A robust a-posteriori error estimator for Divergence-conforming DG methods for Oseen equation

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Motivation

- The finite element approximations of convection-diffusion equations or oseen equations have one drawback due to present layers of small width in solutions where their gradients change very rapidly.
- In general, these layers seem in solution as boundary layers (near the outflow boundary of the domain) or as internal layers (due to non-smooth data near the inflow boundary).
- To resolve these problems, the approach requires adaptive finite element methods which are able to locally refining the meshes in the vicinity of the layers and other singularities.

Oseen equation

$$\begin{aligned} -\nu\Delta\mathbf{u} + \underline{\mathbf{a}}\cdot\nabla\mathbf{u} + \nabla p + b\mathbf{u} &= \mathbf{f} \quad \text{in} \quad \Omega, \\ \nabla\cdot\mathbf{u} &= 0 \quad \text{in} \quad \Omega, \\ \mathbf{u} &= \mathbf{0} \quad \text{on} \quad \Gamma, \\ \int_\Omega p\, dx &= 0. \quad \text{(Compatibility relation)} \end{aligned}$$

Here $\mathbf{u}$, p, $\mathbf{f}$, ν, $\underline{\mathbf{a}}$ and b are the velocity, the pressure, a prescribed external body force, the kinematic viscosity, a convective velocity field and a given scalar function, respectively.

Existence and uniqueness

- Here $\underline{a}(\mathbf{x}) \in \mathbf{W}^{1,\infty}(\Omega)$ and $b(\mathbf{x}) \in L^\infty(\Omega)$. If $\underline{a}(\mathbf{x})$ and the size of domain Ω are of order one, then $\frac{1}{\nu}$ is the reynold number.
- (First case) Assume that $\mathbf{u}$ is the velocity at the current time, $\underline{\mathbf{a}}$ is the velocity at the previous time step and $b = 1/\Delta t$, this imply $b > 0$.
- (Second case) $b = 0$ for the steady-state Navier-Stokes problem.
- (Assumption 1)
$$-\frac{1}{2}\nabla\cdot\underline{\mathbf{a}}(\mathbf{x}) + b(\mathbf{x}) \geq \beta, \quad \mathbf{x}\in\Omega, \quad \|\nabla\cdot\underline{\mathbf{a}}(\mathbf{x}) + b(\mathbf{x})\|_{L^\infty(\Omega)} \leq c_\star\beta.$$
- Assumption 1 guarantees existence and uniqueness of a solution $(\mathbf{u}, p) \in \mathbf{H}_0^1(\Omega)\times L_0^2(\Omega)$.
- If $\beta = 0$ then $\nabla\cdot\underline{a} = b$. Moreover $-\nu\Delta\mathbf{u} + \nabla\cdot(\underline{\mathbf{a}}\mathbf{u}) + \nabla p = \mathbf{f}$. Assumption 1 is satisfied provided that $\nabla\cdot\underline{\mathbf{a}} \geq 0$.

H^{div}-DG formulation for Oseen problem

- First we devide the domain Ω by a subdivision $\mathcal{T}_h$ into a mesh of shape-regular rectangular cell K.
- Let h_K and $\mathcal{E}(\mathcal{T}_h)$ be denoted as the diameter of an element K and the set of edges of $\mathcal{T}_h$, respectively.
- For given mesh $\mathcal{T}_h$, the notions of broken spaces for the continuous and differentiable function spaces are denoted as $\mathcal{C}(\mathcal{T}_h)$ and $H^s(\mathcal{T}_h)$.
- Discrete subspace of $H_0^{div}(\Omega)$
$$\mathbf{V}_h = \{v \in H_0^{\mathrm{div}} \mid \forall K\in\mathcal{T}_h : v|_K \in RT_k \quad \text{for } k\geq 1\},$$
$$\mathbf{V}_h^0 = \{v\in\mathbf{V}_h \mid \nabla\cdot v = 0\}.$$
- Discrete subspace of $L_0^2(\Omega)$
$$Q_h = \{v\in L_0^2 \mid \forall K\in\mathcal{T}_h : v|_K \in Q_k(K) \quad \text{for } k\geq 1\}.$$
- Important property of the pair $\mathbf{V}_h\times Q_h$
$$\nabla\cdot\mathbf{V}_h \subset Q_h.$$
- (Discrete weak formulation) Find $(\mathbf{u}, p)\in\mathbf{V}_h\times Q_h$ such that
$$\mathcal{A}_h(\mathbf{u}, p; \mathbf{v}, q) = (\mathbf{f}, \mathbf{v}), \quad \forall \quad (\mathbf{v}, q)\in\mathbf{V}_h\times Q_h,$$
where
$$\mathcal{A}_h(\mathbf{u}, p; \mathbf{v}, q) = a_h(\mathbf{u}, \mathbf{v}) + o_h(\mathbf{u}, \mathbf{v}) - (p, \nabla.\mathbf{v}) - (q, \nabla.\mathbf{u}).$$
- Details of $a_h(\mathbf{u}, \mathbf{v})$
$$\begin{aligned} a_h(\mathbf{u},\mathbf{v}) &= \nu(\nabla\mathbf{u}, \nabla\mathbf{v})_{\mathcal{T}_h} + a_h^i(\mathbf{u},\mathbf{v}) + a_h^\partial(\mathbf{u},\mathbf{v}), \\ a_h^i(\mathbf{u},\mathbf{v}) &= a_p^i(\mathbf{u},\mathbf{v}) - a_c^i(\mathbf{u},\mathbf{v}) - a_c^i(\mathbf{v},\mathbf{u}), \\ a_h^\partial(\mathbf{u},\mathbf{v}) &= a_p^\partial(\mathbf{u},\mathbf{v}) - a_c^\partial(\mathbf{u},\mathbf{v}) - a_c^\partial(\mathbf{v},\mathbf{u}). \end{aligned}$$
- Interior face terms and Nitsche terms
$$a_c^i(\mathbf{u},\mathbf{v}) = \langle\{\{\nu\nabla\mathbf{u}\}\}, [\![\mathbf{v}\otimes\mathbf{n}]\!]\rangle_{\mathcal{E}^i(\mathcal{T}_h)}, \quad a_p^i(\mathbf{u},\mathbf{v}) = \langle\gamma_h^2[\![\mathbf{u}\otimes\mathbf{n}]\!], [\![\mathbf{v}\otimes\mathbf{n}]\!]\rangle_{\mathcal{E}^i(\mathcal{T}_h)},$$
$$a_c^\partial(\mathbf{u},\mathbf{v}) = \langle\nu\nabla\mathbf{u}, \mathbf{v}\otimes\mathbf{n}\rangle_{\mathcal{E}^\partial(\mathcal{T}_h)}, \quad a_p^\partial(\mathbf{u},\mathbf{v}) = \langle\gamma_h^2\mathbf{u}\otimes\mathbf{n}, \mathbf{v}\otimes\mathbf{n}\rangle_{\mathcal{E}^\partial(\mathcal{T}_h)},$$
- Definition of $o_h(\mathbf{u}, \mathbf{v})$
$$o_h(\mathbf{u},\mathbf{v}) = \sum_{K\in\mathcal{T}_h}\int_K((b-\nabla.\underline{\mathbf{a}})\mathbf{u}\mathbf{v} - (\underline{\mathbf{a}}.\nabla)\mathbf{v}\cdot\mathbf{u})\,\mathrm{dx} + \frac{1}{2}\sum_{K\in\mathcal{T}_h}\int_{\partial K}[\underline{\mathbf{a}}.n_K[\![\mathbf{u}\otimes\mathbf{n}]\!] - |\underline{\mathbf{a}}.n_K|(\mathbf{u}^e - \mathbf{u})]\cdot\mathbf{v}\mathrm{ds}.$$
- DG Norm
$$|||(\mathbf{u}, p)|||^2 = |||\mathbf{u}|||^2 + \nu^{-1}||p||^2_{\mathcal{T}_h},$$
where
$$|||\mathbf{u}|||^2 = \nu||\nabla\mathbf{u}||^2_{\mathcal{T}_h} + a_p^i(\mathbf{u},\mathbf{u}) + a_p^\partial(\mathbf{u},\mathbf{u}) + \beta||\mathbf{u}||^2_{\mathcal{T}_h}.$$
- Semi Norm
$$|\mathbf{u}|_A^2 = |\underline{\mathbf{a}}\mathbf{u}|_\star^2 + \sum_{E\in\mathcal{E}(\mathcal{T}_h)}\left(\beta h_E + \frac{h_E}{\nu}\right)||[\![\mathbf{u}]\!]||^2_{0,E}, \text{ where } |\underline{q}|_\star^2 = \sup_{\phi\in\mathbf{H}_0^1(\Omega)\setminus\{0\}}\frac{\int_\Omega\underline{q}\cdot\nabla\phi d\mathbf{x}}{|||\phi|||}.$$

References

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A posteriori error estimator

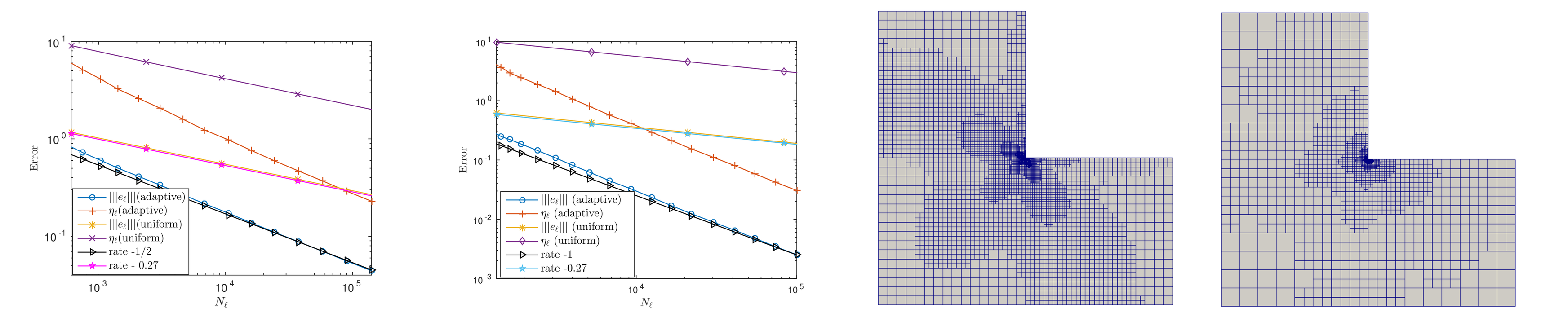
- Parameters
$$\rho_K = \min\{h_K\nu^{-\frac{1}{2}}, \beta^{-\frac{1}{2}}\}, \quad \rho_E = \min\{h_E\nu^{-\frac{1}{2}}, \beta^{-\frac{1}{2}}\}.$$
- Note that if $\beta = 0$ we choose $\rho_K = h_k\nu^{-\frac{1}{2}}$ and $\rho_E = h_E\nu^{-\frac{1}{2}}$.
- Local error indicator
$$\eta_K^2 = \eta_{R_K}^2 + \eta_{E_K}^2 + \eta_{J_K}^2,$$
- Interior residual
$$\eta_{R_K}^2 = \rho_K^2||\mathbf{f}_h + \nu\Delta\mathbf{u}_h - \underline{\mathbf{a}}_h.\nabla\mathbf{u}_h - \nabla p_h - b\mathbf{u}_h||^2_{0,K}.$$
- Edge residual
$$\eta_{E_K}^2 = \frac{1}{2}\sum_{E\in\partial K\setminus\Gamma}\nu^{-\frac{1}{2}}\rho_E||[\![(p_h\underline{\mathbf{I}} - \nu\nabla\mathbf{u}_h)\cdot\mathbf{n}]\!]||^2_{0,E}.$$
- Jump of the approximate solution $\mathbf{u}_h$
$$\eta_{J_K}^2 = \frac{1}{2}\sum_{E\in\partial K\setminus\Gamma}\left(\frac{\gamma\nu}{h_E} + \beta h_E + \frac{h_E}{\nu}\right)||[\![\mathbf{u}_h\otimes\mathbf{n}]\!]||^2_{0,E} + \sum_{E\in\partial K\cap\Gamma}\left(\frac{\gamma\nu}{h_E} + \beta h_E + \frac{h_E}{\nu}\right)||\mathbf{u}_h||^2_{0,E}.$$
- Data oscillation term
$$\Theta_K^2 = \rho_K^2(||\mathbf{f} - \mathbf{f}_h||^2_{0,K} + ||(\underline{\mathbf{a}} - \underline{\mathbf{a}}_h).\nabla\mathbf{u}_h||^2_{0,K} + ||(b - b_h)\mathbf{u}_h||^2_{0,K}.$$
- A-posteriori error estimator and Data oscillation error
$$\eta = \Big(\sum_{K\in\mathcal{T}_h}\eta_K^2\Big)^{\frac{1}{2}}, \quad \Theta = \Big(\sum_{K\in\mathcal{T}_h}\Theta_K^2\Big)^{\frac{1}{2}}.$$
- Reliability
$$|||\mathbf{u} - \mathbf{u}_h||| + \nu^{-1/2}||p - p_h||_0 + |\mathbf{u} - \mathbf{u}_h|_A \lesssim \eta + \Theta.$$
- Efficiency
$$\eta \lesssim |||\mathbf{u} - \mathbf{u}_h||| + \nu^{-1/2}||p - p_h||_0 + |\mathbf{u} - \mathbf{u}_h|_A + \Theta.$$

Remark

- Assume that the error $|||e|||$ converges with optimal order $\mathcal{O}(N^{-k/2})$, where N and k are the number of degree of freedom and the poynomial order of $RT_k\times Q_k$, respectively. Then
$$|\underline{\mathbf{a}}(\mathbf{u} - \mathbf{u}_h)|_\star \lesssim \nu^{-1/2}||\mathbf{u} - \mathbf{u}_h||_0.$$
- Assume that $||\mathbf{u} - \mathbf{u}_h||_0$ converges with with optimal order $\mathcal{O}(N^{-k/2-1/2})$, then
$$|\underline{\mathbf{a}}(\mathbf{u} - \mathbf{u}_h)|_\star \lesssim (N^{-1/2}\nu^{-1})\,\nu^{1/2}N^{-k/2}.$$
- Similarly, we have
$$\Big(\sum_{E\in\mathcal{E}(\mathcal{T}_h)}h_E\nu^{-1}||[\![\mathbf{u} - \mathbf{u_h}]\!]||^2_{0,E}\Big)^{1/2} \lesssim (N^{-1/2}\nu^{-1})\,\nu^{1/2}N^{-k/2},$$
$$\Big(\sum_{E\in\mathcal{E}(\mathcal{T}_h)}h_E\beta||[\![\mathbf{u} - \mathbf{u_h}]\!]||^2_{0,E}\Big)^{1/2} \lesssim \beta^{1/2}N^{-k/2-1/2}.$$
- Hence the same conclusion as for $|\underline{\mathbf{a}}(\mathbf{u} - \mathbf{u}_h)|_\star$ can also prove for the error $|\mathbf{u} - \mathbf{u}_h|_A$.

Numerical Results

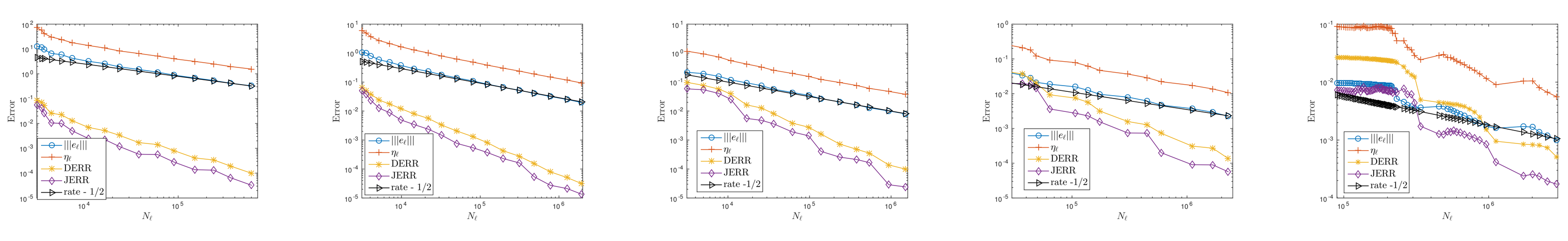
(First Example) L-shape domain (Stokes equation) $\Omega = (-1,1)^2\setminus(0,1)^2$.



(a) (b) (c) (d)

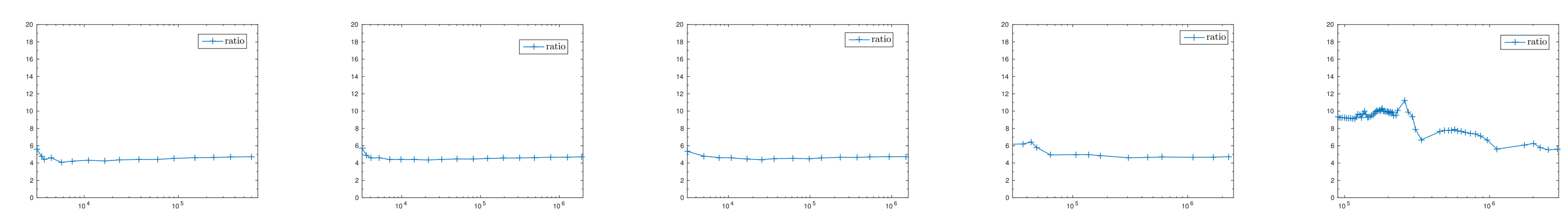
Figure : Convergence history of $|||e_\ell|||$ and η_ℓ (a) $RT_1\times Q_1$ (b) $RT_2\times Q_2$ on uniformly and adaptively refined meshes for the L-shape domain. (c) Adaptive refined meshes for $RT_1\times Q_1$ (d) Adaptive refined meshes for $RT_2\times Q_2$.

(Second Example) Kovasznays solution (Oseen equation) $\Omega = \left(-\frac{1}{2}, \frac{3}{2}\right)\times(0,2)$.



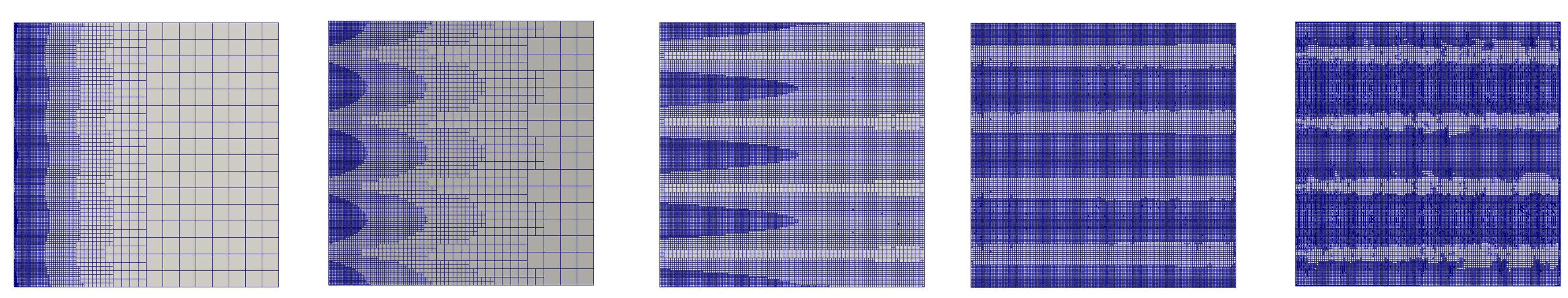
(a) (b) (c) (d) (e)

Figure : Convergence behaviour for (a) $\nu = 1$, (b) $\nu = 10^{-1}$, (c) $\nu = 10^{-2}$ (d) $\nu = 10^{-3}$, (e) $\nu = 10^{-4}$ with $RT_1\times Q_1$ element



(a) (b) (c) (d) (e)

Figure : Ratio for (a) $\nu = 1$, (b) $\nu = 10^{-1}$, (c) $\nu = 10^{-2}$ (d) $\nu = 10^{-3}$, (e) $\nu = 10^{-4}$ with $RT_1\times Q_1$ element



(a) (b) (c) (d) (e)

Figure : Adaptive refined mesh for (a) $\nu = 1$, (b) $\nu = 10^{-1}$, (c) $\nu = 10^{-2}$ (d) $\nu = 10^{-3}$, (e) $\nu = 10^{-4}$ with $RT_1\times Q_1$ element

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