Homogenization problem for random elliptic equations

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Abstract

This poster deals with the homogenization theory for one-dimensional pseudo-elliptic equations with highly oscillatory random coefficients displaying Long-range correlation. We prove that the corrector to homogenization, *i.e.* the difference between the random solution and the homogenized solution, convergence to stochastic integrals with respect to the Hermite process.

Introduction

In the present study, we shall consider the following one-dimensional elliptic equation displaying random coefficients:

$$\begin{cases} P(x,D)u_\varepsilon(x,\omega) + q_\varepsilon(x,\omega)u_\varepsilon(x,\omega) = f(x), & \text{in } (0,1) \\ u_\varepsilon(0,\omega) = u_\varepsilon(1,\omega) = 0, \end{cases} \tag{1}$$

where

- $P(x,D)$ is a deterministic self-adjoint, elliptic, pseudo-differential operator,
- $q(\frac{x}{\varepsilon},\omega)$ is a rescaled version of a bounded stationary stochastic process $q(x,\omega)$ defined on some abstract probability space $\left(\Omega,\mathcal{F},\mathbb{P}\right)$. (For simplicity, we assume that $\mathbb{E}[q(x,\omega)] = 0$)
- the source term $f(x) \in L^2\left((0,1),dx\right)$.

We are interested in the limiting behavior of the solution $u^\varepsilon(x,\omega)$ when $\varepsilon \to 0$ and more precisely in the size of the random fluctuations of $u^\varepsilon(x,\omega)$ and of their limiting distribution after proper rescaling.

Main assumptions

Assumption 1: Stationarity and Ergodicity

There exists an ergodic group of $\mathbb{P}$-preserving transformations $(\tau_x)_{x\in\mathbb{R}}$ on Ω, where ergodicity means that $E \in \mathcal{F}$ and

$$\tau_x E = E, \quad \text{for all } x \in \mathbb{R}$$

imply that $\mathbb{P}(E) \in \{0,1\}$. The random potential $q(y,\omega)$ is given by $\tilde{q}(\tau_y\omega)$ where $\tilde{q} : \Omega \to \mathbb{R}$ is a random variable satisfying

$$0 \le \tilde{q}(\omega) \le M, \quad \text{for all } \omega \in \Omega.$$

The above assumption is sufficient for proving homogenization result and u^ε converges, almost surely in $L^2\left((0,1)\times\Omega\right)$ to the solution of the deterministic homogenized problem:

$$\begin{cases} P(x,D)u_0(x,\omega) = f(x), & x \in (0,1), \\ u_0(0,\omega) = u_0(1,\omega) = 0, \end{cases} \tag{2}$$

Let $\mathcal{G}$ denote the inverse of $P(x,D)$ and let $G(x,y)$ be the Green function associated to $\mathcal{G}$ which is assumed to be non-negative, real valued, symmetric and satisfy the following estimate

$$|G(x,y)| \le \frac{C}{|x-y|^{1-\beta}}, \tag{3}$$

for some universal constant C and some real number $\beta \in (0,1)$, which measures how singular the Green's function is near the diagonal $x = y$. Thus the solution u_0 is given explicitly by

$$u(x) = \mathcal{G}f(x) := \int_0^1 G(x,y)f(y)\,dy,$$

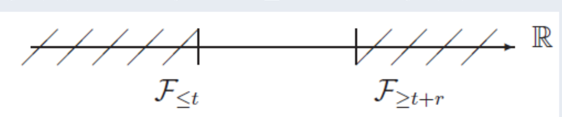
To estimate the size of the homogenization error and to characterize the limiting distribution of the random fluctuation, more assumptions on the random potential $q(.,\omega)$ are necessary. The auto-correlation function $R(x)$ of q is defined as

$$R(x) = \mathbb{E}[q(x+y,\omega)q(y,\omega)], \quad \sigma^2 := \int_{\mathbb{R}} R(x)\,dx.$$

When R is integrable on $\mathbb{R}$, i.e. $\sigma^2 < \infty$, we say that q has short range correlations; we say q has long range correlations if otherwise.

Assumption 2: Short range correlated random potential.

Let $\mathcal{F}_{\le t} := \sigma\{q(x), x \le t\}$, $\quad \mathcal{F}_{\ge t+r} := \sigma\{q(x), x \ge t+r\}$



A standard assumption here is a *strong mixing*: Strong mixing coefficient $\rho(r)$ is a non-negative function s.t.

$$\left|\mathbb{E}(\xi\mu) - \mathbb{E}\xi\mathbb{E}\mu\right| \le \rho(r)\left(\mathrm{Var}\xi\mathrm{Var}\mu\right)^{\frac{1}{2}},$$

for any ξ and μ are $\mathcal{F}_{\le t}$ and $\mathcal{F}_{\ge t+r}$ measurable with finite variance. Assumption: we assume that $\rho(r) \le \mathrm{Cst}\,r^{-\alpha}$ for $\alpha > 1$. In other hand, for any $x \in \mathbb{R}$,

$$|R(x)| = |\mathbb{E}(q(x))(q(0))| \le \rho(r)\mathrm{Var}(q).$$

This implies that the (auto)-correlation function $R(x)$ is integrable.

Assumption 3: Long range correlated potentials constructed from Gaussian fields.

In what follows, we will assume that q has the form

$$q(x,\omega) = \Phi\left(g(x,\omega)\right), \tag{4}$$

where the stochastic process $\left\{g(x)\right\}_{x\in\mathbb{R}_+}$ and the function $\Phi : \mathbb{R} \to \mathbb{R}$ are constructed as follows.

\# *Assumptions on g.*

1. Let $m \in \mathbb{N}^*$ be fixed, let $H_0 \in (1 - \frac{1}{2m}, 1)$, and set $H = 1 + m(H_0 - 1) \in (1/2, 1)$;
2. Fix a slowly varying function $L : (0,+\infty) \to (0,+\infty)$ at $+\infty$. Assume furthermore that L is bounded away from 0 and $+\infty$ on every compact subset of $(0,+\infty)$.
3. Let $e : \mathbb{R} \to \mathbb{R}$ be a square-integrable function such that
   (3a) $\int_{\mathbb{R}} e(u)^2\,du = 1$,
   (3b) $|e(u)| \le Cu^{H_0-\frac{3}{2}}L(u)$ for almost all $u > 0$, for some absolute constant C,
   (3c) $e(u) \sim C_0 u^{H_0-\frac{3}{2}}L(u)$, where $C_0 = \left(\int_0^\infty (u+u^2)^{H_0-\frac{3}{2}}\,du\right)^{-1/2}$,
   (3d) their exist $0 < \gamma < \min\left\{H_0 - (1-\frac{1}{2m}), 1 - H_0\right\}$ such that
   $$\int_{-\infty}^0 |e(u)e(xy+u)|\,du = o(x^{2H_0-2}L(x)^2)y^{2H_0-2-2\gamma}$$
   as $x \to \infty$, uniformly in $y \in (0,t]$ for each given $t > 0$.
4. Finally, let W be a two-sided Brownian motion.

Bearing all these ingredients in mind, we can now set, for $x \in \mathbb{R}_+$,

$$g(x) := \int_{-\infty}^{\infty} e(x-\xi)dW_\xi\ . \tag{5}$$

Exemple: Fractional gaussian noise

$$g_1(x) := W_{H_0}(x) - W_{H_0}(x-1), \quad x \in \mathbb{R}$$

where W_{H_0} is fractional Brownian motion with Hurst index H_0.
\# *Assumptions on Φ.* We assume that $\Phi \in L^2(\mathbb{R},\nu)$ admits the following series expansion

$$\Phi = \sum_{q=0}^{\infty}\frac{V_q}{q!}H_q, \quad \text{with } V_q := \int_{\mathbb{R}}\Phi(x)H_q(x)\nu(dx),$$

and where $H_q(x) = (-1)^q\exp(x^2/2)\frac{d^q}{dx^q}\exp(-x^2/2)$ denotes the qth Hermite polynomial. We assume that $m := \inf\{q \ge 0 : V_q \ne 0\}$ is the *Hermite rank* of Φ (with the convention $\inf\emptyset = +\infty$).

The process $\left\{g(x)\right\}_{x\in\mathbb{R}_+}$, constructed above, exhibits a long-range correlation. In fact, we can show that:

$$R_g(x) := \mathbb{E}\left[g(s)g(s+x)\right] \sim x^{2H_0-2}L(x)^2 \quad \text{as } x \to +\infty.$$

This implies that also the process $q(x,\omega) = \Phi[g(x,\omega)]$ displaying long-range correlation and we have:

$$\left|R_q(x)\right| = \left(o(1) + V_m^2/m!\right)L(|x|)^{2m}|x|^{-2(1-H)},$$

Main results

Results 1: Convergence rate of homogenization.

- *Short range correlation:* Under Assumption 2, the convergence rate is:
$$\mathbb{E}\|u_\varepsilon - u_0\|^2 \le C\|f\|^2 \times \begin{cases} \varepsilon^{2\beta}, & 2\beta < 1 \\ \varepsilon|\log\varepsilon|, & 2\beta = 1 \\ \varepsilon, & 2\beta > 1 \end{cases}$$
- *Long range correlation:* Under Assumption 3 we have, for $2\beta < 1$,
$$\mathbb{E}\|u_\varepsilon - u_0\|^2 \le C\|f\|^2 \times \begin{cases} \varepsilon^{2m(1-H_0)}, & 2m(1-H_0) < 2\beta \\ \varepsilon^{2\beta}|\log\varepsilon|, & 2m(1-H_0) = 2\beta \\ \varepsilon^{2\beta}, & 2m(1-H_0) > 2\beta \end{cases}$$
This bound shows how the competition between the de-correlation rate $m(1-H_0)$ and the Green's function singularity β affects the convergence rate of homogenization.

We formulate the problem for u_ε as follows: $u_\varepsilon = \mathcal{G}(f - q_\varepsilon u_\varepsilon)$, where $\mathcal{G} = (P(x,D))^{-1}$, and thus

$$u_\varepsilon = \mathcal{G}f - \mathcal{G}q_\varepsilon\mathcal{G}f + \mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon u_\varepsilon.$$

Because $u_0 = \mathcal{G}f$, we have

$$\begin{aligned} u_\varepsilon - u_0 &= -\mathcal{G}q_\varepsilon u_0 + \mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon(u_\varepsilon - u_0 + u_0) \\ &= -\mathcal{G}q_\varepsilon u_0 + \mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon u_0 + \mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon(u_\varepsilon - u_0). \end{aligned}$$

Thus, the random corrector problem we deal with will reduce in a careful analysis of the asymptotic behaviour of random quantities of the form

$$\int_{\mathbb{R}} q_\varepsilon(x)h(x)\,dx\ , \tag{6}$$

as $\varepsilon \to 0$, for some suitable (continuous) functions h. This is why we analyze here the convergence of such oscillatory random integrals.

Results 2: Convergence of oscillatory integrals.

- *Short range correlation:* For any $h \in C([0,1])$, we have the following convergence in law:
$$\frac{1}{\sqrt{\varepsilon}}\int_{\mathbb{R}} q^\varepsilon(x)h(x)\,dx \xrightarrow{\varepsilon\downarrow 0} \sigma\int_{\mathbb{R}} h(x)\,dW(x)\,, \tag{7}$$
where W is the standard Wiener process.
- *Long range correlation:* For any $h \in C([0,1])$, we have the following convergence in law:
$$M_h^\varepsilon := \frac{1}{\mathfrak{X}(\varepsilon)}\int_{\mathbb{R}} q^\varepsilon(x)h(x)\,dx \xrightarrow{\varepsilon\downarrow 0} M_h^0 := \frac{V_m}{m!}\int_{\mathbb{R}} h(x)\,dZ(x)\,, \tag{8}$$
where
$$\mathfrak{X}(\varepsilon) = \frac{m!}{\sigma^m K(m,H_0)}\varepsilon^{m(1-H_0)}L(1/\varepsilon)^m, \tag{9}$$
$$K(m,H_0) = \left\{\frac{m!\left[m(H_0-1)+1\right]\left[2m(H_0-1)+1\right]}{\left(\int_0^\infty (u+u^2)^{H_0-\frac{3}{2}}\,du\right)^m}\right\}^{1/2},$$
and Z is the Hermite process of order m and self-similar index $H = m(H_0-1)+1$ (see [4] for the definition).

Now, we use the above convergence to prove the main results in this work

Results 3: Convergence of random corrector's.

\# *More assumptions on Φ and G.*
\- The function Φ satisfies

$$\int_{\mathbb{R}} |\hat{\Phi}(\xi)|\left(1 + |\xi|^3\right)d\xi < \infty, \tag{10}$$

where $\hat{\Phi}$ denotes the Fourier transformation of Φ.
\- We assume that the Green's function $G(x,y)$ is Lipschitz continuous in x with Lipschitz constant $\mathrm{Lip}(G)$ uniform in y.

- *Short range correlation:* Under Assumption 2, the corrector satisfies
$$\frac{u_\varepsilon(x) - u_0(x)}{\sqrt{\varepsilon}} \xrightarrow[\varepsilon\to 0]{\text{distribution}} -\sigma\int_0^1 G(x,y)u_0(y)\,dW_y$$
- *Long range correlation:* Under the above assumption and Assumption 3, the corrector satisfies
$$\frac{u_\varepsilon(x) - u_0(x)}{\mathfrak{X}(\varepsilon)} \xrightarrow[\varepsilon\to 0]{\text{distribution}} -\frac{V_m}{m!}\int_0^1 G(x,y)u_0(y)\,dZ(y).$$

The case of unperturbed oscillatory elliptic equations

Now, we will be interested in the unperturbed equation of (1). We take $P(\frac{x}{\varepsilon},D) = -\frac{d}{dx}a(\frac{x}{\varepsilon},\omega)\frac{d}{dx}$ and we consider the following *one-dimensional* elliptic equation displaying random coefficients:

$$\begin{cases} -\dfrac{d}{dx}\left(a(x/\varepsilon,\omega)\dfrac{d}{dx}u^\varepsilon(x,\omega)\right) = f(x)\,, & x \in (0,1)\,, \quad \varepsilon > 0 \\ u^\varepsilon(0,\omega) = 0\,, \quad u^\varepsilon(1,\omega) = b \in \mathbb{R}, \end{cases} \tag{11}$$

where the random potential $\{a(x)\}_{x\in\mathbb{R}_+}$ is assumed to be a uniformly bounded stationary stochastic process, whereas the function f is assumed to belong to $C([0,1])$. Under ergodic and stationary assumptions on a, the above equations homogenizes i.e. u^ε converges to u^0 which solves the equation with effective coefficient $a^* := 1/\mathbb{E}\left[1/a(0)\right]$.

Let $q^\varepsilon(y) := q(y/\varepsilon) = \frac{1}{a^\varepsilon(y)} - \frac{1}{a^*}\begin{cases} -\text{Short range correlation} \\ -\text{Long range correlation} \end{cases}$

- *Short range correlation:*
$$\frac{u^\varepsilon(x) - u^0(x)}{\sqrt{\varepsilon}} \to \sigma\int_0^1 F(x,y)\,dW_y.$$
- *Long range correlation:*
$$\frac{u_\varepsilon(x) - u_0(x)}{\mathfrak{X}(\varepsilon)} \to \frac{V_m}{m!}\int_0^1 F(x,y)\,dZ(y).$$

Here $F(x,y)$ is some nice function (see [1] for more details).

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