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Mean-square stability analysis of SPDE approximations

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Mean-square stability

$$\mathrm{d}X(t) = (AX(t) + F(X(t)))\,\mathrm{d}t + G(X(t))\,\mathrm{d}L(t), \quad X(0) = X_0$$

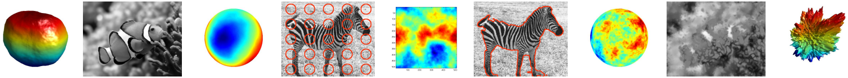
Assume $\mathbb{E}[\|X_0\|_H^2] < \delta$. What do we know about $\mathbb{E}[\|X(t)\|_H^2]$?

$$X_h^{j+1} = D_{\Delta t,h}^{\mathrm{det}} X_h^j + D_{\Delta t,h}^{\mathrm{stoch},j} X_h^j, \quad X_h^0 = X_0$$

Assume $\mathbb{E}[\|X_0\|_H^2] < \delta$. What do we know about $\mathbb{E}[\|X_h^j\|_H^2]$?

Overview

- Stochastic partial differential equations
- Asymptotic mean-square stability
- Application to Galerkin methods
- Simulation



Introduction to SPDEs

H separable Hilbert space

Heat equation

- $D = [0, 1)$
- $H = L^2(D)$

$$\frac{\partial}{\partial t} u(t,x) = \Delta u(t,x) \quad + \dot{\eta}(t,x), \qquad u(0) = u_0$$

$$\frac{\partial}{\partial t} u(t,x) = \Delta u(t,x) \quad + g(u(t,x),t,x)\dot{\eta}(t,x), \qquad u(0) = u_0$$

Stochastic partial differential equations

$$\mathrm{d}X(t) = (AX(t) + F(X(t)))\,\mathrm{d}t + G(X(t))\,\mathrm{d}M(t), \quad X(0) = X_0$$

- Stochastic *integral equation*

$$X(t) = X_0 + \int_0^t (AX(s) + F(X(s)))\,ds + \int_0^t G(X(s))\,dM(s)$$

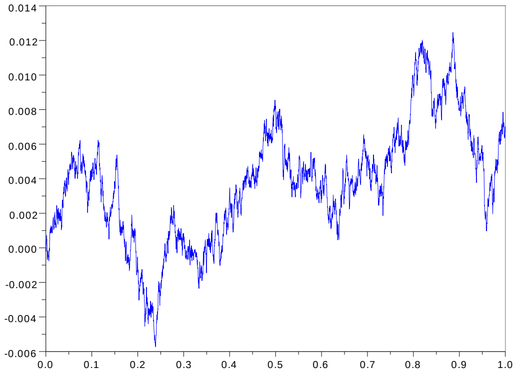
- *Noise*
  - Filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}_t, t \geq 0), P)$
  - Separable Hilbert space U — e. g., $\mathbb{R}^d$, $L^2(D)$, $H^\alpha(D)$
  - Martingale $M = (M(t), t \geq 0) \in \mathcal{M}^2(U)$ with covariance $(Q_t, t \geq 0)$

$$\int_r^t Q_s\, d\langle M, M\rangle_s \leq (t-r)\,Q, \quad r \leq t$$

  - $Q \in L_1^+(U)$ and $\mathcal{H} = Q^{1/2}(U)$

Examples

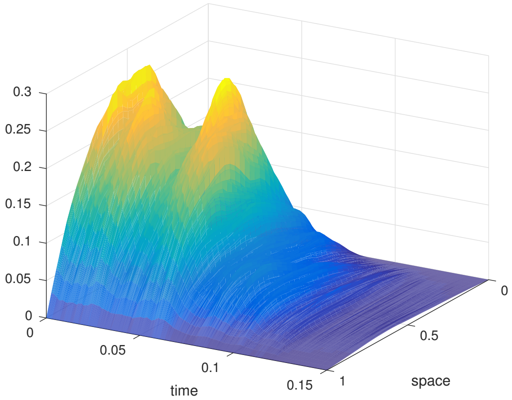
- Brownian motion — Q–Wiener process $W = (W(t), t \geq 0)$



Examples

- *Parabolic* partial differential equation (heat equation)

$$\mathrm{d}X(t) = \Delta X(t)\,\mathrm{d}t + G(X(t))\,\mathrm{d}W(t)$$



Framework

$$\mathrm{d}X(t) = (AX(t) + F(X(t)))\,\mathrm{d}t + G(X(t))\,\mathrm{d}M(t), \quad X(0) = X_0$$

- A generator of C_0-semigroup $S = (S(t), t \geq 0)$
- $F : H \to H$ linear
- $G : H \to L_{HS}(\mathcal{H}; H)$ linear

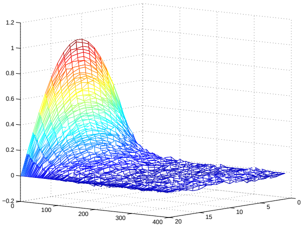
Theorem ([Peszat, Zabczyk 07])

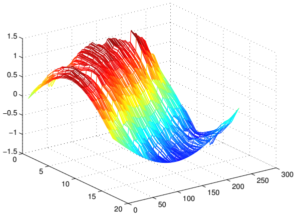
- $\exists!$ *mild solution* X
- X *has càdlàg modification*

$$X(t) = S(t)X_0 + \int_0^t S(t-s)F(X(s))\,\mathrm{d}s + \int_0^t S(t-s)G(X(s))\,\mathrm{d}M(s)$$

Solutions of SPDEs $\longrightarrow$ Numbers on computer

- Discretization in space and time: *fully discrete problem*
- Discretization of the noise: *truncation of Karhunen–Loève expansion*
- “Discretization” of Ω: *path simulation*, *Monte Carlo methods*





Fully discrete approximation of solutions

$$\mathrm{d}X(t) = (AX(t) + F(X(t)))\,\mathrm{d}t + G(X(t))\,\mathrm{d}M(t), \quad X(0) = X_0$$

- Abstract approximation scheme

$$X_h^{j+1} = D_{\Delta t,h}^{\mathrm{det}} X_h^j + D_{\Delta t,h}^{\mathrm{stoch},j} X_h^j, \quad X_h^0 = \tilde{X}_0 \in V_h$$

  - $(V_h, h \in (0,1])$, $V_h \subset H$, $\dim(V_h) = N_h \in \mathbb{N}$
  - P_h orthogonal projection onto V_h
  - $\Delta t > 0$, $t_j = j \cdot \Delta t$
- $D_{\Delta t,h}^{\mathrm{det}} \in L(V_h)$ approximating PDE $\dfrac{\partial}{\partial t} u(t) = Au(t) + F(u(t))$
- $D_{\Delta t,h}^{\mathrm{stoch},j} X_h^j$ approximating stochastic integral
  - $(D_{\Delta t,h}^{\mathrm{stoch},j}, j \in \mathbb{N}_0)$ $\mathcal{F}$*-compatible*
    ($D_{\Delta t,h}^{\mathrm{stoch},j}$ $\mathcal{F}_{t_{j+1}}$-measurable, $\mathbb{E}[D_{\Delta t,h}^{\mathrm{stoch},j} | \mathcal{F}_{t_j}] = 0$)
  - $D_{\Delta t,h}^{\mathrm{stoch},j} \perp\!\!\!\perp \mathcal{F}_{t_j}$

Stability

$$X_h^{j+1} = D_{\Delta t,h}^{\mathrm{det}} X_h^j + D_{\Delta t,h}^{\mathrm{stoch},j} X_h^j, \quad X_h^0 = \tilde{X}_0$$

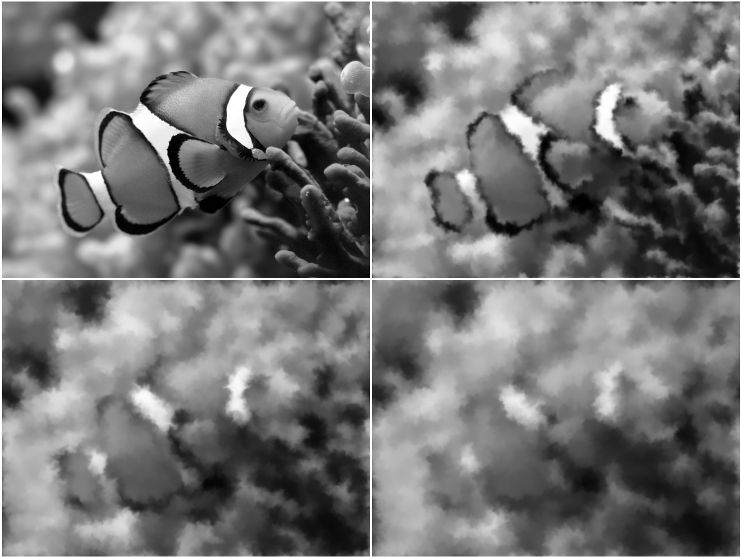
Definition

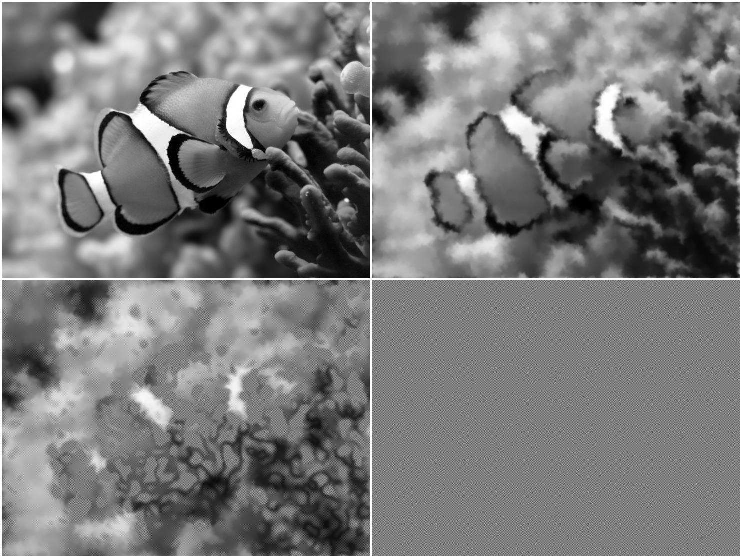
- *Numerical stability:* $\exists\, K > 1 : \forall \Delta t, h > 0,\ j \in \{0, \dots, n\}$:

$$\|(D_{\Delta t,h}^{\mathrm{det}})^j P_h\|_{L(H)} \le K$$

- *Mean-square stability:* $\forall \epsilon > 0 : \exists \delta > 0 : \forall j \in \mathbb{N}_0$:

$$\mathbb{E}[\|X_h^0\|_H^2] < \delta \quad \Longrightarrow \quad \mathbb{E}[\|X_h^j\|_H^2] < \epsilon$$





Lax's equivalence theorem

$$\mathrm{d}X(t) = AX(t)\,\mathrm{d}t + G(X(t))\,\mathrm{d}M(t)$$

Theorem ([L. 10])

Consistency in mean square

$$\Downarrow$$

Stability $\Longleftrightarrow$ *Convergence in mean square*

Mean-square stability: solution

$$\mathrm{d}X(t) = (AX(t) + F(X(t)))\,\mathrm{d}t + G(X(t))\,\mathrm{d}M(t), \quad X(0) = X_0$$

- $(X_e(t) = 0, t \geq 0)$ *equilibrium* solution

Definition

- X_e *mean-square stable:* $\forall \epsilon > 0 : \exists \delta > 0 : \forall t \geq 0 :$

$$\mathbb{E}[\|X(0)\|_H^2] < \delta \quad \Longrightarrow \quad \mathbb{E}[\|X(t)\|_H^2] < \epsilon$$

- X_e *asymptotically mean-square stable:* mean-square stable &

$$\lim_{t\to\infty} \mathbb{E}[\|X(t)\|_H^2] = 0$$

- X_e *asymptotically mean-square unstable:* if not asymptotically mean-square stable

Mean-square stability: approximation

$$X_h^{j+1} = D_{\Delta t,h}^{\mathrm{det}} X_h^j + D_{\Delta t,h}^{\mathrm{stoch},j} X_h^j, \quad X_h^0 = \tilde{X}_0$$

- $(X_{h,e}^j = 0, j \in \mathbb{N}_0)$ *equilibrium* solution

Definition

- $X_{h,e}$ *mean-square stable:* $\forall \epsilon > 0 : \exists \delta > 0 : \forall j \in \mathbb{N}_0 :$

$$\mathbb{E}[\|X_h^0\|_H^2] < \delta \quad \Longrightarrow \quad \mathbb{E}[\|X_h^j\|_H^2] < \epsilon$$

- $X_{h,e}$ *asymptotically mean-square stable:* mean-square stable &

$$\lim_{j\to\infty} \mathbb{E}[\|X_h^j\|_H^2] = 0$$

- $X_{h,e}$ *asymptotically mean-square unstable:* if not asymptotically mean-square stable

Mean-square stability: approximation

$$X_h^{j+1} = D^{\text{det}}_{\Delta t,h} X_h^j + D^{\text{stoch},j}_{\Delta t,h} X_h^j, \quad X_h^0 = \tilde{X}_0$$

Theorem

- $\mathcal{S}_j = D^{\text{det}}_{\Delta t,h} \otimes D^{\text{det}}_{\Delta t,h} + \mathbb{E}[D^{\text{stoch},j}_{\Delta t,h} \otimes D^{\text{stoch},j}_{\Delta t,h}]$
- $\lim_{n\to\infty} \|\mathcal{S}_n \cdots \mathcal{S}_0\|_{L(V_h^{(2)})} = 0$

$\Longrightarrow X_{h,e}$ *asymptotically mean-square stable*

Corollary

- $(D^{\text{stoch},j}_{\Delta t,h}, j \in \mathbb{N}_0)$ *iid*
- $\mathcal{S} = D^{\text{det}}_{\Delta t,h} \otimes D^{\text{det}}_{\Delta t,h} + \mathbb{E}[D^{\text{stoch},1}_{\Delta t,h} \otimes D^{\text{stoch},1}_{\Delta t,h}]$

$X_{h,e}$ *asymptotically mean-square stable* $\Longleftrightarrow \rho(\mathcal{S}) < 1$

Application to Galerkin methods: Euler–Maruyama

$$X_h^{j+1} = \left(R(\Delta t A_h) + r_d^{-1}(\Delta t A_h)(\Delta t P_h F + P_h G(\cdot)\Delta L^j)\right) X_h^j$$
$$X_h^0 = P_h X_0$$

- $R(z) = \dfrac{r_n(z)}{r_d(z)} \approx \exp(z)$ *rational approximation*
- $-A : \mathcal{D}(-A) \subset H \to H$ densely defined, self-adjoint, pos. def.
- $\dot{H}^1 = \mathcal{D}((-A)^{1/2})$, $(V_h, h \in (0,1])$, $V_h \subset \dot{H}^1$
- $-A_h : V_h \to V_h$ given by $\langle -A_h \phi, \psi \rangle_H = \langle (-A)^{1/2}\phi, (-A)^{1/2}\psi \rangle_H$

- $L = (L(t), t \geq 0)$ *Lévy process* with $\Delta L^j = L(t_{j+1}) - L(t_j)$ and

$$L(t) = \sum_{i=1}^{\infty} \sqrt{\mu_i} L_i(t) f_i$$

Application to Galerkin methods: Euler–Maruyama

Proposition

- $C \in L(U; L(V_h))$, $Cu = r_d^{-1}(\Delta t A_h) P_h G(\cdot) u$
- $q = \sum_{k=1}^{\infty} \mu_k f_k \otimes f_k \in U^{(2)}$
- $\mathcal{S} = D_{\Delta t,h}^{\det} \otimes D_{\Delta t,h}^{\det} + \Delta t \, (C \otimes C) q \in L(V_h^{(2)})$

$X_{h,e}$ *asymptotically mean-square stable* $\iff$ $\rho(\mathcal{S}) < 1$

Corollary (sufficient condition)

- $(\lambda_{h,k}, k = 1, \ldots, N_h)$ *eigenvalues of* $-A_h$

$$\left(\max_{k=1,\ldots,N_h} |R(\Delta t \lambda_{h,k})| + \max_{k=1,\ldots,N_h} |r_d^{-1}(\Delta t \lambda_{h,k})| \Delta t \|F\|_{L(H)} \right)^2 + \max_{k=1,\ldots,N_h} |r_d^{-1}(\Delta t \lambda_{h,k})|^2 \Delta t \operatorname{tr}(Q) \|G\|_{L(H;L(U;H))}^2 < 1$$

Examples of rational approximations

Proposition (sufficient conditions)

- *(Backward Euler)* $R(z) = \frac{1}{1+z}$

$$\frac{(1 + \Delta t \|F\|_{L(H)})^2 + \Delta t \operatorname{tr}(Q) \|G\|^2_{L(H;L(U;H))}}{(1 + \Delta t \lambda_{h,1})^2} < 1$$

- *(Crank–Nicolson)* $R(z) = \frac{1-z/2}{1+z/2}$

$$\left(\max_{k \in \{1, N_h\}} \left| \frac{1 - \Delta t \lambda_{h,k}/2}{1 + \Delta t \lambda_{h,k}/2} \right| + \Delta t \frac{\|F\|_{L(H)}}{(1 + \Delta t \lambda_{h,1}/2)} \right)^2 + \Delta t \frac{\operatorname{tr}(Q) \|G\|^2_{L(H;L(U;H))}}{(1 + \Delta t \lambda_{h,1}/2)^2} < 1$$

- *(Forward Euler)* $R(z) = 1 - z$

$$\left(\max_{\ell \in \{1, N_h\}} |1 - \Delta t \lambda_{h,\ell}| + \Delta t \|F\|_{L(H)} \right)^2 + \Delta t \operatorname{tr}(Q) \|G\|^2_{L(H;L(U;H))} < 1$$

Application to Galerkin methods: Milstein

$$X_h^{j+1} = \big(R(\Delta t A_h) + r_d^{-1}(\Delta t A_h)(\Delta t P_h F + P_h G(\cdot)\Delta L^j)\big) X_h^j + \int_{t_j}^{t_{j+1}} r_d^{-1}(\Delta t A_h) P_h G \left(\int_{t_j}^{s} G(X_h^j) \mathrm{d}L(r) \right) \mathrm{d}L(s)$$

- $G(G(\psi)f_j)f_i = G(G(\psi)f_i)f_j$
- $L(t) = \sum_{i=1}^{\infty} \sqrt{\mu_i} L_i(t) f_i$
  - $(L_i, i \in \mathbb{N})$ pairwise independent
  - $L_i(t) \in L^4(\Omega; \mathbb{R})$ of finite activity

Application to Galerkin methods: Milstein

Proposition

- $C \in L(U; L(V_h))$, $Cu = r_d^{-1}(\Delta t A_h) P_h G(\cdot) u$
- $q = \sum_{k=1}^{\infty} \mu_k f_k \otimes f_k \in U^{(2)}$
- $C' \in L(U^{(2)}, L(V_h))$, $C'(u_1, u_2) = (r_d^{-1}(\Delta t A_h) P_h G(G(\cdot) u_1) u_2)$
- $q' = \sum_{k,\ell=1}^{\infty} \mu_k \mu_\ell (f_k \otimes f_\ell) \otimes (f_k \otimes f_\ell) \in U^{(4)}$
- $\mathcal{S} = D^{\det}_{\Delta t,h} \otimes D^{\det}_{\Delta t,h} + \Delta t\,(C \otimes C) q + \frac{\Delta t^2}{2} (C' \otimes C') q'$

$X_{h,e}$ *asymptotically mean-square stable* $\iff$ $\rho(\mathcal{S}) < 1$

Application to Galerkin methods: Milstein

Proposition (sufficient conditions)

- *(backward Euler)* $R(z) = \frac{1}{1+z}$
- $(\lambda_{h,k}, k = 1, \ldots, N_h)$ *eigenvalues of* $-A_h$

$$(1 + \Delta t \|F\|_{L(H)})^2 + \Delta t \operatorname{tr}(Q) \|G\|^2_{L(H;L(U;H))} + \frac{\Delta t^2}{2} \operatorname{tr}(Q)^2 \|G\|^4_{L(H;L(U;H))} < (1 + \Delta t \lambda_{h,1})^2$$

(Crank–Nicolson) and *(forward Euler)* with *restrictions* on h, Δt

Connection to analytical solution

$$\mathrm{d}X(t) = (AX(t) + F(X(t)))\,\mathrm{d}t + G(X(t))\,\mathrm{d}W(t), \quad X(0) = X_0$$

- *Q-Wiener process* $W(t) = \sum_{i=1}^{\infty} \sqrt{\mu_i}\beta_i(t) f_i$

Theorem ([Liu 06])

- $X_0 = x_0 \in \dot{H}^1$ *deterministic*
- $\exists c > 0 : \forall u \in \dot{H}^2 :$

$$2\langle u, Au + F(u)\rangle_H + \operatorname{tr}[G(u)Q(G(u))^*] \leq -c\|u\|_H^2$$

$\Longrightarrow X_e$ *asymptotically mean-square stable*

Connection to analytical solution

$$\mathrm{d}X(t) = (AX(t) + F(X(t)))\,\mathrm{d}t + G(X(t))\,\mathrm{d}W(t), \quad X(0) = x_0$$

Backward Euler

$$X_h^{j+1} = (1 + \Delta t A_h)^{-1}(1 + \Delta t P_h F + P_h G(\cdot)\Delta W^j)X_h^j, \quad X_h^0 = P_h x_0$$

Corollary

$$2\left(\|F\|_{L(H)} - \lambda_1\right) + \mathrm{tr}(Q)\|G\|^2_{L(H;L(U;H))} < 0$$

$\Longrightarrow X_e$ *and* $X_{h,e}$ *asymptotically mean-square stable*

Connection to analytical solution

$$\mathrm{d}X(t) = AX(t)\,\mathrm{d}t + G(X(t))\,\mathrm{d}W(t), \quad X(0) = x_0$$

Milstein, backward Euler

$$\begin{aligned} X_h^{j+1} &= (1+\Delta t A_h)^{-1}(1 + P_h G(\cdot)\Delta W^j) X_h^j \\ &\quad + \int_{t_j}^{t_{j+1}} (1+\Delta t A_h)^{-1} P_h G \left(\int_{t_j}^{s} G(X_h^j)\mathrm{d}W(r) \right) \mathrm{d}W(s) \\ X_h^0 &= P_h x_0 \end{aligned}$$

Corollary

$$-\sqrt{2}\lambda_1 + \mathrm{tr}(Q)\|G\|^2_{L(H;L(U;H))} < 0$$

$\Longrightarrow X_e$ *and* $X_{h,e}$ *asymptotically mean-square stable*

Simulation – Stochastic heat equation

$$\mathrm{d}X(t) = \nu\Delta X(t)\,\mathrm{d}t + G(X(t))\,\mathrm{d}W(t), \quad X(0) = X_0$$

- $U = H = L^2([0,1])$
- $X_0(x) = \sqrt{30}\,x(1-x)$
- $Qe_i = \mu_i e_i$, $\mu_i = C_\mu i^{-3}$
- $\nu\Delta e_i = -\lambda_i e_i = -\nu i^2\pi^2 e_i$, $e_i(y) = \sqrt{2}\sin(i\pi y)$
- $G_1(v)v_0 := \sum_{j=1}^{\infty} \langle v, e_j\rangle_H \langle v_0, e_j\rangle_H e_j$

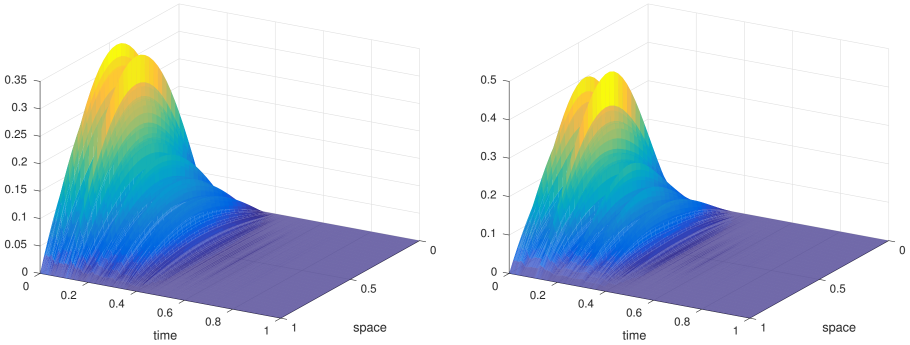
$$X(t) = \sum_{j=1}^{\infty} \langle X_0, e_j\rangle_H \exp\left(-(\lambda_j + \frac{\mu_j}{2})t + \mu_j^{\frac{1}{2}}\beta_j(t)\right) e_j$$

and

$$\mathbb{E}\left[\|X(t)\|_H^2\right] = \sum_{j=1}^{\infty} \langle X_0, e_j\rangle_H^2 \exp\left((-2\lambda_j + \mu_j)t\right)$$

Stochastic heat equation with multiplicative noise

$$\mathrm{d}X(t) = \nu\Delta X(t)\,\mathrm{d}t + G(X(t))\,\mathrm{d}W(t)$$



Spectral Galerkin method

$$X(t) = \sum_{i=1}^{\infty} \langle X(t), e_i \rangle_H e_i = \sum_{i=1}^{\infty} x_i(t) e_i,$$

with *geometric Brownian motions*

$$\mathrm{d}x_i(t) = -\lambda_i x_i(t) \, \mathrm{d}t + \sqrt{\mu_i} x_i(t) \, \mathrm{d}\beta_i(t).$$

Approximation scheme

$$X_h(t) = \sum_{k=1}^{N_h} \langle X(t), e_k \rangle_H e_k = \sum_{k=1}^{N_h} x_k(t) e_k$$

and $x_k(t)$ is approximated by

- Euler–Maruyama, Milstein
- Backward Euler, Crank–Nicolson, forward Euler

Spectral Galerkin method

Euler–Maruyama

$$\begin{aligned}\mathcal{S}(e_k \otimes e_\ell) &= (D^{\det}_{\Delta t,h} \otimes D^{\det}_{\Delta t,h})(e_k \otimes e_\ell) + \Delta t \left((C \otimes C)q\right)(e_k \otimes e_\ell)\\ &= \Lambda_{k,\ell}(e_k \otimes e_\ell)\end{aligned}$$

with

$$\Lambda_{k,\ell} = R(\Delta t\lambda_k)R(\Delta t\lambda_\ell) + \delta_{k,\ell}\,\Delta t\mu_k\, r_d^{-1}(\Delta t\lambda_k)r_d^{-1}(\Delta t\lambda_\ell)$$

Milstein

$$\begin{aligned}\mathcal{S}(e_k \otimes e_\ell) &= \Big(D^{\det}_{\Delta t,h} \otimes D^{\det}_{\Delta t,h} + \Delta t\,(C \otimes C)q + \frac{\Delta t^2}{2}(C' \otimes C')q'\Big)(e_k \otimes e_\ell)\\ &= \Lambda_{k,\ell}(e_k \otimes e_\ell)\end{aligned}$$

with

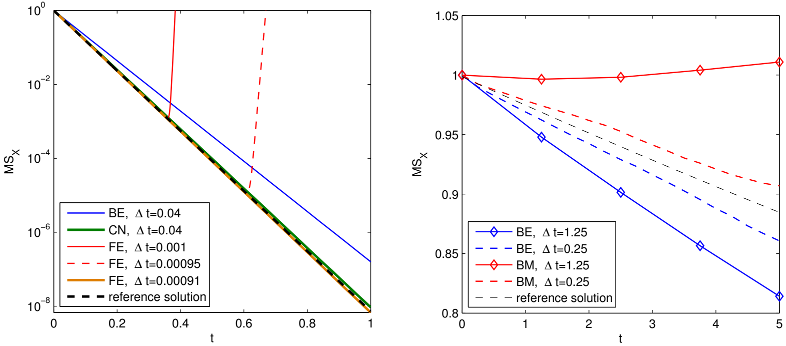
$$\Lambda_{k,\ell} = R(\Delta t\lambda_k)R(\Delta t\lambda_\ell) + \delta_{k,\ell}\, r_d^{-1}(\Delta t\lambda_k)r_d^{-1}(\Delta t\lambda_\ell)\Big(\Delta t\mu_k + \frac{\Delta t^2\mu_k^2}{2}\Big)$$

Spectral Galerkin method

$$\Lambda_{k,\ell} = \begin{cases} R(\Delta t\lambda_k)R(\Delta t\lambda_\ell) + \delta_{k,\ell}\,\Delta t\mu_k\, r_d^{-1}(\Delta t\lambda_k) r_d^{-1}(\Delta t\lambda_\ell) \\ R(\Delta t\lambda_k)R(\Delta t\lambda_\ell) + \delta_{k,\ell}\, r_d^{-1}(\Delta t\lambda_k) r_d^{-1}(\Delta t\lambda_\ell)\left(\Delta t\mu_k + \frac{\Delta t^2\mu_k^2}{2}\right) \end{cases}$$

rational approx./ stochastic approx.	$\Lambda_{k,k}$	$\rho(\mathcal{S})<1 \Leftrightarrow \forall k=1,\dots,N_h:$
backward Euler/EM	$\frac{1+\Delta t\mu_k}{(1+\Delta t\lambda_k)^2}$	$-2\lambda_k+\mu_k-\Delta t\lambda_k^2<0$
backward Euler/Milstein	$\frac{1+\Delta t\mu_k+\Delta t^2\mu_k^2/2}{(1+\Delta t\lambda_k)^2}$	$-2\lambda_k+\mu_k+\Delta t(-\lambda_k^2+\mu_k^2/2)<0$
Crank–Nicolson/EM	$\frac{(1-\Delta t\lambda_k/2)^2+\mu_k\Delta t}{(1+\Delta t\lambda_k/2)^2}$	$-2\lambda_k+\mu_k<0$
forward Euler/EM	$(1-\Delta t\lambda_k)^2+\mu_k\Delta t$	$-2\lambda_k+\mu_k+\Delta t\lambda_k^2<0$

Spectral Galerkin method



Finite element method

$$\mathrm{d}X(t) = \nu\Delta X(t)\,\mathrm{d}t + G(X(t))\,\mathrm{d}W(t), \quad X(0) = X_0$$

- $H = L^2([0,1])$, $U = \dot{H}^1$
- $(G_2(v)v_0)[x] := v(x)v_0(x)$ of *Nemytskii type*
- $h = (N_h + 1)^{-1}$
- $V_h \subset \dot{H}^1$ piecewise linear finite elements

The *eigenvalues* of A_h are

$$\lambda_{h,i} = \frac{4\nu}{h^2}\left(\frac{2}{3} + \frac{1}{3}\cos(i\pi h)\right)^{-1}(\sin(i\pi h/2))^2$$

Finite element method

$$\lambda_{h,i} = \frac{4\nu}{h^2} \left(\frac{2}{3} + \frac{1}{3} \cos(i\pi h) \right)^{-1} (\sin(i\pi h/2))^2$$

$$\|G_2\|_{L(H;L(\dot{H}^1;H))} \leq \left(2 \sum_{i=1}^{\infty} \lambda_i^{-1} \right)^{1/2} = \hat{g}$$

rational approximation	$\rho(\mathcal{S}) < 1 \Leftarrow$
backward Euler	$\rho_{\mathrm{BE}} = \Delta t \operatorname{tr}(Q)\hat{g}^2 - 2\Delta t \lambda_{h,1} - \Delta t^2 \lambda_{h,1}^2 < 0$
Crank–Nicolson	$\rho_{\mathrm{CN}} = \max_{k \in \{1,N_h\}} \left\| \frac{1-\Delta t \lambda_{h,k}/2}{1+\Delta t \lambda_{h,k}/2} \right\|^2 + \frac{\Delta t \operatorname{tr}(Q)\hat{g}^2}{(1+\Delta t \lambda_{h,1}/2)^2} - 1 < 0$
forward Euler	$\rho_{\mathrm{FE}} = \max_{k \in \{1,N_h\}} (1 - \Delta t \lambda_{h,k})^2 + \Delta t \operatorname{tr}(Q)\hat{g}^2 - 1 < 0$

Finite element method

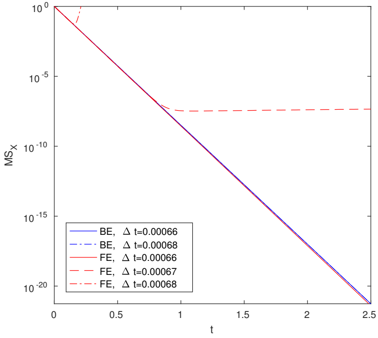
$$\rho_{\mathrm{BE}} = \Delta t \, \mathrm{tr}(Q)\hat{g}^2 - 2\Delta t \lambda_{h,1} - \Delta t^2 \lambda_{h,1}^2$$

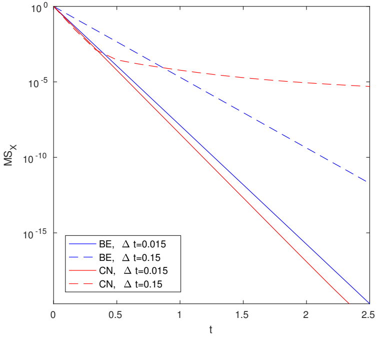
$$\rho_{\mathrm{CN}} = \max_{k \in \{1, N_h\}} \left| \frac{1 - \Delta t \lambda_{h,k}/2}{1 + \Delta t \lambda_{h,k}/2} \right|^2 + \frac{\Delta t \, \mathrm{tr}(Q)\hat{g}^2}{(1 + \Delta t \lambda_{h,1}/2)^2} - 1$$

$$\rho_{\mathrm{FE}} = \max_{k \in \{1, N_h\}} (1 - \Delta t \lambda_{h,k})^2 + \Delta t \, \mathrm{tr}(Q)\hat{g}^2 - 1$$

Δt	ρ_{BE}	ρ_{CN}	ρ_{FE}
0.15	-5.11613e+00	2.08460e-03	1.99602e+05
0.015	-3.13089e-01	-1.58504e-01	1.91542e+03
0.00068	-1.32387e-02	-1.31050e-02	6.09395e-02
0.00067	-1.30434e-02	-1.29135e-02	3.39709e-04
0.00066	-1.28480e-02	-1.27221e-02	-1.27626e-02

Finite element method





Conclusions

- Asymptotic mean-square stability
- Stochastic heat equation with multiplicative noise
- References: `[L., Petersson, Thalhammer 17+, arXiv:soon]`

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Thank you for your attention!