Long-time homogenization of the wave equation

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Genève

Credits

Inspired by

- Dohnal, Lamacz, Schweizer: DLS (2011, 2014, 2015)
- Allaire, Briane, Vanninathan: ABV (2016)

Independent results by

- Allaire, Rauch (in preparation)

Similar results for the Schrödinger equation with potential

- Duerinckx, Gloria, Shirley (in preparation)

Main result in the periodic setting

For all $\varepsilon > 0$,

- $a_\varepsilon := a(\frac{\cdot}{\varepsilon})$, a periodic + symmetric tensor
- $\Box_\varepsilon := \partial^2_{tt} - \nabla \cdot a_\varepsilon \nabla$ the wave operator

There exists a family $\{\bar{a}_j\}_{j\in\mathbb{N}}$ of $j+2$-order tensors (with $\bar{a}_{2j+1} = 0$), and for all $\ell \in \mathbb{N}$ we set

- $\bar{\Box}_{\ell,\varepsilon} := \partial^2_{tt} - \sum_{j=0}^{\ell-1} \varepsilon^j \bar{a}_j \cdot \nabla^{j+2} - K_\ell (i\varepsilon)^{2([\frac{\ell-1}{2}]+1)} \mathrm{Id} \cdot \nabla^{2([\frac{\ell-1}{2}]+2)}$.

For all $\ell \in \mathbb{N}$, well-chosen $K_\ell = K_\ell(\bar{a}_0, \dots \bar{a}_{\ell-1})$, all $u_0 \in \mathcal{S}(\mathbb{R}^d)$, the solutions $u_\varepsilon, u_{\ell,\varepsilon} \in L^\infty(\mathbb{R}_+, L^2(\mathbb{R}^d))$ of

$$\left\{\begin{array}{rcl} \Box_\varepsilon u_\varepsilon &=& 0 \\ u_\varepsilon(0,\cdot) &=& u_0 \\ \partial_t u_\varepsilon(0,\cdot) &=& 0 \end{array}\right. , \left\{\begin{array}{rcl} \bar{\Box}_{\ell,\varepsilon} u_{\ell,\varepsilon} &=& 0 \\ u_{\ell,\varepsilon}(0,\cdot) &=& u_0 \\ \partial_t u_{\ell,\varepsilon}(0,\cdot) &=& 0 \end{array}\right.$$

satisfy for all $T > 0$:

$$\sup_{t\le T} \| u_\varepsilon(t) - u_{\ell,\varepsilon}(t) \|_{L^2(\mathbb{R}^d)} \lesssim C_\ell(u_0)(\varepsilon + \varepsilon^\ell T).$$

Comments

$$\sup_{t\le T}\|u_\varepsilon(t)-u_{\ell,\varepsilon}(t)\|_{L^2(\mathbb{R}^d)}\lesssim C_\ell(u_0)(\varepsilon+\varepsilon^\ell T)$$

- dispersive effect for $\ell=3,4$:
  $\partial^2_{tt}-\bar a_0\cdot\nabla^2-\underbrace{\varepsilon\bar a_1\cdot\nabla^3}_{=0}-\varepsilon^2\bar a_2\cdot\nabla^4-\underbrace{\varepsilon^3\bar a_3\cdot\nabla^5}_{=0}-$(yellow term).
  Estimate new for $\ell=4$ in periodic case (cf. DLS). Possible to use Boussinesq trick instead of regularization ($\bar a_2$ has a sign).
- for $\ell>4$: new (but $\bar a_6$ has no sign: no Boussinesq trick)
- proof is robust: natural norm, no regularity assumption on a, any order ℓ, quasi-periodic coefficients OK, results for random coefficients (subtle), related to localization/delocalization and diffusive/ballistic transport

Outline

Rest of the talk:

- Approach à la DLS: exact spectral theory + Fourier analysis
- Alternative approach: approximate spectral theory + Fourier analysis + energy estimates
- A few words on correctors and quantitative (stochastic) homogenization

Part 1: Approach à la DLS

Consider the wave operator $\Box := \partial^2_{tt} - \triangle$ and let u be the solution in $\mathbb{R}_+ \times \mathbb{R}^d$ of

$$\begin{cases} \Box u &= 0 \\ u(0,\cdot) &= u_0 \in L^2(\mathbb{R}^d) \\ \partial_t u(0,\cdot) &= 0. \end{cases}$$

Exact spectral theory: Fourier transform diagonalizes $-\triangle$

Reformulation of wave equation as a family of ODEs parametrized by frequencies: for all $k \in \mathbb{R}^d$,

$$\begin{cases} \partial^2_{tt}\hat{u}(t,k) - |k|^2\hat{u}(t,k) &= 0 \\ \hat{u}(0,k) &= \hat{u}_0(k) \\ \partial_t \hat{u}(0,k) &= 0. \end{cases}$$

Yields explicit formula by time integration and inverse Fourier transform

$$u(t,x) = \int_{\mathbb{R}^d} e^{ik\cdot x}\hat{u}_0(k)\cos(|k|t)d^*k.$$

Floquet-Bloch analysis

Let a periodic and set $\mathcal{L} := -\nabla \cdot a\nabla$ and $\Box := \partial^2_{tt} + \mathcal{L}$.

Then for all $k \in \mathbb{R}^d$ there exist $\Lambda(k) \geq 0$ and $\psi(\cdot, k) : \mathbb{R}^d \to \mathbb{C}$ periodic (and of norm L^2 unity on the torus) with

$$\mathcal{L}(e^{ik\cdot x}\psi(x,k)) = \Lambda(k)e^{ik\cdot x}\psi(x,k).$$

Bloch-Floquet theory: for all $g \in L^2(\mathbb{R}^d)$:

$$\tilde{g}(k) := \int_{\mathbb{R}^d} g(x)e^{-ik\cdot x}\psi(x,k)^* dx, \quad g(x) = \int_{\mathbb{R}^d} \tilde{g}(x)e^{ik\cdot x}\psi(x,k)d^*k.$$

Diagonalization of $-\nabla \cdot a\nabla$ and explicit ODE integration in frequencies:

$$\left\{\begin{array}{rcl} \Box u &=& 0 \\ u(0,\cdot) &=& u_0 \\ \partial_t u(0,\cdot) &=& 0 \end{array}\right. \implies u(t,x) = \int_{\mathbb{R}^d} e^{ik\cdot x}\psi(x,k)\tilde{u}_0(k)\cos(t\sqrt{\Lambda(k)})d^*k.$$

Strategy of DLS

Rescale a for $\varepsilon > 0$: $\mathcal{L}_\varepsilon := -\nabla \cdot a(\frac{\cdot}{\varepsilon})\nabla$ and $\Box_\varepsilon := \partial_{tt}^2 + \mathcal{L}_\varepsilon$.

Bloch-Floquet analysis yields (after rescaling):

$$\left\{ \begin{array}{rcl} \Box_\varepsilon u_\varepsilon & = & 0 \\ u_\varepsilon(0,\cdot) & = & u_0 \\ \partial_t u_\varepsilon(0,\cdot) & = & 0 \end{array} \right. \implies u_\varepsilon(t,x) = \int_{\mathbb{R}^d} e^{ik\cdot x}\psi(\frac{x}{\varepsilon},\varepsilon k)\tilde{u}_0^\varepsilon(\varepsilon k)\cos(\frac{t}{\varepsilon}\sqrt{\Lambda(\varepsilon k)})d^*k.$$

Strategy of DLS: quantify the convergences

- $\psi(\frac{x}{\varepsilon},\varepsilon k) = 1 + O(\varepsilon)$,
- $\tilde{u}_0^\varepsilon(\varepsilon k) = \hat{u}_0(k) + O(\varepsilon)$,
- $\Lambda(\varepsilon k) = \varepsilon^2 \bar{a}_0 \cdot k^{\otimes 2} + \varepsilon^4 \bar{a}_2 \cdot k^{\otimes 4} + o(\varepsilon^4)$,

which suggest $u_\varepsilon(t,x) \simeq \int_{\mathbb{R}^d} e^{ik\cdot x}\hat{u}_0(k)\cos(t\sqrt{\bar{a}_0\cdot k^{\otimes 2} + \varepsilon^2 \bar{a}_2 \cdot k^{\otimes 4}})d^*k$

Comments on the strategy of DLS

Fundamental observation: need to quantify the convergences of

- $\psi(\frac{x}{\varepsilon}, \varepsilon k) = 1 + O(\varepsilon)$,
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Limitations:

- starting point: Floquet-Bloch theorem only holds for periodic coefficients (wrong for quasi-periodic or random coefficients)
- technique: estimates by hands on Fourier formula are difficult, which yields suboptimal norm ($L^2 + L^\infty$) and estimate in DLS
- deep algebraic structure yet to be understood

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From Bloch waves to homogenization: $|k| \ll 1$

Recall that we have $\mathcal{L}(e^{ik\cdot x}\psi(x,k)) = \Lambda(k)e^{ik\cdot x}\psi(x,k)$ for periodic $\psi(\cdot,k)$. Expand: magnetic eigenvalue problem on the torus

$$-(\nabla + ik)\cdot a(\nabla + ik)\psi(x,k) = \Lambda(k)\psi(x,k) \text{ on } \mathbb{T} = [0,2\pi)^d.$$

Observation: for $k = 0$, $\Lambda(0) = 0$ and $\psi(\cdot,0) \equiv 1$.

Linearization in the regime $k = \kappa e$, e unit vector of $\mathbb{R}^d$ and $0 < \kappa \ll 1$:
$\psi(x,k) = 1 + i\kappa\phi_e(x) + o(\kappa)$, $\Lambda(k) = \kappa^2\lambda_e + o(\kappa^2)$

$$-\nabla\cdot a(\nabla\phi_e(x) + e) = 0 \text{ on } \mathbb{T}$$

Multiply original equation by ψ, integrate over $\mathbb{T}$, use expansion:

$$\lambda_e = \fint_{\mathbb{T}} (\nabla\phi_e + e)\cdot a(\nabla\phi_e + e)$$

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$\psi(x,k) = 1 + i\kappa\phi_e(x) + o(\kappa)$, $\Lambda(k) = \kappa^2\lambda_e + o(\kappa^2)$

$$-\nabla \cdot a(\nabla\phi_e(x) + e) = 0 \text{ on } \mathbb{T} \rightsquigarrow \text{corrector}$$

Multiply original equation by ψ, integrate over $\mathbb{T}$, use expansion:

$$\lambda_e = \fint_{\mathbb{T}} (\nabla\phi_e + e)\cdot a(\nabla\phi_e + e) \rightsquigarrow \text{homogenized coefficient}$$

Justification & limitation

The magnetic eigenvalue problem

$$-(\nabla + ik) \cdot a(\nabla + ik)\psi(x,k) \;=\; \Lambda(k)\psi(x,k) \text{ on } \mathbb{T} = [0,2\pi)^d.$$

has compact resolvent (by Rellich) so that $-(\nabla + ik) \cdot a(\nabla + ik)$ admits (for all $k \in [0,2\pi)^d$) a sequence of eigenvectors and eigenvalues...

In the neighborhood of 0, $k \mapsto \psi(\cdot, k)$ is analytic so that the linearization can be justified. Higher-order expansions yield ϕ_2, ϕ_3,..., $\bar{a}_1, \bar{a}_2$,... And we have quantitative estimates for truncations of the series.

Justifies the DLS strategy: $\psi(x, \varepsilon k) = 1 + \varepsilon\kappa\phi_e(x) + O(\kappa^2)$.

Main limitation: $-(\nabla + ik) \cdot a(\nabla + ik)$ does not have compact resolvent for a quasi-periodic, almost-periodic, or random...

Part 2: Alternative approach

Observation 1: To state the result, we only need "correctors"

Observation 2: To prove the result, we only need to "diagonalize" the elliptic operator at the bottom of the spectrum since frequencies are rescaled by ε

Strategy: Use correctors to develop an approximate spectral theory

- construct correctors at any order and define approximate Bloch waves close to 0 as a jet using the correctors (=Taylor-Bloch waves)
- Taylor-Bloch waves are approximate extended waves: there is an error in the eigenvector/eigenvalue relation (=eigendefect), the structure of which is very special
- control the large-time error due to the eigendefect directly using the wave equation and its special structure (=energy estimates)

New family of correctors $(\phi_j, \sigma_j, \chi_j)$

- $\phi_0 \equiv 1$, and for all $j \geq 1$, ϕ_j is a scalar field solving $-\nabla \cdot a\nabla\phi_j = \nabla \cdot (-\sigma_{j-1} e + ae\phi_{j-1} + \nabla\chi_{j-1})$;
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-
-
-

New family of correctors $(\phi_j, \sigma_j, \chi_j)$

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- for all $j \geq 0$, $\bar{a}_j \cdot e^{\otimes(j+1)} = \int_{\mathbb{T}} a(\nabla \phi_{j+1} + e\phi_j), \lambda_j := \bar{a}_j \cdot e^{\otimes(j+2)}$;
- $\chi_0 \equiv 0$, $\chi_1 \equiv 0$, and for all $j \geq 2$, χ_j is a scalar field solving $-\triangle \chi_j = \nabla \chi_{j-1} \cdot e + \sum_{l=1}^{j-1} \lambda_{j-1-l} \phi_l$;
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New family of correctors $(\phi_j, \sigma_j, \chi_j)$

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- for all $j \geq 1$, q_j is the vector field
  $q_j := a(\nabla\phi_j + e\phi_{j-1}) - \lambda_{j-1}e + \nabla\chi_{j-1} - \sigma_{j-1}e, \quad \int_{\mathbb{T}} q_j = 0;$
- $\sigma_0 \equiv 0$, and for all $j \geq 1$, σ_j is a skew-symmetric matrix field, i.e. $\sigma_{jkl} = -\sigma_{jlk}$, that solves $-\triangle\sigma_j = \nabla \times q_j, \quad \nabla \cdot \sigma_j = q_j$, with the three-dimensional notation: $[\nabla \times q_j]_{mn} = \nabla_m[q_j]_n - \nabla_n[q_j]_m$,

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- for all $j \geq 0$, $\bar{a}_j \cdot e^{\otimes(j+1)} = \int_{\mathbb{T}} a(\nabla\phi_{j+1} + e\phi_j), \lambda_j := \bar{a}_j \cdot e^{\otimes(j+2)}$;
- $\chi_0 \equiv 0$, $\chi_1 \equiv 0$, and for all $j \geq 2$, χ_j is a scalar field solving $-\triangle\chi_j = \nabla\chi_{j-1} \cdot e + \sum_{l=1}^{j-1} \lambda_{j-1-l}\phi_l$;
- for all $j \geq 1$, q_j is the vector field $q_j := a(\nabla\phi_j + e\phi_{j-1}) - \lambda_{j-1}e + \nabla\chi_{j-1} - \sigma_{j-1}e, \quad \int_{\mathbb{T}} q_j = 0$;
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Very subtle algebraic structure.

For periodic coefficients all the correctors exist and are periodic.

Approximate spectral theory

Taylor-Bloch wave $\psi_{k,\ell}$ and Taylor-Bloch eigenvalue $\tilde{\lambda}_{k,\ell}$ of order ℓ in direction $k = \kappa e$ are defined by

$$\psi_{k,\ell} := \sum_{j=0}^{\ell} (i\kappa)^j \phi_j, \quad \tilde{\lambda}_{k,\ell} := \kappa^2 \sum_{j=0}^{\ell-1} (i\kappa)^j \lambda_j \in \mathbb{R}.$$

Almost diagonalization of magnetic Laplacian for $0 < \kappa \ll 1$:

$$-(\nabla + ik) \cdot a(\nabla + ik)\psi_{k,\ell} = \tilde{\lambda}_{k,\ell}\psi_{k,\ell} - (i\kappa)^{\ell+1}\mathfrak{d}_{k,\ell}, \qquad (1)$$

where the Taylor-Bloch eigendefect $\mathfrak{d}_{k,\ell}$ is given by

$$\mathfrak{d}_{k,\ell} = \nabla \cdot (-\sigma_\ell e + ae\varphi_\ell + \nabla\chi_\ell) + i\kappa\Big(e \cdot ae\varphi_\ell - \sum_{j=1}^{\ell} \sum_{l=\ell-j}^{\ell-1} (i\kappa)^{j+l-\ell}\lambda_l \phi_j\Big).$$

Subtle structure: eigendefect = divergence term + higher order term

Approximate solution of the wave equation

Strategy: use the approximate spectral theory to construct an approximate solution of the wave equation

$$\begin{cases} \square_\varepsilon u_\varepsilon &= 0 \\ u_\varepsilon(0,\cdot) &= u_0 \\ \partial_t u_\varepsilon(0,\cdot) &= 0 \end{cases}$$

- well-prepare initial condition (cannot use that u_0 can be expanded on Taylor-Bloch waves)
- use that Taylor-Bloch wave almost diagonalize the elliptic operator and control the error by energy estimates
- reformulate the almost-solution and write an approximate (high-order) homogenized equation

[Estimates are first presented in the periodic and quasi-periodic setting]

Approximate solution of the wave equation

Step 1: replace u_0 by a well-prepared data
$u_{0,\ell,\varepsilon} := \int_{\mathbb{R}^d} \hat{u}_0(k) e^{ik\cdot x} \psi_{\varepsilon k,\ell}(\frac{x}{\varepsilon}) d^* k$.

Energy estimate: Solution $v_{\varepsilon,\ell}$ with initial condition $u_{0,\ell,\varepsilon}$

$$\|u_\varepsilon - v_{\varepsilon,\ell}\|_{L^\infty(\mathbb{R}_+, L^2(\mathbb{R}^d))} \leq \|u_0 - u_{0,\ell,\varepsilon}\|_{L^2(\mathbb{R}^d)} \leq C(u_0)\varepsilon.$$

Step 2: almost diagonalization of the wave equation
Set $\Lambda_\ell(k) := \sqrt{\max\{0, \tilde{\lambda}_{k,\ell}\}}$, and define

$$w_{\varepsilon,\ell}(t,x) = \int_{\mathbb{R}^d} \hat{u}_0(k) e^{ik\cdot x} \psi_{\varepsilon k,\ell}(\frac{x}{\varepsilon}) \cos(\varepsilon^{-1} \Lambda_\ell(\varepsilon k) t) d^* k$$

Energy estimate: for all $T \geq 0$,

$$\|v_{\varepsilon,\ell} - w_{\varepsilon,\ell}\|_{L^\infty([0,T], L^2(\mathbb{R}^d))} \leq C(u_0)(\varepsilon + \varepsilon^\ell T).$$

Sketch of the argument for **step 2**

One of the error terms solves

$$\begin{cases} \Box_\varepsilon \delta v(t,x) &= \varepsilon^\ell \int_{\mathbb{R}^d} G(t,k)\nabla\cdot\Big(g(\tfrac{x}{\varepsilon})e^{ik\cdot x}\Big)\cos(\varepsilon^{-1}\Lambda_\ell(\varepsilon k)t)dk \\ \delta v(0,\cdot) &= \partial_t\delta v(0,\cdot) = 0 \end{cases}$$

Difficulty: how not to lose accuracy in ε?

- wave equation is not regularizing: need to estimate RHS in $L^2(R^d)$
- $\nabla\cdot\Big(g(\frac{x}{\varepsilon})e^{ik\cdot x}\Big)$ is only bounded by ε^{-1} in $L^2(\mathbb{R}^d)$

First need to estimate $\|\partial_t\delta v(t)\|_{L^2(\mathbb{R}^d)} + \|\nabla\delta v(t)\|_{L^2(\mathbb{R}^d)}$.
Multiply by $\partial_t\delta v$ and integrate over $[0,t]\times\mathbb{R}^d$:

$$\|\partial_t\delta v(t)\|^2_{L^2(\mathbb{R}^d)} + \|\nabla\delta v(t)\|^2_{L^2(\mathbb{R}^d)} \lesssim \varepsilon^\ell I,$$
$$I := -\int_{[0,t]\times\mathbb{R}^d}\int_{\mathbb{R}^d} G(s,k)\nabla\cdot\Big(g(\tfrac{x}{\varepsilon})e^{ik\cdot x}\Big)\cos(\varepsilon^{-1}\Lambda_\ell(\varepsilon k)t)dk\,\partial_t\delta v(s,x)dsdx$$

Sketch of the argument for **step 2**

Key observation: integrate by parts in space first, then in time

$$
\begin{aligned}
I &= -\int_{[0,t]\times\mathbb{R}^d}\int_{\mathbb{R}^d} G(s,k)\nabla\cdot\Big(g(\tfrac{x}{\varepsilon})e^{ik\cdot x}\Big)\cos(\varepsilon^{-1}\Lambda_\ell(\varepsilon k)t)dk\partial_t\delta v(s,x)dsdx \\
&= \int_{[0,t]\times\mathbb{R}^d}\int_{\mathbb{R}^d} G(s,k)g(\tfrac{x}{\varepsilon})e^{ik\cdot x}\cos(\varepsilon^{-1}\Lambda_\ell(\varepsilon k)t)dk\nabla\partial_t\delta v(s,x)dsdx \\
&= \int_{[0,t]\times\mathbb{R}^d}\int_{\mathbb{R}^d} \varepsilon^{-1}\Lambda_\ell(\varepsilon k)G(s,k)\Big(g(\tfrac{x}{\varepsilon})e^{ik\cdot x}\Big)\sin(\varepsilon^{-1}\Lambda_\ell(\varepsilon k)t)dk\nabla\delta v(s,x)dsdx \\
&\quad +\int_{\mathbb{R}^d}\nabla\delta v(t,x)\cdot\int_{\mathbb{R}^d} g(\tfrac{x}{\varepsilon})e^{ik\cdot x}\cos(\varepsilon^{-1}\Lambda_\ell(\varepsilon k)t)dkdx
\end{aligned}
$$

Recall that

$$\|\partial_t\delta v(t)\|^2_{L^2(\mathbb{R}^d)}+\|\nabla\delta v(t)\|^2_{L^2(\mathbb{R}^d)}\lesssim -\varepsilon^\ell I,$$

so that by Young's inequality

$$\|\partial_t\delta v(t)\|^2_{L^2(\mathbb{R}^d)}+\|\nabla\delta v(t)\|^2_{L^2(\mathbb{R}^d)}\lesssim C_\ell(u_0)\varepsilon^\ell T.$$

Approximate solution of the wave equation

Step 3: Throw away the correctors

$$\begin{aligned} w_{\varepsilon,\ell}(t,x) &= \int_{\mathbb{R}^d} \hat{u}_0(k) e^{ik\cdot x} \psi_{\varepsilon k,\ell}(\frac{x}{\varepsilon}) \cos(\varepsilon^{-1}\Lambda_\ell(\varepsilon k)t) d^* k \\ &\simeq \int_{\mathbb{R}^d} \hat{u}_0(k) e^{ik\cdot x} \cos(\varepsilon^{-1}\Lambda_\ell(\varepsilon k)t) d^* k \;=:\; v_\varepsilon, \end{aligned}$$

With $\tilde{\Box}_{\ell,\varepsilon} := \partial^2_{tt} - \sum_{j=0}^{\ell-1} \varepsilon^j \bar{a}_j \cdot \nabla^{j+2}$, we have for all $T \geq 0$,

$$\left\{ \begin{array}{rcl} \Box_\varepsilon u_\varepsilon &=& 0 \\ u_\varepsilon(0,\cdot) &=& u_0 \\ \partial_t u_\varepsilon(0,\cdot) &=& 0 \end{array} \right. , \left\{ \begin{array}{rcl} \tilde{\Box}_{\ell,\varepsilon} v_\varepsilon &=& 0 \\ v_\varepsilon(0,\cdot) &=& u_0 \\ \partial_t v_\varepsilon(0,\cdot) &=& 0 \end{array} \right.$$

$$\sup_{t\leq T} \|u_\varepsilon(t) - v_\varepsilon(t)\|_{L^2(\mathbb{R}^d)} \lesssim C_\ell(u_0)(\varepsilon + \varepsilon^\ell T).$$

And it remains to add a regularizing term to $\tilde{\Box}_{\ell,\varepsilon}$ to make it invertible.

Main result in the periodic setting

For all $\varepsilon > 0$,

- $a_\varepsilon := a(\frac{\cdot}{\varepsilon})$, a periodic + symmetric tensor
- $\Box_\varepsilon := \partial_{tt}^2 - \nabla \cdot a_\varepsilon \nabla$ the wave operator

There exists a family $\{\bar{a}_j\}_{j\in\mathbb{N}}$ of $j+2$-order tensors (with $\bar{a}_{2j+1} = 0$), and for all $\ell \in \mathbb{N}$ we set

- $\bar{\Box}_{\ell,\varepsilon} := \partial_{tt}^2 - \sum_{j=0}^{\ell-1} \varepsilon^j \bar{a}_j \cdot \nabla^{j+2} - K_\ell (i\varepsilon)^{2([\frac{\ell-1}{2}]+1)} \mathrm{Id} \cdot \nabla^{2([\frac{\ell-1}{2}]+2)}$.

For all $\ell \in \mathbb{N}$, well-chosen $K_\ell = K_\ell(\bar{a}_0, \dots \bar{a}_{\ell-1})$, all $u_0 \in \mathcal{S}(\mathbb{R}^d)$, the solutions $u_\varepsilon, u_{\ell,\varepsilon} \in L^\infty(\mathbb{R}_+, L^2(\mathbb{R}^d))$ of

$$\left\{\begin{array}{rcl} \Box_\varepsilon u_\varepsilon &=& 0 \\ u_\varepsilon(0,\cdot) &=& u_0 \\ \partial_t u_\varepsilon(0,\cdot) &=& 0 \end{array}\right., \left\{\begin{array}{rcl} \bar{\Box}_{\ell,\varepsilon} u_{\ell,\varepsilon} &=& 0 \\ u_{\ell,\varepsilon}(0,\cdot) &=& u_0 \\ \partial_t u_{\ell,\varepsilon}(0,\cdot) &=& 0 \end{array}\right.$$

satisfy for all $T > 0$:

$$\sup_{t \le T} \|u_\varepsilon(t) - u_{\ell,\varepsilon}(t)\|_{L^2(\mathbb{R}^d)} \lesssim C_\ell(u_0)(\varepsilon + \varepsilon^\ell T).$$

Part 3: Bounds on correctors in the random case

Example of a given by Poisson random inclusions of fixed size we have for the correctors [G-Otto,G-Neukamm-Otto,Armstrong-Kuusi-Mourrat]:

$$
\begin{aligned}
|(\phi_1,\sigma_1,\nabla\chi_1)(x)| &\lesssim_\omega \begin{cases} d=1 & : \quad (1+|x|)^{\frac{1}{2}} \\ d=2 & : \quad \log(2+|x|)^{\frac{1}{2}} \\ d>2 & : \quad 1 \end{cases} \\
|(\phi_2,\sigma_2,\nabla\chi_2)(x)| &\lesssim_\omega \begin{cases} d=3 & : \quad (1+|x|)^{\frac{1}{2}} \\ d=4 & : \quad \log(2+|x|)^{\frac{1}{2}} \\ d>4 & : \quad 1 \end{cases} \\
|(\phi_3,\sigma_3,\nabla\chi_3)(x)| &\lesssim_\omega \begin{cases} d=5 & : \quad (1+|x|)^{\frac{1}{2}} \\ d=6 & : \quad \log(2+|x|)^{\frac{1}{2}} \\ d>6 & : \quad 1 \end{cases}
\end{aligned}
\tag{2}
$$

Apply strategy of Part 2: dispersive effects appear for $d \geq 5$.
In smaller dimensions, homogenization breaks down before the occurence of dispersive effects

Further comments on the random case

- Sharp bounds on the correctors can be proved for correlated fields as well [Duerinckx-G.,G-Neukamm-Otto]
- The main result can be formulated as asymptotic ballistic transport of classical waves at the bottom of the spectrum (for random case, requires $d > 2$)
- Two phenomena could occur when "homogenization breaks down":
  - the transport remains ballistic (as for wave equation), but the effective equation is different (if any)
  - the transport stops being ballistic, and might become diffuse (as for the random Schrödinger equation), radiative transfer?

Summary of the talk

- Main idea 1: develop an approximate spectral theory at the bottom of the spectrum (a lot of structure), cf. Taylor-Bloch waves
- Main idea 2: combine Fourier space (in the form of estimates of Fourier multipliers) with energy estimates given by the wave equation
- Main result: bounds on extended correctors drive long-time homogenization of the wave equation (periodic, quasi-periodic, almost periodic, random...)
- Main challenging problem: what happens when homogenization breaks down? (For Schrödinger, work in progress with Duerinckx & Shirley)