Adaptive time-stepping to control growth

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Joint with : Conall Kelly : UWI

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- Motivation & Taming
- Adaptivity introduction
- General framework for adaptivity & convergence
- Extensions and numerics

Non-convergence: [Hutzenthaler, Jentzen, Kloeden 2011].

$$\text{SDE} \qquad dX = f(X)dt + g(X)dW.$$

Euler-Maruyama method:

$$X_{n+1}^N = X_n^N + hf(X_n^N) + g(X_n^N)(W((n+1)h) - W(nh)).$$

- Drift f and/or diffusion g

  not globally Lipschitz + polynomial growth condition then
  Non-convergence of $\mathbb{E}\|X(t) - X_n\|^2$

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The associated Euler map with stepsize h for deterministic Eq.

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- stable equilibrium solution at 0
- unstable two-cycle at $\left\{\pm \sqrt[\nu]{2/h\beta}\right\}$.

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- Outside of the basin of attraction : oscillation and growth !

Tamed Euler-Maruyama methods

[Hutzenthaler, Jentzen, Kloeden], [Hutzenthaler,Jentzen], [Gyongy, Sabanis, Siska], etc

- Idea : introduce higher order perturbation of the flow

Drift-tamed Euler-Maruyama :

$$Y_{n+1}^N = Y_n^N + \frac{h}{1 + h\|f(Y_n^N)\|} f(Y_n^N) + g(Y_n^N)(W((n+1)h) - W(nh))$$

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Moment bounds

$$\sup_{n \in \mathbb{N}} \sup_{n \in \{0,1,\dots,N\}} \mathbb{E}[\|Y_n^N\|^p] < \infty. \tag{1}$$

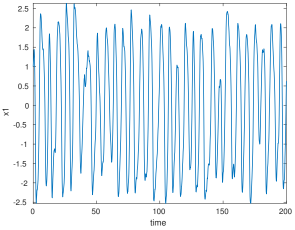
Strong convergence

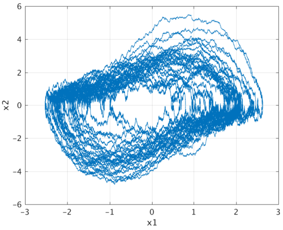
$$\left(\mathbb{E}\left[\sup_{t \in [0,T]} \|X(t) - \bar{Y}_t^N\|^p\right]\right)^{1/p} \leq C_p h^{1/2}$$

- but use a finite h in computations.

Perturbation - large step h

- VdPol equation : True $h = 10^{-4}$





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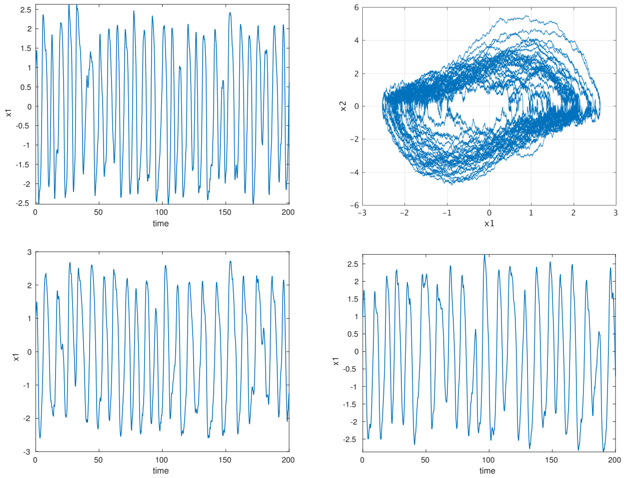

Fixed step approximations : $h = 0.0838$ and $h = 0.1269$
Relative Errors in frequency: ≈ 0.21 & ≈ 0.28

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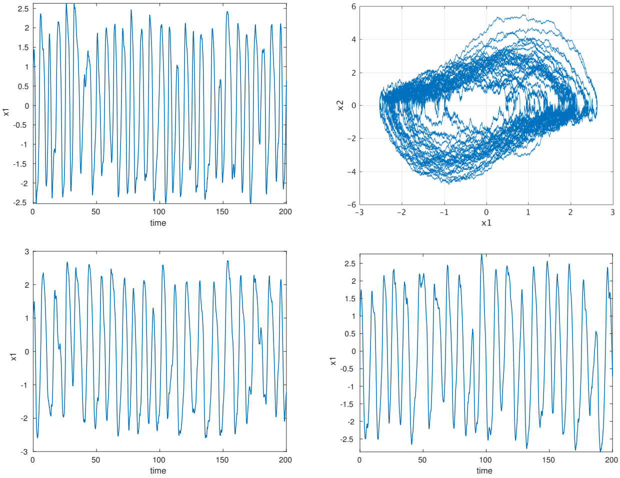

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- Adapt the step. Relative Errors : ≈ 0.09 & ≈ 0.18

SPDE: same issues apply

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- For example $x \in [0, 1]$, $W(t) \in H_0^1(0, 1)$

$$du = \left[\epsilon u_{xx} + u - u^3\right] dt + \sigma dW,$$

$\epsilon = 0.01$, $\sigma = 0.5$
Discretized in space: (Eg FEM)

$$du_h = \left[\epsilon A_h u_h + u_h - u_h^3\right] dt + \sigma dW_h.$$

Large system of SDEs with additive noise : non-convergence.

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Sample solution at $T = 5$. $\Delta t_{ref} = 5 \times 10^{-4}$. 100 SDEs

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- RMS L^2 Error using Fixed step $\Delta t = 0.004555$: 0.157084

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- 2 Ideas to adapt step + general framework

Adaptive time stepping 1 : Tamed

Consider SDE $dX(t) = f(X(t))dt + g(X(t))dW(t)$.
Explicit EM and tamed EM maps associated with the drift are

$$F_h(x) = x + hf(x); \quad \tilde{F}_h(x) = x + \frac{hf(x)}{1 + h\|f(x)\|}.$$

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One approach : apply the Euler map, but at each step to choose a stepsize $h(x)$ so that

$$\|F_h(x_n) - \tilde{F}_h(x_n)\| < \varepsilon. \tag{2}$$

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Then (2) holds iff

$$h < \frac{1}{\|f(x)\|}\left[\frac{\varepsilon + \sqrt{\varepsilon^2 + 4\varepsilon}}{2}\right].$$

Suggest an adaptive stepsize h_{n+1} defined by

$$h_{n+1}(X_n) = \frac{c}{\|f(X_n)\|}\left[\frac{\varepsilon + \sqrt{\varepsilon^2 + 4\varepsilon}}{2}\right]$$

where $c \in (0, 1)$, normally $c = 1/2$.

Adaptive time stepping 2 : Basin

Recall 1D example : $dX = -\beta X|X|^{\nu} dt + g(X)dW$.
The associated Euler map with stepsize h is given by

$$x_{n+1} = F_h(x_n) = x_n - h\beta x_n |x_n|^{\nu}. \tag{3}$$

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- Increase the size of the basin of attraction by choosing h sufficiently small.
- Example. Cubic drift equation : $f(x) = -x^3$.

Basin based adaptation :

$$h_{n+1} = \min\left\{h_{\max}, \frac{1}{2|X_n|^2}\right\}.$$

Taming based adaptation :

$$h_{n+1}(X_n) = \min\left\{h_{\max}, \frac{1}{|X_n|^3}\left[\frac{\varepsilon + \sqrt{\varepsilon^2 + 4\varepsilon}}{4}\right]\right\}$$

For large values of X_n, taming based will select much smaller h_n.

General Framework

$$dX(t) = f(X(t))dt + g(X(t))dW(t), \quad t > 0,$$

Let $(\mathcal{F}_t)_{t\geq 0}$ be the natural filtration of W.
Suppose $f : \mathbb{R}^d \to \mathbb{R}^d$ is continuously differentiable.

$$\|Df(x)\| \leq c(1 + \|x\|^c)$$

and a one-sided Lipschitz condition with constant $\alpha > 0$:

$$\langle f(x) - f(y), x - y \rangle \leq \alpha \|x - y\|^2.$$

For diffusion term : global Lischitz

$$\|g(x) - g(y)\|_F \leq \kappa \|x - y\|.$$

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- Unique strong solution on $[0, T]$, on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.

For each $p > 0$ there is $C = C(p, T, X(0)) > 0$ such that

$$\mathbb{E} \sup_{s\in[0,T]} \|X(s)\|^p \leq C.$$

EM with adaptive step

Euler-type method for SDE over a random mesh $\{t_n\}_{n\in\mathbb{N}}$ on $[0, T]$

$$Y_{n+1} = Y_n + h_{n+1} f(Y_n) + g(Y_n)\left(W(t_{n+1}) - W(t_n)\right)$$

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- $\{h_n\}_{n\in\mathbb{N}}$ sequence of random timesteps: h_{n+1} determined by Y_n.
- Let $\{t_n := \sum_{i=1}^{n} h_i\}_{n=1}^{N}$ with $t_0 = 0$, t_n a $(\mathcal{F}_t)$-stopping time.

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Define discrete-time filtration $\{\mathcal{F}_{t_n}\}_{n\in\mathbb{N}}$ by

$$\mathcal{F}_{t_n} = \{A \in \mathcal{F} \,:\, A \cap \{t_n \le t\} \in \mathcal{F}_t\}, \quad n \in \mathbb{N}.$$

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Let h_n satisfy $h_{\min} < h_n < h_{\max}$ where

$$h_{\max} = \rho h_{\min} \qquad 0 < \rho \in \mathbb{R}$$

- $h_{\min}$ ensures finite number of time steps over $[0, T]$.

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$$h_{\max} = \rho h_{\min} \qquad 0 < \rho \in \mathbb{R}$$

- $h_{\min}$ ensures finite number of time steps over $[0, T]$.
- $h_{\max}$ prevents stepsizes from becoming too large.

Convergence as $h_{\max} \to 0$.

Adaptive timestepping scheme

$$Y_{n+1} = Y_n + h_{n+1}\left[f(Y_n)\mathcal{I}_{\{h_{n+1}>h_{\min}\}} + \frac{f(Y_n)}{1+h_{\min}\|f(Y_n)\|}\mathcal{I}_{\{h_{n+1}=h_{\min}\}}\right] + g(Y_n)\left(W(t_{n+1})-W(t_n)\right), \quad n=0,\ldots,N-1.$$

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- However, if h_{n+1} is an $\mathcal{F}_{t_n}$-stopping time then $W(t_{n+1}) - W(t_n)$ is $\mathcal{F}_{t_n}$-conditionally normally distributed with

$$\begin{aligned} \mathbb{E}\left[\|W(t_{n+1}) - W(t_n)\| \middle| \mathcal{F}_{t_n}\right] &= 0, \quad a.s. \\ \mathbb{E}\left[\|W(t_{n+1}) - W(t_n)\|^2 \middle| \mathcal{F}_{t_n}\right] &= h_{n+1}, \quad a.s. \end{aligned}$$

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\end{aligned}$$

- In practice : replace Wiener increments with i.i.d. $\mathcal{N}(0,1)$ random variables denoted $\{\xi_n\}_{n=1}^N$, scaled at each step by the $\mathcal{F}_{t_n}$-measurable random variable $\sqrt{h_{n+1}}$.

Admissible steps

- *Admissible timestepping strategy* if whenever $h_{\min} \leq h_n \leq h_{\max}$,

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(iii) $h_{n+1} \le \delta\|Y_n\|/(\|f(Y_n)\|)$;
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Proof.

Admissible steps

- *Admissible timestepping strategy* if whenever $h_{\min} \le h_n \le h_{\max}$,

$$\|f(Y_n)\|^2 \le R_1 + R_2\|Y_n\|^2, \quad n = 0, \ldots, N-1.$$

- Lemma : Let $\delta \le h_{\max}$, and c be the constant in bound on Df.

$\{h_n\}_{n\in\mathbb{N}}$ is admissible if, for each $n = 0, \ldots, N-1$, one of the following holds

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and so $R_1 = \rho^2$ and $R_2 = 0$.

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Theorem: Strong Convergence

Let $(X(t))_{t\in[0,T]}$ be solution of the SDE
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Initial value $Y_0 = X_0$.
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There are a.s. finite and $\mathcal{F}_{t_n}$-measurable random variables $\bar{K}_1, \bar{K}_2 > 0$, and constants $K_1, K_2 < \infty$,

$$\mathbb{E}\left\|\int_{t_n}^{t_{n+1}} R_z(s, t_n, X(t_n))ds\right\| \Big| \mathcal{F}_{t_n} \leq \bar{K}_1 h_{n+1}^{3/2}, \quad a.s.$$

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$$\mathbb{E}[\bar{K}_1] \leq K_1, \quad \text{and} \quad \mathbb{E}[\bar{K}_2] \leq K_2.$$

Define the error sequence $\{E_n\}_{n\in\mathbb{N}}$ by $E_{n+1} := Y_{n+1} - X(t_{n+1})$

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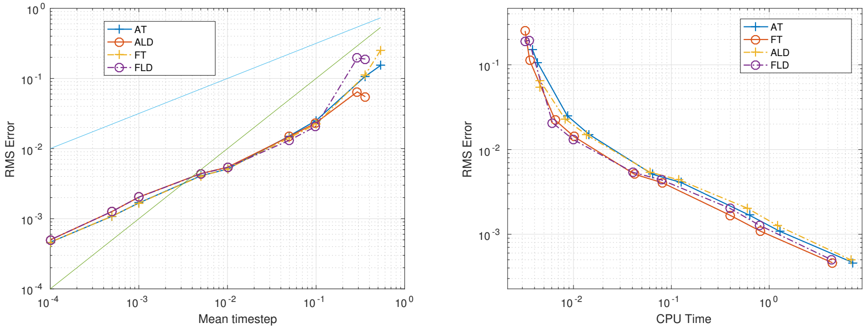
4. Sum, take expectation (Tower property) & discrete Gronwall

Numerical convergence

SDE : SGL equation Multiplicative

$$dX(t) = \left(\left(\eta + \frac{1}{2}\sigma^2\right) X(t) - \lambda X(t)^3\right) dt + \sigma X(t) dW(t)$$



$\rho = 100$. $\eta = 0.1$, $\lambda = 2$ and $\sigma = 0.5$. $T = 2$.

Numerical convergence

SDE : SGL equation Additive

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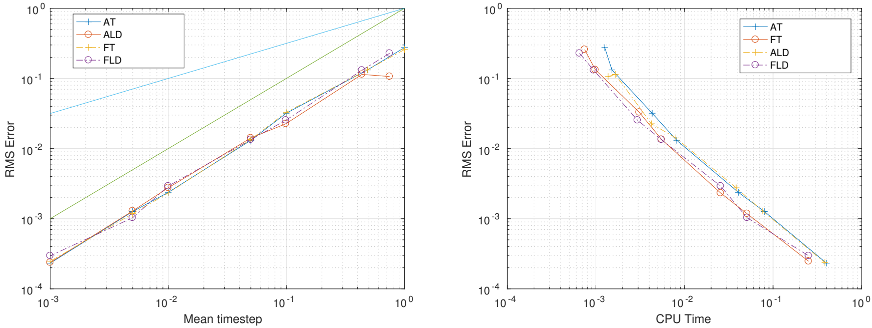

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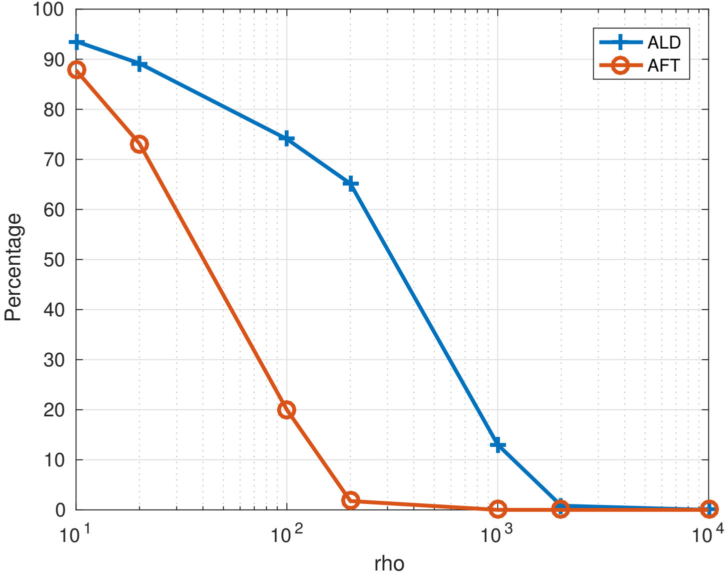
Role of $\rho = h_{\max}/h_{\min}$

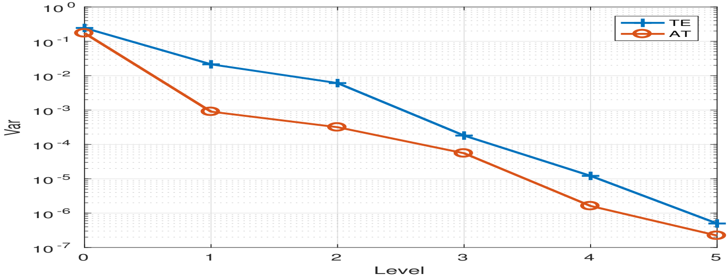


Here $h_{\max} = 2$ and so $h_{\min} = 0.2, \ldots, 0.0002$.

Adaptive time step and MLMC

$$\mathbb{E}(X_L) = \mathbb{E}(X_{L_0}) + \sum_{j=L_0}^{L-1} \mathbb{E}[(X_{j+1}) - (X_j)].$$

Stochastic Ginzburg-Landau (1D) equation : Accuracy $\epsilon = 0.01$
$h^{\ell}_{\max} = h^0_{\max} k^{-\ell}$, with $h^0_{\max} = 1$ and $k = 4$.



[Tempone et al]

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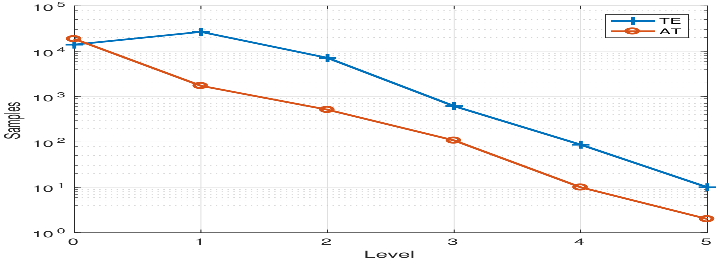

[Tempone et al]

Fang and Giles

$$\langle Y_n, f(Y_n)\rangle + \frac{1}{2}h_{n+1}\|f(Y_n)\|^2 \leq \alpha\|Y_n\|^2 + \beta, \quad n = 0, \ldots, N-1,$$

One sided linear bound $\langle x, f(x)\rangle \leq \alpha\|x\|^2 + \beta$, for $\alpha, \beta > 0$

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Additional upper and lower bounds on each timestep

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- Included in the framework of our proof.

Two Extensions of our Proof

$$\text{SDE} \qquad dX = [AX + f(X)]\,dt + g(X)dW.$$

1. semi-implicit Euler–Maruyama

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More suitable for SPDEs.

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Numerical Results

- SPDEs : 1D, 2D, semi-implicit and multiplicative noise.

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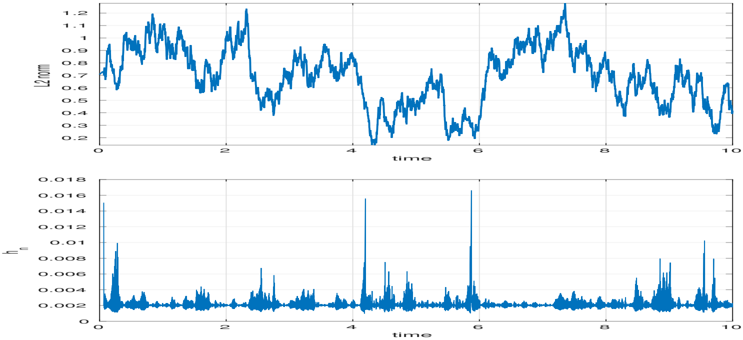

(Explicit time stepping.)

Numerical results : Semi-Implicit adaptive time stepping

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AT = Adaptive Tamed
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$$h_{\min} \leq h_n \leq \max(1, \|X_n\|)/\|f(X_n)\| \leq h_{\max}$$

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H^1	AT	0.009928	0.157084	0.004555
H^1	AM	0.020858	0.968483	0.046765

Reference solution fixed step tamed method with $h = 0.0005$.
100 realizations.

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H^r	Adpt Method	Error Adapt	Error IMPLICIT	hmean
$H^{-1/2}$	AT	0.038453	0.034556	0.001695
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H^1	AT	0.005273	0.034370	0.004605
H^1	AM	0.021265	0.167884	0.046780

Reference solution fixed step tamed method with $h = 0.0005$.
100 realizations.

2D SPDEs additive noise.

Semi-implicit solver.

$$du = \left[\epsilon \Delta u + u - u^3\right] dt + \sigma dW$$

Adpt Method	Error Adapt	Error Fixed	hmean
AT	0.128144	0.130304	0.006200
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$$\text{vorticity} \quad u := \nabla \times \vec{v}$$

$$du = [\varepsilon \Delta u - (\vec{v} \cdot \nabla) u] + \sigma dW \qquad \Delta \psi = -u$$

$\psi(t, \vec{x})$ is scalar stream function, and $\vec{v} = (\psi_y, -\psi_x)$.

Adpt Method	Error Adapt	Error Fixed	hmean
AT	0.136268	0.118946	0.003540
AM	0.112863	0.130985	0.003880

Summary

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4. Extension to diffusion term as SDE system.

- No rejection of steps

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Post-Doc position :
Enabling Quantification of Uncertainty for Inverse Problems
Part of EQUIP Grant with **Prof M. Christie** in IPE - based at Heriot Watt working with Warwick and Imperial.