Exit sets of the Gaussian free field - an Overview

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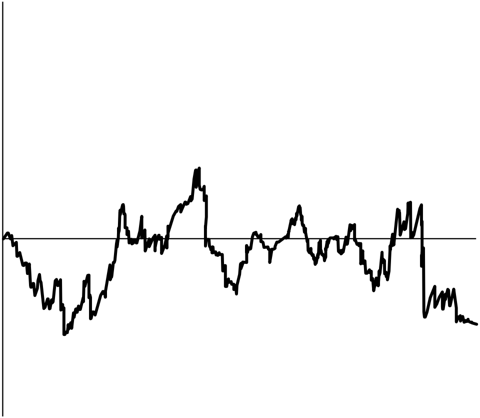
Contents

Plan

The Brownian motion



$$\tau_{-a,b} := \inf\{t \geqslant 0 : -a \leqslant B_t \leqslant b\},$$
$$\tau_{-a} := \inf\{t \geqslant 0 : B_t \geqslant -a\}.$$

Plan

The discrete Gaussian free field

Let $\Lambda \subseteq \mathbb{Z}^2$ be a graph. A GFF is the random function in the vertex of the graph

$$\mathbb{P}(d\phi) \propto \exp\left(-\frac{1}{2}\|\nabla\Phi\|^2\right)\prod d\phi_i$$

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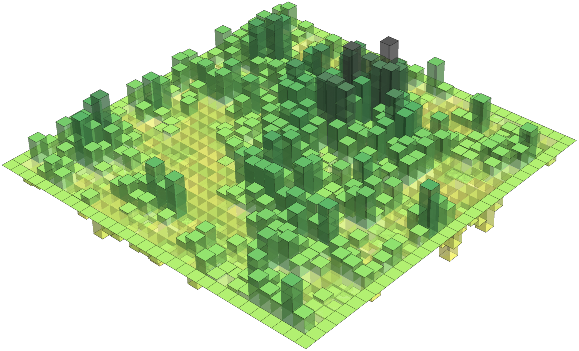
Equivalently the GFF is the centred Gaussian process with covariance

$$\mathbb{E}\left[\phi(k)\phi(j)\right] = G_\Lambda(k,j)$$

Simulation of the discrete Gaussian free field

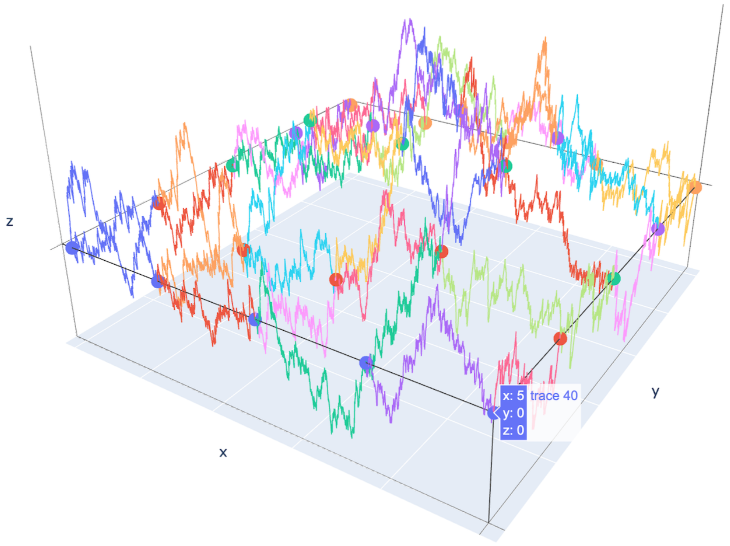


The metric Gaussian free field

Let $\Lambda \subseteq \mathbb{Z}^2$ be a graph. Define $\tilde{\Lambda}$ the metric space where you replace every vertex with a copy of $[0, 1]$. The GFF is the centred Gaussian process with covariance

$$\mathbb{E}\left[\phi(k)\phi(j)\right] = G_{\tilde{\Lambda}}(k, j)$$

Simulation of the metric Gaussian free field



Simulation by Thomas Laengle

The Gaussian free field

Take $D \subseteq \mathbb{C}$, The (continuum) Gaussian free field (GFF) is a centred Gaussian process with covariance given by

$$\mathbb{E}\left[\Phi(x)\Phi(y)\right] = G_D(x,y) \overset{x \to y}{\sim} -\log(\|x-y\|).$$

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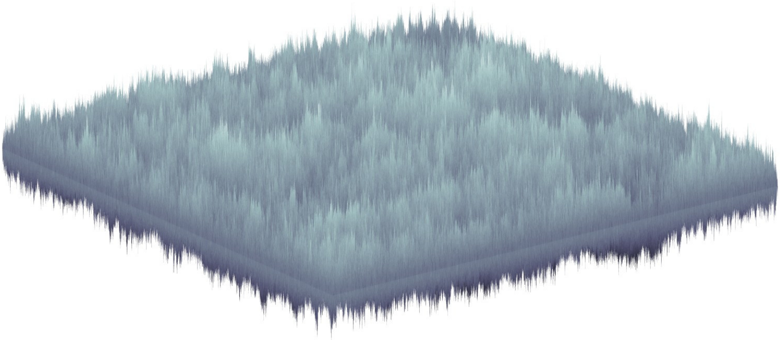
The Gaussian free field, is defined as a random "generalised function" such that $(\Phi, f)_{f \text{ smooth}}$ is a centred Gaussian process with

$$\mathbb{E}\left[(\Phi, f)(\Phi, g)\right] = \iint\limits_{D\times D} f(x) G_D(x,y) g(y) dxdy.$$

Approximation of a continuum Gaussian free field



Conformal invariance

- The Green's function is conformally invariant, i.e., for any conformal transformation $\varphi : D \mapsto D'$

$$G_D(x, y) = G_{D'}(\varphi(x), \varphi(y)).$$

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$$\Phi^D(\cdot) \overset{law}{=} \Phi^{D'}(\varphi(\cdot)).$$

Weak Markov property

Weak Markov property

Let A be a closed set of $D \subseteq \mathbb{C}$. Then there exist two independent "generalized functions" Φ_A and Φ^A such that

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2. Φ_A is harmonic in $D \backslash A$.
3. Φ^A is a GFF in $D \backslash A$.

Stopping (local) set

A is a stopping set of Φ if for all closed sets $C \subseteq D$

$$\{A \subseteq C\} \in \sigma(\Phi_C).$$

Strong Markov property

Let A be a stopping set of Φ. Then, conditionally on A, there exist two conditionally independent "generalised functions" Φ_A and Φ^A such that

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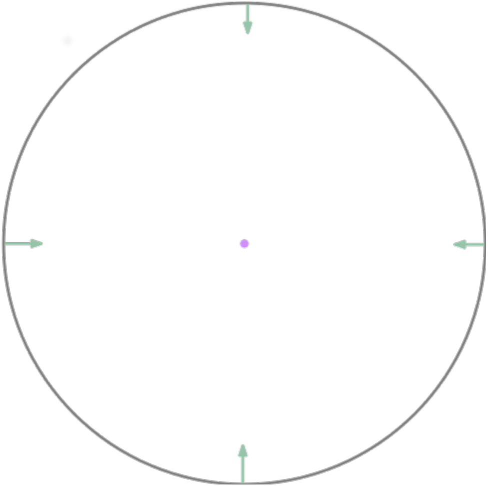
Example of a stopping set



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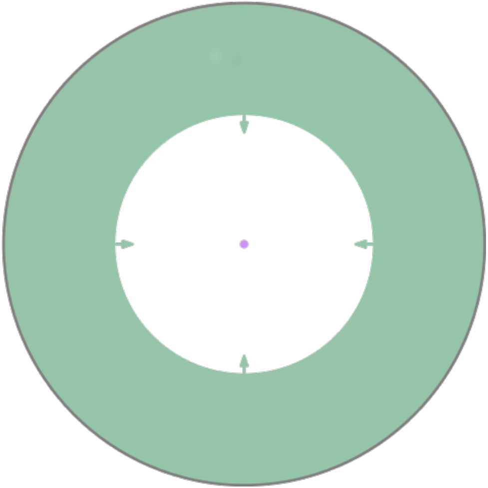
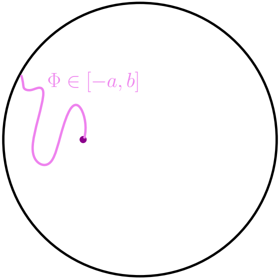


Plan

Two-valued sets



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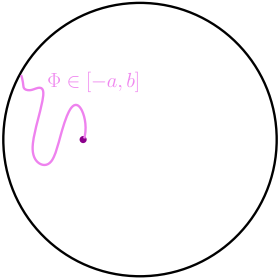
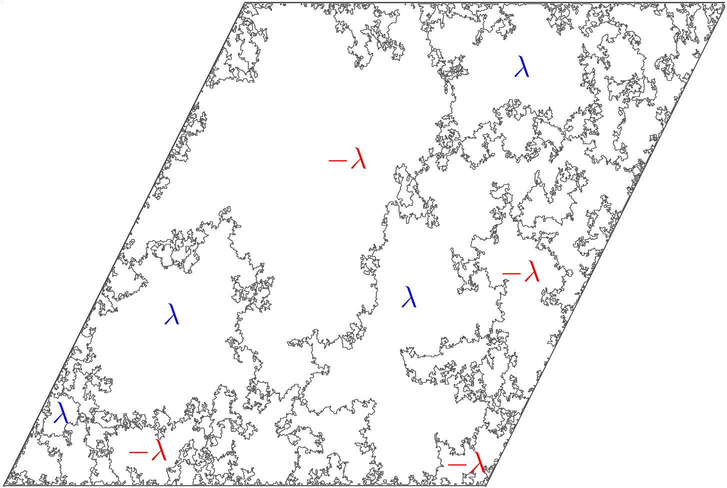


Theorem (Aru-S.-Werner '17)

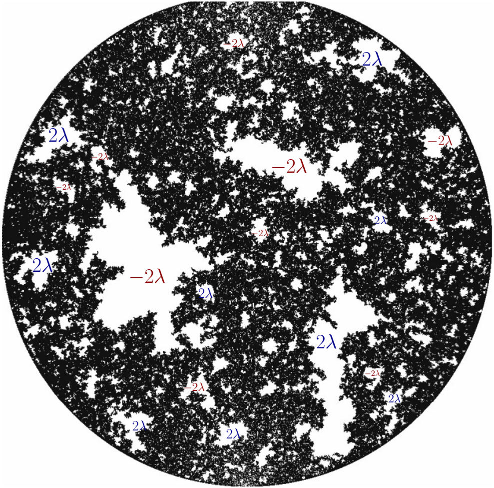
Take $a, b \geqslant 0$, such that $a + b \geqslant 2\lambda := \pi$. There exists a unique stopping set $\mathbb{A}_{-a,b}$ such that $\Phi_{\mathbb{A}_{-a,b}}$ is a harmonic function constant in each connected component taking values in $\{-a, b\}$.

$\mathbb{A}_{-\lambda,\lambda}$



Simulation by B. Werness.

$\mathbb{A}_{-2\lambda,2\lambda}$ =CLE$_4$ (Miller-Sheffield)



Simulation by D. Wilson.

Monotonicity

Proposition (Aru-S.-Werner '17)

Let $a, b \geqslant 0$ *with* $a + b \geqslant 2\lambda$ *and* $a', b' \geqslant 0$ *such that* $[-a, b] \subseteq [-a', b']$ *then* $\mathbb{A}_{-a,b} \subseteq \mathbb{A}_{-a',b'}$.

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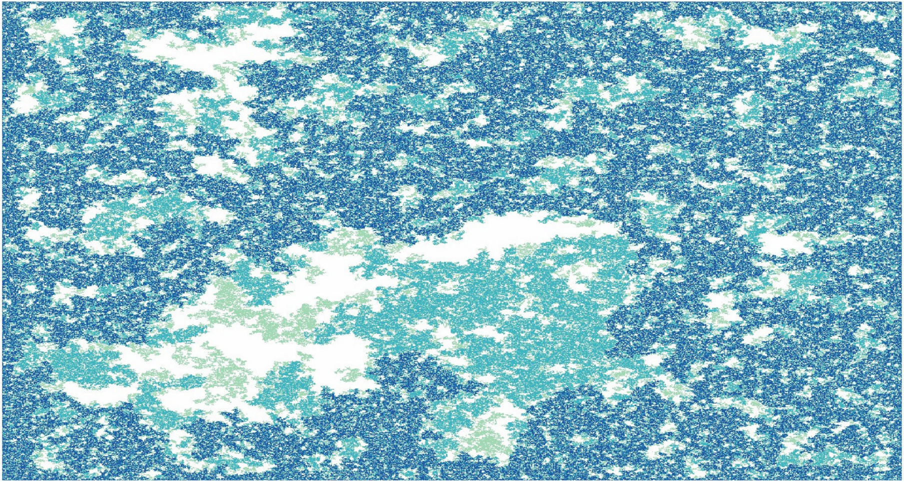
Furthermore, for any fixed a, b *a.s. we have that*

$$\mathbb{A}_{-a,b} := \bigcap_{\substack{a',b' \in \mathbb{Q} \\ [-a,b] \subset [-a',b']}} \mathbb{A}_{-a',b'}$$

Simulation for monotonicity



Non-existence

Proposition (Aru-S.-Werner '17)

Let $a, b \geqslant 0$ *with* $a + b < 2\lambda$. *There is no stopping set* $\mathbb{A}_{-a,b}$ *with the property that* $\Phi_{\mathbb{A}_{-a,b}}$ *is a harmonic function with values in* $\{-a, b\}$.

Explicit laws

Theorem (Aru-S.-Werner'17)

The law of $-\log(CR(0, \mathbb{D} \backslash \mathbb{A}_{-a,b}))$ *is equal to the law of the first time a Brownian motion exits* $[-a, b]$.

Explicit laws

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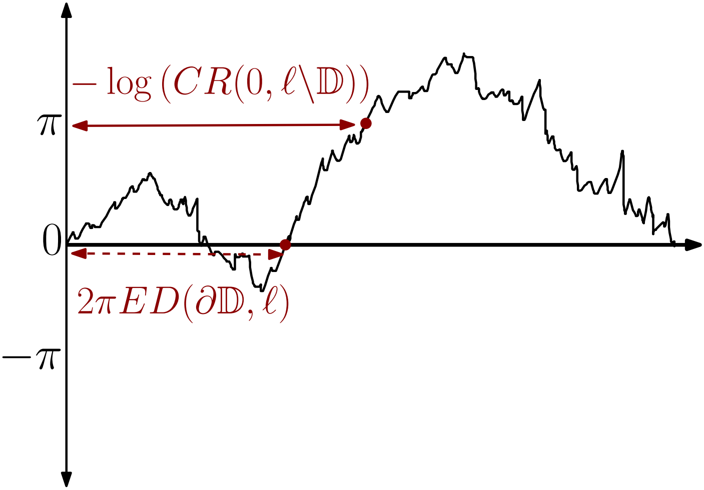
Theorem (Aru-Lupu-S.'22)

Take ℓ *the outer most loop of* $\mathbb{A}_{-2\lambda,2\lambda}$ *the joint law of* $(2\pi ED(\partial B(0,1), -\log(CR(0, \mathbb{D}\backslash\mathbb{A}_{-2\lambda,2\lambda}))$ *is that of* $(\sup\{t < \tau\{-\pi, \pi : B_t = 0\}, \tau_{-\pi,\pi})$.

Representation of the joint law



Multifractal spectrum

Theorem (Schoug-S.-Viklund '19)

A.s. for any $a, b > 0$ *with* $a, b \geqslant 2\lambda$ *the Haussdorf dimension of* $\mathbb{A}_{-a,b}$ *is equal to*

$$2 - \frac{2\lambda^2}{(a+b)^2}.$$

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A.s. for any $a, b > 0$ with $a, b \geqslant 2\lambda$ the Haussdorf dimension of $\mathbb{A}_{-a,b}$ is equal to

$$2 - \frac{2\lambda^2}{(a+b)^2}.$$

This is related to the imaginary chaos

$$:\cos(i\alpha\Phi):|_{\mathbb{A}_{-a,a}} = \mu_{\mathbb{A}_{-a,a}}$$

when $\alpha a = \pi/2$.

Law of the labels in $\mathbb{A}_{-2\lambda,2\lambda}$

Proposition (Miller-Sheffield '11, Aru-S.-Werner '17)

Conditionally on the geometry of $\mathbb{A}_{-2\lambda,2\lambda}$ the harmonic function $h_{\mathbb{A}_{-2\lambda,2\lambda}}$ takes values $\pm 2\lambda$ independently.

Measurability of labels

Proposition (Aru-S. '18)

Let $a, b > 0$ with $a + b \geqslant 2\lambda$ and $a \neq b$, then

- *If $2\lambda \leqslant a + b < 4\lambda$, the harmonic function $h_{\mathbb{A}_{-a,b}}$ is a measurable function of $\mathbb{A}_{-a,b}$.*
- *If $a + b \geqslant 4\lambda$, the harmonic function $h_{\mathbb{A}_{-a,b}}$ is not a function of $\mathbb{A}_{-a,b}$.*

Connectivity of the loops

Consider $G(\mathbb{A}_{-a,b})$ the graph whose vertices are the connected components of $\mathbb{D}\backslash\mathbb{A}_{-a,b}$ and there is an edge between $\mathcal{O}_1$ and $\mathcal{O}_2$ if $\overline{\mathcal{O}}_1 \cap \overline{\mathcal{O}}_2 \neq \emptyset$.

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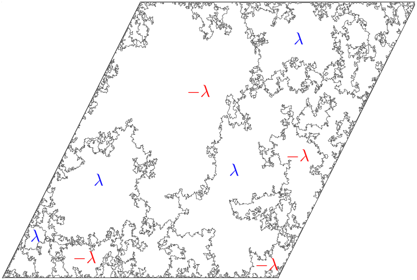
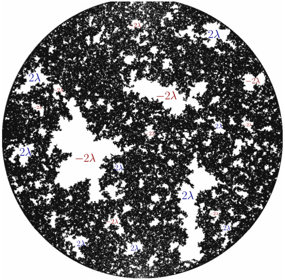
Proposition (Aru-S. '18)

Let $a, b > 0$ with $a + b \geqslant 2\lambda$, then

- *If $2\lambda \leqslant a + b < 4\lambda$, $G(\mathbb{A}_{-a,b})$ is connected and bipartite with colouring given by the value of $h_{\mathbb{A}_{-a,b}}$ in the connected component.*
- *If $a + b \geqslant 4\lambda$, $G(\mathbb{A}_{-a,b})$ is totally disconnected.*

Two critical cases





Percolative properties

Proposition (Aru-S. '18)

Take $x, y \in \partial\mathbb{D}$

1. *If* $a, b \geqslant 2\lambda$, *a.s. there exists a continuous path* η *going from* x *to* y *in* $\mathbb{D} \cap \mathbb{A}_{-a,b}$.
2. *If either* a *or* b *is strictly smaller than* 2λ, *a.s. there is no continous path* η *going from* x *to* y *in* $\mathbb{D} \cap \mathbb{A}_{-a,b}$.

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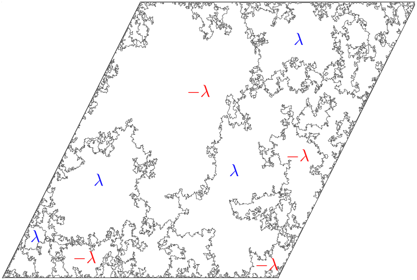
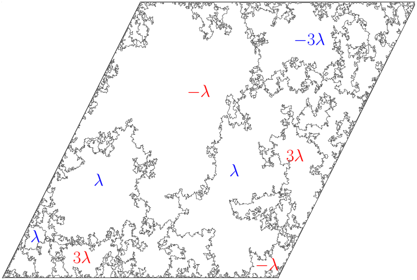
What happens in the discrete/metric case?

Coupling between 0-boundary and free-boundary GFF

Theorem (Qian-Werner '18)

One can couple a free-boundary GFF and a 0-boundary GFF using $\mathbb{A}_{-\lambda,\lambda}$.

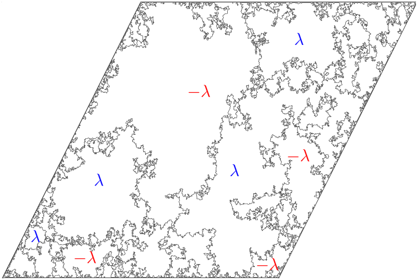




Convergences from the discrete GFF

Theorem (Schramm-Sheffield '11, Aru-S. '18)

The union of all the 0 *level lines of a discrete Gaussian free field converge to* $\mathbb{A}_{-\lambda,\lambda}$



Convergence from the double current model

Theorem (Duminil-Copin - Lis - Qian '21)

Let n^δ be a critical double current model in a graph $\Lambda_\delta \subseteq \delta\mathbb{Z}^2$. Then,

1. *The outer-most boundary of the outermost cluster converges to* $\mathbb{A}_{-2\lambda,2\lambda}$.
2. *The inner-most boundary of the outermost cluster converges to* $\mathbb{A}_{-2\lambda,(2\sqrt{2}-2)\lambda}$ *inside the loops of* $\mathbb{A}_{-2\lambda,2\lambda}$.

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What happens for the XOR-Ising model?

What happens for the Ashkin-Teller model?

Convergence of approximative exit sets

Does the discrete/metric $\mathbb{A}_{-a,a}$ converge to a non-trivial set?

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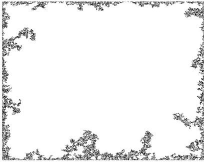
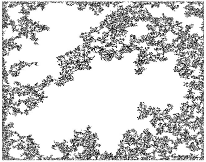
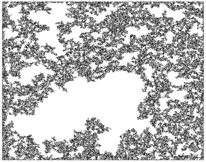


(a) $\mathbb{A}^{discrete}_{-0.95\lambda,0.95\lambda}$



(b) $\mathbb{A}^{discrete}_{-\lambda,\lambda}$



(c) $\mathbb{A}^{discrete}_{-1.05\lambda,1.05\lambda}$

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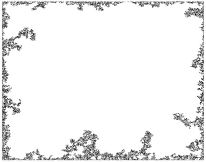

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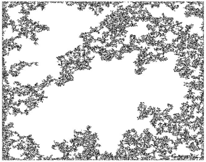

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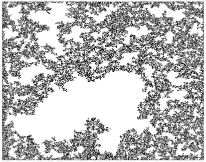

(c) $\mathbb{A}^{discrete}_{-1.05\lambda,1.05\lambda}$

What happens for the level sets of integer-valued GFF? (Bauerschmidt - Park-Rodriguez '22)

However, we now what happens with two GFF

Theorem (Aru-Garban-S. '22)

Let (ϕ^1, ϕ^2) *two independent discrete (or metric) GFF. For any* $a > 0$*, the level set* $\mathbb{A}_{\|\cdot\| \leqslant a}$ *is trivial.*

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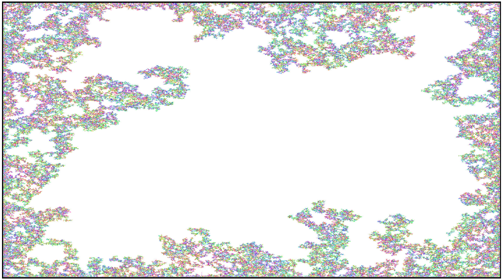

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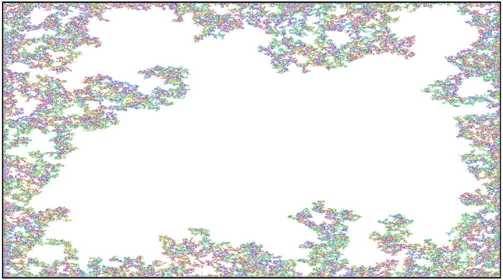
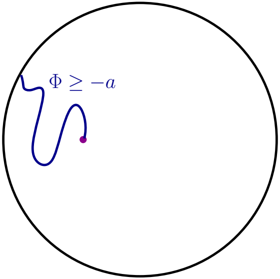


What happens in the continuum?

Plan

Definition



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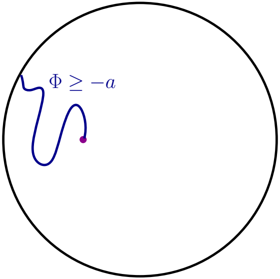
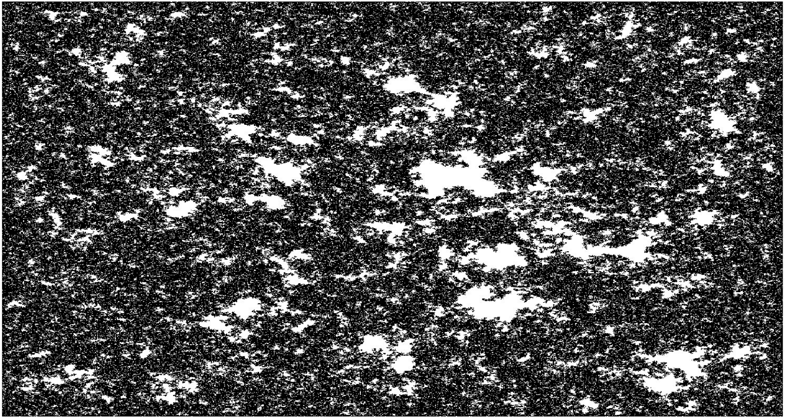


Theorem (Aru-Lupu-S.'19)

Take $a \geqslant 0$. There exists a unique stopping set $\mathbb{A}_{-a}$ such that

1. *Restricted to $D \backslash \mathbb{A}_{-a}$, $\Phi_{\mathbb{A}_{-a}}$ is strictly equal to $-a$.*
2. *$\Phi_{\mathbb{A}_{-a}} + a$ is a positive measure supported in $\mathbb{A}_{-a}$.*

Simulation: $\mathbb{A}_{-2\lambda}$



Monotonicity

Proposition (Aru-Lupu-S. '19)

Take $0 \leqslant a \leqslant a'$, *then* $\mathbb{A}_{-a} \subseteq \mathbb{A}_{-a'}$.

Monotonicity: idea

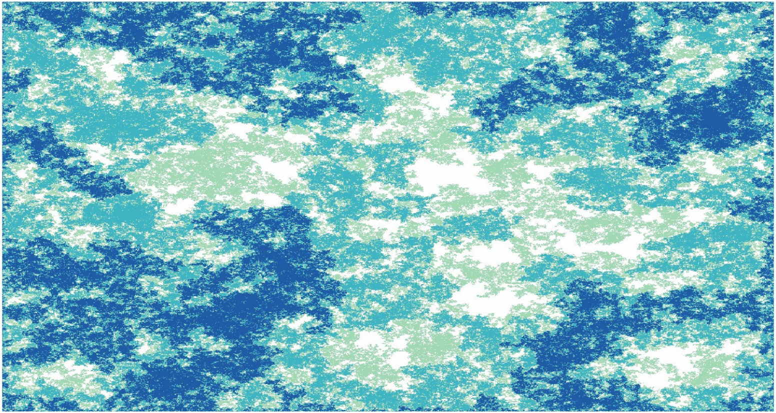


Figure: $\mathbb{A}_{-\lambda}$, $\mathbb{A}_{-2\lambda}$ and $\mathbb{A}_{-3\lambda}$

Relationship with two-valued sets

Theorem (Aru-Lupu-S. '19)

Take $a, b \geqslant 0$ *such that* $a + b \geqslant 2\lambda$, *then*

$$\mathbb{A}_{-a,b}(\Phi) = \mathbb{A}_{-a}(\Phi) \cap \mathbb{A}_{-b}(-\Phi)$$

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What happens when $a + b < 2\lambda$?.

What happens for independent $\mathbb{A}_{-a}$ and $\mathbb{A}_{-b}$?.

Dimension

Theorem (Aru-Lupu-S. '18)

$\mathbb{A}_{-a}$ *has fractal dimension equal to 2. Furthermore, the non-trivial measure* $\Phi_{\mathbb{A}_{-a}} + a$ *correspond to a Minkowski content measure of the gauge* $r \mapsto r^2 |\log r|^{1/2}$.

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This is strongly related with the real exponential of the GFF : $e^{\gamma\phi}$:

Excursion decomposition

Theorem (Aru-Lupu-S. '18-'23+)

There exists a sequence of random pairwise disjoint sets $(e_i)_{i\in\mathbb{N}}$ and an independent i.i.d. centred sequence of signs $(\sigma_i)_{i\in\mathbb{N}}$ such that

$$\sum_{i\in\mathbb{N}} \sigma_i \mu_{e_i} \text{ has the law of a GFF,}$$

where μ_{e_i} is the Minkowski content measure of gauge $r \mapsto r^2|\log r|^{1/2}$ of the set e_i.

Convergences

Theorem (Aru-Lupu- S. ’18)

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End

Gracias!