Weak convergence for semi-linear SPDEs

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Joint work with Arnulf Jentzen, Ryan Kurniawan, and Timo Welti (all ETH Zürich)

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Outline

1. Basic notions and context
2. Our results on the stochastic wave equation
3. Key ingredients for our proof
4. Extensions

Let H, U be a $\mathbb{R}$-Hilbert spaces and let $X \in L^p(\Omega; C([0,T], H))$ be the mild solution to

$$\begin{aligned} dX_t &= AX_t\, dt + F(X_t)\, dt + B(X_t)\, dW_t \quad t \in [0,T]; \\ X_0 &= \xi_0 \end{aligned} \tag{SPDE}$$

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where

- $T \in (0, \infty)$,
- $A\colon D(A) \subset H \to H$ generator of C_0-semigroup $(e^{tA})_{t\in[0,\infty)}$,
- U separable Hilbert spaces;
- $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$ a stochastic basis and $(W_t)_{t\in[0,T]}$ an Id_U-Brownian motion,
- V_F and V_B Hilbert spaces such that e^{tA} 'somehow' defines an element of $L(V_F, H)$ and of $L(V_B, H)$,
- $F\colon H \to V_F$ and $B\colon H \to HS(U, V_B)$ Lipschitz continuous,
- $p \in [2, \infty)$, $\xi_0 \in L^p((\Omega, \mathcal{F}_0, \mathbb{P}); H)$.

I.e., for all $t \in [0, T]$:

$$X_t = e^{tA}\xi_0 + \int_0^t e^{(t-s)A}F(X_s)\,ds + \int_0^t e^{(t-s)A}B(X_s)\,dW_s \quad \text{a.s.}$$

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E.g. $(X^{(n)})_{n\in\mathbb{N}}$ is obtained by either or both of

1. spatial discretization – spectral Galerkin method, finite element method.
2. temporal discretization – Euler-Maruyama method, exponential Euler method.

Weak convergence

Wanted: $\alpha \in (0,\infty)$ and $\mathcal{A} \subseteq C_b(H,\mathbb{R})$, both as large as possible, such that

$$\forall \phi \in \mathcal{A}\, \exists C \in (0,\infty)\, \forall n \in \mathbb{N}: \quad \left| \mathbb{E}\phi(X_T) - \mathbb{E}\phi(X_T^{(n)}) \right| \leq C n^{-\alpha}. \tag{1}$$

("Weak convergence with rate α of $(X_T^{(n)})_{n\in\mathbb{N}}$ against X_T.")

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'Typical' result: $\mathcal{A} = C_b^k(H, \mathbb{R})$ with $k \in \{2, 3, 4\}$ and $\alpha = 2\beta$, where $\beta \in (0, \infty)$ is such that for all $n \in \mathbb{N}$ one has

$$\left(\mathbb{E} \left\| X_T - X_T^{(n)} \right\|^2 \right)^{\frac{1}{2}} \leq C n^{-\beta}.$$

("Strong convergence with rate β of $(X_T^{(n)})_{n\in\mathbb{N}}$ against X_T.")

The Kolmogorov equation

Suppose $A \in L(H)$, $F \in C_b^2(H, H)$, $B \in C_b^2(H, HS(U, H))$, $\phi \in C_b^2(H, \mathbb{R})$ and for $x \in H$ let $X^x \in L^2(\Omega; C([0, T], H))$ satisfy, for all $t \in [0, T]$,

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and define $u(t, x) = \mathbb{E}\phi(X_t^x), \quad (t, x) \in [0, T] \times H.$

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and define $u(t,x) = \mathbb{E}\phi(X_t^x)$, $\quad (t,x) \in [0,T] \times H$.
Then $u \in C^{1,2}([0,T] \times H, \mathbb{R})$, and if $(e_k)_{k\in\mathbb{N}}$ is an ONB for H then

$$\frac{\partial u}{\partial t}(t,x) = \frac{\partial u}{\partial x}(t,x)Ax + F(x) + \frac{1}{2}\sum_{k=1}^{\infty} \frac{\partial^2 u}{\partial x^2}(t,x)(B(x)e_k, B(x)e_k),$$

$$u(0,x) = \phi(x), \quad t \in [0,T], x \in H.$$

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Now apply Itô's formula to u?

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Other techniques have been developed, using e.g. Malliavin calculus, duality arguments for the stochastic integral, or (in our case) the mild Itô formula.

Some works on weak convergence for SPDEs:

Shadlow (2003), Hausenblas (2003/2010), de Bouard and Debussche (2006), Debussche and Printems (2009), Geissert, Kovács, Larsson (2009), Debussche (2011), Kovács, Larsson, Lindgren (2011, 2013), Andersson, Larsson (2012), Lindner and Schilling (2013), Bréhier (2013), Bréhier and Kopec (2014), Andersson, Kruse, Larsson (2013), Wang (2013), Wang and Gan (2013), Conus, Jentzen, Kurniawan (2014), Jentzen and Kurniawan (2015), Jacobe de Naurois, Jentzen, Welti (2015), Bréhier, Hairer, Stuart (2016).

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Jentzen et al.: optimal weak rates for full discretization of semi-linear parabolic SPDEs and spectral Galerkin method for semi-linear hyperbolic SPDE.

Exponential Euler method

For $h \in [0, \infty)$ let $\hat{Y}_0^h = X_0$ and, for $n \in \mathbb{N}_0$,

$$\hat{Y}_{n+1}^h = e^{hA}\left[\hat{Y}_n^h + hF(\hat{Y}_n^h) + B(\hat{Y}_n^h)(W_{(n+1)h} - W_{nh})\right].$$

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$(\hat{Y}_n^h)_{n \in \mathbb{N}_0}$ is the exponential Euler approximation of X with step size h;
also, $\hat{Y}_n^h = Y_{nh}^h$, where $Y^h \colon \mathbb{R} \times \Omega \to H$ satisfies, for all $t \in [0, \infty)$:

$$Y_t^h = e^{tA} X_0 + \int_0^t e^{(t - \lfloor s \rfloor_h)A} F(Y_{\lfloor s \rfloor_h}^h)\, ds + \int_0^t e^{(t - \lfloor s \rfloor_h)A} B(Y_{\lfloor s \rfloor_h}^h)\, dW_s, \tag{3}$$

with

$$\lfloor s \rfloor_h = \sup\{nh \colon n \in \mathbb{N}_0 \text{ and } nh \le s\}.$$

Y^h is the exponential Euler approximation process of X with step size h.

Theorem (C, Jentzen, Welti; 2016)

Let $b_0, b_1 \in \mathbb{R}$ *and consider:*

$$\frac{\partial^2}{\partial t^2} u(t,x) = \frac{\partial^2}{\partial x^2} u(t,x) + [b_0 + b_1 u(t,x)]\xi(t,x) \qquad \text{(1D WAVE)}$$
$$(t,x) \in [0,T] \times [0,1],$$

with suitable initial and Dirichlet boundary conditions, where ξ *is space-time white noise.*

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with suitable initial and Dirichlet boundary conditions, where ξ *is space-time white noise.*
For $h \in [0,\infty)$ *let* $Y^h \colon [0,T] \times \Omega \to H$ *be the exponential Euler approximation process with step size* h *applied to* (1D WAVE) *and let* $\Phi \in C_b^4(L^2(0,1),\mathbb{R})$.

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Then for all $\alpha \in (0,1)$ there exists a constant C such that for all $h \in [0,\infty)$ it holds that

$$\left| \mathbb{E}\phi(u(T,\cdot)) - \mathbb{E}\phi(Y_T^h) \right| \leq Ch^\alpha.$$

Details

Let Δ_d denote the Dirichlet Laplacian on $L^2(0,1)$.
The wave equation fits into the setting with

- $H = L^2(0,1) \times W^{-1,2}(0,1)$;
- $A = \begin{bmatrix} 0 & Id_H \\ \Delta_d & 0 \end{bmatrix}$;
- $U = L^2(0,1)$;
- $B(u,v)h = \begin{bmatrix} 0 \\ (b_0 + b_1 v)h \end{bmatrix}$;
- $dW_t = \xi(t,x)$;

i.e.,

$$d \begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} V \\ \Delta_d U \end{bmatrix} dt + \begin{bmatrix} 0 \\ (b_0 + b_1 U) \end{bmatrix} dW_t. \qquad (4)$$

More details

In fact, we assume $B \in \mathrm{Lip}(H, L_2(U, H))$ such that
$\exists \rho, r, \gamma \in [0, \infty)$, $\beta \in [\gamma/2, \gamma] \cap [\gamma - 1/2, \gamma]$ such that

- $B|_{H_\rho} \in \mathrm{Lip}(H_\rho, L(U, H_\gamma) \cap L_2(U, H_\rho))$,
- $B|_{H_r} \in C_b^4(H_r, L_2(U, H))$,

and non-linear $F \in \mathrm{Lip}(H_0)$ satisfying similar conditions.

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and non-linear $F \in \mathrm{Lip}(H_0)$ satisfying similar conditions.
This gives convergence rate $2(\gamma - \beta)$.

Outline of proof

1. Project problem onto finite dimensional subspaces (Galerkin approximation): this gives processes $Y^{h,N}$ and X^N, $N \in \mathbb{N}$, $h \in [0, \infty)$.

[1] *A mild Itô formula for SPDEs*, arXiv1009.3526.

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3. Obtain appropriate error estimates using, among others, the *mild Itô formula* developed by Da Prato, Jentzen and Röckner[1].

[1] *A mild Itô formula for SPDEs*, arXiv1009.3526.

Mild Itô formula

Let $X\colon [0, T] \times \Omega \to H$ satisfy, for all $t \in [0, \infty)$,

$$X_t = e^{tA} X_0 + \int_0^t e^{(t-s)A} B(X_s)\, dW_s.$$

Mild Itô formula

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$$X_t = e^{tA}X_0 + \int_0^t e^{(t-s)A} B(X_s)\, dW_s.$$

Then for all $\phi \in C^2(H)$ it holds that

$$\begin{aligned}
&\phi(X_t) - \phi(e^{tA}X_t) \\
&= \int_0^t \phi'(e^{(t-s)A}X_s) e^{(t-s)A} B(X_s)\, dW_s \\
&\quad + \tfrac{1}{2}\sum_{k=1}^{\infty} \int_0^t \phi''(e^{(t-s)A}X_s)\left(e^{(t-s)A}B(X_s), e^{(t-s)A}B(X_s)\right) ds.
\end{aligned}$$

Proof (cont'd)

$$\left|\mathbb{E}\left[\phi(Y_T^{h,N}) - \phi(X_T^N)\right]\right|$$
$$= \left|\mathbb{E}\left[u_N(0, Y_T^{h,N}) - u_N(T, Y_0^{h,N})\right]\right|$$

Proof (cont'd)

By (the 'classical') Itô formula

$$
\begin{aligned}
&\left|\mathbb{E}\left[\phi(Y_T^{h,N}) - \phi(X_T^N)\right]\right| \\
&= \left|\mathbb{E}\left[u_N(0, Y_T^{h,N}) - u_N(T, Y_0^{h,N})\right]\right| \\
&= \Bigg|\mathbb{E}\Bigg[-\int_0^T \frac{\partial}{\partial t} u_N(t, Y_t^{h,N})\, dt + \int_0^T \frac{\partial}{\partial x} u_N(t, Y_t^{h,N}) A Y_t^{h,N}\, dt \\
&\quad + \frac{1}{2}\sum_{k=1}^{\infty}\int_0^T \frac{\partial^2}{\partial x^2} u_N(t, Y_t^{h,N}) \left(e^{\delta_h(t)A} B(Y_{\lfloor t \rfloor_h}^{h,N}) e_k, e^{\delta_h(t)A} B(Y_{\lfloor t \rfloor_h}^{h,N}) e_k\right) dt\Bigg]\Bigg|
\end{aligned}
$$

Proof (cont'd)

By (the 'classical') Itô formula and the Kolmogorov equation

$$
\begin{aligned}
&\left|\mathbb{E}\left[\phi(Y_T^{h,N}) - \phi(X_T^N)\right]\right| \\
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&\quad \left.\left. + \frac{1}{2}\sum_{k=1}^{\infty}\int_0^T \frac{\partial^2}{\partial x^2} u_N(t, Y_t^{h,N})\left(e^{\delta_h(t)A} B(Y_{\lfloor t \rfloor_h}^{h,N}) e_k, e^{\delta_h(t)A} B(Y_{\lfloor t \rfloor_h}^{h,N}) e_k\right) dt\right]\right| \\
&= \left|\mathbb{E}\left[\frac{1}{2}\sum_{k=1}^{\infty}\int_0^T \frac{\partial^2}{\partial x^2} u_N(t, Y_t^{h,N})\left(e^{\delta_h(t)A} B(Y_{\lfloor t \rfloor_h}^{h,N}) e_k, e^{\delta_h(t)A} B(Y_{\lfloor t \rfloor_h}^{h,N}) e_k\right)\right.\right. \\
&\quad \left.\left. - \frac{\partial^2}{\partial x^2} u_N(t, Y_t^{h,N})\left(B(Y_t^{h,N}) e_k, B(Y_t^{h,N}) e_k\right) dt\right]\right|.
\end{aligned}
$$

By the triangle inequality we obtain:

$$\Bigg|\mathbb{E}\left[\sum_{k=1}^{\infty}\int_0^T \frac{\partial^2}{\partial x^2}u_N(t,Y_t^{h,N})\left(e^{\delta_h(t)A}B(Y_{\lfloor t\rfloor_h}^{h,N})e_k, e^{\delta_h(t)A}B(Y_{\lfloor t\rfloor_h}^{h,N})e_k\right)\right.$$

$$\left.- \frac{\partial^2}{\partial x^2}u_N(t,Y_t^{h,N})\left(B(Y_t^{h,N})e_k, B(Y_t^{h,N})e_k\right)dt\right]\Bigg|$$

$$\leq \left|\mathbb{E}\left[\sum_{k=1}^{\infty}\int_0^T \frac{\partial^2}{\partial x^2}u_N(t,Y_t^{h,N})\left((e^{\delta_h(t)A}-I)B(Y_{\lfloor t\rfloor_h}^{h,N})e_k,(e^{\delta_h(t)A}+I)B(Y_{\lfloor t\rfloor_h}^{h,N})e_k\right)dt\right]\right|$$

$$+\ldots+\ldots$$

$$+\Bigg|\mathbb{E}\left[\sum_{k=1}^{\infty}\int_0^T \frac{\partial^2}{\partial x^2}u_N(t,e^{\delta_h(t)A}Y_{\lfloor t\rfloor_h}^{h,N})\left(B(e^{\delta_h(t)A}Y_{\lfloor t\rfloor_h}^{h,N})e_k, B(e^{\delta_h(t)A}Y_{\lfloor t\rfloor_h}^{h,N})e_k\right)\right.$$

$$\left.- \frac{\partial^2}{\partial x^2}u_N(t,Y_t^{h,N})\left(B(Y_t^{h,N})e_k, B(Y_t^{h,N})e_k\right)dt\right]\Bigg|.$$

By the triangle inequality we obtain:

$$\Bigg|\mathbb{E}\Bigg[\sum_{k=1}^{\infty}\int_0^T \frac{\partial^2}{\partial x^2}u_N(t, Y_t^{h,N})\left(e^{\delta_h(t)A}B(Y_{\lfloor t\rfloor_h}^{h,N})e_k, e^{\delta_h(t)A}B(Y_{\lfloor t\rfloor_h}^{h,N})e_k\right) - \frac{\partial^2}{\partial x^2}u_N(t, Y_t^{h,N})\left(B(Y_t^{h,N})e_k, B(Y_t^{h,N})e_k\right)dt\Bigg]\Bigg|$$

$$\leq \Bigg|\mathbb{E}\Bigg[\sum_{k=1}^{\infty}\int_0^T \frac{\partial^2}{\partial x^2}u_N(t, Y_t^{h,N})\left((e^{\delta_h(t)A} - I)B(Y_{\lfloor t\rfloor_h}^{h,N})e_k, (e^{\delta_h(t)A} + I)B(Y_{\lfloor t\rfloor_h}^{h,N})e_k\right)dt\Bigg]\Bigg|$$

$$+ \ldots + \ldots$$

$$+ \Bigg|\mathbb{E}\Bigg[\sum_{k=1}^{\infty}\int_0^T \frac{\partial^2}{\partial x^2}u_N(t, e^{\delta_h(t)A}Y_{\lfloor t\rfloor_h}^{h,N})\left(B(e^{\delta_h(t)A}Y_{\lfloor t\rfloor_h}^{h,N})e_k, B(e^{\delta_h(t)A}Y_{\lfloor t\rfloor_h}^{h,N})e_k\right) - \frac{\partial^2}{\partial x^2}u_N(t, Y_t^{h,N})\left(B(Y_t^{h,N})e_k, B(Y_t^{h,N})e_k\right)dt\Bigg]\Bigg|.$$

Recall mild Itô formula:

$$\psi(Y_t^{t,N}) - \psi(e^{tA}Y_t^{t,N}) = \int_0^t \psi'(e^{(t-s)A}Y_s^{t,N})e^{(t-\lfloor s\rfloor_h)A}B(Y_s^{t,N})\,dW_s + \frac{1}{2}\sum_{k=1}^{\infty}\int_0^t \psi''(e^{(t-s)A}Y_s^{t,N})\left(e^{(t-\lfloor s\rfloor_h)A}B(Y_s^{t,N}), e^{(t-\lfloor s\rfloor_h)A}B(Y_s^{t,N})\right)ds.$$

Nemytskii operators

Let $f \in C_b(\mathbb{R}, \mathbb{R})$. The operator $F \in L(L^2(0,1))$ is the *Nemytskii operator associated with* f if for all $g \in L^2(0,1)$ we have $F(g)(x) = f(g(x))$.

Recall that we assumed that $B \in \mathrm{Lip}(H, L_2(U,H))$ such that $\exists \rho, r, \gamma \in [0,\infty)$, $\beta \in [\gamma/2, \gamma] \cap [\gamma - 1/2, \gamma]$ such that

- $B|_{H_\rho} \in \mathrm{Lip}(H_\rho, L(U, H_\gamma) \cap L_2(U, H_\rho))$,
- $B|_{H_r} \in C_b^4(H_r, L_2(U,H))$,

and non-linear $F \in \mathrm{Lip}(H_0)$ satisfying similar conditions.

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and non-linear $F \in \mathrm{Lip}(H_0)$ satisfying similar conditions.
In general, these assumptions are not satisfied by Nemytskii operators.

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Let $f \in C_b(\mathbb{R}, \mathbb{R})$, let $p \in [1, \infty)$ and $F \in L(L^p, L^p)$ be given by $F(g)(x) = f(g(x))$.

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In particular, for $p = 2$:

$$F \in C^k\left(W^{\frac{1}{2} - \frac{1}{2(k+\alpha)}, 2}, L^2\right).$$

Problem: interpreting the Nemytskii operator F as

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Solution: interpret problem in a Banach space V.

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will not give optimal rates.

Solution: interpret problem in a Banach space V.
More precisely, in the space $V = L^p$ for some large $p \in [1,\infty)$:

$$F \in C^k\left(W^{\frac{1}{p}-\frac{1}{(k+\alpha)p},p}, L^p\right).$$

Future work

Extension to the Banach space setting.

Motivation:

- All known results on weak convergence assume F and G to be *at least* twice continuously Fréchet differentiable.

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- Nemytskii operators do have the 'right' Fréchet differentiability properties in the Banach space setting.

Thank you!