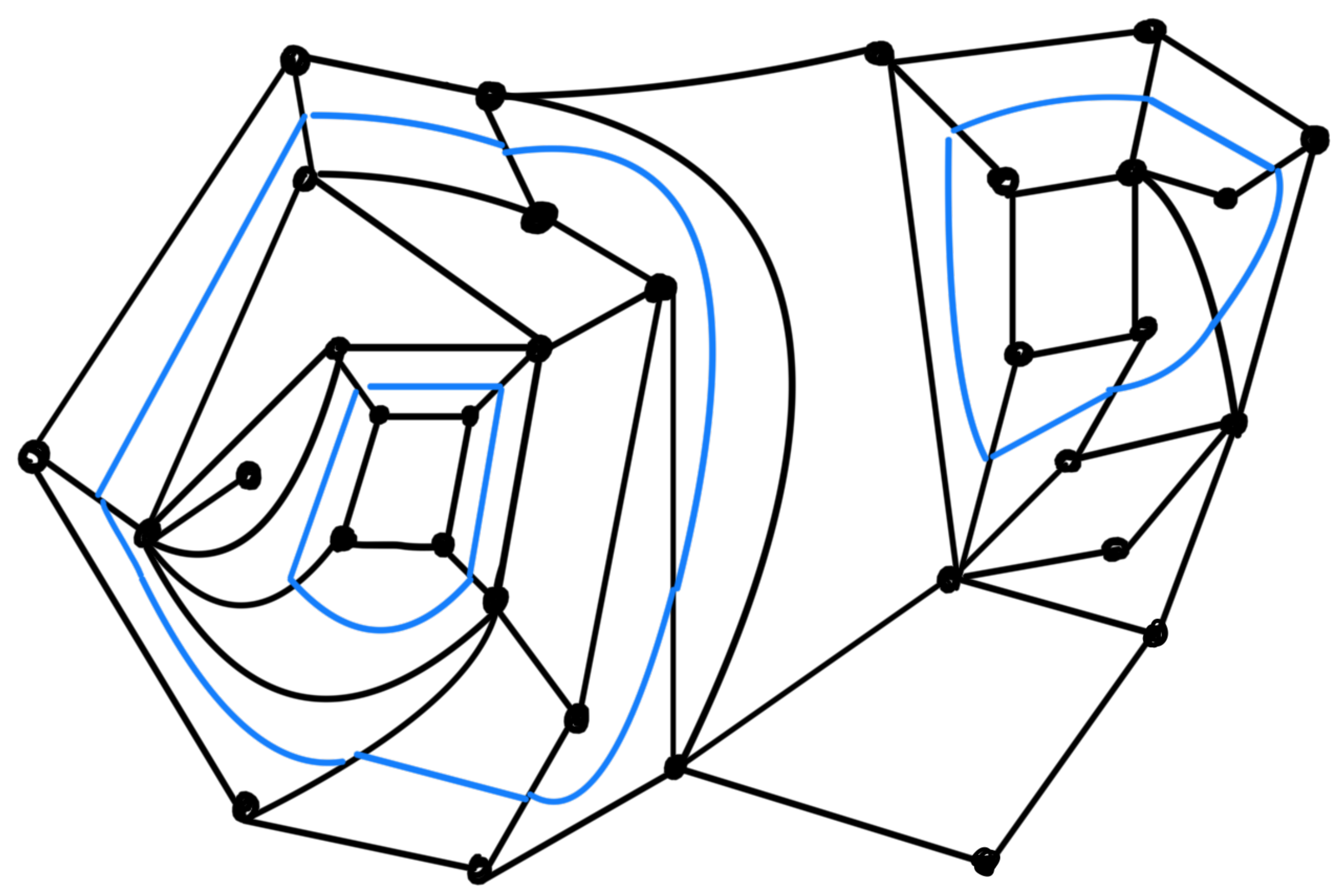

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- $\mathbf{P}_p((q,l)) \propto g^{\#\text{faces } q}\, h^{\text{total length } l} n^{\#l}$
- **Conjecture** $(n \in (0,2))$

$\exists (g^*, h^*)$ "dilute **critical**" values s.t large p scaling limit of (q, l) embedded in $\mathbb{D}$

=independent CLE_κ plus γ-GMC measure

$$\kappa = \gamma^2 = 2 - \frac{1}{\pi}\arccos\left(\frac{n}{2}\right) \in \left(\frac{8}{3}, 4\right)$$

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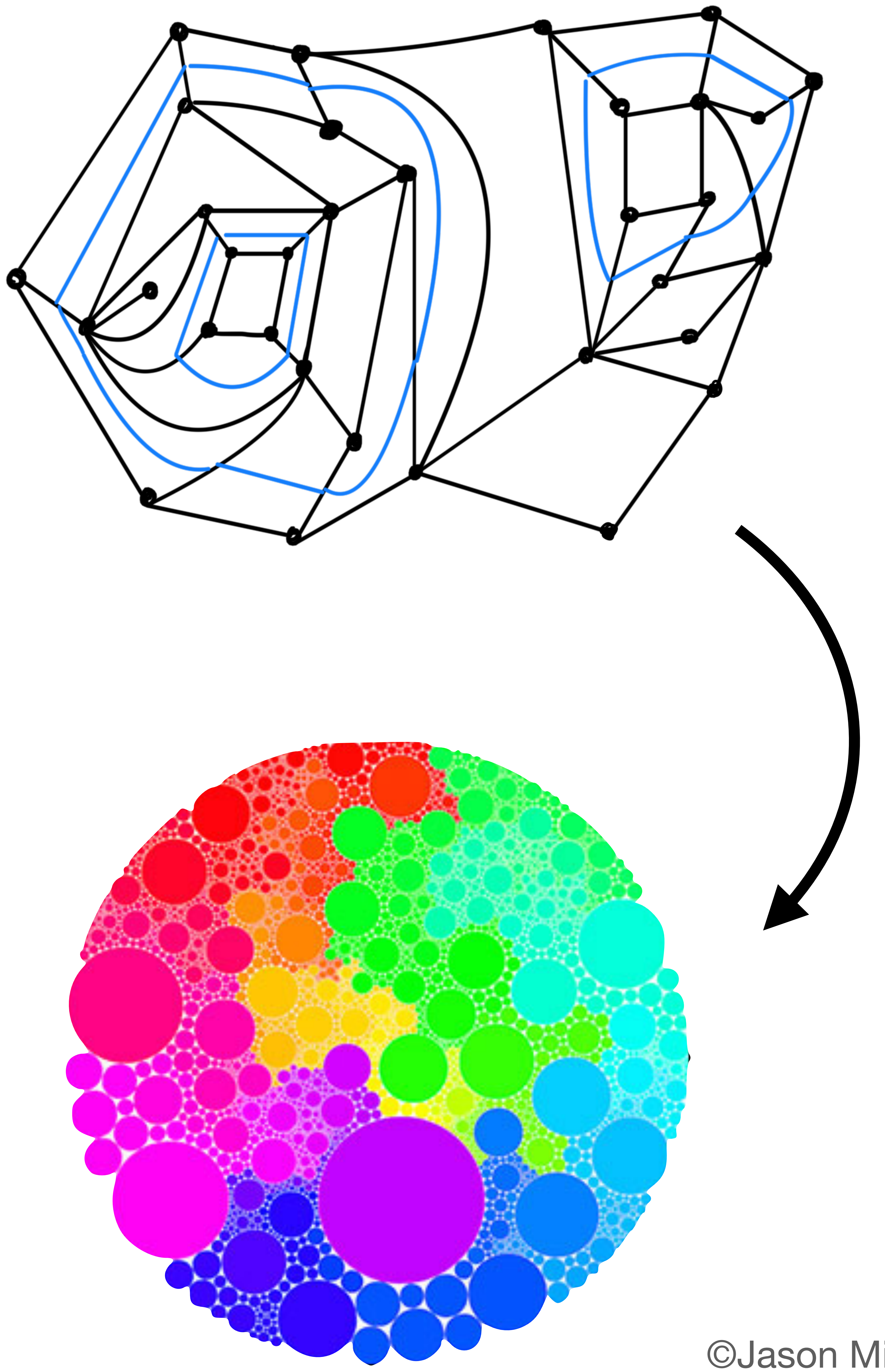

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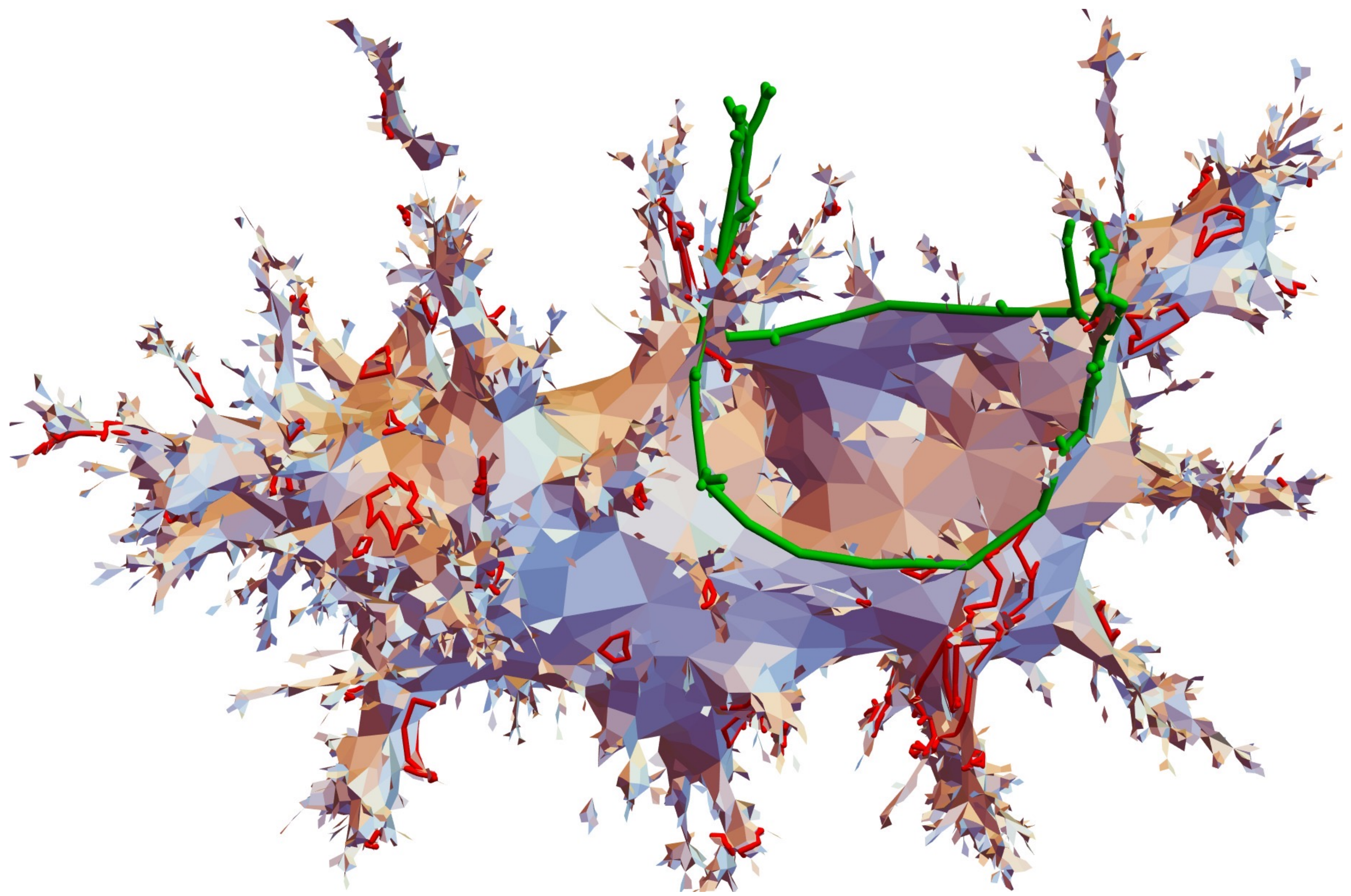

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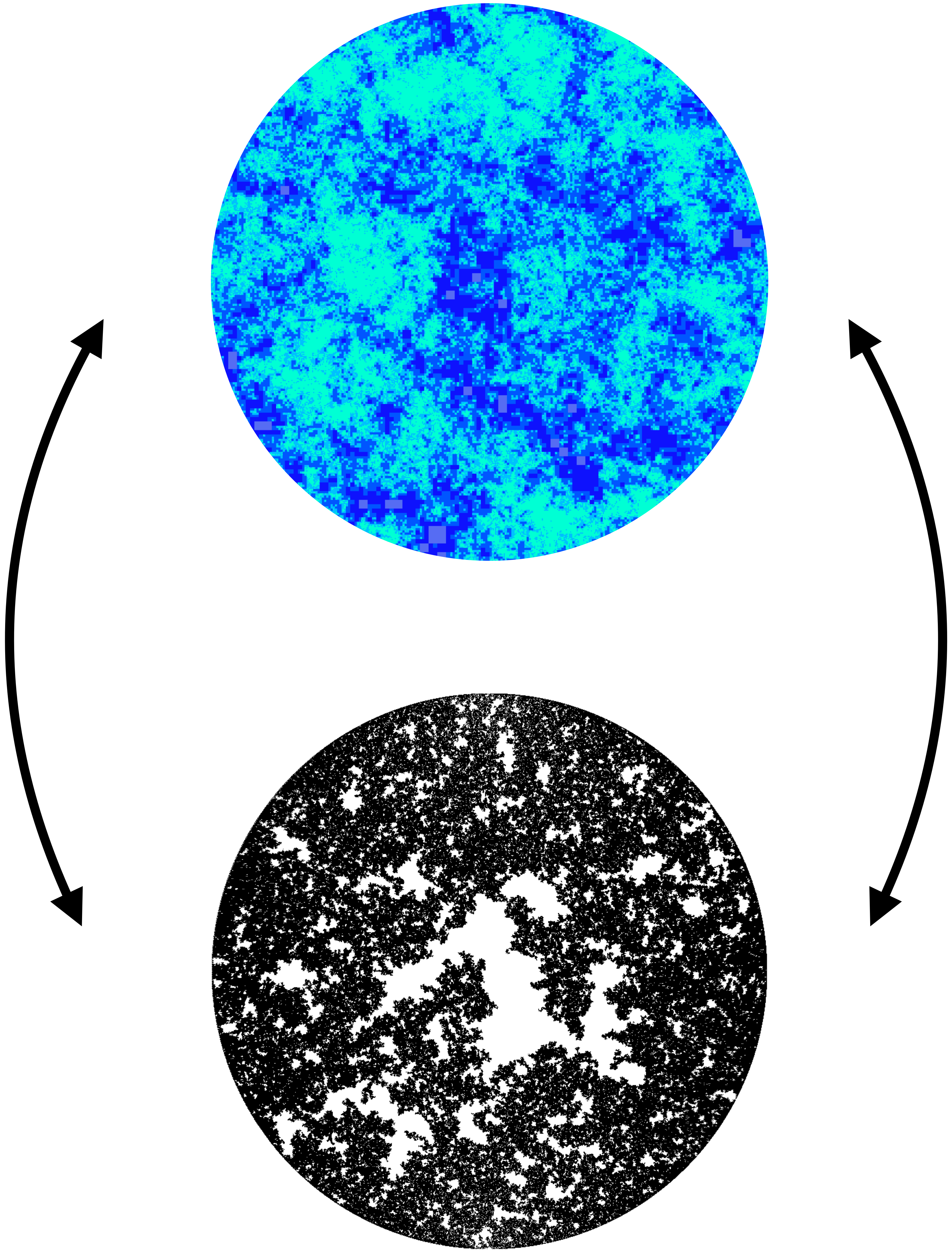
CLE decorated GMC

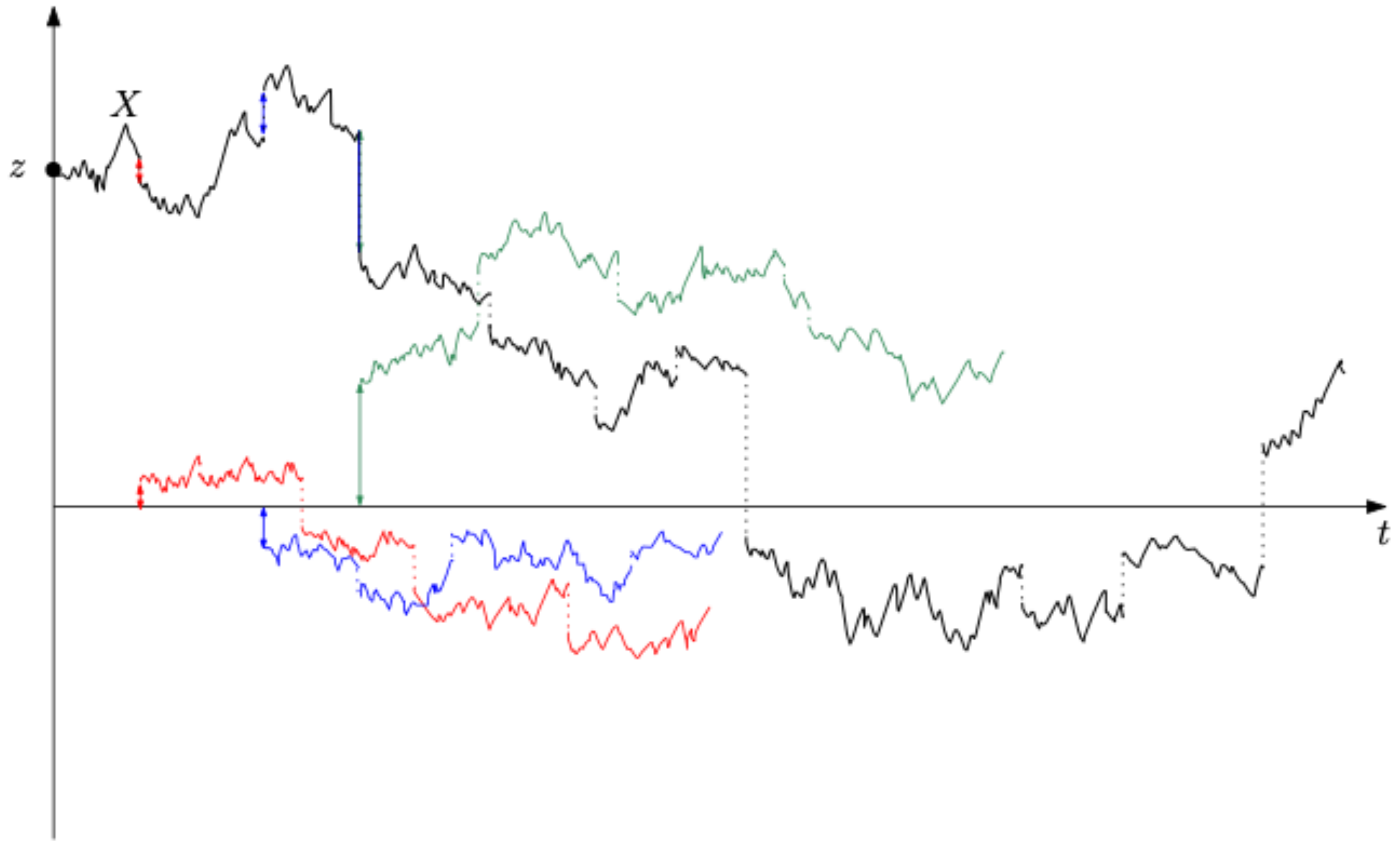
Therefore natural to study in the continuum (on $\mathbb{D}$):

- a conformal loop ensemble with parameter $\kappa \in (8/3,4]$
- a $(\gamma = \sqrt{\kappa}) - \mathrm{GMC}$ measure
- **independent** of one another



So far…

Growth Fragmentations



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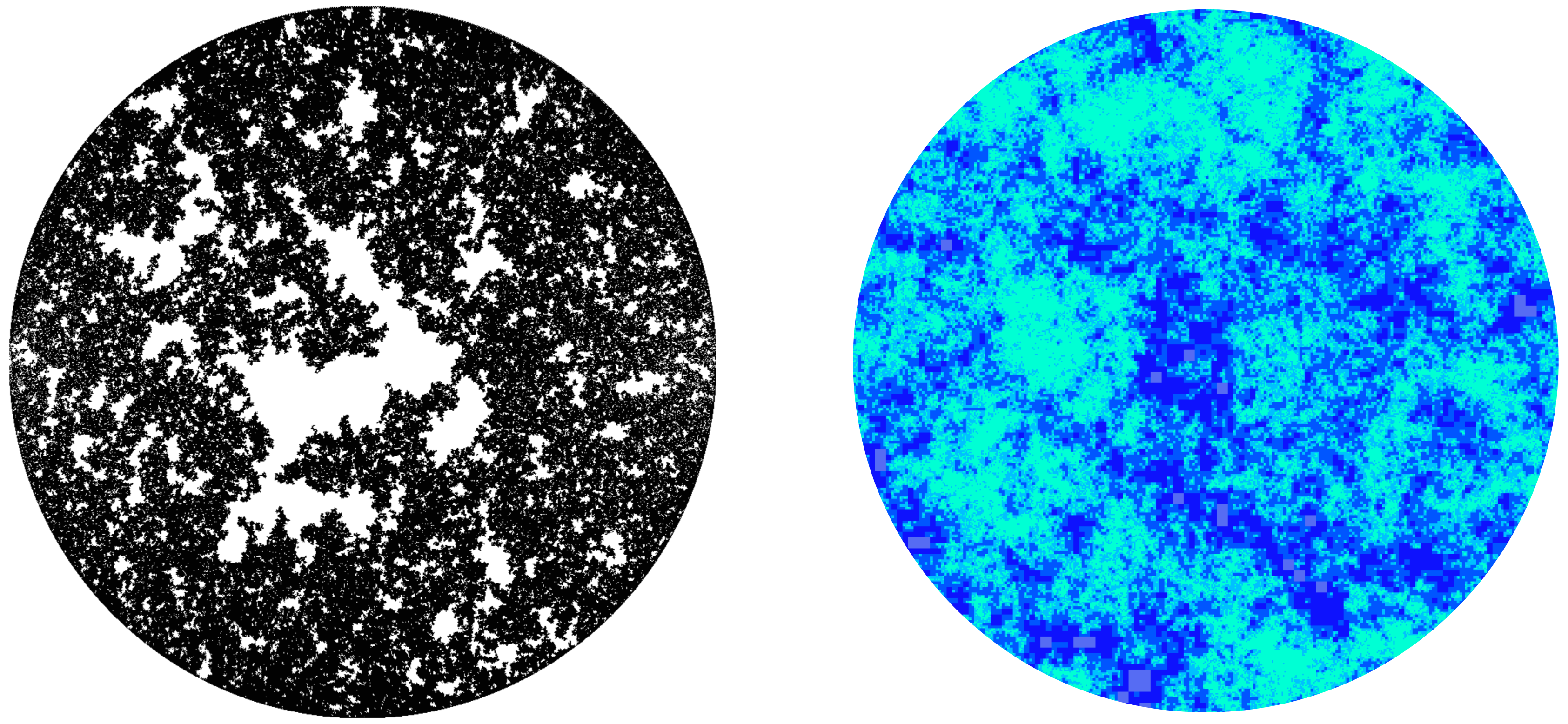
Gaussian Multiplicative Chaos & Conformal Loop Ensembles



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Existing connection with growth fragmentations

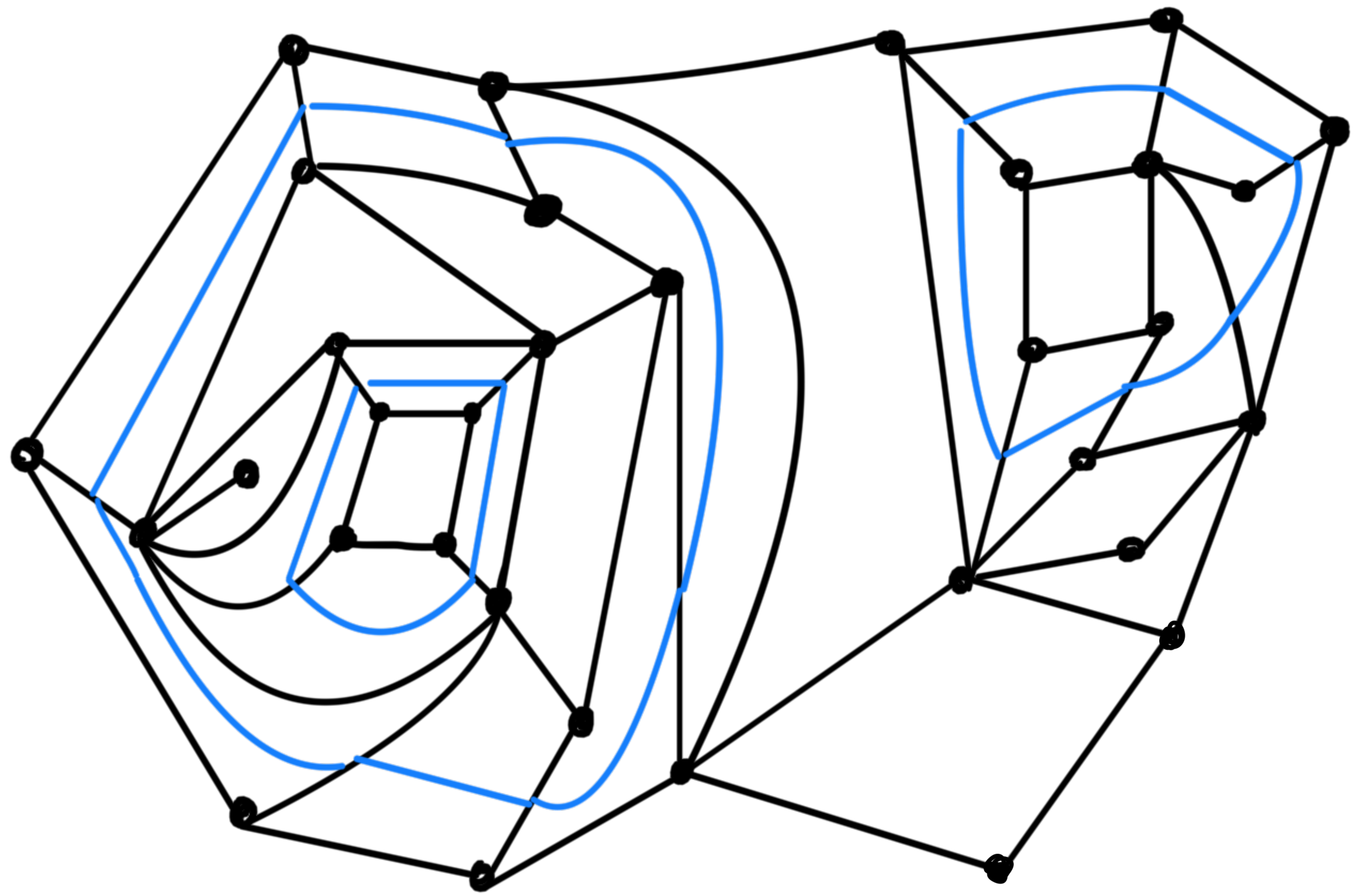
Growth Fragmentations and Random Quadrangulations

- **Example:** $O(n)$ model of random quadrangulation with fixed perimeter p plus loops

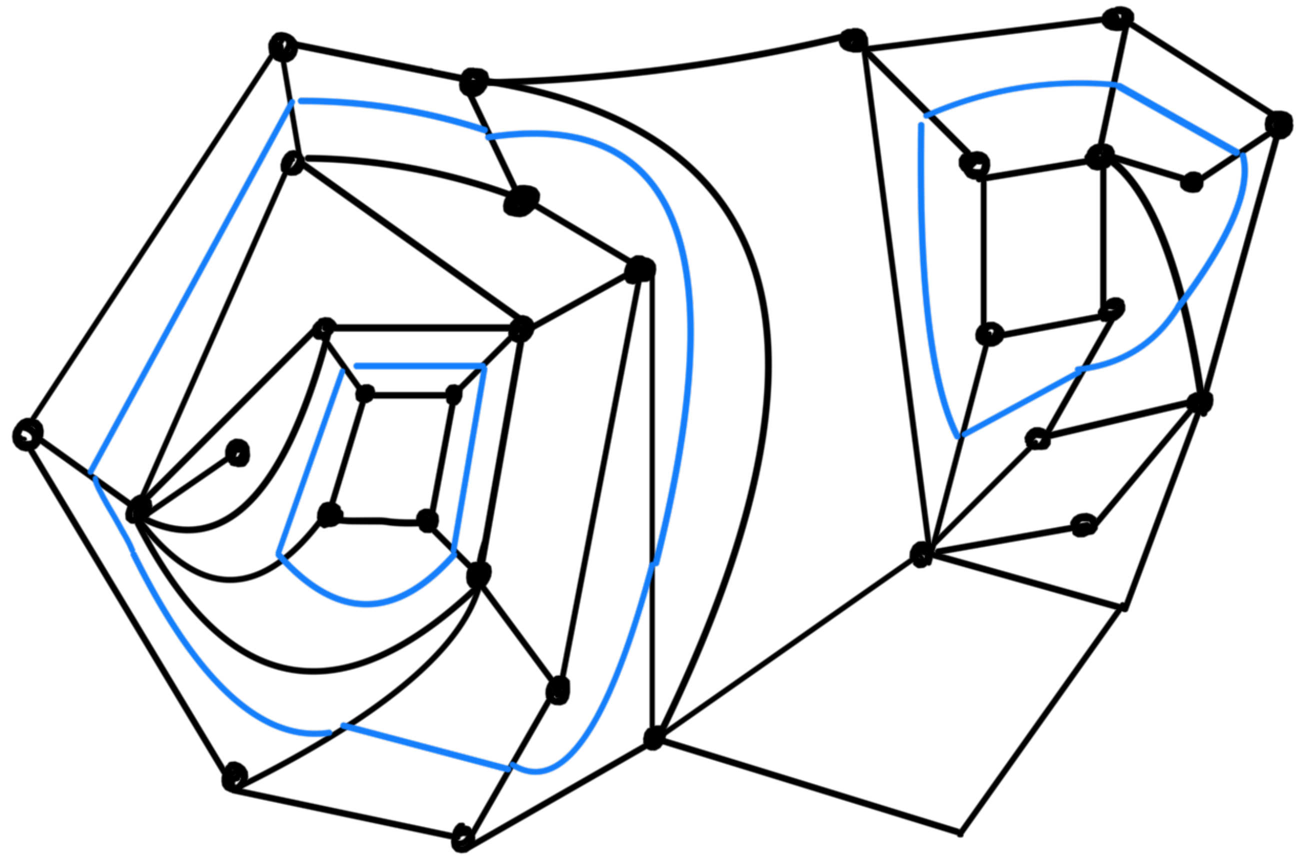
$$(q, l)$$

- $\mathbf{P}((q,l)) \propto g^{\#\text{faces}\, q}\, h^{\text{total length}\, l} n^{\#l}$
- $(g*, h*) = (g*(n), h*(n))$ dilute **critical**, $n \in (0,2)$

Borot-Bouttier-Guittier

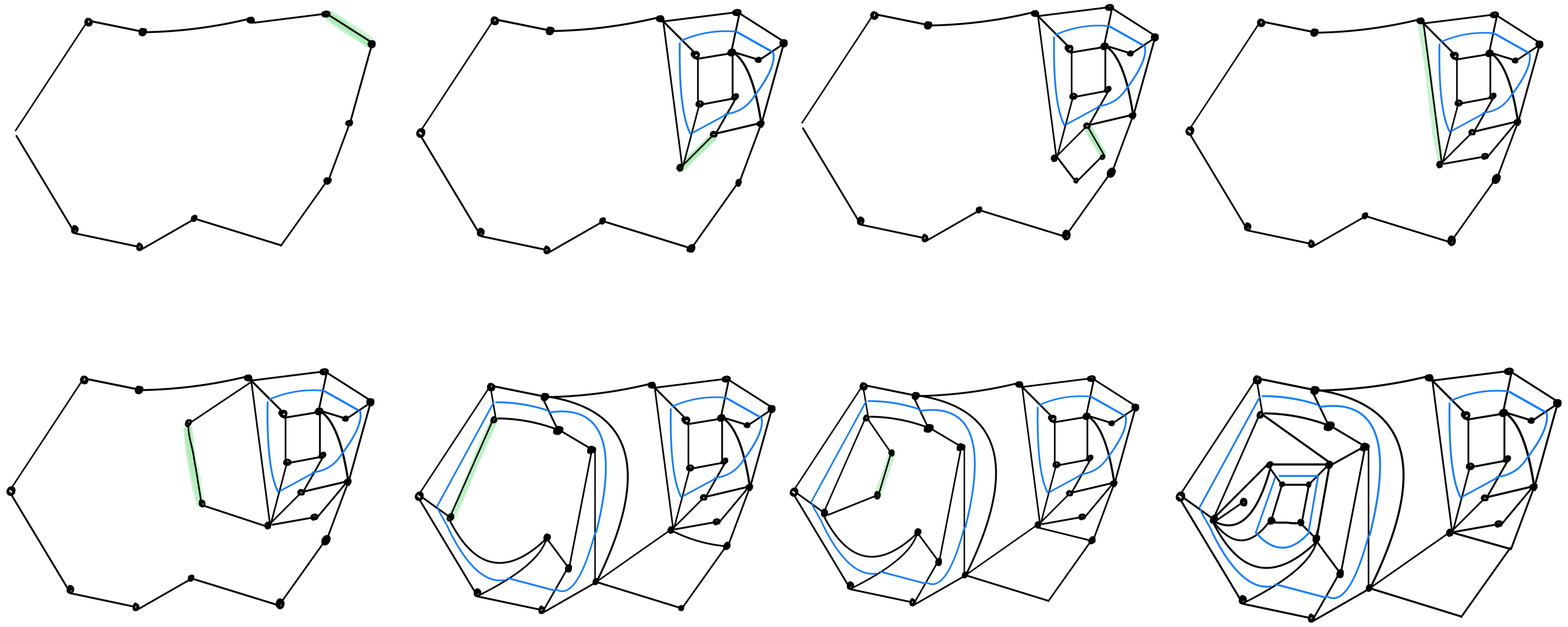


Growth Fragmentations and Random Quadrangulations

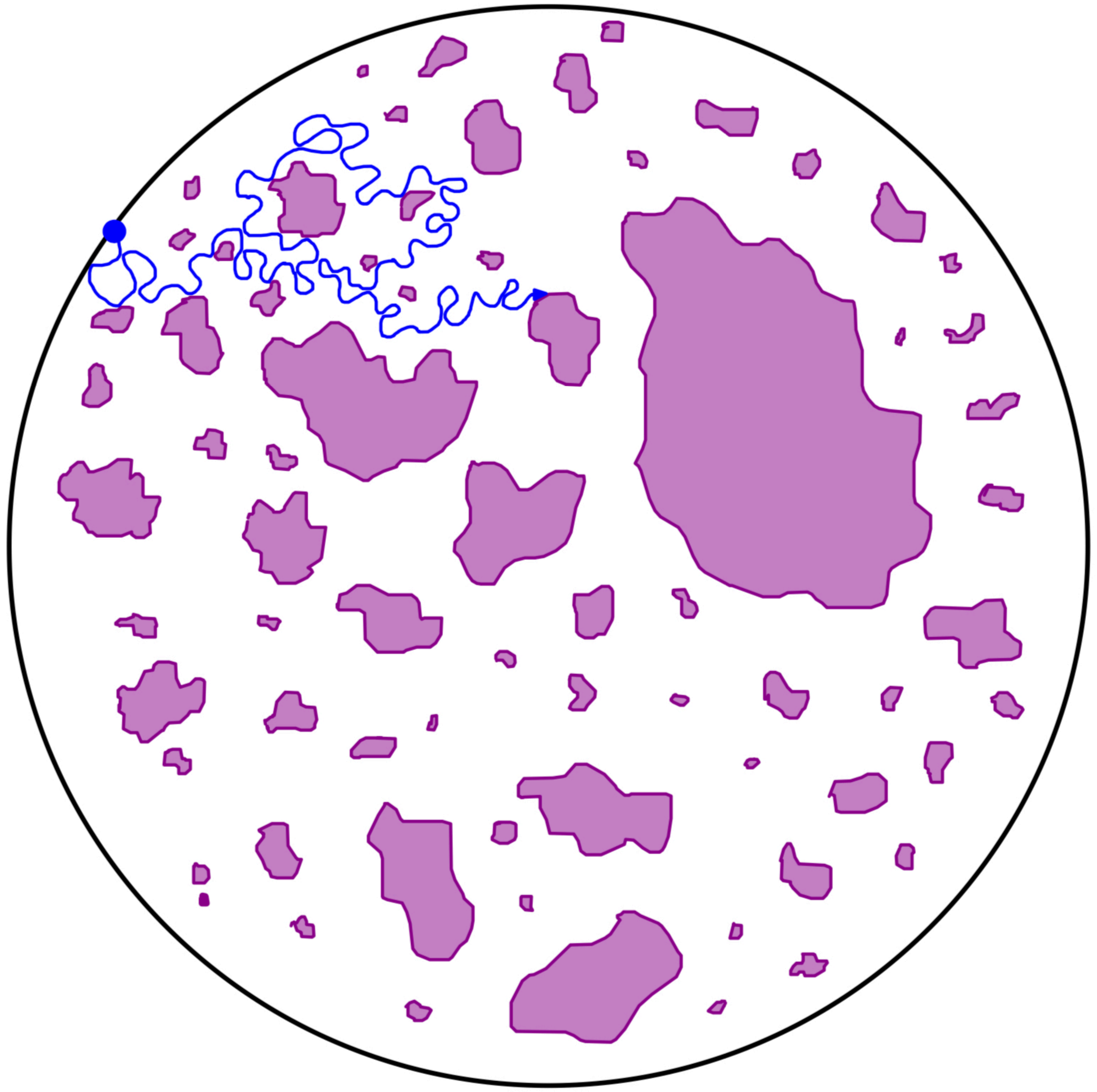


Angel, Bertoin-Curien-Kortchemski, Bertoin-Budd-Curien-Kortchemski, Budd-Curien, Chen-Curien-Maillard, Curien-Le Gall …

- **Peeling processes** explore maps from boundary inwards in a **Markovian** way
- **Branching** variants
- Functional limit theorems as perimeter $p \to \infty$
- Get explicit **growth fragmentation** for perimeters of to-be-explored regions



CLE Growth Fragmentations



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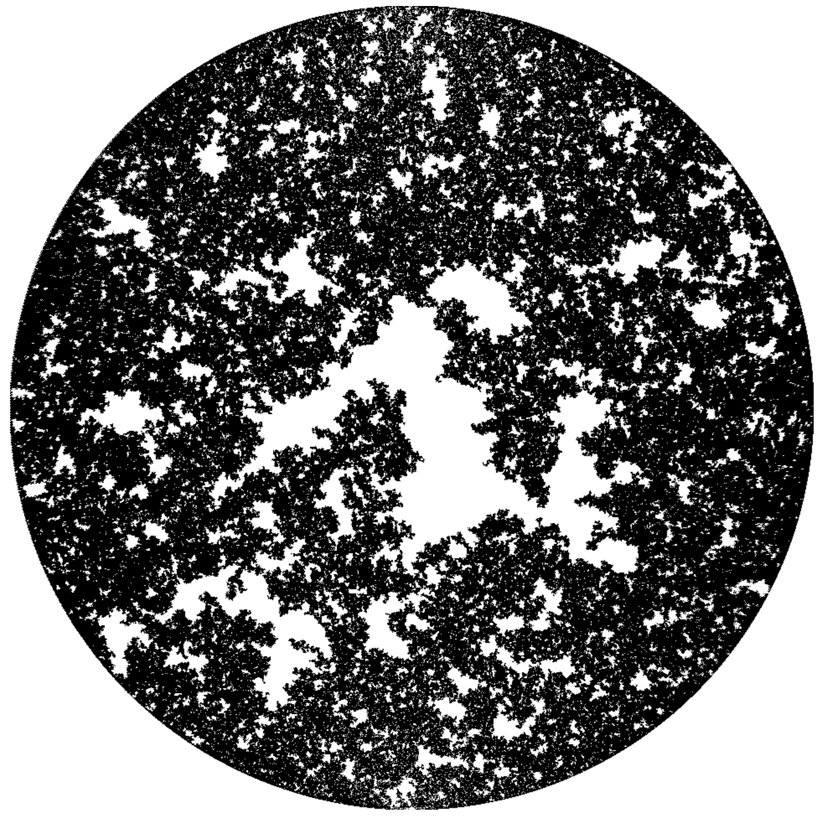
- **Recall** $n \in (0,2)$, with (g^*, h^*) as before: large volume scaling limit of (q, l) should be an independent CLE_κ plus γ-LQG surface with
- $$\kappa = \gamma^2 = 2 - \frac{1}{\pi}\arccos\left(\frac{n}{2}\right) \in \left(\frac{8}{3}, 4\right)$$
- Miller-Sheffield-Werner $\rightsquigarrow$ $\exists$ continuum analogue of a **peeling exploration:** random interface in CLE gasket discovering CLE loops along the way
- Obtain same processes as Bertoin-Budd-Curien-Kortchemski

The $\gamma = 2, \kappa = 4$ case

Special case $(\gamma = 2, \kappa = 4)$

- $\kappa = 4$ is a **critical value** for SLE and CLE; SLE_κ is simple for $\kappa \leq 4$ but self-touching for $\kappa > 4$
- $\gamma = 2$ is **critical** for GMC in the plane; usual definition **doesn't work**.
- $(\gamma = 2)$-GMC can be defined from $(\gamma < 2)$-GMC, but need to **blow up** measures by $1/(2 - \gamma)$
- Miller-Sheffield-Werner's exploration doesn't have a nice limit, but…
- Budd-Curien-Marzouk: peeling the gasket of a(n) (infinite) critical $O(2)$ model $\rightsquigarrow$ Cauchy process



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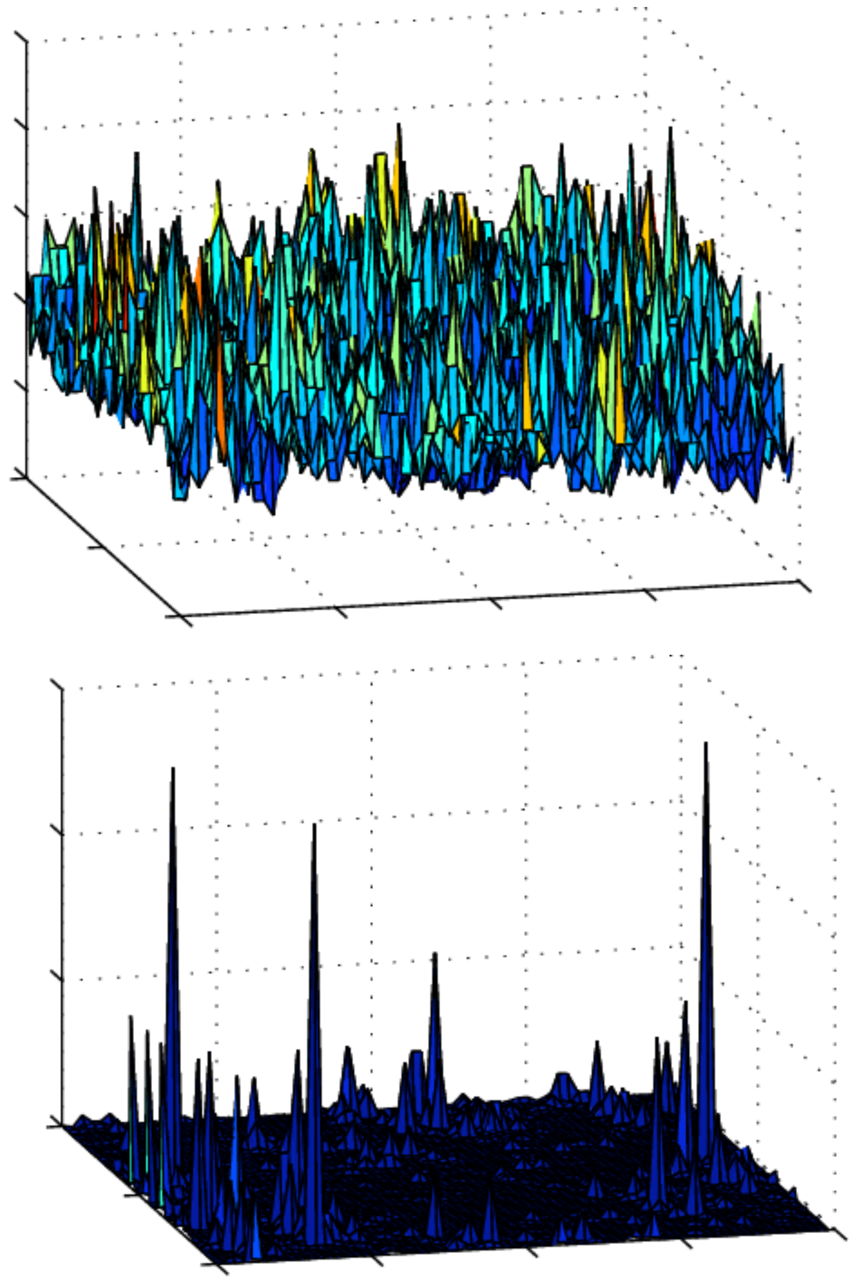

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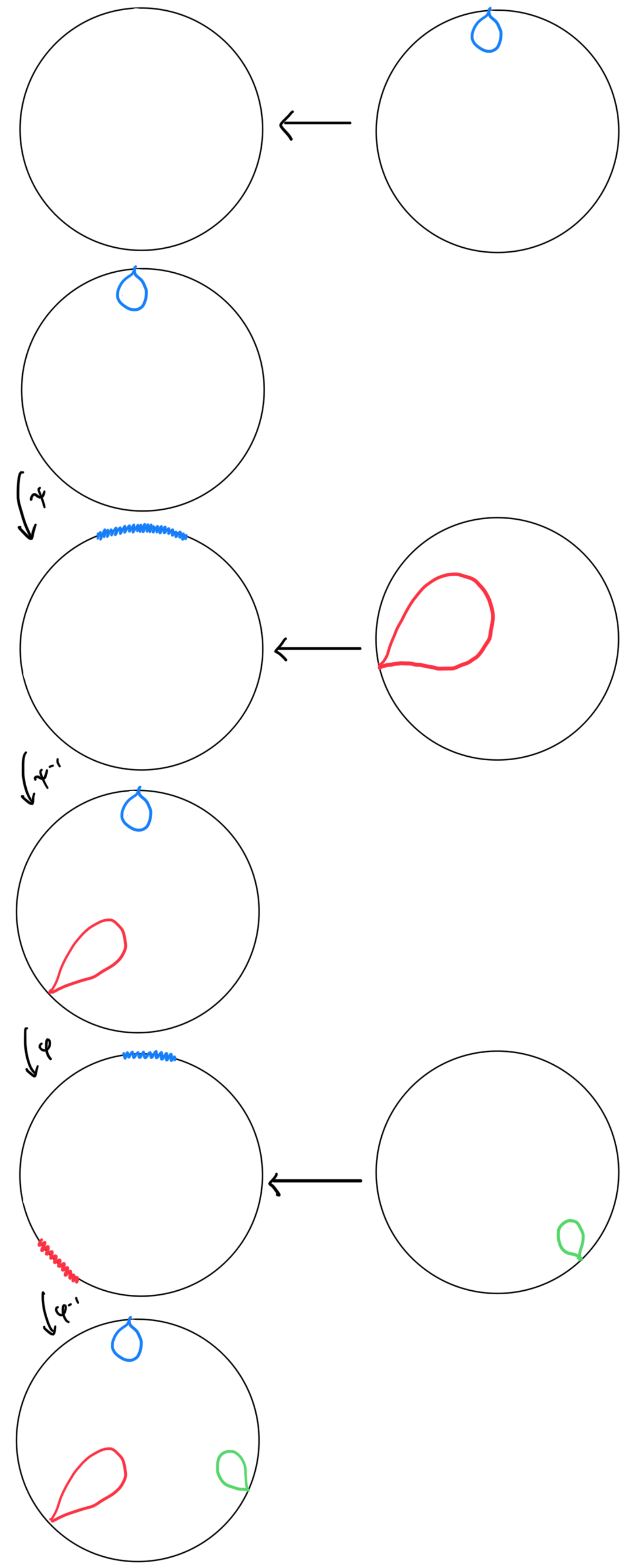
Special case $(\gamma = 2, \kappa = 4)$

<u>Theorem</u> <u>(Aru-Holden-P.-Sun)</u>

Take a **uniform branching exploration*** of a CLE_4 in $\mathbb{D}$ and an independent GFF (variant) on $\mathbb{D}$

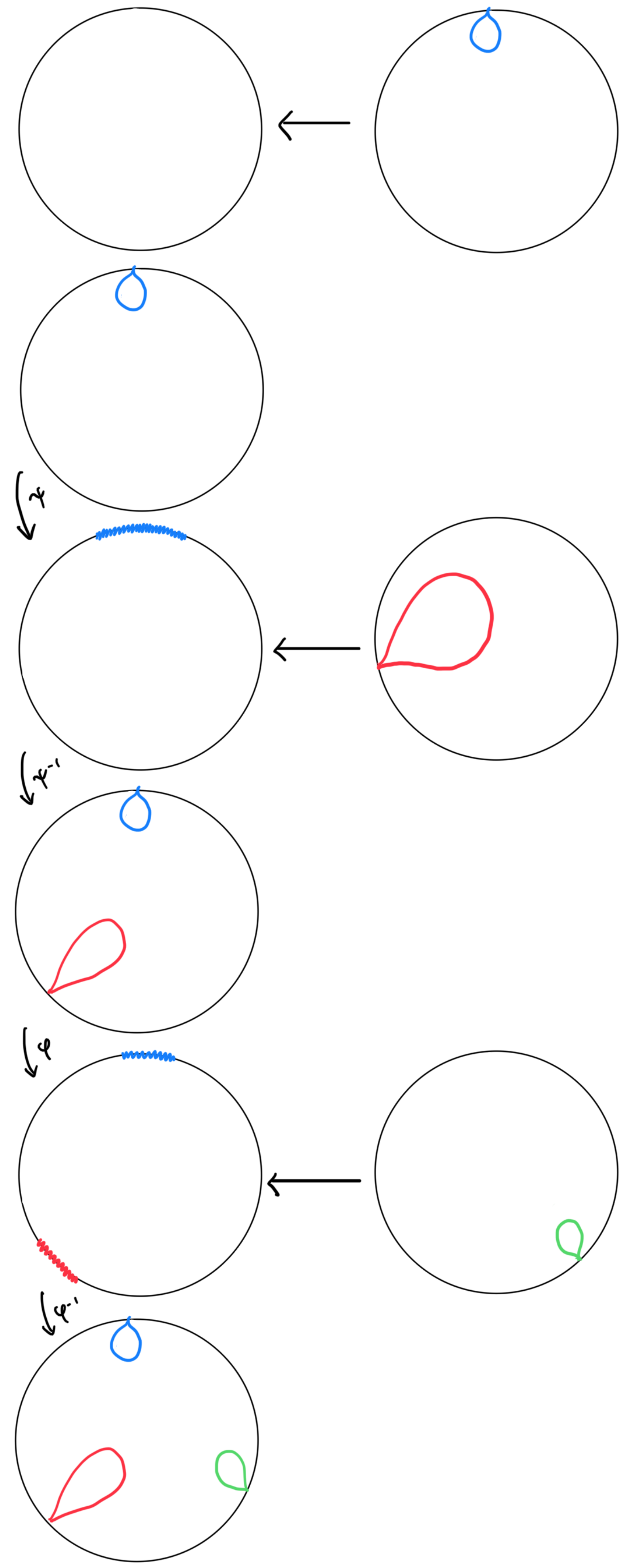
*Roughly: a PPP of SLE_4 type bubbles are "added in" uniformly on the boundary of the to-be-explored domain: see drawing!

Then the critical GMC lengths, as measured by the GFF, of the yet-to-be-explored connected components gives an (explicit) **growth fragmentation process**



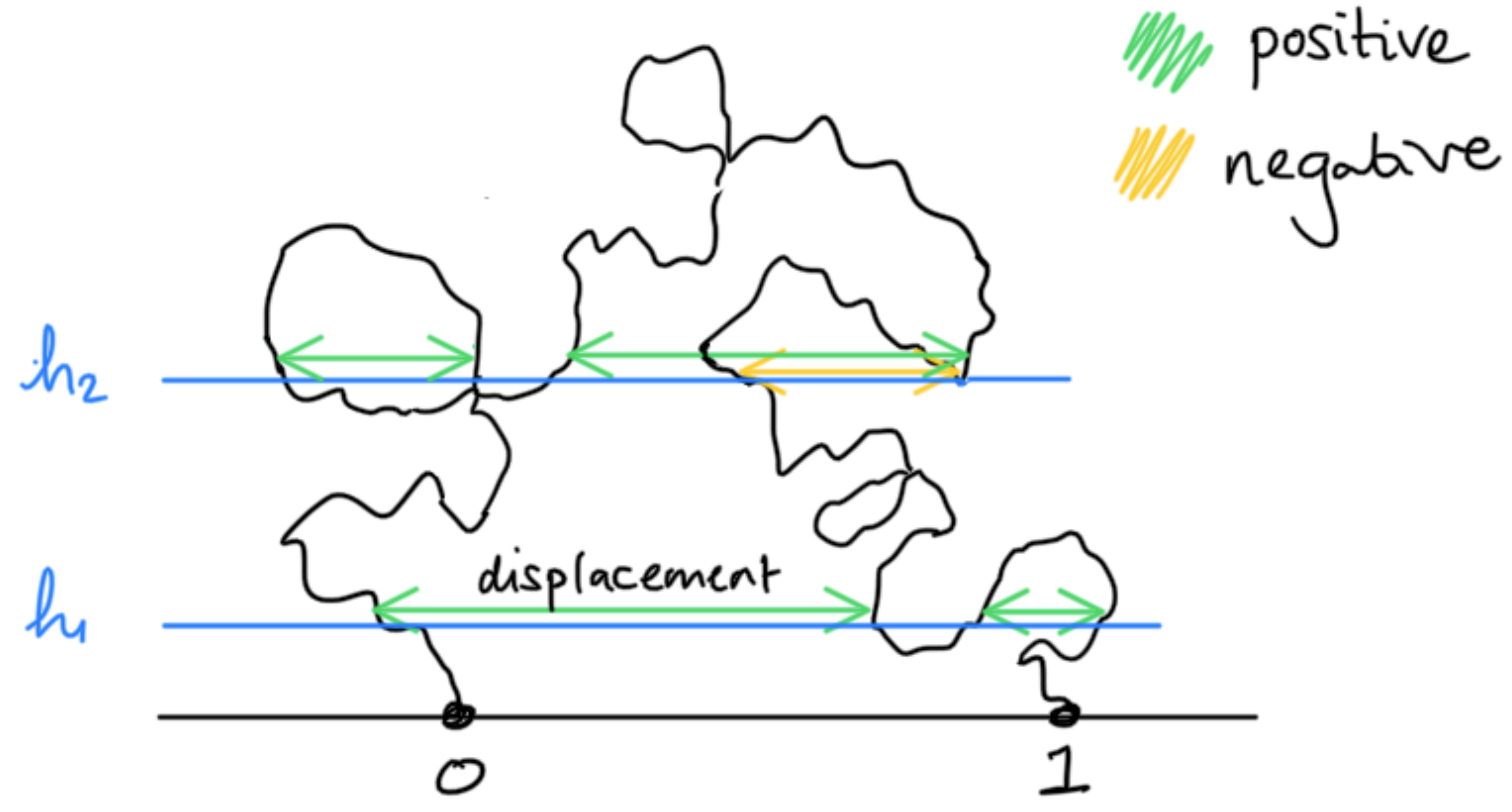
Comments

- The uniform CLE_4 exploration is different to that considered by Miller-Sheffield-Werner in the subcritical case
- The growth fragmentation is explicit and **signed** (signs correspond to level of nesting)
- "Eve cell" (pssMp X from def of GF) is a type of **Cauchy process**
- Time parameterisation = **"quantum distance"** from boundary
- It's exactly the same the signed GF that Aïdékon-Da Silva constructed out of a **Brownian half plane excursion…**



Brownian half-plane excursions

Growth fragmentations and Cauchy processes (Aïdekon & Da Silva)



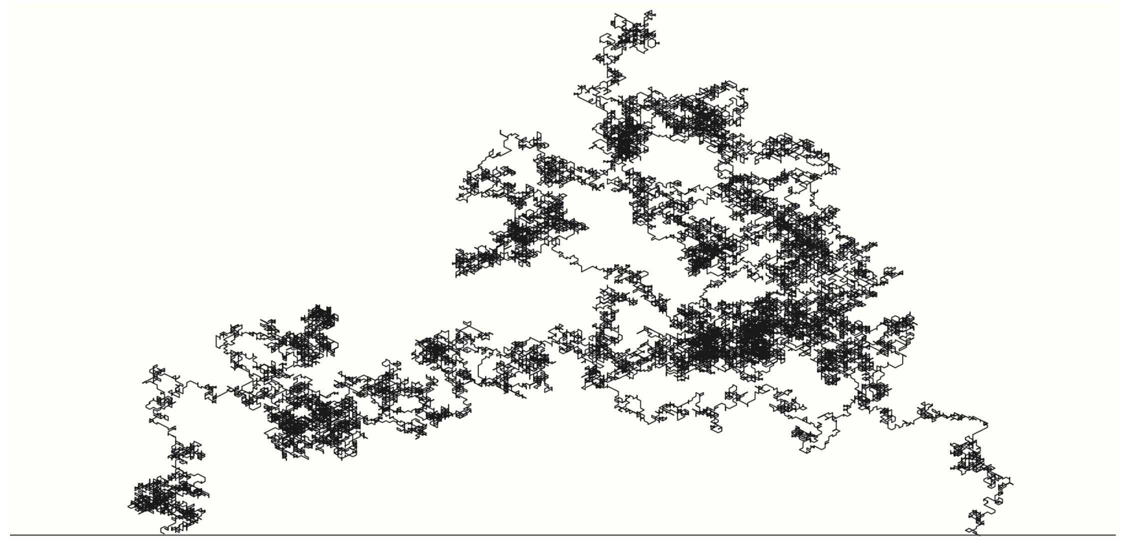
- Start with a half-planar Brownian excursion (given duration, X coordinate is Brownian bridge and Y coordinate is independent Brownian excursion)
- At each height $h \geq 0$ have countable collection of sub-excursions above h
- These have **masses** (widths) with **signs** according to direction traversed by the Brownian half-plane excursion
- Gives a **signed growth fragmentation** with the same law as in our theorem

Our Result

Correspondence:

Brownian half-plane excursion ↔
CLE_4 + "critical quantum disk"

CLE_4 decorated critical quantum disk	Brownian half-plane excursion
Branching structure defined by exploration	Branching structure in the associated CRT
Boundary lengths of discovered disks	Displacements of sub-excursions above heights
Areas of discovered disk	Durations of sub-excursions above heights
Parity of nesting	Sign of subexcursion
Some notion of "quantum" distance from boundary	Height



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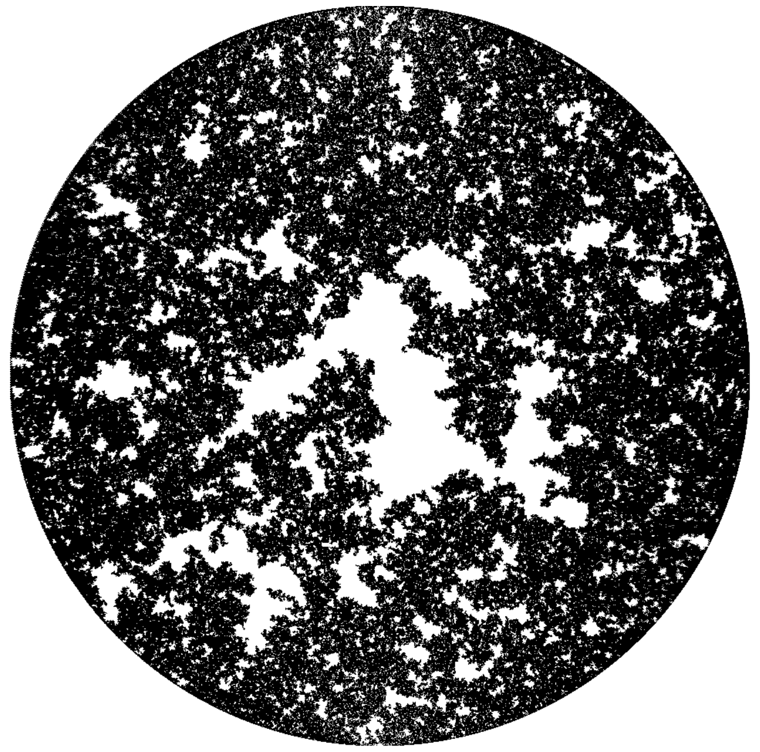

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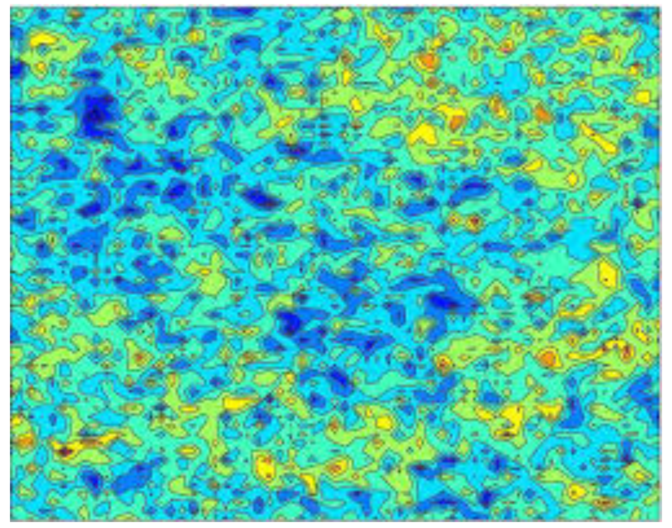
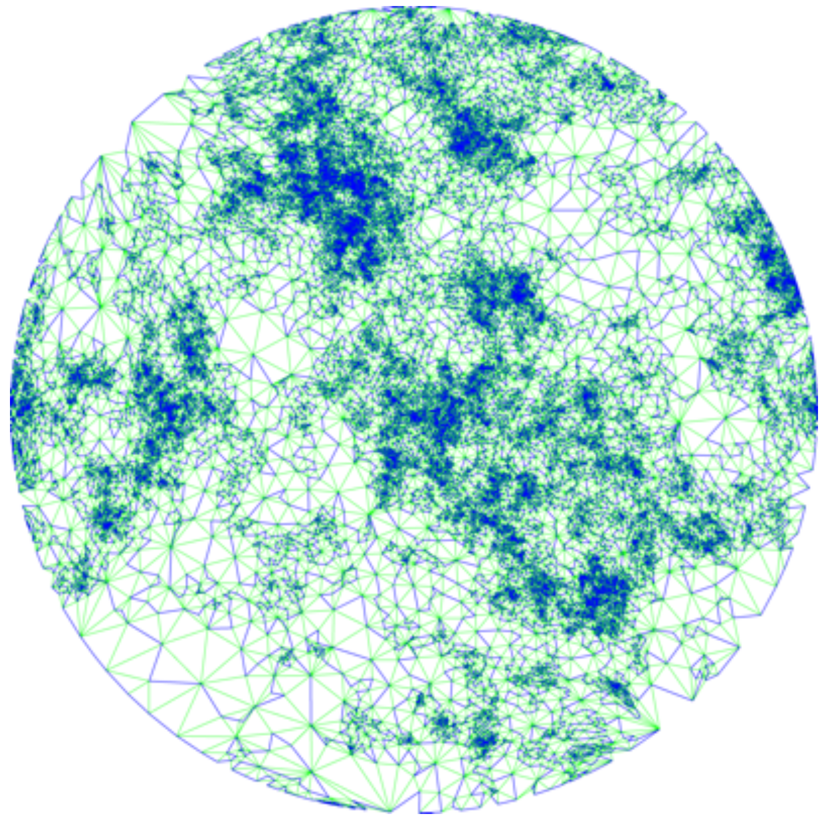
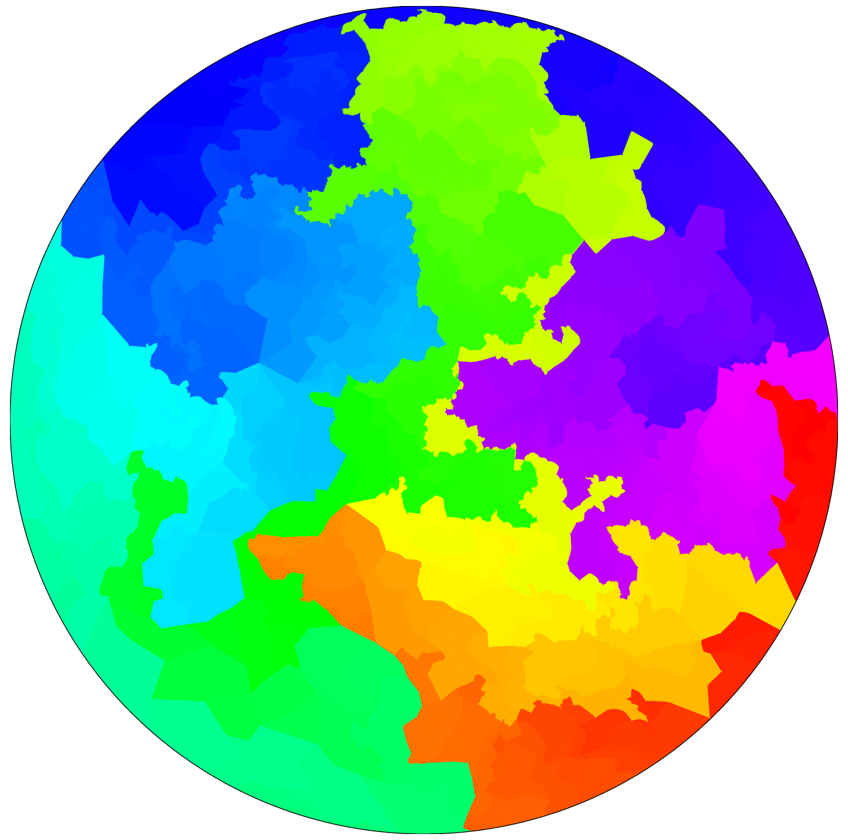
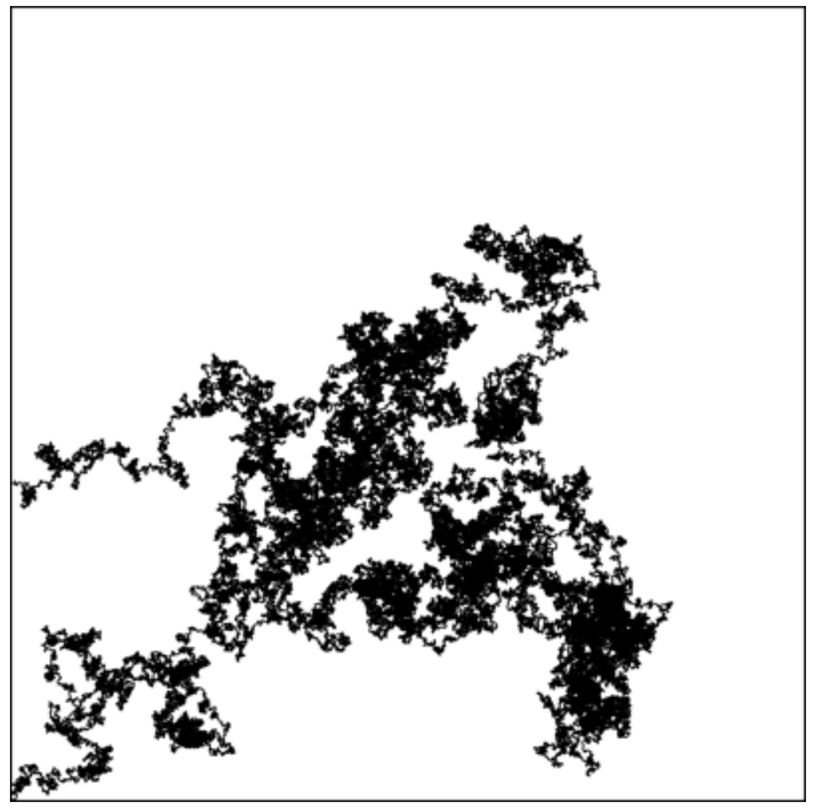


Proof and a Question







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- For $\gamma \neq 2, \kappa \neq 4$, a correspondence between CLE_κ decorated γ-GMC and **Brownian cone excursions** is already known (Duplantier-Miller-Sheffield)
- Our proof is based on taking a limit (of lots of things at once…) in this picture
- **Question** Can you extract a growth fragmentation process directly from correlated BM? Work in progress with Alex Watson and William Da Silva

Thanks!